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Research article

Some new versions of Jensen, Schur and Hermite-Hadamard type inequalities for $(\mathfrak{p}, \mathfrak{J})$-convex fuzzy-interval-valued functions

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Abstract: To create various kinds of inequalities, the idea of convexity is essential. Convexity and integral inequality hence have a significant link. This study's goals are to introduce a new class of generalized convex fuzzy-interval-valued functions (convex $FIVF$s) which are known as $(\mathfrak{p}, \mathfrak{J})$-convex $FIVF$s and to establish Jensen, Schur and Hermite-Hadamard type inequalities for $(\mathfrak{p}, \mathfrak{J})$-convex $FIVF$s using fuzzy order relation. The Kulisch-Miranker order relation, which is based on interval space, is used to define this fuzzy order relation level-wise. Additionally, we have demonstrated that, as special examples, our conclusions encompass a sizable class of both new and well-known inequalities for $(\mathfrak{p}, \mathfrak{J})$-convex $FIVF$s. We offer helpful examples that demonstrate the theory created in this study's application. These findings and various methods might point the way in new directions for modeling, interval-valued functions and fuzzy optimization issues.

Keywords: $(\mathfrak{p}, \mathfrak{J})$-convex fuzzy-interval-valued function; fuzzy Riemann integral; Jensen type inequality; Schur type inequality; Hermite-Hadamard type inequality; Hermite-Hadamard-Fejér type inequality
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1. Introduction

Integral inequality is well recognized to be important in both pure and practical mathematics; for examples, see [1–10]. The behavior of inequalities makes it clear that mathematical approaches are useless in the absence of inequalities. For this reason, exact inequalities are now required in order to demonstrate the validity and uniqueness of the mathematical procedures. Convexity also plays a big part in the subject of inequalities, owing to the behavior of its definition.

Let K be a convex set. Then, a real valued function $\mathfrak{G}: K \to \mathbb{R}$ is named as convex on K if the inequality

$$\mathfrak{G}(\mathfrak{s}\mathfrak{x} + (1-\mathfrak{s})y) \leq \mathfrak{s}\mathfrak{G}(\mathfrak{x}) + (1-\mathfrak{s})\mathfrak{G}(y) \tag{1}$$

holds for all $\mathfrak{x}, y \in K, \mathfrak{s} \in [0,1]$. If $\mathfrak{G}$ is concave, then $-\mathfrak{G}$ is convex. Over the years, convex sets and convex functions have been modified to a remarkable variety of convexities, such as harmonic convexity [11], quasi convexity [12], Schur convexity [13,14], strong convexity [15,16], p-convexity [17], generalized convexity [18] and so on. For more information, see [19–34] and the references therein.

The Jensen inequality [35,36] is one of these inequalities for convex functions, and it can be stated as follows.

Let $\mathfrak{w}_j \in [0,1]$, $\mathfrak{a}_j \in [\mathfrak{a}, \mathfrak{b}]$, $(j = 1, 2, 3, \dots \kappa, \kappa \geq 2)$ and $\mathfrak{G}$ be a convex function. Then,

$$\mathfrak{G}\left(\sum_{j=1}^{\kappa} \mathfrak{w}_j\, \mathfrak{x}_j\right) \leq \left(\sum_{j=1}^{\kappa} \mathfrak{w}_j\, \mathfrak{G}(\mathfrak{x}_j)\right), \tag{2}$$

with $\sum_{j=1}^{\kappa} \mathfrak{w}_j = 1$. If $\mathfrak{G}$ is concave, then inequality (2) is reversed.

Research on the idea of convexity with integral problems is fascinating. As a result, several writers have proved numerous equalities or inequalities as applications of convex functions. Among the notable outcomes are the Gagliardo-Nirenberg inequality [37], the Hardy inequality [38], the Ostrowski inequality [39], the Olsen inequality [40] and the Hermite-Hadamard inequality (HH-inequality, in short) [41]. The HH-inequality is an interesting outcome in convex analysis which is formulated for convex function $\mathfrak{G}: K \to \mathbb{R}^+$ on an interval $K = [\mathfrak{a}, \mathfrak{b}]$ by

$$\mathfrak{G}\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right) \leq \frac{1}{\mathfrak{b}-\mathfrak{a}} \int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{G}(\mathfrak{x})d\mathfrak{x} \leq \frac{\mathfrak{G}(\mathfrak{a}) + \mathfrak{G}(\mathfrak{b})}{2}, \tag{3}$$

for all $\mathfrak{a}, \mathfrak{b} \in K$. If $\mathfrak{G}$ is concave, then inequality (3) is reversed.

Fejér created the Hermite-Hadamard-Fejér inequality (HH-Fejér inequality), which is the most significant weighted extension of the HH-inequality, in [42].

Let $\mathfrak{G}: [\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}^+$ be a convex function on a convex set K and $\mathfrak{a}, \mathfrak{b} \in K$ with $\mathfrak{a} \leq \mathfrak{b}$. Then,

$$\mathfrak{G}\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right) \leq \frac{1}{\int_{\mathfrak{a}}^{\mathfrak{b}} \Omega(\mathfrak{x})d\mathfrak{x}} \int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{G}(\mathfrak{x})\Omega(\mathfrak{x})d\mathfrak{x} \leq \frac{\mathfrak{G}(\mathfrak{a}) + \mathfrak{G}(\mathfrak{b})}{2} \int_{\mathfrak{a}}^{\mathfrak{b}} \Omega(\mathfrak{x}))d\mathfrak{x}. \tag{4}$$

If $\Omega(\mathfrak{x}) = 1$, then we obtain (3) from (4). With the assistance of inequality (4), many inequalities can be obtained through special symmetric function $\Omega(\mathfrak{x})$ for convex functions.

Meanwhile, to increase the accuracy of measurement findings and to carry out error analysis automatically, interval analysis has been proposed and researched by Moore [43], Kulish and Miranker [44], and they have substituted interval operations with real operations. In this field, an interval of real numbers is used to represent an uncertain variable. Following their research, numerous authors concentrated on the literature and employed this idea in various contexts. An h-convex interval-valued function (h-convex IVF) was first proposed in 2018 by Zhao et al. [45], who demonstrated that convex IVF is a specific example of the HH-inequality.

Let $\mathfrak{G}: [\mathfrak{a}, \mathfrak{b}] \subset \mathbb{R} \to \mathcal{K}_C^{+}$ be a convex IVF given by $\mathfrak{G}(\mathfrak{x}) = [\mathfrak{G}_*(\mathfrak{x}), \mathfrak{G}^*(\mathfrak{x})]$ for all $\mathfrak{x} \in [\mathfrak{a}, \mathfrak{b}]$, where $\mathfrak{G}_*(\mathfrak{x})$ is a convex function, and $\mathfrak{G}^*(\mathfrak{x})$ is a concave function. If $\mathfrak{G}$ is Riemann integrable, then

$$\mathfrak{G}\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right) \supseteq \frac{1}{\mathfrak{b}-\mathfrak{a}}\ (IR)\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{G}(\mathfrak{x})d\mathfrak{x} \supseteq \frac{\mathfrak{G}(\mathfrak{a})+\mathfrak{G}(\mathfrak{b})}{2}. \tag{5}$$

We refer readers to [46–63] and the references therein for more examination of the literature on the uses and characteristics of generalized convex functions and HH-integral inequalities.

Operation research, computer science, management sciences, artificial intelligence, control engineering and decision sciences are just a few of the applied sciences and pure mathematics problems that are studied in [64], where a large amount of research on fuzzy sets and systems has been devoted to the development of various fields. Similar to this, the concepts of convexity and non-convexity are crucial in optimization in the fuzzy domain because they allow us to distinguish the optimality condition of convexity and produce fuzzy variational inequalities. As a result, the theories of variational inequality and fuzzy complementary problems have powerful mechanisms of mathematical problems and a cordial relationship. This field is fascinating and has produced many writers. Additionally, the concepts of convex fuzzy mapping and finding its optimality condition with the aid of fuzzy variational inequality were studied by Nanda and Kar [65] and Chang [66]. Fuzzy convexity's generalization and extension are crucial to its application in a variety of contexts. Let's remark that preinvex fuzzy mapping is one of the most often discussed kinds of nonconvex fuzzy mapping. This concept was first proposed by Noor [67], who also demonstrated some findings that show how fuzzy variational-like inequality distinguishes the fuzzy optimality condition of differentiable fuzzy preinvex mappings. For a more in-depth review of the literature on the uses and characterization of generalized convex fuzzy mappings and variational-like inequalities, see [68–85] and the references therein.

Fuzzy-interval-valued functions are the fuzzy mappings. There are certain integrals that deal with $FIVF$s and have $FIVF$s as their integrands. For instance, Oseuna-Gomez et al. [86] and Costa et al. [87] created the Kulisch-Miranker order relation to establish Jensen's integral inequality for $FIVF$s. Costa and Floures provided Minkowski and Beckenbach's inequalities, where the integrands are $FIVF$s, using the same methodology. Because Costa et al. [88] established a relationship between the components of fuzzy-interval space and interval space and introduced a level-wise fuzzy order relation on fuzzy-interval space through the Kulisch-Miranker order relation defined on interval space, these authors were particularly inspired by their work. By creating a fuzzy-interval integral inequality for extended convex $FIVF$, where the integrands are $(\mathfrak{p}, \mathfrak{J})$-convex $FIVF$, we generalize the integral inequalities (2)–(4). For more information related to fuzzy sets, fuzzy functions and inequalities, see [89–101] and the references therein.

The structure of this study is as follows: Preliminary ideas and findings in interval space, the space of fuzzy intervals and convex analysis are presented in Section 2. Additionally, the brand-new idea of $(\mathfrak{p}, \mathfrak{J})$-convex $FIVF$s is also presented. For $(\mathfrak{p}, \mathfrak{J})$-convex $FIVF$s, Section 3 derives discrete Jensen and Schur type inequalities. Using fuzzy Riemann integrals, Section 4 derives fuzzy-interval HH-inequalities for $(\mathfrak{p}, \mathfrak{J})$-convex $FIVF$s. To support our findings, some compelling instances are also provided. Section 5 gives conclusions and future plans.

2. Definitions and basic results

In this section, we first provide a few definitions, rough notations and findings that will be beneficial for future research. Then, we give new $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$s definitions and properties.
Let $\mathcal{K}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $\mathfrak{o} \in \mathcal{K}_C$ be defined by

$$\mathfrak{o} = [\mathfrak{o}_*, \mathfrak{o}^*] = \{\mathfrak{x} \in \mathbb{R} |\ \mathfrak{o}_* \leq \mathfrak{x} \leq \mathfrak{o}^*\},\ (\mathfrak{o}_*, \mathfrak{o}^* \in \mathbb{R}).$$

If $\mathfrak{o}_* = \mathfrak{o}^*$, then $\mathfrak{o}$ is named as degenerate. In this article, all intervals will be non-degenerate intervals. If $\mathfrak{o}_* \geq 0$, then $[\mathfrak{o}_*, \mathfrak{o}^*]$ is named as a positive interval. The set of all positive intervals is denoted by $\mathcal{K}_C^+$ and defined as $\mathcal{K}_C^+ = \{[\mathfrak{o}_*, \mathfrak{o}^*]: [\mathfrak{o}_*, \mathfrak{o}^*] \in \mathbb{R}_I \text{ and } \mathfrak{o}_* \geq 0\}$.

