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Research article

Solvability of a nonlinear integro-differential equation with fractional order using the Bernoulli matrix approach

Raniyah E. Alsulaiman[1], Mohamed A. Abdou[2,*], Eslam M. Youssef[2] and Mai Taha[2]

[1] Department of Mathematics, College of Science, Jouf University, Sakaka 2014, Saudi Arabia
[2] Department of Mathematics, Faculty of Education, Alexandria University, Alexandria 21256, Egypt

* **Correspondence:** Email: abdella_777@alexu.edu.eg; Tel: +201273285125.

Abstract: Under some suitable conditions, we study the existence and uniqueness of a solution to a new modification of a nonlinear fractional integro-differential equation (**NFIDEq**) in dual Banach space C_E (E, [0, T]), which simulates several phenomena in mathematical physics, quantum mechanics, and other domains. The desired conclusions are demonstrated with the use of fixed-point theorems after applying the theory of fractional calculus. The validation of the provided strategy has been done by utilizing the Bernoulli matrix approach (**BMA**) method as a numerical method. The major motivation for selecting the **BMA** approach is that it combines Bernoulli polynomial approximation with Caputo fractional derivatives and numerical integral transformation to reduce the **NFIDEq** to an algebraic system and then derive the numerical solution; additionally, the convergence analysis indicated that the proposed strategy has more precision than other numerical methods. Finally, as a verification of the theoretical work, we apply two examples with numerical results by using [Matlab R2022b], illustrating the comparisons between the exact solutions and numerical solutions, as well as the absolute error in each case is computed.

Keywords: fractional integro-differential equations; Caputo fractional derivative; fixed point theorem; Gauss quadrature formula; Bernoulli matrix approach
Mathematics Subject Classification: 26A33, 34A08, 34A12, 37C25, 45G15

1. Introduction

Fractional calculus is a study of the fractional order of integral and derivative operators. Several experts define the fractional integral and derivative as excellent for modelling the memory and heredity features of diverse substances or systems and other real world challenges. The use of fractional differentiation for the mathematical modelling of real world physical models has been extended in recent years, e.g., the modelling of national disasters, the fluid dynamic traffic model with fractional derivatives, the measurement of viscoelastic material properties, oil pollution, etc. In recent years, scientists have drawn the attention of many applications related to integro-differential equations by improving their results in modelling real-world problems, as seen in Abdou et al. [1,2], Jalili et al. [3–5], Kilbas et al. [6], Miller and Ross [7], Podulbny [8], Zhou et al. [8–10], and Trujillo [11]. Additionally, in recent years, qualitative evaluations of fractional calculus have drawn the interest of several scientists, for example, Tunç et al. [12–14], Bohner et al. [15], and Haibu et al. [16]. Moreover, there are only a few techniques for the approximate solution of fractional integro-differential equations. Some of these methods are: the Adomian decomposition method (ADM), illustrated by Mittal et al. [17], the fractional differential transform method (FDTM), shown by Davoud et al. [18], the collocation method by Yang et al. [19], and the Sumudo transform method (STM), illustrated by Amer et al. [20].

Furthermore, in the previous years, numerous authors examined the existence of solutions of abstract fractional integro-differential equations.

Recently, Baleanu et al. [21] studied the FPDE

$$^{c}D^{v}x(t)=f(t,x(t)),t\in J=[0,T],\ 0<v<1, \tag{1.1}$$

$$x(0)=x(T),x(0)=\beta_1 x(\eta),x(T)=\beta_2 x(\eta),0<\eta<T,0<\beta_1<\beta_2<1.$$

Using fixed-point methods, explored the existence and uniqueness of a solution for the nonlinear fractional boundary value problem raised by Devi and Sreedhar [22] utilised the monotonic iterative technique to the Caputo fractional integro-differential equation of the type

$$^{c}D^{v}x(t)=f(t,x(t),I^{v}x(t)),t\in J=[0,T],\ 0<v<1,\quad x(0)=x_0. \tag{1.2}$$

Dong et al. [23] showed the existence and uniqueness of solutions via Banach and Schauder fixed point techniques for the issue presented by

$$^{c}D^{v}_{0^+}x(t)=f(t,x(t))+\int_0^t G(t,s,x(s))ds,t\in J=[0,T],\ 0<v<1,\quad x(0)=\xi. \tag{1.3}$$

Benchohra et al. [24] examined existence and stability of solutions for a class of boundary value issue for implicit Caputo fractional differential equations of the type

$$^{c}D^{v}x(t)=f(t,x(t),{}^{c}D^{v}x(t)),t\in J=[0,T],T>0,\ 0<v<1,x(0)+g(x)=x_0. \tag{1.4}$$

Hussain [25] developed some additional requirements for the existence and uniqueness of solutions for a Caputo fractional Volterra integro-differential equations with nonlocal conditions of the type

$$ {}^{c}D^{v}x(t)=f\left(t,x(t),\int_{0}^{t}k(t,s)x(s)ds,\int_{0}^{T}h(t,s)x(s)ds\right),t\in J=[0,T],T>0,\ 0<v<1, \tag{1.5}$$

$$x(0)+g(x)=x_{0}.$$

Moreover, Abdou et al. [26] employed the semi-group technique to investigate the existence and uniqueness of solutions for fractional and partial integro differential equations of heat type in Banach space E provided by

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{\partial^{2}u(x,t)}{\partial x^{2}}+\int_{0}^{t}k(x,y)u(x,y)dy+h(x,t),\ t\in J=[0,T],\ x\geq 0,\ 0<\alpha<1,$$

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^{2}u(x,t)}{\partial x^{2}}+\int_{0}^{t}k(x,y)u(x,y)dy+h(x,t),\ u(x,0)=u_{0}(x). \tag{1.6}$$

Motivated by the references [21–26, 31–32] and as a generalized case of the previous Eqs (1.1)–(1.6), we explore the following nonlinear fractional integro-differential equation **NFIDEq**

$$D_{t}^{\alpha}u(x,t)=f\left(x,t,D_{x}^{\beta}u(x,t),\int_{t_0}^{t}g(x,t,s,u(x,s))ds\right), \tag{1.7}$$

with initial condition

$$u^{m}(x,0)=u^{m}(0,t)=u_{m},\ m=0,1,...,n-1,$$

where $x\in[a,b]$, $t\in I=[0,T]$, D_t^{α} and D_x^{β} be standard Caputo fractional derivatives with orders α, β respectively such that $n-1<\beta<\alpha<n$, $n\in N$ and $u(x,t)\in C_{E}(E\times[0,T])$, where C_E (E × [0,T]) be a dual Banach space, the functions $f(x,t)$, $g(x,t)$ are continuous functions identified as

$$(f,g):([a,b]\times I)\times((R\times I)\times(R\times I))\longrightarrow R\times I.$$

Using the previous information, Eq (1.7) can be considered a new modification. The goal of this article is to provide novel results related to the existence and uniqueness solution of **NFIDEq** and also give a numerical solution using the approach of the Bernoulli matrix. The results will be helpful to researchers working on fractional calculus, especially the solvability study of **NFIDEq**, and they provide some new improvements on the topic.

