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Research article

On the reducibility of a class of almost-periodic linear Hamiltonian systems and its application in Schrödinger equation

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Abstract: In the present paper, we focus on the reducibility of an almost-periodic linear Hamiltonian system

$$\frac{dX}{dt} = J[A + \varepsilon Q(t)]X, X \in \mathbb{R}^{2d},$$

where J is an anti-symmetric symplectic matrix, A is a symmetric matrix, $Q(t)$ is an analytic almost-periodic matrix with respect to t, and ε is a parameter which is sufficiently small. Using some non-resonant and non-degeneracy conditions, rapidly convergent methods prove that, for most sufficiently small ε, the Hamiltonian system is reducible to a constant coefficients Hamiltonian system through an almost-periodic symplectic transformation with similar frequencies as $Q(t)$. At the end, an application to Schrödinger equation is given.

Keywords: reducibility; almost-periodic; KAM method; Hamiltonian system
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1. Introduction

In this paper, we will focus on the reducibility of an almost-periodic linear Hamiltonian system

$$\frac{dX}{dt} = J\,[A + \varepsilon Q(t)]\,X, \qquad X \in \mathbb{R}^{2d}, \tag{1.1}$$

where A is a symmetric $2d \times 2d$ constant matrix with possible multiple proper-values, $Q(t)$ is an almost-periodic analytic symmetric matrix with respect to t, $J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$, where I_d is an identity matrix of order d, and ε is a sufficiently small parameter.

Let $A(t)$ be a quasi-periodic matrix of order d, and the differential equation

$$\frac{dX}{dt} = A(t)X, \qquad X \in \mathbb{R}^{d}, \tag{1.2}$$

is known as reducible if there exists a nonsingular quasi-periodic (q-p) Lyapunov-Perron (L-P) change of variables $X = \phi(t)Y$, where $\phi(t)$ and $\phi^{-1}(t)$ are quasi-periodic and bounded, which transforms (1.2) into

$$\frac{dY}{dt} = BY, \tag{1.3}$$

where B is a constant matrix.

Over recent years, the reducibility of differential systems has been studied widely by a lot of researchers [1–12]. The earliest result in this field is the well known Floquet Theory, which states that every periodic differential equation (1.2) can be reduced to a constant coefficient differential equation (1.3) by means of a periodic change of variables with the same period as $A(t)$. However, the result is no longer always true for quasi-periodic systems. A counterexample was provided by Palmer [2].

For example, the quasi-periodic linear systems which come from the quasi-periodic Schrödinger operators, which are defined on $L^2(\mathbb{R})$ as

$$(LY)(t) = -\frac{d^2Y}{dt^2} + q(\theta + \omega t)Y(t), \tag{1.4}$$

where $\theta \in T^n$ is known as phase, and $q : T^n \to \mathbb{R}$ is known as the potential. It is notable that the spectrum of L does not depend on the phase when ω is rationally independent, yet it is closely related to the dynamics of Schrödinger equation

$$(LY)(t) = -\frac{d^2Y}{dt^2} + q(\theta + \omega t)Y(t) = EY(t), \tag{1.5}$$

or, on the other hand, the dynamics of the linear differential systems

$$\frac{dX}{dt} = V_{E,q}(\theta)X, \qquad \frac{d\theta}{dt} = \omega, \tag{1.6}$$

where

$$V_{E,q}(\theta) = \begin{pmatrix} 0 & 1 \\ q(t) - E & 0 \end{pmatrix} \in sl(2, \mathbb{R}). \tag{1.7}$$

Dinaburg and Sinai [10] showed that linear system (1.6) is reducible for most $E > E^*(q, \alpha, \tau)$, which are sufficiently large, if ω is fixed and fulfills the non-resonance condition

$$|\langle k, \omega\rangle| \geq \frac{\alpha}{|k|^\tau}, \qquad k \in \mathbb{Z}^r \backslash \{0\},$$

where $\alpha > 0, \tau > 0$. The result of [10] was generalized by Rüssmann [7], in which ω satisfied the Brjuno condition.

Eliasson [11] showed the full measure reducibility result for quasi-periodic linear Schrödinger equations. Specifically, he showed that (1.6) is reducible for almost all $E > E^*(q, \omega)$ in the Lebesgue measure sense, where ω is the Diophantine vector which is fixed.

Jorba and Simó [1] considered the differential equations

$$\frac{dX}{dt} = [A + \varepsilon Q(t)]\, X, \qquad X \in \mathbb{R}^d, \tag{1.8}$$

where A is a constant matrix of order d with d distinct proper-values. They showed that under the non-resonant conditions and non-degeneracy conditions, there exists a non-empty Cantor subset E, such that for $\varepsilon \in E$, the system (1.8) is reducible.

Xu [3] considered the case that A has multiple eigenvalues and showed the system (1.8) is reducible for $\varepsilon \in E$.

Recently, Xue and Zhao [9] considered the linear q-p Hamiltonian system

$$\frac{dX}{dt} = [A + \varepsilon Q(t)]\, X, \tag{1.9}$$

where A is a constant matrix with possible multiple proper-values, and $Q(t)$ is an analytic matrix with respect to t and with frequencies $\omega = (\omega_1, \omega_2, \ldots, \omega_r)$. Under some nonresonant conditions, using KAM iterations and for most sufficiently small parameters ε they proved that the system (1.9) is reducible by means of a quasi-periodic symplectic change of variables with the same basic frequencies as $Q(t)$.

Rather than the reducibility of a q-p system to a constant coefficient system, Xu and You [5] investigated the reducibility of the following almost-periodic linear differential equations:

$$\frac{dX}{dt} = [A + \varepsilon Q(t)]\, X, \qquad X \in \mathbb{R}^d, \tag{1.10}$$

where A is a constant matrix with distinct proper-values, and $Q(t)$ is an almost periodic analytic matrix of order d with frequencies $\omega = (\omega_1, \omega_2, \ldots)$. Under some small divisor conditions, using KAM iterations and the "spatial structure" of almost periodic functions, they proved that for most sufficiently small ε, Eq (1.10) is reducible.

Inspired by [5,8], in this paper, we extend the results of [9] to almost-periodic Hamiltonian systems instead of quasi-periodic Hamiltonian systems. Here the related LP change of variables should not only be almost-periodic but also be symplectic.

To state our problem, we should present some notations and definitions.

A function $f(t)$ is said to be a quasi-periodic function with essential frequencies $\omega = (\omega_1, \omega_2, \ldots, \omega_d)$, if $f(t) = F(\theta_1, \theta_2, \ldots, \theta_d)$, where F is 2π periodic in all its arguments, and

$\theta_i = \omega_i t$ for $i = 1, 2, \ldots, d$. $f(t)$ will be known as an analytic q-p in a strip of width ϱ if F is analytical on $D_\varrho = \{\theta | |\Im\theta_l| \leq \varrho, l = 1, 2, \ldots, n\}$. For the present case, we denote the norm of $f(t)$ as $\|f\|_\varrho = \sum_{k\in\mathbb{Z}^n} |F_k| e^{\varrho|k|}$. $f(t)$ is almost-periodic, if $f(t) = \sum_{m=1}^{\infty} f_m(t)$ where $f_m(t)$ $(m = 1, 2, 3, \ldots)$ are all quasi-periodic.

Definition 1.1. *Let $A(t) = (a_{lj}(t))$ be a quasi-periodic $d \times d$ matrix. If every $a_{lj}(t)$ is analytic in D_ϱ, then we call $A(t)$ analytic on D_ϱ. The norm of $A(t)$ is defined as*

$$\|A(t)\|_\varrho = d \times \max_{1\leq l,j\leq d} \|a_{lj}(t)\|_\varrho.$$

If A is a constant matrix, the norm of A is defined as:

$$\|A\| = d \times \max_{1\leq l,j\leq d} |a_{lj}|.$$

In [5], we have noticed that "spatial structure" and "approximation function" are valuable tools to study the almost-periodic systems. To overcome the difficulties from infinite frequency which generate the small divisors problems, we require much stronger norms. So, let's introduce these notations from [6, 7].