Let $\rho \in \mathbb{R}$ and $\rho\mathfrak{o}$ be defined by

$$\rho.\mathfrak{o} = \begin{cases} [\rho\mathfrak{o}_*, \rho\mathfrak{o}^*] \text{ if } \rho \geq 0, \\ [\rho\mathfrak{o}^*, \rho\mathfrak{o}_*] \text{ if } \rho < 0. \end{cases}$$

Then, the Minkowski difference $\mathfrak{q} - \mathfrak{o}$, addition $\mathfrak{o} + \mathfrak{q}$ and $\mathfrak{o} \times \mathfrak{q}$ for $\mathfrak{o}, \mathfrak{q} \in \mathcal{K}_C$ are defined by

$$[\mathfrak{q}_*, \mathfrak{q}^*] - [\mathfrak{o}_*, \mathfrak{o}^*] = [\mathfrak{q}_* - \mathfrak{o}_*, \quad \mathfrak{q}^* - \mathfrak{o}^*],$$

$$[\mathfrak{q}_*, \mathfrak{q}^*] + [\mathfrak{o}_*, \mathfrak{o}^*] = [\mathfrak{q}_* + \mathfrak{o}_*, \quad \mathfrak{q}^* + \mathfrak{o}^*],$$

and $[\mathfrak{q}_*, \mathfrak{q}^*] \times [\mathfrak{o}_*, \mathfrak{o}^*] = [min\{\mathfrak{q}_*\mathfrak{o}_*, \mathfrak{q}^*\mathfrak{o}_*, \mathfrak{q}_*\mathfrak{o}^*, \mathfrak{q}^*\mathfrak{o}^*\},\ max\{\mathfrak{q}_*\mathfrak{o}_*, \mathfrak{q}^*\mathfrak{o}_*, \mathfrak{q}_*\mathfrak{o}^*, \mathfrak{q}^*\mathfrak{o}^*\}]$. The inclusion "⊆" means that $\mathfrak{q} \subseteq \mathfrak{o}$ If and only if, $[\mathfrak{q}_*, \mathfrak{q}^*] \subseteq [\mathfrak{o}_*, \mathfrak{o}^*]$, if and only if $\mathfrak{o}_* \leq \mathfrak{q}_*,\ \mathfrak{q}^* \leq \mathfrak{o}^*$.

Remark 2.1. [44] The relation " $\leq_I$ " is defined on $\mathcal{K}_C$ by

$$[\mathfrak{q}_*, \mathfrak{q}^*] \leq_I [\mathfrak{o}_*, \mathfrak{o}^*] \text{ if and only if } \mathfrak{q}_* \leq \mathfrak{o}_*,\ \mathfrak{q}^* \leq \mathfrak{o}^*,$$

for all $[\mathfrak{q}_*, \mathfrak{q}^*],\ [\mathfrak{o}_*, \mathfrak{o}^*] \in \mathcal{K}_C$, and it is an order relation. For given $[\mathfrak{q}_*, \mathfrak{q}^*],\ [\mathfrak{o}_*, \mathfrak{o}^*] \in \mathcal{K}_C$, we say that $[\mathfrak{q}_*, \mathfrak{q}^*] \leq_I [\mathfrak{o}_*, \mathfrak{o}^*]$ if and only if $\mathfrak{q}_* \leq \mathfrak{o}_*,\ \mathfrak{q}^* \leq \mathfrak{o}^*$ or $\mathfrak{q}_* \leq \mathfrak{o}_*,\ \mathfrak{q}^* < \mathfrak{o}^*$.

For $[\mathfrak{q}_*, \mathfrak{q}^*],\ [\mathfrak{o}_*, \mathfrak{o}^*] \in \mathbb{R}_I$, the Hausdorff-Pompeiu distance between intervals $[\mathfrak{q}_*, \mathfrak{q}^*]$ and $[\mathfrak{o}_*, \mathfrak{o}^*]$ is defined by

$$d([\mathfrak{q}_*, \mathfrak{q}^*], [\mathfrak{o}_*, \mathfrak{o}^*]) = max\{[\mathfrak{q}_*, \mathfrak{q}^*], [\mathfrak{o}_*, \mathfrak{o}^*]\}.$$

It is a familiar fact that $(\mathbb{R}_I, d)$ is a complete metric space.
The concept of a Riemann integral for IVF first introduced by Moore [43] is defined as follows:

Theorem 2.2. [43] If $\mathfrak{G}: [\mathfrak{a}, \mathfrak{b}] \subset \mathbb{R} \to \mathbb{R}_I$ is an IVF such that $\mathfrak{G}(\mathfrak{x}) = [\mathfrak{G}_*(\mathfrak{x}), \mathfrak{G}^*(\mathfrak{x})]$, then $\mathfrak{G}$ is Riemann integrable over $[\mathfrak{a}, \mathfrak{b}]$ if and only if $\mathfrak{G}_*(\mathfrak{x})$ and $\mathfrak{G}^*(\mathfrak{x})$ both are Riemann integrable over $[\mathfrak{a}, \mathfrak{b}]$ such that

$$(IR)\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{G}(\mathfrak{x})d\mathfrak{x} = \left[(R)\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{G}_*(\mathfrak{x})d\mathfrak{x}, (R)\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{G}^*(\mathfrak{x})d\mathfrak{x}\right].$$

The collections of all Riemann integrable real valued functions and Riemann integrable IVFs are denoted by $\mathcal{R}_{[\mathfrak{a},\mathfrak{b}]}$ and $\mathcal{IR}_{[\mathfrak{a},\mathfrak{b}]}$, respectively.

Let $\mathbb{R}$ be the set of real numbers. A fuzzy subset set $\mathcal{A}$ of $\mathbb{R}$ is distinguished by a function $\mathfrak{g}: \mathbb{R} \to [0,1]$ called the membership function. In this study, this depiction is approved. Moreover, the collection of all fuzzy subsets of $\mathbb{R}$ is denoted by $\mathbb{F}(\mathbb{R})$.

A real fuzzy-interval $\mathfrak{g}$ is a fuzzy set in $\mathbb{R}$ with the following properties:

(1) $\mathfrak{g}$ is normal, i.e., there exists $\mathfrak{x} \in \mathbb{R}$ such that $\mathfrak{g}(\mathfrak{x}) = 1$.

(2) $\mathfrak{g}$ is upper semi continuous, i.e., for given $\mathfrak{x} \in \mathbb{R}$, and for every $\mathfrak{x} \in \mathbb{R}$ there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|\mathfrak{x} - y| < \delta$.

(3) $\mathfrak{g}$ is fuzzy convex, i.e., $\mathfrak{g}\big((1-\mathfrak{s})\mathfrak{x} + \mathfrak{s}y\big) \geq min\big(\mathfrak{g}(\mathfrak{x}), \mathfrak{g}(y)\big),\ \forall\ \mathfrak{x}, y \in \mathbb{R}$ and $\mathfrak{s} \in [0,1]$.

(4) $\mathfrak{g}$ is compactly supported, i.e., $cl\{\mathfrak{x} \in \mathbb{R}|\ \mathfrak{g}(\mathfrak{x}) > 0\}$ is compact.
The collection of all real fuzzy-intervals is denoted by $\mathbb{F}_C(\mathbb{R})$.
Since $\mathbb{F}_C(\mathbb{R})$ denotes the set of all real fuzzy-intervals and let $\mathfrak{g} \in \mathbb{F}_C(\mathbb{R})$ be real fuzzy-interval, if and only if, $\jmath$-levels $[\mathfrak{g}]^{\jmath}$ is a nonempty compact convex set of $\mathbb{R}$. This is represented by

$$[\mathfrak{g}]^{\jmath} = \{\mathfrak{x} \in \mathbb{R}|\ \mathfrak{g}(\mathfrak{x}) \geq \jmath\},$$

and from these definitions, we have

$$[\mathfrak{g}]^{\jmath} = [\mathfrak{g}_*(\jmath), \mathfrak{g}^*(\jmath)],$$

where

$$\mathfrak{g}_*(\jmath) = inf\{\mathfrak{x} \in \mathbb{R}|\ \mathfrak{g}(\mathfrak{x}) \geq \jmath\},\ \mathfrak{g}^*(\jmath) = sup\{\mathfrak{x} \in \mathbb{R}|\ \mathfrak{g}(\mathfrak{x}) \geq \jmath\}.$$

Proposition 2.3. [88] Let $\mathfrak{g}, \mathfrak{d} \in \mathbb{F}_C(\mathbb{R})$. Then, relation "$\preccurlyeq$" given on $\mathbb{F}_C(\mathbb{R})$ by

$$\mathfrak{g} \preccurlyeq \mathfrak{d} \text{ if and only if } [\mathfrak{g}]^{\jmath} \leq_I [\mathfrak{d}]^{\jmath} \text{ for all } \jmath \in [0, 1]$$

is a partial order relation.

We now discuss some properties of real fuzzy-intervals under addition, scalar multiplication, multiplication and division. If $\mathfrak{g}, \mathfrak{d} \in \mathbb{F}_C(\mathbb{R})$ and $\rho \in \mathbb{R}$, then arithmetic operations are defined by

$$[\mathfrak{g} \widetilde{+} \mathfrak{d}]^{\jmath} = [\mathfrak{g}]^{\jmath} + [\mathfrak{d}]^{\jmath}, \tag{6}$$

$$[\mathfrak{g} \widetilde{\times} \mathfrak{d}]^{\jmath} = [\mathfrak{g}]^{\jmath} \times [\mathfrak{d}]^{\jmath}, \tag{7}$$

$$[\rho.\mathfrak{g}]^{\jmath} = \rho.[\mathfrak{g}]^{\jmath}. \tag{8}$$

Remark 2.4. Obviously, $\mathbb{F}_C(\mathbb{R})$ is closed under addition and nonnegative scalar multiplication, and the above defined properties on $\mathbb{F}_C(\mathbb{R})$ are equivalent to those derived from the usual extension principle. Furthermore, for each scalar number $\rho \in \mathbb{R}$,

$$[\rho \widetilde{+} \mathfrak{g}]^{\jmath} = \rho + [\mathfrak{g}]^{\jmath}. \tag{9}$$