The outline of this article is organized as follows: essential topics are discussed in Section 2. In Section 3, we formulate the sufficient conditions for the existence and uniqueness of a solution to Eq (1.7). The Bernoulli matrix approach **BMA** method is used in Section 4 to obtain the numerical solution of Eq (1.7), and the convergence analysis is also proven. Afterward, in Section 5, we explain numerical examples related to what we introduced in Section 4 to demonstrate the preciseness of the method and also compute the absolute error of the case. Finally, in Section 6, a conclusion is given.

2. Basic concepts

Definition 2.1. (*Odibat et al. [27]***).** Riemann-Liouville fractional integral and derivative operator of order $\alpha \in (C \text{ or } R)$ is respectively defined by:

$$\left(\mathrm{I}_{a+}^{\alpha}\ f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{f(s)}{(x-s)^{1-\alpha}}\,ds, \quad x > a;$$

and

$$\left(\mathrm{D}_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^{n}}{dx^{n}}\int_{a}^{x}\frac{f(s)}{(x-s)^{\alpha-n+1}}\,ds = \frac{d^{n}}{dx^{n}}\left(\mathrm{I}_{a+}^{n-\alpha}\ f\right)(x). \tag{2.1}$$

Where $n = [R(\alpha)+1]$, $R(\alpha)$ indicates the integer part of α, while Caputo fractional derivative of order $\alpha \in (N \text{ or } R)$ is defined by

$$\left(\mathrm{D}_{a+}^{\alpha}\ y\right)(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-s)^{n-\alpha-1}\ y^{(n)}(s)\,ds,$$

or

$$\left(\mathrm{D}_{a+}^{\alpha}\ y\right)(x) = \left(\mathrm{D}^{\alpha}\left(y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^{k}\right)\right)(x). \tag{2.2}$$

Proposition 2.1. (*Burton et al. [28]*). Let $R(\alpha), R(\beta) > 0$, $x \in [a,b]$ and $f(x) \in C[a,b]$, then we have

(i) $\left(\mathrm{I}_{a+}^{\alpha}\ \mathrm{I}_{a+}^{\beta}\ f\right)(x) = \left(\mathrm{I}_{a+}^{\alpha+\beta}\ f\right)(x),$

(ii) $\left(\mathrm{D}_{a+}^{\beta}\ \mathrm{I}_{a+}^{\alpha}\ f\right)(x) = \left(\mathrm{I}_{a+}^{\alpha-\beta}\ f\right)(x),$ for $R(\alpha) > R(\beta),$

(iii) $\left(\mathrm{D}_{a+}^{\alpha}\ \mathrm{I}_{a+}^{\alpha}\ f\right)(x) = f(x),$

(iv) $\left(\mathrm{I}_{a+}^{\alpha}\ \mathrm{D}_{a+}^{\alpha}\ f\right)(x) = f(x) - \sum_{k=1}^{n}\frac{f_{n-\alpha}^{(n-k)}(a)}{\Gamma(\alpha-k+1)}(x-a)^{\alpha-k}.$

Lemma 2.1. (*Burton et al. [28]*). Let $n-1 < \beta < \alpha < n$, for $x \in [a,b]$, if we have $y \in C_E[a,b]$ and

$$\left(\mathrm{D}_{a+}^{\alpha}\ y\right)(x) \in C_E[a,b], \text{ then } \left(\mathrm{D}_{a+}^{\beta}\ y\right)(x) \in C_E[a,b].$$

Lemma 2.2. (*Karthikeyan et al. [29]*). Let $n \in N_0$, the space $C^{n}[a,b]$ is composed of continuous functions, which are represented in the form

$$f(x)=\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\, f^n(t)\,dt + \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{2.3}$$

Definition 2.2. (*Boas et al. [35]***)**. Bernoulli polynomials of order m can be stated as

$$B_m(x)=\sum_{i=0}^{m}\binom{m}{i}B_{m-r}x^i,\ \ x\in[0,1]. \tag{2.4}$$

Where $B_i = B_i(0)$, $i=0,1,2,...m$, are Bernoulli numbers.

Proposition 2.2. (*Boas et al. [35]***).** The standard Bernoulli polynomials are commonly identified by the following relation

$$\begin{cases}\dfrac{d\,B_m(x)}{dx}=mB_{m-1}(x),\ \ \forall m\geq 1.\\ \mathrm{B}_0(\mathrm{x})=1.\end{cases} \tag{2.5}$$

Proposition 2.3. (*Samadi et al. [36]*). Bernoulli polynomials have a full basis on the interval [0, 1].

Definition 2.3. (*Samadi et al. [36]***).** Legendre Gauss quadrature formula can be specifically defined as

$$\int_0^1 g(s)\,ds \ \approx \sum_{i=0}^{N}\omega_i\, g(s_i), \tag{2.6}$$

where $t_i for i = 0,1,2,\ldots N,$ are the roots of the (N+1) Legendre polynomial $P_{N+1}(t)$ in the interval (-1,1) where $p_n(t)=\frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n$ and ω_{i} be the corresponding weights such that $\omega_{\mathrm{i}} = 1\big/(1-t_i^2)P_{N+1}^{\backslash}(t_i)$.

Theorem 2.1. (*Odibat et al. [27]***).** Let X be a Banach space, $S=\{s(t)\}$ be a family of continuous mappings $s: J \to X$. If S is uniformly bounded and equicontinuous, and for any $t^*\in J$, the set S^* is relatively compact, then there exists a uniformly convergent function sequence in S.

Theorem 2.2. (*Odibat et al. [27]***).** Let X be a Banach space, K a convex subset of X, A an open set of K. Suppose that $T:\overline{A}\to K$ is a continuous and compact operator where $\overline{A}$ is closure of A. Then either T has a fixed point in $\overline{A}$, or $\exists\, v\in\partial A$ such that $v=\lambda T v$ for $\lambda\in(0,1)$.

Theorem 2.3. (*Karthikeyan et al. [29]***).** Every contraction mapping on a Banach space admits a unique fixed point.