Definition 1.2. *[6] Suppose that $\mathbb{N}$ is the natural number set, τ is the set of a few subsets of $\mathbb{N}$. Then, $(\tau, [\cdot])$ is known as a finite spatial structure in $\mathbb{N}$ if τ fulfills*

(1) $\emptyset \in \tau$,

(2) if $\Lambda_1, \Lambda_2 \in \tau$, then $\Lambda_1 \cup \Lambda_2 \in \tau$,

(3) $\cup_{\Lambda\in\tau}\Lambda = \mathbb{N}$,
and a weight function $[\cdot]$ is defined on τ, such that $[\emptyset] = 0, [\Lambda_1 \cup \Lambda_2] \leq [\Lambda_1] + [\Lambda_2]$.

Consider $k \in \mathbb{Z}^{\mathbb{N}}$. Indicate k as the support set, and, is defined as

$$\text{supp } k = \{(l_1, l_2, \ldots, l_n) | k_i \neq 0,\ i = l_1, l_2, \ldots, l_n;\ \text{otherwise } k_i = 0\}.$$

The weight value is denoted by $[k]$, and $[k] = \inf_{\text{supp}k\subset\Lambda,\Lambda\in\tau}[\Lambda]$. Write

$$|k| = \sum_{l=1}^{\infty} |k_l|.$$

Definition 1.3. *[7] In the following, the non-resonance conditions are provided for the supposed approximation functions. Δ is called an approximation function, if*

- *$\Delta : [0, \infty) \to [1, \infty)$, is an increasing function, and fulfills $\Delta(0) = 1$;*
- *$\frac{\log \Delta(t)}{t}$ is decreasing on $[0, \infty)$;*
- *$\int_0^\infty \frac{\log \Delta(t)}{t^2} dt < \infty$.*

It is clear that if $\Delta(t)$ is an approximation function, then so is $\Delta^3(t)$.

Definition 1.4. *If $Q(t) = \sum_{\Lambda\in\tau} Q_\Lambda(t)$, where $Q_\Lambda(t)$ are quasi-periodic matrices having frequencies $\omega_\Lambda = \{\omega_l | l \in \Lambda\}$, then $Q(t)$ is called an almost-periodic matrix having the spatial structure $(\tau, [\cdot])$ and* *frequency ω of $Q(t)$, which is the maximum subset of $\cup\omega_\Lambda$ in the sense of integer modular. Denote $\overline{Q} = (\overline{q}_{lj})$ as the average of $Q(t) = (q_{lj}(t))$, and*

$$\overline{q}_{lj} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} q_{lj}(t)dt.$$

For $\varrho > 0$, $m > 0$, the weighted norm of $Q(t)$ with spatial structure $(\tau, [.])$ is defined as:

$$|||Q(t)|||_{m,\varrho} = \sum_{\Lambda\in\tau} e^{m[\Lambda]} \|Q_\Lambda(t)\|_\varrho.$$

In our paper, the non-resonant condition is

$$|\lambda_l - \lambda_j - \sqrt{-1}\langle k, \omega\rangle| \geq \frac{\alpha_0}{\Delta^3(|k|)\Delta^3([k])},\ l \neq j,$$

$\forall\ 1 \leq l, j \leq 2d$, and $k \in \mathbb{Z}^{\mathbb{N}}\backslash\{0\}$, where $\alpha_0 > 0$ is the small constant $\lambda_1, \lambda_2, \ldots, \lambda_{2d}$ are the proper-values of JA, $\omega = (\omega_1, \omega_2, \ldots)$ is the frequency of $Q(t)$, and $\Delta(t)$ is an approximation function which fulfills $\sum_{k\in\mathbb{Z}^{\mathbb{N}}} \frac{1}{\Delta(|k|)\Delta([k])} < +\infty$. From [6], it is assumed that

$$[\Lambda] = 1 + \sum_{l\in\Lambda} \log^r(1 + |l|), r > 2.$$

So, we are in a position to state our main result.

Theorem 1.1. *Consider the Hamiltonian system (1.1) in which JA is the Hamiltonian matrix with possible multiple proper-values $\lambda_1, \lambda_2, \ldots, \lambda_{2d}$, and $JQ(t) = \sum JQ_\Lambda(t)$ is analytic almost-periodic on D_ϱ with frequencies $\omega = (\omega_1, \omega_2, \ldots)$ and has spatial structure $(\tau, [\cdot])$, which depends continuously upon the small parameter ε. Suppose that*

$A_{1.}$ $\exists\, m > 0$, s.t. $|||Q(t)|||_{m,\varrho} < +\infty$.

$A_{2.}$ (Non-resonant Conditions) Suppose that $\lambda = (\lambda_1, \ldots, \lambda_{2d})$ and $\omega = (\omega_1, \omega_2, \ldots)$ fulfill

$$|\lambda_l - \lambda_j - \sqrt{-1}\langle k, \omega\rangle| \geq \frac{\alpha_0}{\Delta(|k|)^3\Delta([k])^3}, \forall 1 \leq l, j \leq 2d, l \neq j,$$

$\forall\ k \in \mathbb{Z}^{\mathbb{N}}\backslash\{0\}$, where $\alpha_0 > 0$, and $\Delta(t)$ is an approximation function.

$A_{3.}$ (Non-degeneracy Conditions) Let $\lambda_l^1(\varepsilon)$ $(1 \leq l \leq 2d)$ be $2d$ distinct proper-values of $J(A + \varepsilon\overline{Q})$ with $|\lambda_l^1| \geq 2\eta\varepsilon$, $|\lambda_l^1 - \lambda_j^1| \geq 2\eta\varepsilon$, $l \neq j$ $0 \leq l, j \leq 2d$, a constant $\eta > 0$ independent from ε, and $\overline{Q}$ is the average of $Q(t)$ which is given in definition 1.4.

Then, there exists some sufficiently small $\varepsilon_* > 0$ and a positive measure non-empty Cantor subset $E_* \subset (0, \varepsilon_*)$, s.t. for $\varepsilon \in E_*$, there is an analytic almost-periodic symplectic change $X = \psi(t)Y$ with the same frequencies and finite spatial structure like $Q(t)$, which changes (1.1) into the Hamiltonian system $\dot{Y} = BY$, where B is a constant matrix. Additionally, means $(\frac{(0,\varepsilon_*)}{E_*})$ approaches 1 as ε_* goes to 0.

Remark 1.1. *Here, as we are dealing with the Hamiltonian system, we need to find the symplectic change, which is not the same as that in [1].*

Remark 1.2. *We allow matrix JA to have multiple eigen-values. Obviously, if the eigen-values of JA are distinct, the non-degeneracy condition holds naturally.*

As an example, we apply the Theorem 1.1 to the following Schrödinger equation:

$$\frac{d^2X}{dt^2} + \varepsilon Ja(t)X = 0, \tag{1.11}$$

where $Ja(t) = \sum Ja_\Lambda(t)$ is an almost-periodic function which is analytic on D_ϱ with frequencies ω and has spatial structure $(\tau, [\cdot])$, which is persistently dependent on small parameter ε. $\overline{a}$ is the average of $a(t)$. If $\overline{a} > 0$ and the frequency ω of $Ja(t) = \sum Ja_\Lambda(t)$ fulfills the non-resonance condition

$$|\langle k, \omega\rangle| \geq \frac{\alpha_0}{\Delta(|k|)^3\Delta([k])^3}, \qquad k \in \mathbb{Z}^{\mathbb{N}}\backslash\{0\}, \tag{1.12}$$

where $\alpha_0 > 0$ is a small constant and $\Delta(t)$ is an approximation function, then there exists some sufficiently small $\varepsilon_* > 0$, the system (1.11) is reducible, and the equilibrium of (1.11) is stable in the sense of Lyapunov for generally sufficiently small $\varepsilon \in (0, \varepsilon_*)$. In addition, all solutions of Eq (1.11) are quasi-periodic with the frequency $\Omega = (\sqrt{b}, \omega_1, \omega_2, \ldots)$ for generally sufficiently small $\varepsilon \in (0, \varepsilon_*)$, where $b = \overline{a}\varepsilon + O(\varepsilon^2)$ as ε approaches 0. Here, we can see that if we rewrite the system (1.11) into the system (1.1), we have

$$JA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which has various proper-values $\lambda_1 = \lambda_2 = 0$. One can see Section 5 for much more details about this example.