Theorem 2.5. [71,74] The space $\mathbb{F}_C(\mathbb{R})$ dealing with a supremum metric, i.e., for $\psi, \mathfrak{d} \in \mathbb{F}_C(\mathbb{R})$

$$\mathfrak{D}(\psi, \mathfrak{d}) = \sup_{0 \leq \jmath \leq 1} H([\mathfrak{g}]^{\jmath}, [\mathfrak{d}]^{\jmath}), \tag{10}$$

is a complete metric space, where H denotes the well-known Hausdorff metric on the space of intervals.
Definition 2.6. [88] A mapping $\mathfrak{G}: K \subset \mathbb{R} \to \mathbb{F}_C(\mathbb{R})$ is named as $FIVF$. For each $\jmath \in [0, 1]$, whose $\jmath$-levels define the family of IVFs $\mathfrak{G}_{\jmath}: K \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\mathfrak{G}_{\jmath}(\mathfrak{x}) = [\mathfrak{G}_*(\mathfrak{x}, \jmath), \mathfrak{G}^*(\mathfrak{x}, \jmath)]$ for all $\mathfrak{x} \in K$. Here, for each $\jmath \in [0, 1]$, the end point real functions $\mathfrak{G}_*(\mathfrak{x}, \jmath), \mathfrak{G}^*(\mathfrak{x}, \jmath): K \to \mathbb{R}$ are called lower and upper functions of $\mathfrak{G}$.
Remark 2.7. Let $\mathfrak{G}: K \subset \mathbb{R} \to \mathbb{F}_C(\mathbb{R})$ be a $FIVF$. Then, $\mathfrak{G}(\mathfrak{x})$ is named as continuous at $\mathfrak{x} \in K$ if for each $\jmath \in [0, 1]$, both end point functions $\mathfrak{G}_*(\mathfrak{x}, \jmath)$ and $\mathfrak{G}^*(\mathfrak{x}, \jmath)$ are continuous at $\mathfrak{x} \in K$.
From the above literature review, the following results can be concluded (see [88,44,43,86]):
Definition 2.8. Let $\mathfrak{G}: [c, d] \subset \mathbb{R} \to \mathbb{F}_C(\mathbb{R})$ be a $FIVF$. The fuzzy Riemann integral of $\mathfrak{G}$ over $[c, d]$, denoted by $(FR) \int_c^d \mathfrak{G}(\mathfrak{x}) d\mathfrak{x}$, is defined level-wise by

$$\left[(FR) \int_c^d \mathfrak{G}(\mathfrak{x}) d\mathfrak{x}\right]^{\jmath} = (IR) \int_c^d \mathfrak{G}_{\jmath}(\mathfrak{x}) d\mathfrak{x} = \left\{\int_c^d \mathfrak{G}(\mathfrak{x}, \jmath) d\mathfrak{x} : \mathfrak{G}(\mathfrak{x}, \jmath) \in \mathcal{R}_{[c,d]}\right\}, \tag{11}$$

for all $\jmath \in [0, 1]$, where $\mathcal{R}_{[c,d]}$ is the collection of end point functions of IVFs. $\mathfrak{G}$ is (FR)-integrable over $[c, d]$ if $(FR) \int_c^d \mathfrak{G}(\mathfrak{x}) d\mathfrak{x} \in \mathbb{F}_C(\mathbb{R})$. Note that if both end point functions are

Lebesgue-integrable, then $\mathfrak{G}$ is fuzzy Aumann-integrable, see [81].

Theorem 2.9. Let $\mathfrak{G}: [c, d] \subset \mathbb{R} \to \mathbb{F}_C(\mathbb{R})$ be a $FIVF$, whose $\jmath$-levels define the family of IVFs $\mathfrak{G}_\jmath: [c, d] \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\mathfrak{G}_\jmath(\mathfrak{x}) = [\mathfrak{G}_*(\mathfrak{x}, \jmath), \mathfrak{G}^*(\mathfrak{x}, \jmath)]$ for all $\mathfrak{x} \in [c, d]$ and for all $\jmath \in [0, 1]$. Then, $\mathfrak{G}$ is (FR)-integrable over $[c, d]$ if and only if, $\mathfrak{G}_*(\mathfrak{x}, \jmath)$ and $\mathfrak{G}^*(\mathfrak{x}, \jmath)$ both are R-integrable over $[c, d]$. Moreover, if $\mathfrak{G}$ is (FR)-integrable over $[c, d]$, then

$$\left[(FR)\int_c^d \mathfrak{G}(\mathfrak{x})d\mathfrak{x}\right]^\jmath = \left[(R)\int_c^d \mathfrak{G}_*(\mathfrak{x},\jmath)d\mathfrak{x}, (R)\int_c^d \mathfrak{G}^*(\mathfrak{x},\jmath)d\mathfrak{x}\right] = (IR)\int_c^d \mathfrak{G}_\jmath(\mathfrak{x})d\mathfrak{x}, \tag{12}$$

for all $\jmath \in [0, 1]$.

The families of all (FR)-integrable $FIVF$s and R-integrable functions over $[c, d]$ are denoted by $\mathcal{FR}_{([c,d],\jmath)}$ and $\mathcal{R}_{([c,d],\jmath)}$, for all $\jmath \in [0, 1]$.

Definition 2.10. [51] A function $\mathfrak{G}: [\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}^+$ is named as a P-convex function if

$$\mathfrak{G}(\mathfrak{s}\mathfrak{x} + (1 - \mathfrak{s})y) \le \mathfrak{G}(\mathfrak{x}) + \mathfrak{G}(y), \tag{13}$$

for all $\mathfrak{x}, y \in [\mathfrak{a}, \mathfrak{b}], \mathfrak{s} \in [0, 1]$. If (13) is reversed, then $\mathfrak{G}$ is named as P-concave.

Definition 2.11. [47] A function $\mathfrak{G}: K \to \mathbb{R}^+$ is named as an s-convex function in the second sense if

$$\mathfrak{G}(\mathfrak{s}\mathfrak{x} + (1 - \mathfrak{s})y) \le \mathfrak{s}^s\mathfrak{G}(\mathfrak{x}) + (1 - \mathfrak{s})^s\mathfrak{G}(y), \tag{14}$$

for all $\mathfrak{x}, y \in [\mathfrak{a}, \mathfrak{b}], \mathfrak{s} \in [0, 1]$, where $s \in (0, 1)$. If (14) is reversed, then $\mathfrak{G}$ is named as s-concave in the second sense.

Definition 2.12. [55] A function $\mathfrak{G}: [\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}^+$ is named as a $\mathfrak{J}$-convex function if for all $\mathfrak{x}, y \in [\mathfrak{a}, \mathfrak{b}], \mathfrak{s} \in [0, 1]$, we have

$$\mathfrak{G}(\mathfrak{s}\mathfrak{x} + (1 - \mathfrak{s})y) \le \mathfrak{J}(\mathfrak{s})\,\mathfrak{G}(\mathfrak{x}) + \mathfrak{J}(1 - \mathfrak{s})\,\mathfrak{G}(y), \tag{15}$$

where $\mathfrak{J}: \mathcal{L} \to \mathbb{R}^+$ such that $\mathfrak{J} \not\equiv 0$, $[0, 1] \subseteq \mathcal{L}$. If (15) is reversed, then $\mathfrak{G}$ is named as $\mathfrak{J}$-concave in the second sense.

A function $\mathfrak{J}: \mathcal{L} \to \mathbb{R}^+$ is named as super multiplicative if for all $\mathfrak{x}, y \in \mathcal{L}$, we have

$$\mathfrak{J}(\mathfrak{x}y) \ge \mathfrak{J}(\mathfrak{x})\mathfrak{J}(y). \tag{16}$$

If (16) is reversed, then $\mathfrak{J}$ is known as sub multiplicative. If the equality holds in (16), then $\mathfrak{J}$ is named as multiplicative.

Definition 2.13. [17] Let $\mathfrak{p} \in \mathbb{R}$ with $\mathfrak{p} \neq 0$. Then, the interval $K_\mathfrak{p}$ is named as $\mathfrak{p}$-convex if

$$[\mathfrak{s}\mathfrak{x}^\mathfrak{p} + (1 - \mathfrak{s})y^\mathfrak{p}]^{\frac{1}{\mathfrak{p}}} \in K_\mathfrak{p}, \tag{17}$$

for all $\mathfrak{x}, y \in K_\mathfrak{p}, \mathfrak{s} \in [0, 1]$, where $\mathfrak{p} = 2n + 1$ and $n \in N$.

Definition 2.14. [17] Let $\mathfrak{p} \in \mathbb{R}$ with $\mathfrak{p} \neq 0$ and $K_\mathfrak{p} = [\mathfrak{a}, \mathfrak{b}] \subseteq \mathbb{R}$. Then, the function $\mathfrak{G}: [\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}^+$is named as a $\mathfrak{p}$-convex function if

$$\mathfrak{G}\left([\mathfrak{s}\mathfrak{x}^\mathfrak{p} + (1 - \mathfrak{s})y^\mathfrak{p}]^{\frac{1}{\mathfrak{p}}}\right) \le \mathfrak{s}\mathfrak{G}(\mathfrak{x}) + (1 - \mathfrak{s})\mathfrak{G}(y), \tag{18}$$

for all $\mathfrak{x}, y \in [\mathfrak{a}, \mathfrak{b}], \mathfrak{s} \in [0, 1]$. If the inequality (18) is reversed, then $\mathfrak{G}$ is named as a $\mathfrak{p}$-concave function.

Definition 2.15. [52] Let $K_\mathfrak{p}$ be a $\mathfrak{p}$-convex set and $\mathfrak{J}: [0, 1] \subseteq \mathcal{L} \to \mathbb{R}^+$ be a nonnegative real-valued function such that $\mathfrak{J} \not\equiv 0$, where $\mathcal{L} \subseteq \mathbb{R}$. Then, function $\mathfrak{G}: K_\mathfrak{p} \to \mathbb{R}$ is named as $(\mathfrak{p}, \mathfrak{J})$-convex on $K_\mathfrak{p}$ such that

$$\mathfrak{G}\left(\left[\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)\leq \mathfrak{J}(\mathfrak{s})\,\mathfrak{G}(\mathfrak{x})+\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}(y), \tag{19}$$

for all $\mathfrak{x},y\in K_{\mathfrak{p}}=[\mathfrak{a},\mathfrak{b}],\mathfrak{s}\in[0,1]$, where $\mathfrak{G}(\mathfrak{x})\geq 0$ and $\mathfrak{J}:\mathcal{L}\to\mathbb{R}^{+}$ such that $\mathfrak{J}\not\equiv 0$ and $[0,1]\subseteq\mathcal{L}$. If (19) is reversed, then $\mathfrak{G}$ is named as $(\mathfrak{p},\mathfrak{J})$-concave on $[\mathfrak{a},\mathfrak{b}]$. $\mathfrak{G}$ is $(\mathfrak{p},\mathfrak{J})$-affine if and only if it is both a $(\mathfrak{p},\mathfrak{J})$-convex and $(\mathfrak{p},\mathfrak{J})$-concave function.

Definition 2.16. [65,66] Let K be a convex set. Then, $FIVF$ $\mathfrak{G}:K\to\mathbb{F}_{C}(\mathbb{R})$ is named as convex on K if

$$\mathfrak{G}(\mathfrak{s}\mathfrak{x}+(1-\mathfrak{s})y)\preccurlyeq \mathfrak{s}\mathfrak{G}(\mathfrak{x})\tilde{+}(1-\mathfrak{s})\mathfrak{G}(y), \tag{20}$$

for all $\mathfrak{x},y\in K,\mathfrak{s}\in[0,1]$, where $\mathfrak{G}(\mathfrak{x})\succcurlyeq\tilde{0}$. If (20) is reversed, then $\mathfrak{G}$ is named as concave on $[\mathfrak{a},\mathfrak{b}]$. $\mathfrak{G}$ is affine if and only if it is both a convex and concave function.