3. Existence and uniqueness solution of NFIDEq

To verify the existence and uniqueness of the solution of Eq (1.7), we first assume the following conditions:

C₁: $f(x,t)$ is a continuous function and $\exists\, P_1, P_2 \in R^+$ such that

$\left|f\left(s,t,u,v\right)\right|\le P_1\left(|u|+|v|\right)+P_2$, where $s\in[a,b]$, $t\in I$ and $u,v\in R$;

C2: $g(x,t)$ is a continuous function and $\exists\, q_1,q_2\in R^+$ such that

$\left|g\left(s_1,t,s_2,u\right)\right|\le q_1|u|+q_2$, where $s_1,s_2\in[a,b], t\in I, u\in R$;

C3: there exist $N_1\in R^+$ such that

$|f(s,t,u,v)-f(s,t,w,z)|\le N_1(|u-w|+|v-z|)$, where $u,v,w,z\in R$, $s\in[a,b]$, $t\in I$;

C4: there exist $N_2\in R^+$ such that

$\left|g\left(s_1,t,s_2,u\right)-g\left(s_1,t,s_2,v\right)\right|\le N_2|u-v|$, where $u,v\in R$, $N_2=\left|\mathrm{N}(s_1,t,s_2,\nu)\right|$.

Before proving the theory of existence and uniqueness of the solution, we must prove the following principle lemma.

Lemma 3.1. If $u\in C_E(E\times[0,T])$, then $u(x,t)$ can be written in the following form $u(x,t)=\frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\frac{y(x,s)}{(t-s)^{1-\beta}}ds+\sum_{m=0}^{n-1}\frac{u^{(m)}(x,t_0)}{m!}(t-t_0)^m$, where; $y(x,t)\in C_E(E\times[0,T])$ satisfy the fractional integral equation

$$y(x,t)=I^{\alpha-\beta}\left(f\left(x,t,y(x,t),\int_{t_0}^{t}g\left(x,t,s,\sum_{m=0}^{n-1}\frac{u^{(m)}(x,t_0)}{m!}(s-t_0)^m+I^{\beta}u(x,s)ds\right)\right)\right).$$

Proof. Consider $u(x,t)\in C_E(E\times[0,T])$ be a solution of Eq (1.7), from Lemma 2.1, we have that

$D_t^{\alpha}u(x,t)\in C_E(E\times[0,T])$,

Then

$$D_t^{\alpha}u(x,t)=\frac{\partial^n}{\partial t^n}\left(I^{n-\alpha}\left(u(x,t)-\sum_{m=0}^{n-1}\frac{u^{(m)}(x,t_0)}{m!}(t-t_0)^m\right)\right).$$

Using Lemma 2.2, we get

$$I^{\alpha}D_t^{\alpha}u(x,t)=\left(I^{\alpha}D^{\alpha}\left(u(x,t)-\sum_{m=0}^{n-1}\frac{u^{(m)}(x,t_0)}{m!}(t-t_0)^m\right)\right),$$

$$=u(x,t)-\sum_{m=0}^{n-1}\frac{u^{(m)}(x,t_0)}{m!}(t-t_0)^m-\sum_{m=0}^{n-1}\frac{u_{n-\alpha}{}^{(n-m)}(x,t_0)}{m!}(t-t_0)^{\alpha-m},$$

where,

$$u_{n-\alpha}=I^{n-\alpha}\left(u(x,t)-\sum_{m=0}^{n-1}\frac{u^{(m)}(x,t_0)}{m!}(t-t_0)^m\right).$$

But

$$u_{n-\alpha}{}^{(m)}(x,t_0)=o, \, for\, m=0,1,\ldots,n-1. \tag{3.1}$$

So

$$I^{\alpha}\ D_t^{\alpha}\ \mathrm{u(x,\,t)}\ =\ u\ \mathrm{(x\,,\,t)}\ -\sum_{\mathrm{m=0}}^{\mathrm{n-1}}\frac{\mathrm{u^{(m)}\ (x\,,\,t_0)}}{\mathrm{m\,!}}\ \mathrm{(t-t_0)^m}. \tag{3.2}$$

From Lemma 2.2 we get
$D_x{}^{\beta}u(x,t)\in C_E(E\times[0,T])$, so, we can apply I^{α} to both sides of Eq (1.7) and using Eq (3.2) we obtain

$$\mathrm{u\,(x,\,t)}\ =\ \sum_{\mathrm{m=0}}^{\mathrm{n-1}}\frac{\mathrm{u^{(m)}\ (x\,,\,t_0)}}{\mathrm{m\,!}}\ \mathrm{(t-t_0)^m}\ +\ \left[\mathrm{I}^{\alpha}\ \mathrm{f}\left(\mathrm{x,\,t,\,D}_x^{\beta}\ \mathrm{u(x,\,t)},\ \int_{\mathrm{t_0}}^{\mathrm{t}}\mathrm{g\,(x,\,t,\,s,\,u(x,\,s))\ ds}\right)\right]. \tag{3.3}$$

Set $y(x,t)=D_x{}^{\beta}u(x,t)$, then $y\in C_E(E\times[0,T])$ and similar to Eq (3.3) we obtain

$$u\mathrm{(x,t=)}\sum_{\mathrm{m=0}}^{\mathrm{n-1}}\frac{\mathrm{u^{(m)}(x,}t_0)}{\mathrm{m\,!}}\ \mathrm{(t-t_0)^m}\ +\ \frac{1}{\Gamma\,(\beta)}\int_{\mathrm{t_0}}^{\mathrm{t}}\frac{\mathrm{y\,(x\,,\,t)}}{\mathrm{(t-s)}^{1-\beta}}\ \mathrm{ds}. \tag{3.4}$$

From Lemma 2.2, we have

$$D_x{}^{\beta}u(x,t)=\left(D_x{}^{\beta}\sum_{m=0}^{n-1}\frac{u_m}{m\,!}\,(t-t_0)^m\right)(x,t)+I^{\alpha-\beta}\left(f\left(x,t,D_x^{\beta}u(x,t),\int_{t_0}^{t}g(x,t,s,u(x,s))ds\right)\right),$$

using Lemma 2.1 and Eq (3.4), we get

$$y(x,t)=I^{\alpha-\beta}\left(f\left(x,t,y(x,t),\int_{t_0}^{t}g\left(x,t,s,\sum_{m=0}^{n-1}\frac{u^{(m)}(x,t_0)}{m\,!}(s-t_0)^m+I_x{}^{\beta}y(x,s)\right)ds\right)\right). \tag{3.5}$$

Now, we will show that $u^{(m)}(x,0)=u_m$, for $m=0,1,2,\ldots n-1$. For n = 1, it is easy to see that $u(x,0)=u_0$.
Now for $n\geq 2$, using Proposition 2.1 and Eq (3.2) we get

$$u(x,t)=\sum_{m=0}^{n-2}\frac{u^{(m)}(x,t_0)}{m\,!}(x-t_0)^m+I^{m-1}\left(u_{m-1}+I^{\alpha-m+1}f\left(x,t,D_x^{\beta}u(x,t),\int_{t_0}^{t}g\left(x,t,s,u(x,s)\right)ds\right)\right).$$