This paper is organized as follows:

- In Section 2, some Lemmas are given.
- In Section 3, we will prove the first KAM step.
- In Section 4, we will prove the main Theorem 1.1.
- Finally, in Section 5, we will analyze the Eq (1.11).

2. The Lemmas

Lemma 2.1. *[5]. Assume that T and R are almost-periodic matrices with similar frequencies and similar spatial structures. If $|||T|||_{m,\varrho} < +\infty$, $|||R|||_{m,\varrho} < +\infty$, then TR is an almost-periodic matrix with similar frequencies and similar spatial structure like T and R,*

$$|||TR|||_{m,\varrho} \leq |||T|||_{m,\varrho}|||R|||_{m,\varrho},$$

and for the average of T, we have $\|\overline{T}\| \leq |||T|||_{m,\varrho}$.

Lemma 2.2. *[1]. Assume that C_0 is a $2d \times 2d$ matrix with distinct non-zero proper-values $\mu_1^0, \ldots, \mu_{2d}^0$ satisfying $|\mu_l^0| > \gamma$, $|\mu_l^0 - \mu_j^0| > \gamma$, $l \neq j$, $0 \leq l, j \leq 2d$ and a regular matrix B_0 s.t. $B_0^{-1}C_0B_0 = diag(\mu_1^0, \ldots, \mu_{2d}^0)$. Choose $\beta_0 = \max\{\|B_0\|, \|B_0^{-1}\|\}$, and pick b s.t. $0 < b < \frac{\gamma}{(6d-1)\beta_0^2}$. If C_1 confirms $\|C_1 - C_0\| \leq b$, then, at that point, the accompanying conclusions hold:*

(1) C_1 has $2d$ distinct non-zero proper-values $\mu_1^1, \ldots, \mu_{2d}^1$;

(2) $\exists$ the regular matrix B_1 such that $B_1^{-1}C_1B_1 = diag(\mu_1^1, \ldots, \mu_{2d}^1)$, which confirms $\|B_1\|, \|B_1^{-1}\| \leq \beta_1$, where $\beta_1 = 2\beta_0$.

The next lemma is the inductive lemma which is used for the inductive procedure in the proof of Theorem 1.1.

Lemma 2.3. *Consider the differential equation of the matrix*

$$\dot{S} = (JA)S - S(JA) + Q, \tag{2.1}$$

where $(JA)_{2d\times 2d}$ is a Hamiltonian matrix, the proper-values of JA are $\lambda_1, \lambda_2, \ldots, \lambda_{2d}$ with $|\lambda_j| > \zeta$ and $|\lambda_j - \lambda_l| > \zeta$ for $j \neq l$, and $\zeta > 0$ is constant. Also, $Q(t) = \sum_{\Lambda\in\tau} Q_\Lambda(t)$ is an almost-periodic Hamiltonian matrix in t, is analytic on D_ϱ with frequencies $\omega = (\omega_1, \omega_2, \ldots)$ and has finite spatial structure $(\tau, [\cdot])$. $\overline{Q} = 0$, where $\overline{Q}$ is the average of $Q(t)$. Let

$$|\lambda_j - \lambda_l - \sqrt{-1}\langle k, \omega\rangle| \geq \frac{\alpha_0}{\Delta^3(|k|)\Delta^3([k])},\ \forall\, k \in \mathbb{Z}^{\mathbb{N}}\backslash\{0\}, \tag{2.2}$$

with $\alpha_0 > 0$ a constant and with the approximation function $\Delta(t)$. Consider $0 < \overline{\varrho} < \varrho$, $0 < \overline{m} < m$. Then, $\exists$ a unique analytic almost-periodic Hamiltonian matrix $S(t)$ with similar finite spatial structure and with similar frequency as $Q(t)$, which gives the solution of Eq (2.1) and fulfills

$$|||S|||_{m-\overline{m},\varrho-\overline{\varrho}} \leq c\frac{\Gamma(\overline{m})\Gamma(\overline{\varrho})}{\alpha_0}|||Q|||_{m,\varrho},$$

where $\Gamma(\varrho) = \sup_{t\geq 0}[\Delta^3(t)e^{-\varrho t}]$, and $c > 0$ is the constant.

Proof: Setting S such that $S^{-1}JAS = D = dia(\lambda_1, \lambda_2, \ldots, \lambda_{2d})$, making transformation $S(t) = BV(t)B^{-1}$ and $R(t) = B^{-1}QB(t)$, Eq (2.1) becomes

$$\dot{V} = DV - VD + R.$$

Consider $V = \sum_{\Lambda\in\tau} V_\Lambda$, $R = \sum_{\Lambda\in\tau} R_\Lambda$, and

$$R_\Lambda = (r_\Lambda^{jl}), \qquad (r_{\Lambda k}^{jl}) = \sum_{\mathrm{supp}k\subset\Lambda} r_{\Lambda k}^{jl} e^{\sqrt{-1}\langle k,\theta\rangle},$$

$$V_\Lambda = (v_\Lambda^{jl}), \qquad (v_{\Lambda k}^{jl}) = \sum_{\mathrm{supp}k\subset\Lambda} v_{\Lambda k}^{jl} e^{\sqrt{-1}\langle k,\theta\rangle},$$

with $\theta = \omega t$.

Substituting above into $\dot{V}_\Lambda = DV_\Lambda - V_\Lambda D + R_\Lambda$ and by comparing the coefficients on both sides, we obtain $v_{\Lambda 0}^{jl} = 0$; or for $k \neq 0$,

$$v_{\Lambda k}^{jl} = \frac{r_{\Lambda k}^{jl}}{\lambda_j - \lambda_l - \sqrt{-1}\langle k, \omega\rangle}.$$

Since Q is analytic on D_ϱ, $R = B^{-1}QB$ is also analytic on D_ϱ. So, using Eq (2.2), we have

$$\|v_\Lambda^{jl}\|_{\varrho-\overline{\varrho}} \leq \sum_{\mathrm{supp}k\subset\Lambda} \frac{\Delta^3(|k|)e^{-\overline{\varrho}|k|}}{\alpha_0}\Delta^3([k])|r_{\Lambda k}^{jl}|e^{\varrho|k|}, \\ \leq \frac{\Gamma(\overline{\varrho})\Delta^3([\Lambda])}{\alpha_0}\|r_{\Lambda k}^{jl}\|_\varrho.$$

Thus,

$$\|V_\Lambda\|_{\varrho-\overline{\varrho}} \leq \frac{\Gamma(\overline{\varrho})\Delta^3([\Lambda])}{\alpha_0}\|R_\Lambda\|_\varrho.$$

Let $V = \sum_{\Lambda\in\tau} V_\Lambda$. From Definition 1.2, we have

$$|||V|||_{m-\overline{m},\varrho-\overline{\varrho}} = \sum_{\Lambda\in\tau}\|V_\Lambda\|_{\varrho-\overline{\varrho}}e^{(m-\overline{m})[\Lambda]}, \\ \leq \sum_{\Lambda\in\tau}\frac{\Gamma(\overline{\varrho})\Delta^3([\Lambda])}{\alpha_0}\|R_\Lambda\|_\varrho e^{m[\Lambda]-\overline{m}[\Lambda]}, \\ \leq \frac{\Gamma(\overline{\varrho})\Gamma(\overline{m})}{\alpha_0}|||R|||_{m,\varrho}.$$

Then, by utilizing Lemmas 2.1 and 2.2, we can write

$$|||S|||_{m-\overline{m},\varrho-\overline{\varrho}} \leq \|B\||||V|||_{m-\overline{m},\varrho-\overline{\varrho}}\|B^{-1}\|,$$

and

$$|||R|||_{m,\varrho} \leq \|B^{-1}\||||Q|||_{m,\varrho}\|B\|.$$

So,

$$|||S|||_{m-\overline{m},\varrho-\overline{\varrho}} \leq c\frac{\Gamma(\overline{m})\Gamma(\overline{\varrho})}{\alpha_0}|||Q|||_{m,\varrho}.$$