Definition 2.17. Let $K_{\mathfrak{p}}$ be a $\mathfrak{p}$-convex set and $\mathfrak{J}:[0,1]\subseteq\mathcal{L}\to\mathbb{R}^{+}$ be a nonnegative real-valued function such that $\mathfrak{J}\not\equiv 0$, where $\mathcal{L}\subseteq\mathbb{R}$. Then, $FIVF$ $\mathfrak{G}:K_{\mathfrak{p}}\to\mathbb{F}_{C}(\mathbb{R})$ is named as $(\mathfrak{p},\mathfrak{J})$-convex on $K_{\mathfrak{p}}$ such that

$$\mathfrak{G}\left(\left[\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)\preccurlyeq \mathfrak{J}(\mathfrak{s})\,\mathfrak{G}(\mathfrak{x})\tilde{+}\,\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}(y), \tag{21}$$

for all $\mathfrak{x},y\in K_{\mathfrak{p}},\mathfrak{s}\in[0,1]$, where $\mathfrak{G}(\mathfrak{x})\succcurlyeq\tilde{0}$ and $\mathfrak{J}:\mathcal{L}\to\mathbb{R}^{+}$ such that $\mathfrak{J}\not\equiv 0$ and $[0,1]\subseteq\mathcal{L}$. If (21) is reversed, then $\mathfrak{G}$ is named as $(\mathfrak{p},\mathfrak{J})$-concave on $[\mathfrak{a},\mathfrak{b}]$. $\mathfrak{G}$ is $(\mathfrak{p},\mathfrak{J})$-affine if and only if it is both $(\mathfrak{p},\mathfrak{J})$-convex and $(\mathfrak{p},\mathfrak{J})$-concave $FIVF$.

Remark 2.18. The $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$s have some very nice properties similar to convex $FIVF$:

If $\mathfrak{G}$ is $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$, then $\Upsilon\mathfrak{G}$ is also $(\mathfrak{p},\mathfrak{J})$-convex for $\Upsilon\geq 0$.

If $\mathcal{F}$ and $\mathfrak{G}$ both are $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$s, then $\max(\mathcal{F}(\mathfrak{x}),\mathfrak{G}(\mathfrak{x}))$ is also $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$.

We now discuss some new and known special cases of $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$s:

If $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}^{s}$ with $s\in(0,1)$, then $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$ becomes $(\mathfrak{p},s)$-convex $FIVF$, that is,

$$\mathfrak{G}\left(\left[\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)\preccurlyeq \mathfrak{s}^{s}\mathfrak{G}(\mathfrak{x})\tilde{+}(1-\mathfrak{s})^{s}\mathfrak{G}(y),\forall\,\mathfrak{x},y\in K_{\mathfrak{p}},\mathfrak{s}\in[0,1]. \tag{22}$$

If $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}$, then $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$ becomes $\mathfrak{p}$-convex $FIVF$, that is,

$$\mathfrak{G}\left(\left[\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)\preccurlyeq \mathfrak{s}\mathfrak{G}(\mathfrak{x})\tilde{+}(1-\mathfrak{s})\mathfrak{G}(y),\forall\,\mathfrak{x},y\in K_{\mathfrak{p}},\mathfrak{s}\in[0,1]. \tag{23}$$

If $\mathfrak{p}\equiv 1$, then $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$ becomes $\mathfrak{J}$-convex $FIVF$, that is,

$$\mathfrak{G}(\mathfrak{s}\mathfrak{x}+(1-\mathfrak{s})y)\preccurlyeq \mathfrak{J}(\mathfrak{s})\,\mathfrak{G}(\mathfrak{x})\tilde{+}\,\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}(y),\forall\,\mathfrak{x},y\in K,\mathfrak{s}\in[0,1]. \tag{24}$$

If $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}^{s}$ with $s\in(0,1)$ and $\mathfrak{p}\equiv 1$, then $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$ becomes s-convex $FIVF$, that is,

$$\mathfrak{G}(\mathfrak{s}\mathfrak{x}+(1-\mathfrak{s})y)\preccurlyeq \mathfrak{s}^{s}\mathfrak{G}(\mathfrak{x})\tilde{+}(1-\mathfrak{s})^{s}\mathfrak{G}(y),\forall\,\mathfrak{x},y\in K,\mathfrak{s}\in[0,1]. \tag{25}$$

If $\mathfrak{p}\equiv 1$ and $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}$, then $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$ becomes convex $FIVF$, see [65,66], that is,

$$\mathfrak{G}(\mathfrak{s}\mathfrak{x}+(1-\mathfrak{s})y)\preccurlyeq \mathfrak{s}\mathfrak{G}(\mathfrak{x})\tilde{+}(1-\mathfrak{s})\mathfrak{G}(y),\forall\,\mathfrak{x},y\in K,\mathfrak{s}\in[0,1]. \tag{26}$$

Theorem 2.19. Let $K_{\mathfrak{p}}$ be a $\mathfrak{p}$-convex set, non-negative real valued function $\mathfrak{J}:[0,1]\subseteq K_{\mathfrak{p}}\to\mathbb{R}$ such that $\mathfrak{J}\not\equiv 0$, and let $\mathfrak{G}:K_{\mathfrak{p}}\to\mathbb{F}_{C}(\mathbb{R})$ be a $FIVF$, whose $\jmath$-levels define the family of IVFs $\mathfrak{G}_{\jmath}:K_{\mathfrak{p}}\subset\mathbb{R}\to{\mathcal{K}_{C}}^{+}\subset\mathcal{K}_{C}$ are given by

$$\mathfrak{G}_{\jmath}(\mathfrak{x})=[\mathfrak{G}_{*}(\mathfrak{x},\jmath),\mathfrak{G}^{*}(\mathfrak{x},\jmath)], \tag{27}$$

for all $\mathfrak{x} \in K_{\mathfrak{p}}$ and for all $\mathscr{j} \in [0,1]$. Then, $\mathfrak{G}$ is $(\mathfrak{p},\mathfrak{J})$-convex on $K_{\mathfrak{p}}$ if and only if, for all $\mathscr{j} \in [0,1]$, $\mathfrak{G}_*(\mathfrak{x},\mathscr{j})$ and $\mathfrak{G}^*(\mathfrak{x},\mathscr{j})$ both are $(\mathfrak{p},\mathfrak{J})$-convex functions.

Proof. Assume that for each $\mathscr{j} \in [0,1]$, $\mathfrak{G}_*(\mathfrak{x},\mathscr{j})$ and $\mathfrak{G}^*(\mathfrak{x},\mathscr{j})$ are $(\mathfrak{p},\mathfrak{J})$-convex on $K_{\mathfrak{p}}$. Then, from (19), we have

$$\mathfrak{G}_*\left([\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right) \le \mathfrak{J}(\mathfrak{s})\,\mathfrak{G}_*(\mathfrak{x},\mathscr{j})+\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}_*(y,\mathscr{j}),\ \ \forall\,\mathfrak{x},y\in K_{\mathfrak{p}},\mathfrak{s}\in[0,1],$$

and

$$\mathfrak{G}^*\left([\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right) \le \mathfrak{J}(\mathfrak{s})\,\mathfrak{G}^*(\mathfrak{x},\mathscr{j})+\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}^*(y,\mathscr{j}),\forall\,\mathfrak{x},y\in K_{\mathfrak{p}},\mathfrak{s}\in[0,1].$$

Then, by (27), (6) and (8), we obtain

$$\mathfrak{G}_{\mathscr{j}}\left([\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)=\left[\mathfrak{G}_*\left([\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right),\mathfrak{G}^*\left([\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\right],$$

$$\le_I [\mathfrak{J}(\mathfrak{s})\,\mathfrak{G}_*(\mathfrak{x},\mathscr{j}),\mathfrak{J}(\mathfrak{s})\,\mathfrak{G}^*(\mathfrak{x},\mathscr{j})]+[\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}_*(y,\mathscr{j}),\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}^*(y,\mathscr{j})],$$

that is,

$$\mathfrak{G}\left([\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right) \preccurlyeq \mathfrak{J}(\mathfrak{s})\,\mathfrak{G}(\mathfrak{x})\tilde{+}\,\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}(y),\ \ \forall\,\mathfrak{x},y\in K_{\mathfrak{p}},\mathfrak{s}\in[0,1].$$

Hence, $\mathfrak{G}$ is $(\mathfrak{p},\mathfrak{J})$-convex *FIVF* on $K_{\mathfrak{p}}$.

Conversely, let $\mathfrak{G}$ be $(\mathfrak{p},\mathfrak{J})$-convex *FIVF* on $K_{\mathfrak{p}}$. Then, for all $\mathfrak{x},y\in K_{\mathfrak{p}}$ and $\mathfrak{s}\in[0,1]$, we have $\mathfrak{G}\left([\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right) \preccurlyeq \mathfrak{J}(\mathfrak{s})\,\mathfrak{G}(\mathfrak{x})\tilde{+}\,\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}(y)$. Therefore, from (27), we have

$$\mathfrak{G}_{\mathscr{j}}\left([\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)=\left[\mathfrak{G}_*\left([\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right),\mathfrak{G}^*\left([\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\right].$$

Again, from (27), (6) and (8), we obtain

$$\mathfrak{J}(\mathfrak{s})\,\mathfrak{G}_{\mathscr{j}}(\mathfrak{x})\tilde{+}\,\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}_{\mathscr{j}}(\mathfrak{x})$$
$$=[\mathfrak{J}(\mathfrak{s})\,\mathfrak{G}_*(\mathfrak{x},\mathscr{j}),\mathfrak{J}(\mathfrak{s})\,\mathfrak{G}^*(\mathfrak{x},\mathscr{j})]+[\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}_*(y,\mathscr{j}),\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}^*(y,\mathscr{j})],$$

Then by $(\mathfrak{p},\mathfrak{J})$-convexity of $\mathfrak{G}$, we have

$$\mathfrak{G}_*\left([\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right) \le \mathfrak{J}(\mathfrak{s})\,\mathfrak{G}_*(\mathfrak{x},\mathscr{j})+\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}_*(y,\mathscr{j}),$$

and

$$\mathfrak{G}^*\left([\mathfrak{s}\mathfrak{x}^{\mathfrak{p}}+(1-\mathfrak{s})y^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right) \le \mathfrak{J}(\mathfrak{s})\,\mathfrak{G}^*(\mathfrak{x},\mathscr{j})+\mathfrak{J}(1-\mathfrak{s})\,\mathfrak{G}^*(y,\mathscr{j}),$$

for each $\mathscr{j} \in [0,1]$. Hence, the result follows.