Thus, from Lemma 2.1 we have
$u(x,t)\in C_E(E\times[0,T])$ and $u^{(m)}$ (x, 0) = u_m, for $m=0,1,2,\ldots n-2$.
At $m=n-1$, we have

$$u^{(n-1)}(x,t_0)=u_{n-1}+I^{\alpha-m+1}f\left(x,t,D_x^{\beta}u(x,t),\int_{t_0}^{t}g\left(x,t,s,u(x,s)\right)ds\right).$$

Using proposition 2.1 we obtain

$$I^{\alpha-m+1}f\left(x,t,D_x^{\beta}u(x,t),\int_{t_0}^{t}g\left(x,t,s,u(x,s)\right)ds\right)=0.$$

Then $u^{(n-1)}(x,0)=u_{n-1}$. ■

Theorem 3.1. Assume that the conditions (C_1–C_4) hold, then Eq (1.7) has a solution

$$u(x,t)\in C_E(E\times[0,T]) \ \text{ if}\ \mu :=\frac{N_1\,(\text{b - a})^{\alpha-\beta}}{\Gamma\,(\alpha\text{ - }\beta+1)}+\frac{N_2\,(\text{b - a})^{\alpha+1}}{\Gamma\,(\alpha+2)}<1. \tag{3.6}$$

Proof. From Lemma 3.1, it is sufficient to show that Eq (1.7) have a solution $u(x,t)\in C_E(E\times[0,T])$.

Using Eq (3.4) to define $(T\ y)(x,t)$ as

$$(T\ y)(x,t)=I^{\alpha-\beta}\left(f\left(x,t,y(x,t),\int_{t_0}^{t}g\left(x,t,s,\sum_{m=0}^{n-1}\frac{u^{(m)}}{m!}(s-t_0)^m+I^{\beta}y(x,s)\right)ds\right)\right)(x,t).$$

Also, set $r=\alpha-\beta,\ s.t\ n-1\prec\alpha-\beta\prec n$.
Consider $\overline{\mathrm{B}_\mathrm{r}}=\left\{\mathrm{y}\in \mathrm{C}_E\ (\mathrm{E}\times[0,\mathrm{T}];\ \|\mathrm{u}\|_{C_E}<\mathrm{r}\right\}$.
Now we need to show that $T:\overline{\mathrm{B}_\mathrm{r}}\to \mathrm{C}_\mathrm{E}$ is a continuous and compact operator.

From the continuity of functions $f(x,t)$ *and* $g(x,t)$ *and* the operators $I^{\alpha-\beta},I^{\beta}$ on $C[a,b]$, it is easy to say that $(T\ y)(x,t)\in C_E(E\times[0,T])$, for $y\in\overline{B}_r$.

Consider $y,z\in\overline{B}_r$ and using conditions C_3, C_4 we obtain

$$\left|(\mathrm{T\ y})(\mathrm{x,t})-(\mathrm{T\ z})(\mathrm{x},\mathrm{t})\right|\le\left(I^{\alpha-\beta}N_1\left|y(x,t)-z(x,t)\right|+I^{\beta}N_2\left|y(x,s)-v(x,s)\right|\right),$$

$$\le\|y-z\|_{C_E}\left(I^{\alpha-\beta}N_1(x,t)+I^{\alpha+\beta}N_2(x,t)\right),$$

$$\le\|y-z\|_{C_E}\left(\frac{N_1(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{N_2(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}\right).$$

Set $\mu=\dfrac{N_1(b-a)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\dfrac{N_2(b-a)^{\alpha+1}}{\Gamma(\alpha+2)}$, therefore we find that T is a continuous operator on $\overline{\mathrm{B}_\mathrm{r}}$ when $\mu<1$. The set $A=\{Tu:u\in\overline{B}_r\}$ is uniformly bounded where $\|\ T\ y\ \|\le\ \mathrm{M}$, such that

$$M=\sup\{|f(s,t,u,v|)\}. \tag{3.7}$$

Let $\theta_1, \theta_2 \in [0, T]$ and $y \in \bar{B}_r$, we have

$$\left| (\mathrm{T\ y})(\mathrm{x}, \theta_1) - (\mathrm{T\ y})(\mathrm{x}, \theta_2) \right| \le \frac{1}{\Gamma(\alpha-\beta)} \left| \int_{t_o}^{\theta_1} \frac{Tu(x,s)}{(\theta_1 - s)^{1-\alpha+\beta}} ds - \int_{t_o}^{\theta} \frac{Tu(x,s)}{(\theta_2 - s)^{1-\alpha+\beta}} ds \right| + \frac{1}{\Gamma(\alpha-\beta)} \left| \int_{\theta_1}^{\theta_2} \frac{Tu(x,s)}{(\theta_2 - \theta_1)^{1-\alpha+\beta}} ds \right|, \ for\ \theta_1 \le \theta_2.$$

From Eq (3.6) we have

$$\left| (\mathrm{T\ y})(\mathrm{x}, \theta_1) - (\mathrm{T\ y})(\mathrm{x}, \theta_2) \right| \le \frac{M\left[(\theta_2 - \theta_1)^{\alpha-\beta} + (\theta_1 - t_0)^{\alpha-\beta} - (\theta_2 - t_0)^{\alpha-\beta} \right]}{(\alpha-\beta)\Gamma(\alpha-\beta)} + \frac{M\left[(\theta_2 - \theta_1)^{\alpha-\beta} \right]}{(\alpha-\beta)\Gamma(\alpha-\beta)},$$

$$\le \frac{2M\left[(\theta_2 - \theta_1)^{\alpha-\beta} \right]}{\Gamma(\alpha-\beta+1)}. \tag{3.8}$$

Then $T : \overline{\mathrm{B_r}} \to \mathrm{C_E}$ is uniformly bounded and equicontinuous; also, E is a relatively compact subset of C_E (E × [0, T]), and hence we can say that the operator T satisfies Arzela Weierstrass's theory (Theorem 2.1). If we show that $u = \lambda Tu$ doesn't have any solution in ∂B_r for some λ, then by (Theorem 2.2), T has a fixed point in $\overline{\partial B_r}$. Hence, the Eq (1.7) has a solution

$$u(x,t) \in C_E(E \times [0,T]). \blacksquare$$

Theorem 3.2. Equation (1.7) has a unique solution $\mathrm{u(x,t)} \in \mathrm{C}_E (\mathrm{E} \times [0,\mathrm{T}]$.

Proof. We can prove the uniqueness of the solution of Eq (1.7) using Banach contraction principle as follow.