To show that $S = \sum_{\Lambda\in\tau} S_\Lambda$ is Hamiltonian, we simply need to make sure that $S_l = J^{-1}S$ is symmetric. Since we have that JA is Hamiltonian and $Q = \sum_{\Lambda\in\tau} Q_\Lambda$ is Hamiltonian, using the definition, A is symmetric, and we can denote $Q = JQ_l$, where Q_l is symmetric. Putting $S = JS_l$ and $Q = JQ_l$ into Eq (2.1), we get

$$\dot{S}_l = AJS_l - S_lJA + Q_l. \tag{2.3}$$

Taking the transpose on the two sides of Eq (2.3), we have

$$\dot{S}_l^t = AJS_l^t - S_l^tAJ + Q_l. \tag{2.4}$$

Multiplying both sides of Eqs (2.3) and (2.4) by J, we get $J\dot{S}_l = (JA)JS_l - JS_l(JA) + Q$, and $J\dot{S}_l^t = (JA)JS_l^t - JS_l^t(AJ) + Q$. This shows that JS_l and JS_l^t are solutions of Eq (2.1). As $v_{\Lambda 0}^{lj} =$, $1 \leq l, j \leq 2d$, we have $\bar{V} = 0$, and so $\overline{S} = 0$. Thus, $J\bar{S}_l = J\bar{S}_l^t = 0$. As Eq (2.1) has unique solution with $\bar{S} = 0$, we get $JS_l = JS_l^t$; and this implies that $S_l = S_l^t$, which shows that S is the Hamiltonian.

□

3. The first KAM step

Choose $A_0 = JA$, $Q_0(t) = JQ(t)$. By condition A_3 of Theorem 1.1, $(A_0 + \varepsilon\overline{Q_0})$ is the Hamiltonian matrix with $2d$ distinct proper-values λ_l^1, $(1 \leq l \leq 2d)$ with $|\lambda_l^1| \geq 2\eta\varepsilon$, and $(0 \leq l, j \leq 2d)$ with $|\lambda_l^1 - \lambda_j^1| \geq 2\eta\varepsilon$, where $\eta > 0$ is the constant independent from ε. Thus, Hamiltonian system (1.1) can be rewritten in the form:

$$\frac{dX}{dt} = \left[A_1 + \varepsilon\widetilde{Q}(t)\right]X, \qquad X \in \mathbb{R}^{2d}, \tag{3.1}$$

where $A_1 = J(A + \varepsilon\overline{Q})$, $\widetilde{Q}(t) = J(Q(t) - \overline{Q})$, $\overline{\widetilde{Q}} = 0$, and A_1 and $\widetilde{Q}(t)$ are the Hamiltonian matrices. Let regular matrix B_1 be such that $B_1^{-1}A_1B_1 = diag(\lambda_1^1, \ldots, \lambda_{2d}^1)$, which fulfills $\beta_1 = \max\{\|B_1\|, \|B_1^{-1}\|\}$. Using symplectic change of variables $X = e^{\varepsilon S(t)}X_1$, where $S(t)$ will be found later, the system (3.1) is converted into

$$\frac{dX_1}{dt} = \left[e^{-\varepsilon S(t)}(A_1 + \varepsilon\widetilde{Q}(t) - \varepsilon\dot{S})e^{\varepsilon S(t)} + e^{-\varepsilon S(t)}(\varepsilon\dot{S}\,e^{\varepsilon S(t)} - \frac{d}{dt}e^{\varepsilon S(t)})\right]X_1. \tag{3.2}$$

By series expansion, we can indicate

$$e^{\varepsilon S} = I + \varepsilon S + W,$$

and

$$e^{-\varepsilon S} = I - \varepsilon S + \widetilde{W},$$

where

$$W = \frac{(\varepsilon S)^2}{2!} + \frac{(\varepsilon S)^3}{3!} + \ldots, \qquad \widetilde{W} = \frac{(\varepsilon S)^2}{2!} - \frac{(\varepsilon S)^3}{3!} + \ldots.$$

Then, the Hamiltonian system (3.2) can be rewritten as

$$\begin{aligned}\frac{dX_1}{dt} &= [(I - \epsilon S + \widetilde{W})(A_1 + \varepsilon\widetilde{Q}(t) - \varepsilon\dot{S})(I + \epsilon S + W) + e^{-\varepsilon S(t)}(\varepsilon\dot{S}\,e^{\varepsilon S(t)} - \frac{d}{dt}e^{\varepsilon S(t)})]X_1,\\ &= [A_1 + \varepsilon\widetilde{Q} - \varepsilon\dot{S} + \varepsilon A_1 S - \varepsilon S A_1 + \varepsilon^2 Q_1]X_1,\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}Q_1 = &- S(\widetilde{Q} - \dot{S}) + (\widetilde{Q} - \dot{S})S - S(A_1 + \varepsilon\widetilde{Q} - \varepsilon\dot{S})S\\ &+ (I - \epsilon S)(A_1 + \varepsilon\widetilde{Q} - \varepsilon\dot{S})\frac{W}{\varepsilon^2} + \frac{\widetilde{W}}{\varepsilon^2}(A_1 + \varepsilon\widetilde{Q} - \varepsilon\dot{S})e^{\varepsilon S}\\ &+ \frac{1}{\varepsilon^2}e^{-\varepsilon S(t)}(\varepsilon\dot{S}\,e^{\varepsilon S(t)} - \frac{d}{dt}e^{\varepsilon S(t)}).\end{aligned}$$

We would like to have

$$\widetilde{Q} - \dot{S} + A_1 S - S A_1 = 0,$$

or, we have

$$\dot{S} = A_1 S - S A_1 + \widetilde{Q}. \tag{3.4}$$

By the condition A_3 of Theorem 1.1, it is not difficult to see that the inequalities

$$|\lambda_l^1| \geq \eta\varepsilon, \quad |\lambda_l^1 - \lambda_j^1| \geq \eta\varepsilon, \quad l \neq j, \quad 0 \leq l, j \leq 2d,$$

hold. By using Lemma 2.3, if

$$|\lambda_l^1 - \lambda_j^1 - \sqrt{-1}\langle k, \omega\rangle| \geq \frac{\alpha_1}{\Delta^3(|k|)\Delta^3([k])}, l \neq j, k \in \mathbb{Z}^{\mathbb{N}} \backslash \{0\}, \tag{3.5}$$

also holds, where $\alpha_1 = \frac{\alpha_0}{4}$, then Eq (3.4) can be solved for a unique almost-periodic Hamiltonian matrix $S = \sum S_\Lambda$ on $D_{\varrho-\overline{\varrho}}$ with similar frequencies and similar spatial structure$(\tau, [\cdot])$ as $\widetilde{Q}$, which fulfills $\overline{S} = 0$ and

$$|||S|||_{m-\overline{m},\varrho-\overline{\varrho}} \leq c\frac{\Gamma(\overline{m})\Gamma(\overline{\varrho})}{\alpha_0}|||Q(t)|||_{m,\varrho}. \tag{3.6}$$

Therefore, by using (3.4), the system (3.3) can be written as

$$\frac{dX_1}{dt} = \left[A_1 + \varepsilon^2 Q_1\right] X_1, \tag{3.7}$$

where,

$$\begin{aligned} Q_1 =& S(A_1 S - S A_1) + (S A_1 - A_1 S)S - S(A_1 + \varepsilon(S A_1 - A_1 S))S \\ &+ (I - \epsilon S)(A_1 + \varepsilon(S A_1 - A_1 S))\frac{W}{\varepsilon^2} + \frac{\widetilde{W}}{\varepsilon^2}(A_1 + \varepsilon(S A_1 - A_1 S))e^{\varepsilon S} \\ &+ \frac{1}{\varepsilon^2}e^{-\varepsilon S(t)}(\varepsilon\dot{S}e^{\varepsilon S(t)} - \frac{d}{dt}e^{\varepsilon S(t)}). \end{aligned}$$

Consequently, under the symplectic transformation $X = e^{\varepsilon S(t)}X_1$, system (3.1) is converted into system (3.7).