Remark 2.20. If $\mathfrak{G}_*(\mathfrak{x},\mathscr{j})=\mathfrak{G}^*(\mathfrak{x},\mathscr{j})$ with $\mathscr{j}=1$, then $(\mathfrak{p},\mathfrak{J})$-convex *FIVF* reduces to the classical $(\mathfrak{p},\mathfrak{J})$-convex function (see [52]).

If $\mathfrak{G}_*(\mathfrak{x},\mathscr{j})=\mathfrak{G}^*(\mathfrak{x},\mathscr{j})$ with $\mathscr{j}=1$ and $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}^s$ with $s\in(0,1)$, then $(\mathfrak{p},\mathfrak{J})$-convex *FIVF* reduces to the classical $(\mathfrak{p},s)$-convex function (see [47]).

If $\mathfrak{G}_*(\mathfrak{x},\mathscr{j})=\mathfrak{G}^*(\mathfrak{x},\mathscr{j})$ with $\mathscr{j}=1$ and $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}$, then $(\mathfrak{p},\mathfrak{J})$-convex *FIVF* reduces to the classical $\mathfrak{p}$-convex function (see [17]).

If $\mathfrak{G}_*(\mathfrak{x},\mathscr{j})=\mathfrak{G}^*(\mathfrak{x},\mathscr{j})$ with $\mathscr{j}=1$ and $\mathfrak{p}=1$, then $(\mathfrak{p},\mathfrak{J})$-convex *FIVF* reduces to the classical

$\mathfrak{J}$-convex function (see [55]).

Example 2.21. We consider $\mathfrak{J}(\mathfrak{s}) = \mathfrak{s}$, for $\mathfrak{s} \in [0,1]$, and the $FIVF$ $\mathfrak{G}: [0,1] \to \mathbb{F}_C(\mathbb{R})$ defined by

$$\mathfrak{G}(\mathfrak{x})(\sigma) = \begin{cases} \frac{\sigma}{2\mathfrak{x}^{\mathfrak{p}}} & \sigma \in [0, 2\mathfrak{x}^{\mathfrak{p}}] \\ \frac{4\mathfrak{x}^{\mathfrak{p}}-\sigma}{2\mathfrak{x}^{2}} & \sigma \in (2\mathfrak{x}^{\mathfrak{p}}, 4\mathfrak{x}^{\mathfrak{p}}] \\ 0 & otherwise, \end{cases} \tag{28}$$

Then, for each $\jmath \in [0,1]$, we have $\mathfrak{G}_{\jmath}(\mathfrak{x}) = [2\jmath\mathfrak{x}^{\mathfrak{p}}, (4-2\jmath)\mathfrak{x}^{\mathfrak{p}}]$. Since end point functions $\mathfrak{G}_*(\mathfrak{x},\jmath)$ and $\mathfrak{G}^*(\mathfrak{x},\jmath)$ both are $(\mathfrak{p},\mathfrak{J})$-convex functions for each $\jmath \in [0,1]$. Hence, $\mathfrak{G}(\mathfrak{x})$ is $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$.

3. Discrete Jensen and Schur type inequalities

We introduce the idea of discrete Jensen and Schur type inequality for $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$s in this section. The discrete Jensen type inequality is further refined in several ways. The discrete Jensen type inequality for $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$ is shown first in the following result.

Theorem 3.1. (Discrete Jensen type inequality for $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$) Let $\mathfrak{w}_j \in \mathbb{R}^+$, $\mathfrak{x}_j \in [\mathfrak{a},\mathfrak{b}]$, $(j = 1,2,3,\ldots\kappa, \kappa \geq 2)$ and $\mathfrak{G}: [\mathfrak{a},\mathfrak{b}] \to \mathbb{F}_C(\mathbb{R})$ be a $(\mathfrak{p},\mathfrak{J})$-convex fuzzy $FIVF$ with non-negative real valued function $\mathfrak{J}: [0,1] \to \mathbb{R}$, whose $\jmath$-levels define the family of IVFs $\mathfrak{G}_{\jmath}: [\mathfrak{a},\mathfrak{b}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathfrak{G}_{\jmath}(\mathfrak{x}) = [\mathfrak{G}_*(\mathfrak{x},\jmath), \mathfrak{G}^*(\mathfrak{x},\jmath)]$ for all $\mathfrak{x} \in [\mathfrak{a},\mathfrak{b}]$ and for all $\jmath \in [0,1]$. If $\mathfrak{J}$ is a super-multiplicative function on $\mathcal{L}$, then

$$\mathfrak{G}\left(\left[\frac{1}{W_\kappa}\sum_{j=1}^{\kappa} \mathfrak{w}_j\, \mathfrak{x}_j{}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right) \preccurlyeq \sum_{j}^{\kappa} \mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right) \mathfrak{G}(\mathfrak{x}_j), \tag{29}$$

where $W_\kappa = \sum_{j=1}^{\kappa} \mathfrak{w}_j$. If function $\mathfrak{J}$ is sub-multiplicative and $\mathfrak{G}$ is $(\mathfrak{p},\mathfrak{J})$-concave, then inequality (29) is reversed.

Proof. When $\kappa = 2$, inequality (29) is true. Consider that inequality (19) is true for $\kappa = n-1$, and then

$$\mathfrak{G}\left(\left[\frac{1}{W_{n-1}}\sum_{j=1}^{n-1} \mathfrak{w}_j\, \mathfrak{x}_j{}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right) \preccurlyeq \sum_{j=1}^{n-1} \mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_{n-1}}\right) \mathfrak{G}(\mathfrak{x}_j).$$

Now, let us prove that inequality (29) holds for $\kappa = n$.

$$\mathfrak{G}\left(\left[\frac{1}{W_n}\sum_{j=1}^{n} \mathfrak{w}_j\, \mathfrak{x}_j{}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right) = \mathfrak{G}\left(\left[\frac{1}{W_n}\sum_{j=1}^{n-2} \mathfrak{w}_j\, \mathfrak{x}_j{}^{\mathfrak{p}} + \frac{\mathfrak{w}_{n-1}+\mathfrak{w}_n}{W_n}\left(\frac{\mathfrak{w}_{n-1}}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\mathfrak{x}_{n-1}{}^{\mathfrak{p}} + \frac{\mathfrak{w}_n}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\mathfrak{x}_n{}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right).$$

Therefore, for each $\jmath \in [0,1]$, we have

$$\mathfrak{G}_*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n} \mathfrak{w}_j\, \mathfrak{x}_j{}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right)$$
$$\mathfrak{G}^*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n} \mathfrak{w}_j\, \mathfrak{x}_j{}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right)$$

$$= \mathfrak{G}_*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n-2}\mathfrak{w}_j\,\mathfrak{x}_j^{\mathfrak{p}} + \frac{\mathfrak{w}_{n-1}+\mathfrak{w}_n}{W_n}\left(\frac{\mathfrak{w}_{n-1}}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\mathfrak{x}_{n-1}^{\mathfrak{p}} + \frac{\mathfrak{w}_n}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\mathfrak{x}_n^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right),$$
$$= \mathfrak{G}^*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n-2}\mathfrak{w}_j\,\mathfrak{x}_j^{\mathfrak{p}} + \frac{\mathfrak{w}_{n-1}+\mathfrak{w}_n}{W_n}\left(\frac{\mathfrak{w}_{n-1}}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\mathfrak{x}_{n-1}^{\mathfrak{p}} + \frac{\mathfrak{w}_n}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\mathfrak{x}_n^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right),$$
$$\leq \sum_{j=1}^{n-2}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_n}\right)\mathfrak{G}_*(\mathfrak{x}_j, \jmath) + \mathfrak{J}\left(\frac{\mathfrak{w}_{n-1}+\mathfrak{w}_n}{W_n}\right)\mathfrak{G}_*\left(\left[\frac{\mathfrak{w}_{n-1}}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\mathfrak{x}_{n-1}^{\mathfrak{p}} + \frac{\mathfrak{w}_n}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\mathfrak{x}_n^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right),$$
$$\leq \sum_{j=1}^{n-2}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_n}\right)\mathfrak{G}^*(\mathfrak{x}_j, \jmath) + \mathfrak{J}\left(\frac{\mathfrak{w}_{n-1}+\mathfrak{w}_n}{W_n}\right)\mathfrak{G}^*\left(\left[\frac{\mathfrak{w}_{n-1}}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\mathfrak{x}_{n-1}^{\mathfrak{p}} + \frac{\mathfrak{w}_n}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\mathfrak{x}_n^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right),$$
$$\leq \sum_{j=1}^{n-2}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_n}\right)\mathfrak{G}_*(\mathfrak{x}_j, \jmath) + \mathfrak{J}\left(\frac{\mathfrak{w}_{n-1}+\mathfrak{w}_n}{W_n}\right)\left[\mathfrak{J}\left(\frac{\mathfrak{w}_{n-1}}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\right)\mathfrak{G}_*(\mathfrak{x}_{n-1}, \jmath) + \mathfrak{J}\left(\frac{\mathfrak{w}_n}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\right)\mathfrak{G}_*(\mathfrak{x}_n, \jmath)\right],$$
$$\leq \sum_{j=1}^{n-2}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_n}\right)\mathfrak{G}^*(\mathfrak{x}_j, \jmath) + \mathfrak{J}\left(\frac{\mathfrak{w}_{n-1}+\mathfrak{w}_n}{W_n}\right)\left[\mathfrak{J}\left(\frac{\mathfrak{w}_{n-1}}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\right)\mathfrak{G}^*(\mathfrak{x}_{n-1}, \jmath) + \mathfrak{J}\left(\frac{\mathfrak{w}_n}{\mathfrak{w}_{n-1}+\mathfrak{w}_n}\right)\mathfrak{G}^*(\mathfrak{x}_n, \jmath)\right],$$
$$\leq \sum_{j=1}^{n-2}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_n}\right)\mathfrak{G}_*(\mathfrak{x}_j, \jmath) + \left[\mathfrak{J}\left(\frac{\mathfrak{w}_{n-1}}{W_n}\right)\mathfrak{G}_*(\mathfrak{x}_{n-1}, \jmath) + \mathfrak{J}\left(\frac{\mathfrak{w}_n}{W_n}\right)\mathfrak{G}_*(\mathfrak{x}_n, \jmath)\right],$$
$$\leq \sum_{j=1}^{n-2}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_n}\right)\mathfrak{G}^*(\mathfrak{x}_j, \jmath) + \left[\mathfrak{J}\left(\frac{\mathfrak{w}_{n-1}}{W_n}\right)\mathfrak{G}^*(\mathfrak{x}_{n-1}, \jmath) + \mathfrak{J}\left(\frac{\mathfrak{w}_n}{W_n}\right)\mathfrak{G}^*(\mathfrak{x}_n, \jmath)\right],$$
$$= \sum_{j=1}^{n}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_n}\right)\mathfrak{G}_*(\mathfrak{x}_j, \jmath),$$
$$= \sum_{j=1}^{n}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_n}\right)\mathfrak{G}^*(\mathfrak{x}_j, \jmath).$$

From that, we have

$$\left[\mathfrak{G}_*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\mathfrak{w}_j\,\mathfrak{x}_j^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right), \mathfrak{G}^*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\mathfrak{w}_j\,\mathfrak{x}_j^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right)\right] \leq_I \left[\sum_{j=1}^{n}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_n}\right)\mathfrak{G}_*(\mathfrak{x}_j, \jmath), \sum_{j=1}^{n}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_n}\right)\mathfrak{G}^*(\mathfrak{x}_j, \jmath)\right],$$

that is,

$$\mathfrak{G}\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\mathfrak{w}_j\,\mathfrak{x}_j^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right) \preccurlyeq \sum_{j=1}^{n}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_n}\right)\mathfrak{G}(\mathfrak{x}_j),$$

and the result follows.