Using the conditions (C_1–C_4), and Eq (3.7) we have

$$\left| (\mathrm{T\ y})(\mathrm{x}, t) - (\mathrm{T\ z})(\mathrm{x}, t) \right| \le \left(I^{\alpha-\beta} N_1 \left| y(x,t) - z(x,t) \right| + I^{\alpha+\beta} N_2 \left| y(x,s) - z(x,s) \right| \right),$$

$$\le \left\| y - z \right\|_{C_E} \left(I^{\alpha-\beta} N_1(x,t) + I^{\alpha+1} N_2(x,t) \right),$$

$$\le \mu \left\| y - z \right\|_{C_E}, \ for\ \mu \in (0,1).$$

Thus, T is a contraction operator, from the Banach Contraction Principle (Theorem 2.3), we get the existence and uniqueness of the solution of the Eq (1.7). ■

4. Bernoulli matrix approach (BMA)

In this Section, we present a computational approach for solving the **NFIDEq** which is based on the Bernoulli polynomials approximation. (See Tohidi et al. [33] and Hassani et al. [34]).

We need to approximate the solution of $F(x,t) \in C_E(E \times [0, 1])$ by the truncated Bernoulli series $F(x,t) = \sum_{m=0}^{N} \sum_{n=0}^{N} f_{m,n} B_m(x) B_n(t)$, where the coefficients $f_{m,n}$ extracted a

$$f_{m,n}=\frac{1}{m!n!}\int_0^1\int_0^1\frac{\partial^{m+n}F(x,t)}{\partial x^m\partial t^n}dxdt. \tag{4.1}$$

Lemma 4.1. Let $D_t^{\alpha}u(x,t)$ be approximated by the Bernoulli polynomials as

$$D_t^{\alpha}u(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}B_m(x)B_n(t),\text{ suppose } 0\prec\alpha\prec 1,$$

then we have

$$u(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b_{n,r}^{\backslash}t^{r+\alpha}B_m(x)+\sum_{i=0}^{n-1}u^{(i)}(x,0)\frac{t^i}{i!}.$$

Proof. Applying operator I_t^{α} on both sides of $D_t^{\alpha}u(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}B_m(x)B_n(t)$, we have

$$u(x,t)-\sum_{i=0}^{n-1}u^{(i)}(x,0)\frac{t^i}{i!} = I_t^{\alpha}\left[\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}B_m(x)B_n(t)\right],$$

$$=I_t^{\alpha}\left[\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}\left(\sum_{r=0}^{m}\binom{n}{r}B_{n-r}t^r\right)B_m(x)\right],$$

$$=\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}\left(\sum_{r=0}^{m}\binom{n}{r}B_{n-r}\frac{\Gamma(r+1)}{\Gamma(r+1+\alpha)}t^{r+\alpha}\right)B_m(x),$$

$$=\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b_{n,r}^{\backslash}x^{r+\alpha}B_m(x).$$

Where $b_{n,r}^{\backslash}=\binom{n}{r}B_{n-r}\frac{\Gamma(r+1)}{\Gamma(r+1+\alpha)}$, then we obtain

$$u(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b_{n,r}^{\backslash}t^{r+\alpha}B_m(x)+\sum_{i=0}^{n-1}u^{(i)}(x,0)\frac{t^i}{i!}. \tag{4.2}$$

4.1. Numerical solution of Eq (1.7) using BMA

The BMA approach turns the **NFIDEq** to a system of algebraic equations by extending the relevant approximate solutions as the linear combination of the Bernoulli polynomials.

According to Lemma 4.1, we can get an approximation solution of Eq (1.7) as follow

$$\mathrm{u}(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b^{\backslash}_{n,r}t^{r+\alpha}B_m(x)+\sum_{i=0}^{n-1}u^{(i)}(x,0)\frac{t^i}{i!}=$$

$$f\left(x,t,y(x,t),\int_0^1 \mathrm{g}(\mathrm{x,\,t,\,s},\ \sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b^{\backslash}_{n,r}s^{r+\alpha}B_m(x)\ )\,\mathrm{ds}\right),\quad (4.3)$$

where $y(x,t)=\mathrm{D}_x^{\beta}\ \mathrm{u(x,\,t)}\ =\ \sum_{i=0}^{M}\sum_{j=0}^{M}u_{i,j}B_i(x)B_j(t).$

Furthermore, we apply Legendre Gauss collocation nodes and also Legendre Gauss quadrature rule for approximating the existing integrals. By collocating the Eq (4.3) at (N+1) points x_p s.t $0=x_0<x_1<x_2<...<x_p<..x_N=1$, then we have

$$\mathrm{u}(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b^{\backslash}_{n,r}t^{r+\alpha}B_m(x_p)+\sum_{i=0}^{n-1}u^{(i)}(x_p,0)\frac{t^i}{i!}\ ,$$

$$=f\left(x,t,y(x_p,t),\int_0^1 \mathrm{g}(x_p,\mathrm{\,t,\,s},\ \sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b^{\backslash}_{n,r}s^{r+\alpha}B_m(x_p)\ )\,\mathrm{ds}\right).\quad (4.4)$$

Where $x_p\ \ p=0,12,...N$, indicates the roots of the shifted Legendre polynomial $P_{N+1}(x)$ in the interval (0,1). Also to apply the Legendre Gauss quadrature for estimating the following equation involves integrals, we should convert s-interval [0, 1] into τ-interval [−1, 1] by the following change of variable: $\tau=2s-1$, then Eq (4.4) will be transformed to

$$\mathrm{u}(\mathrm{x_p},\mathrm{t_i})=\sum_{m=0}^{N}\sum_{n=0}^{N}c_{m,n}b^{\backslash}_{n,r}t^{r+\alpha}B_m(x_p)+\sum_{i=0}^{n-1}u^{(i)}(x_p,0)\frac{t^i}{i!},$$

$$=f\left(x_p,t_i,y(x_p,t),\int_{-1}^{1} \mathrm{g}(x_p,\mathrm{t_i},\ \frac{1}{2}(\tau+1),(\frac{1}{2})^{r+\alpha-1}\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b^{\backslash}_{n,r}(\tau+1)^{r+\alpha}B_m(x_p)\ )\,\mathrm{d}\tau\right),$$

$$=f\left(x_p,t_i,y(x_p,t),\sum_{i=0}^{N}\omega_{\mathrm{i}}\,\mathrm{g}(x_p,\mathrm{t_i},\ \frac{1}{2}(\tau_i+1),(\frac{1}{2})^{r+\alpha-1}\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b^{\backslash}_{n,r}(\tau_i+1)^{r+\alpha}B_m(x_p)\ )\right).$$

(4.5)

Where all of the τ_q's are the N+1 zeroes of the Legendre polynomial P $_{N+1}$ (τ) and the w_q's are the corresponding weights. The solutions of the nonlinear algebraic system (4.5) are the coefficients of the truncated double Bernoulli series, which are defined in the interval [0, 1].