For sufficiently small ε, we have $|||\varepsilon S|||_{m-\overline{m},\varrho-\overline{\varrho}} < 1$; thus, from

$$W = \frac{(\varepsilon S)^2}{2!} + \frac{(\varepsilon S)^3}{3!} + \ldots, \quad \widetilde{W} = \frac{(\varepsilon S)^2}{2!} - \frac{(\varepsilon S)^3}{3!} + \ldots,$$

we have

$$\begin{aligned} |||W|||_{m-\overline{m},\varrho-\overline{\varrho}} \leq& \frac{|||\varepsilon S|||^2_{m-\overline{m},\varrho-\overline{\varrho}}}{2!} + \frac{|||\varepsilon S|||^3_{m-\overline{m},\varrho-\overline{\varrho}}}{3!} + \ldots, \\ &= |||\varepsilon S|||^2_{m-\overline{m},\varrho-\overline{\varrho}}(\frac{1}{2!} + \frac{||\varepsilon S|||_{m-\overline{m},\varrho-\overline{\varrho}}}{3!} + \ldots), \\ &\leq L|||\varepsilon S|||^2_{m-\overline{m},\varrho-\overline{\varrho}}, \end{aligned}$$

where $L = \frac{1}{2!} + \frac{|||\varepsilon S|||_{m-\overline{m},\varrho-\overline{\varrho}}}{3!} + \ldots$.
In the same way, we can get $|||\overline{W}|||_{m-\overline{m},\varrho-\overline{\varrho}} \leq L|||\varepsilon S|||^2_{m-\overline{m},\varrho-\overline{\varrho}}$.

Thus, for sufficiently small ε

$$|||Q_1|||_{m-\overline{m},\varrho-\overline{\varrho}} \leq C_0|||\varepsilon S|||^2_{m-\overline{m},\varrho-\overline{\varrho}} \leq C_0^* \frac{\Gamma(\overline{m})^2\Gamma(\overline{\varrho})^2}{\alpha_0^2}|||Q(t)|||^2_{m,\varrho},$$

where $C_0 > 0, C_0^* > 0$ are constants. That is the end of the first KAM step.

4. Proof of Theorem 1.1

Now, we consider the iteration step. At the n^{th} step, suppose the Hamiltonian system

$$\frac{dX_n}{dt} = \left[A_n + \varepsilon^{2^n} Q_n(t)\right] X_n, n \geq 1, \tag{4.1}$$

where A_n is the Hamiltonian matrix, and $Q_n(t)$ is an analytic almost-periodic Hamiltonian matrix on D_{ϱ_n} with basic frequencies $\omega = (\omega_1, \omega_2, \ldots)$ and has spatial structure $(\tau, [\cdot])$. λ_l^n are eigenvalues of A_n with $|\lambda_l^n| \geq \eta\varepsilon$, $|\lambda_l^{n+1} - \lambda_j^{n+1}| \geq \eta\varepsilon$, $l \neq j$, $0 \leq l, j \leq 2d$, where $\eta > 0$ is independent from ε. By defining the average of $Q_n(t)$ as $\overline{Q}_n$, the system (4.1) is rewritten as

$$\frac{dX_n}{dt} = \left[A_{n+1} + \varepsilon^{2^n} \widetilde{Q}_n(t)\right] X_n, \quad n \geq 1, \tag{4.2}$$

where $A_{n+1} = (A_n + \varepsilon^{2^n}\overline{Q}_n)$ $\widetilde{Q}_n(t) = Q_n(t) - \overline{Q}_n$.

Presently, by making the symplectic change $X_n = e^{\varepsilon^{2^n} S_n(t)} X_{n+1}$, where $S_n(t)$ will be found later, the system (4.2) becomes

$$\begin{aligned}\frac{dX_{n+1}}{dt} =&[e^{-\varepsilon^{2^n} S_n}(A_{n+1} + \varepsilon^{2^n}\widetilde{Q}_n - \varepsilon^{2^n}\dot{S}_n)e^{\varepsilon^{2^n} S_n} \\ &+ e^{-\varepsilon^{2^n} S_n}(\varepsilon^{2^n}\dot{S}_n e^{\varepsilon^{2^n} S_n} - \frac{d}{dt}e^{\varepsilon^{2^n} S_n(t)})]X_{n+1}.\end{aligned} \tag{4.3}$$

By series expansion, we can indicate

$$\begin{aligned} e^{\epsilon^{2^n} S_n} &= I + \epsilon^{2^n} S_n + W_m, \\ e^{-\epsilon^{2^n} S_n} &= I - \epsilon^{2^n} S_m + \widetilde{W}_n \end{aligned}$$

where

$$\begin{aligned} W_m &= \frac{(\epsilon^{2^n} S_n)^2}{2!} + \frac{(\epsilon^{2^n} S_n)^3}{3!} + \ldots, \\ \widetilde{W}_n &= \frac{(\epsilon^{2^n} S_n)^2}{2!} - \frac{(\epsilon^{2^n} S_n)^3}{3!} + \ldots. \end{aligned}$$

Then, the system (4.3) can be rewritten as

$$\frac{dX_{n+1}}{dt} = \left[A_{n+1} + \varepsilon^{2^n}\widetilde{Q}_n - \varepsilon^{2^n}\dot{S}_n + \varepsilon^{2^n} A_{n+1} S_n - \varepsilon^{2^n} S_n A_{n+1} + \varepsilon^{2^{n+1}} Q_{n+1}(t)\right] X_{n+1}, \tag{4.4}$$

where

$$Q_{n+1}(t) = -S_n(\widetilde{Q}_n - \dot{S}_n) + (\widetilde{Q}_n - \dot{S}_n)S_n - S_n(A_{n+1} + \varepsilon^{2^n}(\widetilde{Q}_n - \dot{S}_n))S_m$$
$$+ (I - \epsilon^{2^n}S_n)(A_{n+1} + \varepsilon^{2^n}(\widetilde{Q}_n - \dot{P}_n))\frac{W_n}{\varepsilon^{2^{n+1}}} + \frac{\widetilde{W}_n}{\varepsilon^{2^{n+1}}}(A_{n+1} + \varepsilon^{2^n}(\widetilde{Q}_m - \dot{S}_n))e^{\varepsilon^{2^n}S_n}$$
$$+ \frac{1}{\varepsilon^{2^{n+1}}}e^{-\varepsilon^{2^n}S_n}(\varepsilon^{2^n}\dot{S}_m e^{\varepsilon^{2^n}S_n} - \frac{d}{dt}e^{\varepsilon^{2^n}S_n(t)}).$$

We would like to have

$$\widetilde{Q}_n - \dot{S}_n + A_{n+1}S_n - S_nA_{n+1} = 0,$$

or we have

$$\dot{S}_n = A_{n+1}S_n - S_nA_{n+1} + \widetilde{Q}_n. \tag{4.5}$$

Since $A_n + \varepsilon^{2^n}\overline{Q}_n$ and $Q_n(t) - \overline{Q}_n$ are Hamiltonian, A_{n+1} and $\widetilde{Q}_n(t)$ are Hamiltonian. If

$$|\lambda_l^{n+1} - \lambda_j^{n+1} - \sqrt{-1}\langle k, \omega\rangle| \geq \frac{\alpha_n}{\Delta^3(|k|)\Delta^3([k])}, \quad l \neq j, \quad k \in \mathbb{Z}^{\mathbb{N}}\backslash\{0\},$$

and A_{n+1} has $2d$ distinct proper-values $\lambda_1^{n+1}, \ldots, \lambda_{2d}^{n+1}$ with $|\lambda_l^{n+1}| \geq \eta\varepsilon$, $|\lambda_l^{n+1} - \lambda_j^{n+1}| \geq \eta\varepsilon$, $l \neq j$ $0 \leq l, j \leq 2d$, by Lemma 2.3, there is a unique almost-periodic matrix $S_n(t)$ on $D_{\varrho_n - \overline{\varrho}_{n+1}}$ having frequencies ω and with finite spatial structure $(\tau, [\cdot])$, which fulfills $\overline{S}_n = 0$ and

$$|||S_n|||_{m_n - \overline{m}_{n+1}, \varrho_n - \overline{\varrho}_{n+1}} \leq c\frac{\Gamma(\overline{m}_n)\Gamma(\overline{\varrho}_n)}{\alpha_n}|||Q_n|||_{m_n, \varrho_n}. \tag{4.6}$$