If $\mathfrak{w}_1 = \mathfrak{w}_2 = \mathfrak{w}_3 = \cdots = \mathfrak{w}_\kappa = 1$, then Theorem 3.1 reduces to the following result:

Corollary 3.2. Let $\mathfrak{x}_j \in [\mathfrak{a}, \mathfrak{b}]$, $(j = 1, 2, 3, \ldots \kappa, \kappa \geq 2)$ and $\mathfrak{G}: [\mathfrak{a}, \mathfrak{b}] \to \mathbb{F}_C(\mathbb{R})$ be a $(\mathfrak{p}, \mathfrak{J})$-convex fuzzy $FIVF$ with non-negative real valued function $\mathfrak{J}: [0, 1] \to \mathbb{R}$, whose $\jmath$-levels define the family of IVFs $\mathfrak{G}_\jmath: [\mathfrak{a}, \mathfrak{b}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathfrak{G}_\jmath(\mathfrak{x}) = [\mathfrak{G}_*(\mathfrak{x}, \jmath), \mathfrak{G}^*(\mathfrak{x}, \jmath)]$ for all $\mathfrak{x} \in [\mathfrak{a}, \mathfrak{b}]$ and for all $\jmath \in [0, 1]$. If $\mathfrak{J}$ is a super-multiplicative function, then

$$\mathfrak{G}\left(\left[\frac{1}{W_\kappa}\sum_{j=1}^{\kappa}\mathfrak{w}_j\,\mathfrak{x}_j^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right) \preccurlyeq \sum_{j=1}^{\kappa}\mathfrak{J}\left(\frac{1}{\kappa}\right)\mathfrak{G}(\mathfrak{x}_j). \tag{30}$$

If function $\mathfrak{J}$ is sub-multiplicative, and $\mathfrak{G}$ is a $(\mathfrak{p}, \mathfrak{J})$-concave, then inequality (30) is reversed.

Next, Theorem 3.3 gives the Schur-type inequality for $(\mathfrak{p}, \mathfrak{J})$-convex $FIVF$s.

Theorem 3.3. (Discrete Schur-type inequality for $(\mathfrak{p}, \mathfrak{J})$-convex $FIVF$) Let $\mathfrak{G}: [\mathfrak{a}, \mathfrak{b}] \to \mathbb{F}_C(\mathbb{R})$ be a $(\mathfrak{p}, \mathfrak{J})$-convex $FIVF$ with non-negative real valued function $\mathfrak{J}: [0, 1] \to \mathbb{R}$, whose $\jmath$-levels define the family of IVFs $\mathfrak{G}_\jmath: [\mathfrak{a}, \mathfrak{b}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathfrak{G}_\jmath(\mathfrak{x}) = [\mathfrak{G}_*(\mathfrak{x}, \jmath), \mathfrak{G}^*(\mathfrak{x}, \jmath)]$ for all $\mathfrak{x} \in [\mathfrak{a}, \mathfrak{b}]$ and for all $\jmath \in [0, 1]$. If $\mathfrak{J}: \mathcal{L} \to \mathbb{R}^+$ is a nonnegative super-multiplicative function, then for $\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3 \in [\mathfrak{a}, \mathfrak{b}]$, such that $\mathfrak{x}_1 < \mathfrak{x}_2 < \mathfrak{x}_3$ and $\mathfrak{x}_3^{\mathfrak{p}} - \mathfrak{x}_1^{\mathfrak{p}}$, $\mathfrak{x}_3^{\mathfrak{p}} - \mathfrak{x}_2^{\mathfrak{p}}$, $\mathfrak{x}_2^{\mathfrak{p}} - \mathfrak{x}_1^{\mathfrak{p}} \in \mathcal{L}$, we have

$$\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})\, \mathfrak{G}(\mathfrak{x}_2) \preccurlyeq \mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}})\, \mathfrak{G}(\mathfrak{x}_1) + \mathfrak{J}(\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})\, \mathfrak{G}(\mathfrak{x}_3). \tag{31}$$

If the function $\mathfrak{J}$ is a nonnegative sub-multiplicative function, and $\mathfrak{G}$ is a $(\mathfrak{p}, \mathfrak{J})$-concave, then inequality (31) is reversed.

Proof. Let $\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3 \in [\mathfrak{a}, \mathfrak{b}]$ and $\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}) > 0$. Then, by hypothesis, we have

$$\mathfrak{J}\left(\frac{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}}}{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}\right) = \frac{\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}})}{\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})} \text{ and } \mathfrak{J}\left(\frac{\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}\right) = \frac{\mathfrak{J}(\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})}{\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})}.$$

Consider $\mathfrak{s} = \frac{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}}}{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}$, and then $\mathfrak{x}_2{}^{\mathfrak{p}} = \mathfrak{s}\mathfrak{x}_1{}^{\mathfrak{p}} + (1 - \mathfrak{s})\mathfrak{x}_3{}^{\mathfrak{p}}$. Since $\mathfrak{G}$ is a $(\mathfrak{p}, \mathfrak{J})$-convex *FIVF*, by hypothesis, we have

$$\mathfrak{G}(\mathfrak{x}_2) \preccurlyeq \mathfrak{J}\left(\frac{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}}}{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}\right) \mathfrak{G}(\mathfrak{x}_1) + \mathfrak{J}\left(\frac{\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}\right) \mathfrak{G}(\mathfrak{x}_3).$$

Therefore, for each $\jmath \in [0, 1]$, we have

$$\begin{aligned} \mathfrak{G}_*(\mathfrak{x}_2, \jmath) &\leq \mathfrak{J}\left(\frac{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}}}{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}\right) \mathfrak{G}_*(\mathfrak{x}_1, \jmath) + \mathfrak{J}\left(\frac{\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}\right) \mathfrak{G}_*(\mathfrak{x}_3, \jmath), \\ \mathfrak{G}^*(\mathfrak{x}_2, \jmath) &\leq \mathfrak{J}\left(\frac{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}}}{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}\right) \mathfrak{G}^*(\mathfrak{x}_1, \jmath) + \mathfrak{J}\left(\frac{\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}{\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}}}\right) \mathfrak{G}^*(\mathfrak{x}_3, \jmath), \end{aligned} \tag{32}$$

$$\begin{aligned} &= \frac{\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}})}{\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})} \mathfrak{G}_*(\mathfrak{x}_1, \jmath) + \frac{\mathfrak{J}(\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})}{\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})} \mathfrak{G}_*(\mathfrak{x}_3, \jmath), \\ &= \frac{\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}})}{\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})} \mathfrak{G}^*(\mathfrak{x}_1, \jmath) + \frac{\mathfrak{J}(\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})}{\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})} \mathfrak{G}^*(\mathfrak{x}_3, \jmath). \end{aligned} \tag{33}$$

From (33), we have

$$\begin{aligned} \mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})\, \mathfrak{G}_*(\mathfrak{x}_2, \jmath) &\leq \mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}})\, \mathfrak{G}_*(\mathfrak{x}_1, \jmath) + \mathfrak{J}(\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})\, \mathfrak{G}_*(\mathfrak{x}_3, \jmath), \\ \mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})\, \mathfrak{G}^*(\mathfrak{x}_2, \jmath) &\leq \mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}})\, \mathfrak{G}^*(\mathfrak{x}_1, \jmath) + \mathfrak{J}(\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})\, \mathfrak{G}^*(\mathfrak{x}_3, \jmath), \end{aligned}$$

that is,

$$[\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})\, \mathfrak{G}_*(\mathfrak{x}_2, \jmath),\ \mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})\, \mathfrak{G}^*(\mathfrak{x}_2, \jmath)]$$

$$\leq_I [\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}})\, \mathfrak{G}_*(\mathfrak{x}_1, \jmath) + \mathfrak{J}(\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})\, \mathfrak{G}_*(\mathfrak{x}_3, \jmath),\ \mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}})\, \mathfrak{G}^*(\mathfrak{x}_1, \jmath) + \mathfrak{J}(\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})\, \mathfrak{G}^*(\mathfrak{x}_3, \jmath)].$$

Hence,

$$\mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})\, \mathfrak{G}(\mathfrak{x}_2) \preccurlyeq \mathfrak{J}(\mathfrak{x}_3{}^{\mathfrak{p}} - \mathfrak{x}_2{}^{\mathfrak{p}})\, \mathfrak{G}(\mathfrak{x}_1) + \mathfrak{J}(\mathfrak{x}_2{}^{\mathfrak{p}} - \mathfrak{x}_1{}^{\mathfrak{p}})\, \mathfrak{G}(\mathfrak{x}_3).$$

A refinement of a Jensen type inequality for $(\mathfrak{p}, \mathfrak{J})$-convex *FIVF* is given in the following theorem.

Theorem 3.4. Let $\mathfrak{w}_j \in \mathbb{R}^+$, $\mathfrak{x}_j \in [\mathfrak{a}, \mathfrak{b}]$, $(j = 1, 2, 3, \ldots \kappa, \kappa \geq 2)$ and $\mathfrak{G}: [\mathfrak{a}, \mathfrak{b}] \to \mathbb{F}_C(\mathbb{R})$ be a $(\mathfrak{p}, \mathfrak{J})$-convex *FIVF* with non-negative real valued function $\mathfrak{J}: [0, 1] \to \mathbb{R}$, whose $\jmath$-levels define the family of *IVF*s $\mathfrak{G}_\jmath: [\mathfrak{a}, \mathfrak{b}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathfrak{G}_\jmath(\mathfrak{x}) = [\mathfrak{G}_*(\mathfrak{x}, \jmath), \mathfrak{G}^*(\mathfrak{x}, \jmath)]$ for all $\mathfrak{x} \in [\mathfrak{a}, \mathfrak{b}]$ and for all $\jmath \in [0, 1]$. If $(L, U) \subseteq [\mathfrak{a}, \mathfrak{b}]$ and $\mathfrak{J}$ is a nonnegative super-multiplicative function, then

$$\sum_{j=1}^{\kappa} \mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right) \mathfrak{G}(\mathfrak{x}_j) \preccurlyeq \sum_{j=1}^{\kappa}\left(\mathfrak{J}\left(\frac{U^{\mathfrak{p}} - \mathfrak{x}_j{}^{\mathfrak{p}}}{U^{\mathfrak{p}} - L^{\mathfrak{p}}}\right) \mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right) \mathfrak{G}(L, \jmath) + \mathfrak{J}\left(\frac{\mathfrak{x}_j{}^{\mathfrak{p}} - L^{\mathfrak{p}}}{U^{\mathfrak{p}} - L^{\mathfrak{p}}}\right) \mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right) \mathfrak{G}(U, \jmath)\right), \tag{34}$$

where $W_\kappa = \sum_{j=1}^{\kappa} \mathfrak{w}_j$. If $\mathfrak{J}$ is a sub-multiplicative function and $\mathfrak{G}$ is $(\mathfrak{p}, \mathfrak{J})$-concave, then inequality (34) is reversed.