4.2. Convergence analysis of the nonlinear algebraic system (4.5)

Also, convergence analysis associated to the presented idea is provided as follow
For $u(x_p,t_i)\in\ \mathrm{c_E}([0\,,\,1]\times[0\,,\,1])$, $\mathrm{x}\in[0,1]$, define an operator
$A:c_{\mathrm{E}}\ ([0\,,\,1]\times[0\,,\,1])\ \rightarrow\ \mathrm{c_E}\ ([0\,,\,1]\times[0\,,\,1])$, such that

$$\mathrm{A\,u}(x_p,t_i)=\mathrm{I}^{\alpha-\beta}\left(f\left(x_p,t_i,y(x_p,t_i),\sum_{i=0}^{N}\omega_i\,g\left(x_p,t_i,\frac{1}{2}(\tau_i+1),\frac{1}{2}^{r+\alpha-1}\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b^{\backslash}_{n,r}(\tau_i+1)^{r+\alpha}B_m(x_p)\right)\right)\right).$$

According to conditions (C_1–C_4) which listed above, we get

$$\left\|\mathrm{A\,u}(x_p,t_i)\right\|=\mathrm{I}^{\alpha-\beta}\left|f\left(x_p,t_i,y(x_p,t_i),\sum_{i=0}^{N}\omega_i\,g\left(x_p,t_i,\frac{1}{2}(\tau_i+1),(\frac{1}{2})^{r+\alpha-1}\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b^{\backslash}_{n,r}(\tau_i+1)^{r+\alpha}B_m(x_p)\right)\right)\right|,$$

$$\leq\mathrm{I}^{\alpha-\beta}\left(P_1\left(\left|y(x_p,t_i)\right|+\sum_{i=0}^{N}\left|\omega_i\,g\left(x_p,t_i,\frac{1}{2}(\tau_i+1),(\frac{1}{2})^{r+\alpha-1}\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b^{\backslash}_{n,r}(\tau_i+1)^{r+\alpha}B_m(x_p)\right)\right|\right)+P_2\right),$$

$$\leq\mathrm{I}^{\alpha-\beta}\left(P_1\left(\left|y(x_p,t_i)\right|+q_1\left|(\frac{1}{2})^{r+\alpha-1}\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b^{\backslash}_{n,r}(\tau_i+1)^{r+\alpha}B_m(x_p)\right|\right)+q_2\right),$$

$$\leq\mathrm{I}^{\alpha-\beta}\left(P_1\left(\left|y(x_p,t_i)\right|+q_1\left|y(x_p,\tau_i)\right|\right)+q_2\right)\leq\eta_{p,i}.$$

Consider $u(x_p,t_i),v(x_p,t_i)\in c_E([0,1]\times[0,1])$, where $x_p\in[0,1]$. Then we obtain

$$\left\|\mathrm{A\,u}(x_p,t_i)-\mathrm{Av}(x_p,t_i)\right\|=$$

$$\mathrm{I}^{\alpha-\beta}\left|f\left(x_p,t_i,y(x_p,t_i),\sum_{i=0}^{N}\omega_i\,g\left(x_p,t_i,\frac{1}{2}(\tau_i+1),(\frac{1}{2})^{r+\alpha-1}\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b^{\backslash}_{n,r}(\tau_i+1)^{r+\alpha}B_m(x_p)\right)\right)-\right.$$

$$\left.f\left(x_p,t_i,z(x_p,t_i),\sum_{i=0}^{N}\omega_i\,g\left(x_p,t_i,\frac{1}{2}(\tau_i+1),(\frac{1}{2})^{r+\alpha-1}\sum_{m=0}^{N}\sum_{n=0}^{N}u_{m,n}b^{\backslash}_{n,r}(\tau_i+1)^{r+\alpha}B_m(x_p)\right)\right)\right|,$$

$$\leq\mathrm{I}^{\alpha-\beta}\left(P_1\left(\left|y(x_p,t_i)-z(x_p,t_i)\right|+q_1\left|y(x_p,\tau_i)-z(x_p,\tau_i)\right|\right)\right),$$

$$\leq\eta_{p,i}\|u-v\|_{C_E}. \tag{4.6}$$

Thus, A is a contraction operator in the case of $\eta_{p,i}\in(0.1)$, and from the Banach contraction. Principle, we get the existence of a unique solution for the Eq (1.7).

5. Numerical examples

In this Section, we present two numerical examples to make a verification of the theoretical work which presented in Section 4 by using **BMA** method.

Example 5.1. Consider the following nonlinear fractional integro-differential equation

$$D_t^{\alpha}u(x,t)=\frac{x^3}{e^t}D_x^{0.4}u(x,t)+\int_0^t\frac{x-s}{1+s}\tan\left(u(x,s)\right)ds, \tag{5.1}$$

with initial condition $u(o,x)=u(0,t)=0,\ x\in(0,1)$.

Observe that example (5.1) is a special case of Eq (1.7) with β =0.4 and functions f and g determined as: $\mathrm{f(x,t,y,v)}=\frac{\mathrm{x}^3}{\mathrm{e}^{\mathrm{t}}}$ y + v, y, v ∈R and $g(x,t,y,v)=\frac{x-t}{1+t}tany$. The exact solution of Eq (5.1) is $u(x,t)=x^2e^t$. The results of exact solution, approximate solutions and the absolute error between them are obtained in Table 1.

Table 1. Represents the exact solution and approximate solutions for example (5.1) at α = 0.5 and α = 0.95.

x	t	Exact Solution	$\alpha = 0.5$		$\alpha = 0.95$	
			Numerical solution	Abs. Error	Numerical solution	Abs. Error
0.2	0.33	0.055638725	0.0556387189	6.02e−08	0.0556387238	1.2e−08
	0.66	0.077391693	0.0773900142	1.55e−06	0.0773916847	8.3e−08
	0.99	0.107649378	0.1076492547	1.24e−07	0.1074693216	5.72e−07
0.4	0.33	0.22255490	0.222534436	2.04e−05	0.22253443435	3.2 e−07
	0.66	0.30956677	0.3095698121	3.04e−06	0.30956981232	4.7 e−09
	0.99	0.430597515	0.430597516	1.12e−09	0.4305975121	9.1 e−09
0.6	0.33	0.500748526	0.5007435324	4.99e−06	0.50074353232	5.2 e−08
	0.66	0.696525240	0.6965352467	1.01e−07	0.69653524672	5.76 e−09
	0.99	0.968844410	0.968844364	4.62e−08	0.96884436443	1.34 e−10
0.8	0.33	0.980219602	0.980214141	5.46e−08	0.98021414098	9.44 e−7
	0.66	1.238267094	1.2382670076	8.64e−08	1.23826700753	2.11 e−8
	0.99	1.722390067	1.72239006007	6.93e−09	1.722390060069	5.87 e−10

Example 5.2. Consider the following nonlinear fractional integro-differential equation

$$D_t^{\alpha} u(x,t) = \frac{x}{4}\cos\left(D_x^{0.02} u(x,t)\right) + \int_0^t (x-s)^2 u(x,s)\, ds, \tag{5.2}$$

with initial condition $u(o,x) = u(0,t) = 0, \quad x \in (0,1]$.