Then, the Hamiltonian system (4.4) becomes

$$\frac{dX_{n+1}}{dt} = \left[A_{n+1} + \varepsilon^{2^{n+1}}Q_{n+1}(t)\right]X_{n+1}. \tag{4.7}$$

where,

$$\begin{aligned} Q_{n+1}(t) =& S_n(A_{n+1}S_n - S_nA_{n+1}) + (S_nA_{n+1} - A_{n+1}S_n)S_n \\ & - S_n(A_{n+1} + \varepsilon^{2^n}(S_nA_{n+1} - A_{n+1}S_n))S_n \\ & + (I - \epsilon^{2^n}S_n)(A_{n+1} + \varepsilon^{2^n}(S_nA_{n+1} - A_{n+1}S_n))\frac{W_n}{\varepsilon^{2^{n+1}}} \\ & + \frac{\widetilde{W}_n}{\varepsilon^{2^{n+1}}}(A_{n+1} + \varepsilon^{2^n}(S_nA_{n+1} - A_{n+1}S_n))e^{\varepsilon^{2^n}S_n} \\ & + \frac{1}{\varepsilon^{2^{n+1}}}e^{-\varepsilon^{2^n}S_n}(\varepsilon^{2^n}\dot{S}_n e^{\varepsilon^{2^n}S_m} - \frac{d}{dt}e^{\varepsilon^{2^n}S_n(t)}). \end{aligned} \tag{4.8}$$

Thus, under the symplectic change $X_n = e^{\varepsilon^{2^n}S_n(t)}X_{n+1}$, system (4.1) is transformed into system (4.7). Let regular matrix B_{n+1} be such that $B_{n+1}^{-1}A_{n+1}B_{n+1} = diag(\lambda_1^{n+1}, \ldots, \lambda_{2d}^{n+1})$ and $\beta_{n+1} = \max\{\|B_{n+1}\|, \|B_{n+1}^{-1}\|\}$. Then, from Lemma 2.2, we can suppose $\beta_{n+1} = 2\beta_n$, and so $\beta_n = 2^{n-1}\beta_1$.

Iteration:

Now, by the KAM iteration, we prove that the iteration is convergent as $n \to \infty$.

From Lemma 2.3, $\overline{m}$ and $\overline{\varrho}$ are taken to be arbitrary, so we can set m_n and ϱ_n as follows: Let

$$m_n = m - \sum_{\nu=1}^{n} \overline{m}_\nu \quad \text{and} \quad \varrho_n = \varrho - \sum_{\nu=1}^{n} \overline{\varrho}_\nu.$$

where $\overline{m}_\nu \to 0$ and $\overline{\varrho}_\nu \to 0$ fulfill $\sum_{\nu=0}^{\infty} \overline{m}_\nu = \frac{1}{2}m_0$ and $\sum_{\nu=0}^{\infty} \overline{\varrho}_\nu = \frac{1}{2}\varrho_0$.

Consider that

$$\varphi(\varrho) = \inf_{\varrho_1+\varrho_2+\ldots<\varrho} \prod_{\nu=1}^{\infty} [\Gamma(\varrho_\nu)]^{2^{-\nu-1}}.$$

Then, from [6], we see

$$\varphi(\frac{1}{2}m_0) = \prod_{\nu=1}^{\infty} [\Gamma(\overline{m}_\nu)]^{2^{-\nu-1}},$$

and

$$\varphi(\frac{1}{2}\varrho_0) = \prod_{\nu=1}^{\infty} [\Gamma(\overline{\varrho}_\nu)]^{2^{-\nu-1}}.$$

In system (4.2), as A_{n+1} has $2d$ distinct proper-values which fulfills the states of the hypothesis, then by using Lemma 2.3, $\exists$ a symplectic change $X_n = e^{\varepsilon^{2^n} S_n} X_{n+1}$, so that $S_n(t) = \sum_{\Lambda\in\tau} S_{\Lambda n}(t)$ is the unique almost-periodic matrix having similar frequencies and similar finite spatial structure like $Q_n(t)$, which fulfills (4.5) and so that the system (4.2) is converted into the system (4.7). Before estimating $|||Q_{n+1}|||_{m_n-\overline{m}_{n+1},\varrho_n-\overline{\varrho}_{n+1}}$, we should see that if $|||\varepsilon^{2^n} S_n|||_{m_n-\overline{m}_{n+1},\varrho_n-\overline{\varrho}_{n+1}} \leq \frac{1}{2}$, it follows that

$$|||e^{\pm\varepsilon^{2^n} S_n}|||_{m_n-\overline{m}_{n+1},\varrho_n-\overline{\varrho}_{n+1}} \leq 1 + |||\varepsilon^{2^n} S_n|||_{m_n-\overline{m}_{n+1},\varrho_n-\overline{\varrho}_{n+1}} + \frac{|||\varepsilon^{2^n} S_n|||^2_{m_n-\overline{m}_{n+1},\varrho_n-\overline{\varrho}_{n+1}}}{2!} + \ldots \leq 2.$$

From the representation of W_n and $\widetilde{W}_n$, we get

$$|||W_n|||_{m_n-\overline{m}_{n+1},\varrho_n-\overline{\varrho}_{n+1}}, |||\widetilde{W}_n|||_{m_n-\overline{m}_{n+1},\varrho_n-\overline{\varrho}_{n+1}} \leq C_n |||\varepsilon^{2^n} S_n|||^2_{m_n-\overline{m}_{n+1},\varrho_n-\overline{\varrho}_{n+1}}, \tag{4.9}$$

where $0 < C_n < 1$. By Eqs (4.7) and (4.8), if $\varepsilon > 0$ is small enough, we get

$$|||Q_{n+1}|||_{m_n-\overline{m}_{n+1},\varrho_n-\overline{\varrho}_{n+1}} \leq C|||S_n|||^2_{m_n-\overline{m}_{n+1},\varrho_n-\overline{\varrho}_{n+1}}.$$

So, by Eq (4.6), we get

$$|||Q_{n+1}|||_{m_n-\overline{m}_{n+1},\varrho_n-\overline{\varrho}_{n+1}} \leq C\varepsilon^{2^{n+1}} (\frac{\Gamma(\overline{\varrho}_{n+1})\Gamma(\overline{m}_{n+1})}{\alpha_n})^2 |||Q_n|||^2_{m_n,\varrho_n}, \tag{4.10}$$

where C is a constant. Pick

$$C_1 = \max\left\{1, \frac{C}{\alpha_0^2}\right\}, \; C_n = \left[(n+1)^{2^{-(n+1)}} n^{2^{-n}} \ldots 2^{2^{-2}} \cdot 1^{2^{-1}}\right]^2,$$

$$\Phi_n(m) = \prod_{\nu=1}^{n+1} \left[\Gamma(\overline{m}_\nu)\right]^{2^{-\nu}}, \; \Phi_n(\varrho) = \prod_{\nu=1}^{n+1} \left[\Gamma(\overline{\varrho}_\nu)\right]^{2^{-\nu}}.$$

From [7], $C_n, \Phi_n(m), \Phi_n(\varrho)$ are all convergent when $n \to +\infty$.