Proof. Consider $L = \mathfrak{x}_1, \mathfrak{x}_j = \mathfrak{x}_2$, $(j = 1, 2, 3, \ldots \kappa)$, $U = \mathfrak{x}_3$. Then, by hypothesis and inequality (32), we have

$$\mathfrak{G}(\mathfrak{x}_j) \le \mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{G}(L,\jmath) + \mathfrak{J}\left(\frac{\mathfrak{x}_j^{\mathfrak{p}}-L^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{G}(U,\jmath).$$

Therefore, for each $\jmath \in [0,1]$, we have

$$\mathfrak{G}_*(\mathfrak{x}_j,\jmath) \le \mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{G}_*(L,\jmath) + \mathfrak{J}\left(\frac{\mathfrak{x}_j^{\mathfrak{p}}-L^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{G}_*(U,\jmath),$$
$$\mathfrak{G}^*(\mathfrak{x}_j,\jmath) \le \mathfrak{J}\left(\frac{U-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{G}^*(L,\jmath) + \mathfrak{J}\left(\frac{\mathfrak{x}_j^{\mathfrak{p}}-L^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{G}^*(U,\jmath).$$

The above inequality can be written as

$$\begin{aligned}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}_*(\mathfrak{x}_j,\jmath) &\le \mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}_*(L,\jmath) + \mathfrak{J}\left(\frac{\mathfrak{x}_j^{\mathfrak{p}}-L^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}_*(U,\jmath),\\ \mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}^*(\mathfrak{x}_j,\jmath) &\le \mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}^*(L,\jmath) + \mathfrak{J}\left(\frac{\mathfrak{x}_j^{\mathfrak{p}}-L^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}^*(U,\jmath).\end{aligned} \tag{35}$$

Taking the sum of all inequalities (35) for $j = 1,2,3,\dots\kappa,$ we have

$$\sum_{j=1}^{\kappa}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}_*(\mathfrak{x}_j,\jmath) \le \sum_{j=1}^{\kappa}\left(\mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}_*(L,\jmath) + \mathfrak{J}\left(\frac{\mathfrak{x}_j^{\mathfrak{p}}-L^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}_*(U,\jmath)\right),$$
$$\sum_{j=1}^{\kappa}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}^*(\mathfrak{x}_j,\jmath) \le \sum_{j=1}^{\kappa}\left(\mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}^*(L,\jmath) + \mathfrak{J}\left(\frac{\mathfrak{x}_j^{\mathfrak{p}}-L^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}^*(U,\jmath)\right).$$

That is,

$$\begin{aligned}\sum_{j=1}^{\kappa}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}(\mathfrak{x}_j) &= \left[\sum_{j=1}^{\kappa}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}_*(\mathfrak{x}_j,\jmath),\ \sum_{j=1}^{\kappa}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}^*(\mathfrak{x}_j,\jmath)\right]\\ &\le_I \left[\sum_{j=1}^{\kappa}\begin{pmatrix}\mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}_*(L,\jmath)\\ +\mathfrak{J}\left(\frac{\mathfrak{x}_j^{\mathfrak{p}}-L^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}_*(U,\jmath)\end{pmatrix},\ \sum_{j=1}^{\kappa}\begin{pmatrix}\mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}^*(L,\jmath)\\ +\mathfrak{J}\left(\frac{\mathfrak{x}_j^{\mathfrak{p}}-L^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}^*(U,\jmath)\end{pmatrix}\right],\\ &\le_I \sum_{j=1}^{\kappa}\mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)[\mathfrak{G}_*(L,\jmath),\ \mathfrak{G}^*(L,\jmath)] + \sum_{j=1}^{\kappa}\mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)[\mathfrak{G}_*(U,\jmath),\\ &\quad \mathfrak{G}^*(U,\jmath)].\\ &= \sum_{j=1}^{\kappa}\mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}(L,\jmath) + \sum_{j=1}^{\kappa}\mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}(U,\jmath).\end{aligned}$$

Thus,

$$\sum_{j=1}^{\kappa}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}(\mathfrak{x}_j) \preccurlyeq \sum_{j=1}^{\kappa}\left(\mathfrak{J}\left(\frac{U^{\mathfrak{p}}-\mathfrak{x}_j^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}(L) + \mathfrak{J}\left(\frac{\mathfrak{x}_j^{\mathfrak{p}}-L^{\mathfrak{p}}}{U^{\mathfrak{p}}-L^{\mathfrak{p}}}\right)\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}(U)\right),$$

and this completes the proof.

We now consider some special cases of Theorems 3.1 and 3.4.

If $\mathfrak{G}_*(\mathfrak{x},\jmath) = \mathfrak{G}_*(\mathfrak{x},\jmath)$, then Theorems 3.1 and 3.4 reduce to the following results:

Corollary 3.5. [52] (Jensen inequality for $(\mathfrak{p},\mathfrak{J})$-convex function) Let $\mathfrak{w}_j \in \mathbb{R}^+$, $\mathfrak{x}_j \in [\mathfrak{a},\mathfrak{b}]$, $(j = 1,2,3,\dots\kappa, \kappa \ge 2)$ and let $\mathfrak{G}: [\mathfrak{a},\mathfrak{b}] \to \mathbb{R}^+$ be a non-negative real-valued function. If $\mathfrak{G}$ is a $(\mathfrak{p},\mathfrak{J})$-convex function, and $\mathfrak{J}$ is a nonnegative super-multiplicative function on $\mathcal{L}$, then

$$\mathfrak{G}\left(\left[\frac{1}{W_\kappa}\sum_{j=1}^{\kappa}\mathfrak{w}_j\,\mathfrak{x}_j^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right) \le \sum_{j=1}^{\kappa}\mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right)\mathfrak{G}(\mathfrak{x}_j), \tag{36}$$

where $W_\kappa = \sum_{j=1}^{\kappa}\mathfrak{w}_j$. If $\mathfrak{J}$ is a sub-multiplicative function, and $\mathfrak{G}$ is $(\mathfrak{p},\mathfrak{J})$-concave function, then inequality (36) is reversed.

Corollary 3.6. Let $\mathfrak{w}_j \in \mathbb{R}^+$, $\mathfrak{x}_j \in [\mathfrak{a}, \mathfrak{b}]$, $(j = 1, 2, 3, \dots \kappa, \kappa \geq 2)$, $\mathfrak{J}$ be a nonnegative super-multiplicative function on $\mathcal{L}$ and $\mathfrak{G}: [\mathfrak{a}, \mathfrak{b}] \to \mathbb{R}^+$ be a non-negative real-valued function. If $\mathfrak{G}$ is a $(\mathfrak{p}, \mathfrak{J})$-convex function, and $\mathfrak{x}_1, \mathfrak{x}_2, \dots, \mathfrak{x}_j \in (L, U) \subseteq [\mathfrak{a}, \mathfrak{b}]$, then

$$\sum_{j=1}^{\kappa} \mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right) \mathfrak{G}(\mathfrak{x}_j) \leq \sum_{j=1}^{\kappa} \left(\mathfrak{J}\left(\frac{U^{\mathfrak{p}} - \mathfrak{x}_j{}^{\mathfrak{p}}}{U^{\mathfrak{p}} - L^{\mathfrak{p}}}\right) \mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right) \mathfrak{G}(L) + \mathfrak{J}\left(\frac{\mathfrak{x}_j{}^{\mathfrak{p}} - L^{\mathfrak{p}}}{U^{\mathfrak{p}} - L^{\mathfrak{p}}}\right) \mathfrak{J}\left(\frac{\mathfrak{w}_j}{W_\kappa}\right) \mathfrak{G}(U) \right), \tag{37}$$

where $W_\kappa = \sum_{j=1}^{\kappa} \mathfrak{w}_j$. If $\mathfrak{J}$ is a sub-multiplicative function, and $\mathfrak{G}$ is a $(\mathfrak{p}, \mathfrak{J})$-concave function, then inequality (37) is reversed.

4. Hermite-Hadamard type inequalities

In view of the HH-inequality in Eq (3) and HH-inequality in Eq (4), we can deduce the following version of the HH-inequalities for $(\mathfrak{p}, \mathfrak{J})$-concave $FIVF$s.

Theorem 4.1. Let $\mathfrak{G}: [\mathfrak{a}, \mathfrak{b}] \to \mathbb{F}_C(\mathbb{R})$ be a $(\mathfrak{p}, \mathfrak{J})$-convex $FIVF$ with non-negative real valued function $\mathfrak{J}: [0, 1] \to \mathbb{R}^+$ and $\mathfrak{J}\left(\frac{1}{2}\right) \neq 0$, whose $\jmath$-levels define the family of IVFs $\mathfrak{G}_\jmath: [\mathfrak{a}, \mathfrak{b}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathfrak{G}_\jmath(\mathfrak{x}) = [\mathfrak{G}_*(\mathfrak{x}, \jmath), \mathfrak{G}^*(\mathfrak{x}, \jmath)]$ for all $\mathfrak{x} \in [\mathfrak{a}, \mathfrak{b}]$ and for all $\jmath \in [0, 1]$. If $\mathfrak{G} \in \mathcal{FR}_{([\mathfrak{a}, \mathfrak{b}], \jmath)}$, then

$$\frac{1}{2\,\mathfrak{J}\left(\frac{1}{2}\right)} \mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}} + \mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right) \preccurlyeq \frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}} - \mathfrak{a}^{\mathfrak{p}}} \, (FR) \int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{x}^{\mathfrak{p}-1} \mathfrak{G}(\mathfrak{x}) d\mathfrak{x} \leq_{\mathfrak{p}} [\mathfrak{G}(\mathfrak{a}) \tilde{+} \mathfrak{G}(\mathfrak{b})] \int_0^1 \mathfrak{J}(\mathfrak{s}) \, d\mathfrak{s}. \tag{38}$$

If $\mathfrak{G}$ is a $(\mathfrak{p}, \mathfrak{J})$-concave $FIVF$, then

$$\frac{1}{2\,\mathfrak{J}\left(\frac{1}{2}\right)} \mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}} + \mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right) \succcurlyeq \frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}} - \mathfrak{a}^{\mathfrak{p}}} \, (FR) \int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{x}^{\mathfrak{p}-1} \mathfrak{G}(\mathfrak{x}) d\mathfrak{x} \succcurlyeq [\mathfrak{G}(\mathfrak{a}) \tilde{+} \mathfrak{G}(\mathfrak{b})] \int_0^1 \mathfrak{J}(\mathfrak{s}) \, d\mathfrak{s}. \tag{39}$$