Also, we observe that example (5.2) is a special case of Eq (1.7) with β =0.02 and functions f and g determined as: $\mathrm{f}(\mathrm{x},\mathrm{t},\mathrm{y},\mathrm{v}) = \frac{x}{4}\cos \mathrm{y} + \frac{1}{2}\mathrm{v}$, and $g(x,t,y,v) = (x-t)^2 u$, y, v ∈ R. The exact solution of Eq (5.2) is $u(x,t) = x\, sint$. The results of exact solution, approximate solutions and the absolute error between them are obtained in Table 2.

Table 2. Represents the exact solution and approximate solutions for example (5.2) at α = 0.5 and α = 0.95.

x	t	Exact Solution	$\alpha = 0.5$ Numerical solution	Abs. Error	$\alpha = 0.95$ Numerical solution	Abs. Error
0.2	0.33	0.19999668273	0.199996682716	1.39e−11	0.1999966827159	3.4 e−10
	0.66	0.19998673101	0.199986731072	2.8e−11	0.19998673107162	2.6 e−11
	0.99	0.19997014519	0.199970145180	9.99e−12	0.199970145180	9.01 e−13
0.4	0.33	0.39999336545	0.399993365395	4.1e−10	0.39999336539476	2.43 e−11
	0.66	0.39997346202	0.399973462014	5.2e−11	0.39997346201401	2.49 e−12
	0.99	0.39994029038	0.399940290375	4.33e−12	0.39994029037499	3.56 e−14
0.6	0.33	0.59999004818	0.599990048154	2.52e−11	0.59999004815391	8.65 e−13
	0.66	0.59996019304	0.599960193024	1.57e−11	0.5999601930233	3.69 e−13
	0.99	0.59991043557	0.599910435563	5.71e−12	0.59991043556284	9.54 e−14
0.8	0.33	0.7999867309	0.799986730857	4.3e−11	0.7999867308521	1.06 e−11
	0.66	0.79994692405	0.799946924047	2.2e−12	0.7999469240469	7.54 e−13
	0.99	0.79988058076	0.799880580759	2.12e−12	0.79988058075892	6.94 e−14

6. Conclusions

The major purpose of this study is to prove the existence and uniqueness of the solution of a nonlinear fractional integro-differential equation in dual Banach space. The desired findings are demonstrated by applying fixed-point theorems after employing fractional calculus. Also, we use the Bernoulli matrix approach method by reducing the NFIDEq to an algebraic system and deriving the numerical solution. We also observed that the matrix approach method is very efficient by verifying the conversion analysis of the numerical solution. Finally, we have chosen two examples as a verification of the theoretical work. The difference between exact solutions and approximate solutions for different levels of α and t are computed as shown in Figures 1–4, and concluded the following:

Throughout Example 5.1, we deduced that

(1) In Table 1, at α = 0.5, the minimum error is 6.93 e−09 at x = 0.8, t = 0.99, and the maximum error is 1.55 e-06 at x = 0.2, t = 0.66.

(2) In Table 1, at α = 0.95, the minimum error is 5.87 e−10 at x = 0.8, t = 0.99, and the maximum error is 3.2 e−07 at x = 0.4, t = 0.33.

Throughout Example 5.2, we deduced that

(3) In Table 2, at α = 0.5, the minimum error 9.99 e−12 at x = 0.2, t = 0.99, and the maximum error is 4.1e−10 at x = 0.4, t = 0.33.

(4) In Table 2, at α = 0.95, the minimum error is 6.94 e−14 at x = 0.8, t = 0.99, and the maximum error is 3.4 e−10 at x = 0.2, t = 0.33.

We can deduce that when the value of α increases to α = 0.8 and the value of t reaches t = 0.99 in each case, the approximation solutions are convergent to exact solutions, hence the error is small. When the value of α decreases to α = 0.2 and the value of t = 0.33, then the approximate solutions are divergent away from exact solutions and the difference between the two solutions is increase.

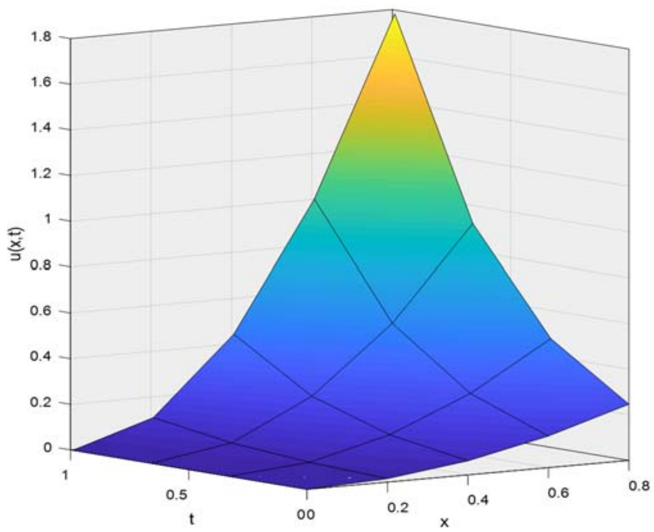

Figure 1. Represent the numerical solutions for example (5.1) at α = 0.5.

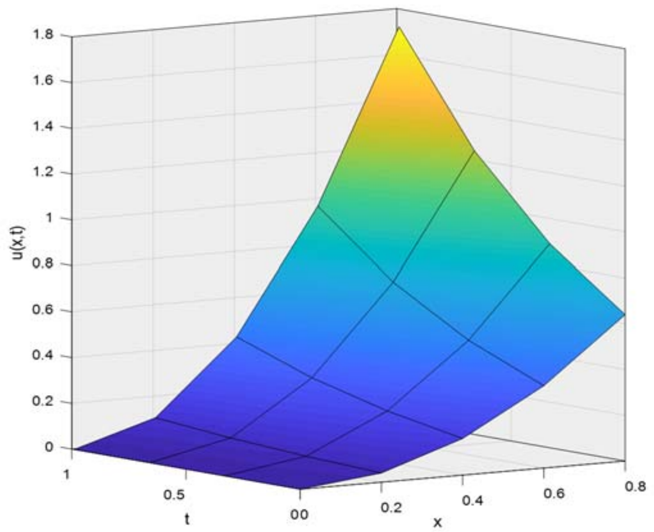

Figure 2. Represent the numerical solutions for example (5.1) at α = 0.95.