Consider

$$N = \max\left\{1, \sup_n (C_1 C_n \Phi_n(m)\Phi_n(\varrho))\right\} |||Q|||_{m_0,\rho_0}.$$

Then, we have $|||Q_{n+1}|||_{m_n-\overline{m}_{n+1},\varrho_n-\overline{\varrho}_{n+1}} \le N^{2^{n+2}}$. From Equation (4.6), it follows that

$$|||\epsilon^{2^n} S_n|||_{m_n-\overline{m}_n,\varrho_n-\overline{\varrho}_n} \le (\varepsilon N^2)^{2^n}. \tag{4.11}$$

Thus, if $\varepsilon N^2 < \frac{1}{2}$, then

$$|||e^{\pm\epsilon^{2^n}} S_n|||_{m_n}, \varrho_n \le 2.$$

Since

$$||A_{n+1} - A_n|| = ||\epsilon^{2^n}\overline{Q}_n|| \le |||\epsilon^{2^n} Q_n|||_{m_n}, \varrho_n < (\varepsilon N^2)^{2^n}, \tag{4.12}$$

if

$$(\varepsilon N^2)^{2^n} \le \frac{\eta\varepsilon}{(6d-1)\beta_n^2} = \frac{\eta\varepsilon}{2^{2n}(6d-1)\beta_1^2}, \tag{4.13}$$

it follows from Eq (4.13) that

$$||A_{n+1} - A_n|| \le \frac{\eta\varepsilon}{2^{2n}(6d-1)\beta_1^2},$$

for all $n \ge 1$. From Lemma 2.2, we notice that A_{n+1} has $2d$ distinct proper values $\lambda_1^{n+1}, \ldots, \lambda_{2d}^{n+1}$.

So, we get

$$|\lambda_l^{n+1} - \lambda_j^{n+1}| \ge \eta\varepsilon, l \ne j, 1 \le l, j \le 2d,$$

and

$$|\lambda_l^{n+1}| \ge \eta\varepsilon, l = 1, \ldots, 2d.$$

Actually, we have

$$\begin{aligned} |\lambda_l^{n+1} - \lambda_j^{n+1} &\ge |\lambda_l^1 - \lambda_j^1| - \sum_{s=1}^{n}(|\lambda_l^{s+1} - \lambda_l^s| + |\lambda_j^{s+1} - \lambda_j^s|), \\ &\ge |\lambda_l - \lambda_j^1| - 2\sum_{s=1}^{n} ||A_{s+1} - A_s||, \\ &\ge 2\eta\epsilon - 2(\varepsilon N^2)^{2^n}, \\ &\ge 2\eta\epsilon - 4(\varepsilon N^2)^2. \end{aligned}$$

So, if $\varepsilon \le \frac{\eta}{4N^4}$, then we obtain $2\eta\epsilon - 4(\varepsilon N^2)^2 \ge \eta\varepsilon$, and thus, we get

$$|\lambda_l^{n+1} - \lambda_j^{n+1}| \ge \eta\varepsilon, \quad l \ne j, \quad 1 \le l, j \le 2d.$$

Similarly, we can prove

$$|\lambda_l^{n+1}| \geq \eta\varepsilon, \quad 1 \leq l \leq 2d.$$

Let $D_{\frac{1}{2}m,\frac{1}{2}\varrho} = \cap_{n=0}^{\infty} D_{m_n,\varrho_n}$. Using the condition A_1 of Theorem 1.1, Eqs (4.6) and (4.11), the composition of all the transformations $e^{\varepsilon^{2^n}} S_n$ is convergent to $\psi(t)$ as $n \to \infty$.

In this way, we get

$$|||\varepsilon^{2^n} Q_n|||_{\frac{1}{2}m_0,\frac{1}{2}\varrho_0} \leq (\varepsilon N^2)^{2^n}. \tag{4.14}$$

If $0 < \varepsilon N^2 < 1$, we have that

$$\lim_{n\to\infty} (\varepsilon N^2)^{2^n} = 0.$$

Moreover, it follows from (4.12) that A_n converges always as $n \to \infty$. Define $B = \lim_{n\to\infty} A_n$. Then, at that point, using symplectic change $X = \psi(t)Y$, the Hamiltonian system (1.1) is transformed into $\dot{Y} = BY$ with constant coefficient matrix B.

Measure Estimate:

Using the iteration above, we currently demonstrate that when ε_0 is sufficiently small, non-resonant conditions

$$|\lambda_l^{n+1} - \lambda_j^{n+1} - \sqrt{-1}\langle k, \omega\rangle| \geq \frac{\alpha_n}{\Delta^3(|k|)\Delta^3([k])}, \tag{4.15}$$

$\forall\ k \in \mathbb{Z}^{\mathbb{N}} \backslash \{0\}$ and $1 \leq l, j \leq 2d$, where $n = 0, 1, 2, \dots$ and Δ is an approximation function, hold for some sufficiently small $\varepsilon \in (0, \varepsilon_*)$.

In [5], using Theorem B, Eq (4.15) holds for $n = 0$, and see that $\exists\ \bar{\varepsilon}_*$ and a non empty set $E_* \in (0, \bar{\varepsilon}_*)$ s.t. for each $\varepsilon \in E_*$, we get

$$|\lambda_l^{n+1} - \lambda_j^{n+1} - \sqrt{-1}\langle k, \omega\rangle| \geq \frac{\alpha_n}{2\Delta^3(|k|)\Delta^3([k])},$$

and $\lim_{\bar{\varepsilon}_0\to 0} \frac{meas(E_*)}{\bar{\varepsilon}_0} = 1$. Clearly, (4.15) holds.

Thus, E_* is a non-empty subset of $(0, \varepsilon_*)$. Hence, for $\varepsilon \in E_*$, $\exists$ an almost-periodic symplectic change $X = \psi(t)Y$, s.t. system (1.1) is transformed into system $\dot{Y} = BY$. Thus, the proof of Theorem 1.1 is finished. □

5. Application (Schrödinger equation)

For instance, we apply Theorem 1.1 to the following almost-periodic Schrödinger equation:

$$\frac{d^2X}{dt^2} + \varepsilon Ja(t)X = 0, \tag{5.1}$$

in which $Ja(t) = \sum Ja_\Lambda(t)$ is an almost-periodic function which is analytic on D_ϱ with frequencies $\omega = (\omega_1, \omega_2, \dots)$ and has finite spatial structure $(\tau, [\cdot])$, which depends continuously upon small parameter ε. $\bar{a}$ denotes average of $a(t)$, and suppose $\bar{a} > 0$. Consider $\frac{dx}{dt} = y$, and then at that point (5.1) can be rewritten in the same structure as

$$\frac{dX}{dt} = Y, \qquad \frac{dY}{dt} = -\varepsilon Ja(t). \tag{5.2}$$

To apply Theorem 1.1, (5.2) can be revised in the form as

$$\frac{dv}{dt} = J[A + \varepsilon Q(t)]v, \tag{5.3}$$

where

$$v = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad JA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad JQ(t) = \begin{pmatrix} 0 & -a(t) \\ 0 & 0 \end{pmatrix}. \tag{5.4}$$

It is not difficult to see that JA has multiple proper-values $\lambda_1 = \lambda_2 = 0$, and $J(A + \varepsilon\overline{Q})$ has two distinct proper values $\mu_1 = \iota\sqrt{\overline{a}\varepsilon}$, $\mu_2 = -\iota\sqrt{\overline{a}\varepsilon}$, where $\overline{Q}$ denotes the average of $Q(t)$ and $\iota = \sqrt{-1}$. Obviously, we have

$$|\mu_i| = \sqrt{\overline{a}\varepsilon} \geq \eta\varepsilon, \quad i = 1, 2, \tag{5.5}$$

$$|\mu_1 - \mu_2| = 2\sqrt{\overline{a}\varepsilon} \geq \eta\varepsilon. \tag{5.6}$$

We choose $\eta = \sqrt{\overline{a}} > 0$ as a constant which is independent from ε. Applying Theorem 1.1, the following result holds.

Theorem 5.1. *Suppose $Ja(t) = \sum Ja_\Lambda(t)$ is an almost-periodic function which is an analytic on D_ϱ with frequencies $\omega = (\omega_1, \omega_2, \ldots)$ and has finite spatial structure $(\tau, [\cdot])$, which relies upon the small parameter ε and $J\overline{a} > 0$.*

Suppose the frequencies $\omega = (\omega_1, \omega_2, \ldots)$ of $Ja(t) = \sum Ja_\Lambda(t)$ fulfill non-resonance conditions

$$|\langle k, \omega\rangle| \geq \frac{\alpha_0}{\Delta(|k|)^3\Delta([k])^3}, \quad k \in \mathbb{Z}^{\mathbb{N}}\backslash\{0\}, \tag{5.7}$$

where $\alpha_0 > 0$ is the small constant, $\tau > \mathbb{N} - 1$, and $\Delta(t)$ is the approximation function.