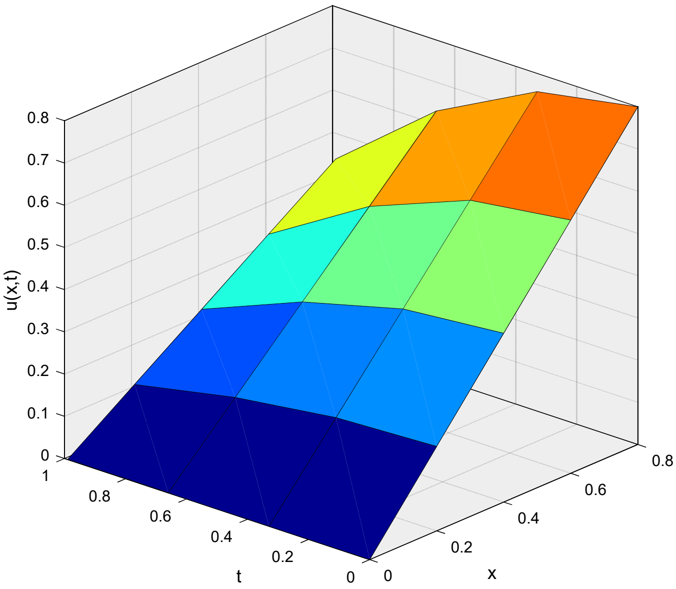

Figure 3. Represent the numerical solutions for example (5.2) at α = 0.5.

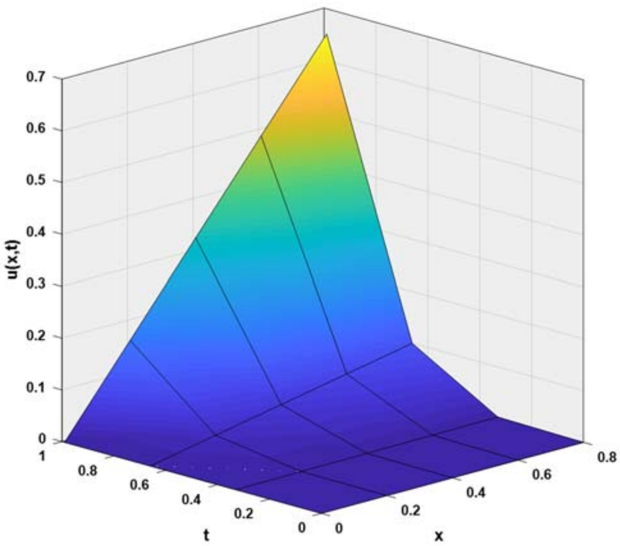

Figure 4. Represent the numerical solutions for example (5.2) at α = 0.95.

From Figures 5 and 6, we concluded the following:

(I) The **BMA** method is very powerful in finding precise numerical solutions, which appear clearly in the comparison between the exact solutions and the numerical solutions as shown.

(II) The numerical solutions at α *=0.95* are more accurate than the numerical solutions at α *=0.5*. The interpretation of it is that the behavior of the function *u(x, t)* at α *=0.5* represents a potential function that causes a slight perturbation in the values of u. We can declare that we obtain the equilibrium state when $\alpha \to 1$, we have a singular case, which is called Cauchy kernel (see [26,37]).

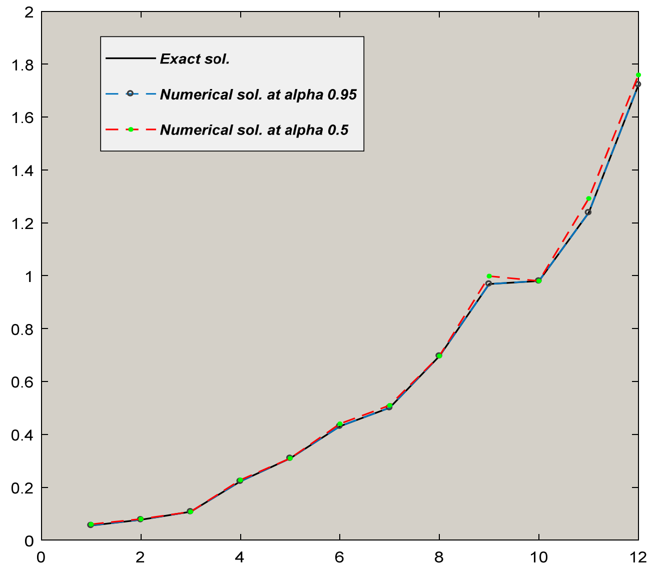

Figure 5. Represent the comparison between the exact solution and the numerical solutions for example (5.1).

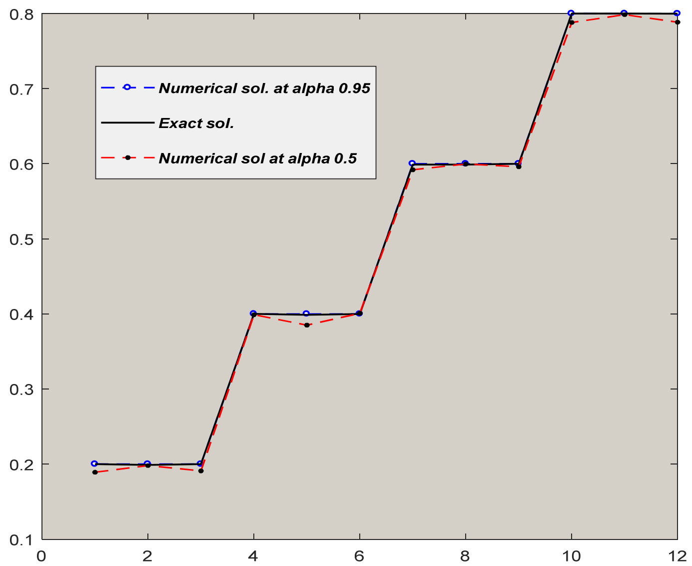

Figure 6. Represent the comparison between the exact solution and the numerical solutions for example (5.2).

Future work: We would like to expand on this work to investigate the optimality conditions for solving this fractional optimal control problem: $J(x(t),u(t),t)=\int_a^T L(x(t),u(t),t)\,dt+\phi(x(t),t)$, subject to a dynamical constraint on the form: $D_t^{\alpha}x(t)=f\left(x(t),t,D_x^{\beta}x(t),\int_{t_0}^{t}g(x,s,u(s))ds\right)$.

Conflict of interest

The authors declare that there is no conflict of interest.

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