Then, $\exists$ some sufficiently small $\varepsilon_* > 0$, and $E_* \neq \phi$ is the positive measure Cantor subset of $(0, \varepsilon_*)$ s.t. for $\varepsilon \in E_*$, Eq (5.1) is always reducible. Also, if ε_* is sufficiently small, meas$(\frac{(0,\varepsilon_*)}{E_*})$ is nearly 1.

Note: From Theorem 5.1, it is clear that Eq (5.1) is transformed into the constant coefficient system for generally sufficiently small $\varepsilon > 0$.

Stability criterion: Presently we need to study the Lyapunov stability of the equilibrium of (5.1), using the results obtained in previous Section. If $a(t)$ is periodic in time, one well known stability criterion was discussed by Magnus and Winkler in [13] for Hills equation

$$\frac{d^2X}{dt^2} + a(t)X = 0, \tag{5.8}$$

i.e., Eq (5.8) is stable if

$$a(t) > 0, \int_0^T a(t)dt \leq \frac{4}{T}, \tag{5.9}$$

which can be proven using a Poincare' inequality. In [14], Zhang and Li generalized and improved the stability criteria which are known as L^p criteria. In [15] Zhang discussed the L^p criteria to the linear planar Hamiltonian system

$$\frac{dX}{dt} = f(t)Y, \quad \frac{dY}{dt} = -g(t)X, \tag{5.10}$$

where $f(t)$, $g(t)$ are continuous and T -periodic functions.

For quasi-periodic systems, Xue and Zhao in [9] proved the stability of the equilibrium of Eq (5.1). However, for almost-periodic Eq (5.1), the above results can not be applied straightforwardly. Then, we get an outcome about the stability of the equilibrium of (5.1).

Theorem 5.2. *Using the conditions of Theorem 5.1, in the sense of Lyapunov, the equilibrium of Eq (5.1) is stable for generally sufficiently small* $\varepsilon > 0$.

Proof: We know that from Theorem 5.1, for generally sufficiently small $\varepsilon > 0, \varepsilon \in (0, \ \varepsilon_*)$, $\exists$ an analytic symplectic change $v = \psi(t)v_1$, in which $\psi(t) = \sum \psi_\Lambda(t)$ has similar frequencies and finite spatial structure $(\tau, [\cdot])$ like $Q(t)$, which converts Eq (5.3) into the equation

$$\frac{dv_1}{dt} = Bv_1, \tag{5.11}$$

where B is the constant matrix. In addition, from proof of Theorem 1.1, it follows that B has two distinct proper values λ_1^1, λ_2^1 fulfilling

$$|\lambda_i^1| \geq \eta\varepsilon \ i = 1, 2, \ |\lambda_1^1 - \lambda_2^1| \geq \eta\varepsilon. \tag{5.12}$$

Moreover, from the proof of Theorem 1.1, we get

$$||B - J(A + \varepsilon\overline{Q})|| \leq (\varepsilon N^2)^2 = O(\varepsilon^2). \tag{5.13}$$

Subsequently, the matrix B has two distinct pure imaginary proper values and can be written as:

$$\lambda_i^1 = \pm\iota\sqrt{b}, \ i = 1, 2, \tag{5.14}$$

where b can be written in the following form:

$$b = \overline{a}\varepsilon + O(\varepsilon^2), \tag{5.15}$$

which relies upon $\overline{a}$ and ε only. Hence, $\exists$ a particular symplectic matrix S such that

$$S^{-1}BS = diag(\iota\sqrt{b}, -\iota\sqrt{b}). \tag{5.16}$$

Let $v_\infty = S\overline{v}_\infty$, and using the symplectic change $v_\infty = S\overline{v}_\infty$, system (5.11) is changed as

$$\frac{d\overline{v}_\infty}{dt} = S^{-1}BS\overline{v}_\infty = \begin{pmatrix} \iota\sqrt{b} & 0 \\ 0 & -\iota\sqrt{b} \end{pmatrix}\overline{v}_\infty. \tag{5.17}$$

Subsequently, by an analytic almost-periodic symplectic change, Eq (5.1) is transformed into

$$\frac{d^2X_\infty}{dt^2} + bX_\infty = 0. \tag{5.18}$$

It is not difficult to see that (5.18) is elliptic. Accordingly, equilibrium of (5.1) is stable in the sense of Lyapunov for generally sufficiently small $\varepsilon > 0$. □

See the quasi-periodic solution of equation of (5.1) in [9]. Lastly, for the presence of almost-periodic solution of Eq (5.1), we have the following result:

Theorem 5.3. *Using the conditions of Theorem 5.1, all solutions of equation (5.1) are almost-periodic with frequencies* $\Omega = (\sqrt{b}, \omega_1, \omega_2, \ldots)$ *for generally sufficiently small* $\varepsilon > 0$, *where* b *can be seen in (5.15).*

Proof: Using Theorem 5.1, we know that, for generally sufficiently small $\varepsilon \in (0, \varepsilon_*)$, $\exists$ an analytic almost-periodic symplectic change having similar frequencies and finite spatial structure like $Ja(t)$, by this change, Eq (5.1) is converted into (5.18). Then again, it is not difficult to see that all solutions of Eq (5.18) are periodic, and the frequency of these solutions is $\sqrt{b}$.

Now, we just have to show that, for generally sufficiently small $\varepsilon \in (0, \varepsilon_*)$, the accompanying non-resonant condition

$$|k_1\omega_1 + k_2\omega_2 + \ldots + k_{\mathbb{N}}\omega_{\mathbb{N}} + k_{\mathbb{N}+1}\sqrt{b}| \geq \frac{\alpha_1}{\Delta^3(|k|)\Delta^3([k])} \tag{5.19}$$

holds for all $k \in \mathbb{Z}^{\mathbb{N}+1}\backslash\{0\}$ and for generally sufficiently small $\varepsilon \in (0, \varepsilon_*)$, where $\alpha_1 = \frac{\alpha_0}{4}$, $\Delta(t)$ is an approximation function, and $(\sqrt{b}, \omega_1, \omega_2, \ldots)$ are basic frequencies of $Ja(t)$. If $k_{\mathbb{N}+1} = 0$, then from the non-resonance condition (5.7), it follows that (5.19) holds.

If $k_{\mathbb{N}+1} \neq 0$, from Theorem B in [5], Eq (5.19) holds; and it can be seen that $\exists\, \bar{\varepsilon}_*$ and a non empty set $E_* \in (0, \bar{\varepsilon}_*)$ s.t. for each $\varepsilon \in E_*$, we get

$$|k_1\omega_1 + k_2\omega_2 + \ldots + k_{\mathbb{N}}\omega_{\mathbb{N}} + k_{\mathbb{N}+1}\sqrt{b}| \geq \frac{\alpha_0}{4\Delta^3(|k|)\Delta^3([k])},$$

and $\lim_{\bar{\varepsilon}_* \to 0} \frac{meas(E_*)}{\bar{\varepsilon}_*} = 1$. Clearly, (5.19) holds.

Hence, all solutions of Equation (5.1) are almost-periodic with frequencies $\Omega = (\sqrt{b}, \omega_1, \omega_2, \ldots)$ for generally sufficiently small $\varepsilon > 0$.

6. Conclusions

In this research work, we discussed the reducibility of almost-periodic Hamiltonian systems and proved that the almost-periodic linear Hamiltonian system (1.1) is reduced to a constant coefficients Hamiltonian system by means of an almost-periodic symplectic transformation. The result was proved for sufficiently small parameter ε by using some non-resonant conditions, non-degeneracy conditions and the rapidly convergent method that is KAM iterations. The result was also verified for Schrödinger equation.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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