Proof. Let $\mathfrak{G}$ be a $(\mathfrak{p}, \mathfrak{J})$-convex $FIVF$. Then, by hypothesis, we have

$$\frac{1}{\mathfrak{J}\left(\frac{1}{2}\right)} \mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}} + \mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right) \preccurlyeq \mathfrak{G}\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}} + (1 - \mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right) \tilde{+} \mathfrak{G}\left([(1 - \mathfrak{s})\mathfrak{a}^{\mathfrak{p}} + \mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right).$$

Therefore, for each
$\jmath \in [0, 1]$, we have

$$\frac{1}{\mathfrak{J}\left(\frac{1}{2}\right)} \mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}} + \mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right) \leq \mathfrak{G}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}} + (1 - \mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}, \jmath\right) + \mathfrak{G}_*\left((1 - \mathfrak{s})\mathfrak{a}^{\mathfrak{p}} + \mathfrak{s}\mathfrak{b}^{\mathfrak{p}}, \jmath\right),$$
$$\frac{1}{\mathfrak{J}\left(\frac{1}{2}\right)} \mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}} + \mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right) \leq \mathfrak{G}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}} + (1 - \mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}, \jmath\right) + \mathfrak{G}^*\left((1 - \mathfrak{s})\mathfrak{a}^{\mathfrak{p}} + \mathfrak{s}\mathfrak{b}^{\mathfrak{p}}, \jmath\right).$$

Then,

$$\frac{1}{\mathfrak{J}\left(\frac{1}{2}\right)} \int_0^1 \mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}} + \mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right) d\mathfrak{s} \leq \int_0^1 \mathfrak{G}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}} + (1 - \mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}, \jmath\right) d\mathfrak{s} + \int_0^1 \mathfrak{G}_*\left((1 - \mathfrak{s})\mathfrak{a}^{\mathfrak{p}} + \mathfrak{s}\mathfrak{b}^{\mathfrak{p}}, \jmath\right) d\mathfrak{s},$$
$$\frac{1}{\mathfrak{J}\left(\frac{1}{2}\right)} \int_0^1 \mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}} + \mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right) d\mathfrak{s} \leq \int_0^1 \mathfrak{G}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}} + (1 - \mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}, \jmath\right) d\mathfrak{s} + \int_0^1 \mathfrak{G}^*\left((1 - \mathfrak{s})\mathfrak{a}^{\mathfrak{p}} + \mathfrak{s}\mathfrak{b}^{\mathfrak{p}}, \jmath\right) d\mathfrak{s}.$$

It follows that

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\leq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)d\mathfrak{x},$$
$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\leq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)d\mathfrak{x}.$$

That is,

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\left[\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right),\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\right]\leq_I\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\left[\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)d\mathfrak{x},\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)d\mathfrak{x}\right].$$

Thus,

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)\preccurlyeq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(FR)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x})d\mathfrak{x}. \tag{40}$$

In a similar way as above, we have

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(FR)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x})d\mathfrak{x}\preccurlyeq[\mathfrak{G}(\mathfrak{a})\tilde{+}\mathfrak{G}(\mathfrak{b})]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s}. \tag{41}$$

Combining (40) and (41), we have

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)\preccurlyeq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(FR)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x})d\mathfrak{x}\preccurlyeq[\mathfrak{G}(\mathfrak{a})\tilde{+}\mathfrak{G}(\mathfrak{b})]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s}.$$

Hence, we have the required result.

Remark 4.2. If $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}^s$, then Theorem 4.1 reduces to the result for $(\mathfrak{p},s)$-convex $FIVF$ (see [85]):

$$2^{s-1}\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)\preccurlyeq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(FR)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x})d\mathfrak{x}\preccurlyeq\frac{1}{s+1}[\mathfrak{G}(\mathfrak{a})\tilde{+}\mathfrak{G}(\mathfrak{b})]. \tag{42}$$

If $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}$, then Theorem 4.1 reduces to the result for $\mathfrak{p}$-convex $FIVF$ (see [85]):

$$\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)\preccurlyeq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(FR)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x})d\mathfrak{x}\preccurlyeq\frac{\mathfrak{G}(\mathfrak{a})\tilde{+}\mathfrak{G}(\mathfrak{b})}{2}. \tag{43}$$

If $\mathfrak{G}_*(\mathfrak{a},\jmath)=\mathfrak{G}^*(\mathfrak{b},\jmath)$ with $\jmath=1$, then Theorem 4.1 reduces to the result for classical $(\mathfrak{p},\mathfrak{J})$-convex function (see [52]):

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)\leq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(R)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x})d\mathfrak{x}\leq[\mathfrak{G}(\mathfrak{a})+\mathfrak{G}(\mathfrak{b})]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s}. \tag{44}$$

If $\mathfrak{G}_*(\mathfrak{a},\jmath)=\mathfrak{G}^*(\mathfrak{b},\jmath)$ with $\jmath=1$ and $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}$, then Theorem 4.1 reduces to the result for classical $\mathfrak{p}$-convex function (see [17]):

$$\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)\leq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(R)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x})d\mathfrak{x}\leq\frac{\mathfrak{G}(\mathfrak{a})+\mathfrak{G}(\mathfrak{b})}{2}. \tag{45}$$

If $\mathfrak{G}_*(\mathfrak{a},\jmath)=\mathfrak{G}^*(\mathfrak{b},\jmath)$ with $\jmath=1$, $\mathfrak{p}=1$ and $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}$, then Theorem 4.1 reduces to the result for classical convex function (see [41]):

$$\mathfrak{G}\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)\leq\frac{1}{\mathfrak{b}-\mathfrak{a}}\,(R)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{G}(\mathfrak{x})d\mathfrak{x}\leq\frac{\mathfrak{G}(\mathfrak{a})+\mathfrak{G}(\mathfrak{b})}{2}. \tag{46}$$

Example 4.3. Let $\mathfrak{p}$ be an odd number and $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}$ for $\mathfrak{s}\in[0,1]$, and the *FIVF* $\mathfrak{G}:[\mathfrak{a},\mathfrak{b}]=[2,3]\to\mathbb{F}_C(\mathbb{R})$ is defined by

$$\mathfrak{G}(\mathfrak{x})(\sigma)=\begin{cases}\frac{\sigma}{\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)} & \sigma\in\left[0,2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right]\\ \frac{2\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)-\sigma}{\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)} & \sigma\in(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}},2\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)]\\ 0 & \text{otherwise.}\end{cases} \tag{47}$$

Then, for each $j\in[0,1]$, we have $\mathfrak{G}_j(\mathfrak{x})=\left[j\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right),\ (2-j)\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)\right]$. Since end point functions $\mathfrak{G}_*(\mathfrak{x},j)=j\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)$, $\mathfrak{G}^*(\mathfrak{x},j)=(2-j)\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)$ are $(\mathfrak{p},\mathfrak{J})$-convex functions for each $j\in[0,1]$. Then, $\mathfrak{G}(\mathfrak{x})$ is a $(\mathfrak{p},\mathfrak{J})$-convex *FIVF*. We now compute the following:

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},j\right)=\frac{4-\sqrt{10}}{2}j,$$

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},j\right)=\frac{4-\sqrt{10}}{2}(2-j),$$

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},j)d\mathfrak{x}=j\int_2^3\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)d\mathfrak{x}=\frac{21}{50}j,$$

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},j)d\mathfrak{x}=(2-j)\int_2^3\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)d\mathfrak{x}=\frac{21}{50}(2-j),$$

$$[\mathfrak{G}_*(\mathfrak{a},j)+\mathfrak{G}_*(\mathfrak{b},j)]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s}=\frac{4-\sqrt{2}-\sqrt{3}}{2}j,$$

$$[\mathfrak{G}^*(\mathfrak{a},j)+\mathfrak{G}^*(\mathfrak{b},j)]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s}=\frac{4-\sqrt{2}-\sqrt{3}}{2}(2-j),$$

for all $j\in[0,1]$. That means
$\left[\frac{4-\sqrt{10}}{2}j,\frac{4-\sqrt{10}}{2}(2-j)\right]\leq_I\left[\frac{21}{50}j,\frac{21}{50}(2-j)\right]\leq_I\left[\frac{4-\sqrt{2}-\sqrt{3}}{2}j,\frac{4-\sqrt{2}-\sqrt{3}}{2}(2-j)\right]$, for all $j\in[0,1]$, and the Theorem has been demonstrated.

Theorem 4.4. Let $\mathfrak{G}:[\mathfrak{a},\mathfrak{b}]\to\mathbb{F}_C(\mathbb{R})$ be a $(\mathfrak{p},\mathfrak{J})$-convex *FIVF* with non-negative real valued function $\mathfrak{J}:[0,1]\to\mathbb{R}^+$ and $\mathfrak{J}\left(\frac{1}{2}\right)\neq 0$, whose j-levels define the family of *IVF*s $\mathfrak{G}_j:[\mathfrak{a},\mathfrak{b}]\subset\mathbb{R}\to\mathcal{K}_C{}^+$ are given by $\mathfrak{G}_j(\mathfrak{x})=[\mathfrak{G}_*(\mathfrak{x},j),\mathfrak{G}^*(\mathfrak{x},j)]$ for all $\mathfrak{x}\in[\mathfrak{a},\mathfrak{b}]$ and for all $j\in[0,1]$. If $\mathfrak{G}\in\mathcal{FR}_{([\mathfrak{a},\mathfrak{b}],j)}$, then

$$\frac{1}{4\left[\mathfrak{J}\left(\frac{1}{2}\right)\right]^2}\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)\preccurlyeq\rhd_2\preccurlyeq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(FR)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x})d\mathfrak{x}\preccurlyeq\rhd_1\preccurlyeq[\mathfrak{G}(\mathfrak{a})\tilde{+}\mathfrak{G}(\mathfrak{b})]\left[\frac{1}{2}+\mathfrak{J}\left(\frac{1}{2}\right)\right]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s}, \tag{48}$$

where

$$\rhd_1=\left[\frac{\mathfrak{G}(\mathfrak{a})\tilde{+}\mathfrak{G}(\mathfrak{b})}{2}\tilde{+}\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)\right]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s},$$

$$\rhd_2=\frac{1}{4\mathfrak{J}\left(\frac{1}{2}\right)}\left[\mathfrak{G}\left(\left[\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}}\right)\tilde{+}\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+3\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}}\right)\right],$$

and $\rhd_1 = [\rhd_{1_*}, \rhd_1{}^*]$, $\rhd_2 = [\rhd_{2_*}, \rhd_2{}^*]$.

Proof. Take $\left[\mathfrak{a}^{\mathfrak{p}}, \frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]$, and we have

$$\frac{1}{\mathfrak{I}\left(\frac{1}{2}\right)} \mathfrak{G}\left(\left[\frac{\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}}{2}+\frac{(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)$$

$$\preccurlyeq \mathfrak{G}\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right) \tilde{+} \mathfrak{G}\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right).$$

Therefore, for each $\jmath \in [0,1]$, we have

$$\frac{1}{\mathfrak{I}\left(\frac{1}{2}\right)} \mathfrak{G}_*\left(\left[\frac{\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}}{2}+\frac{(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}}{2}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right)$$

$$\leq \mathfrak{G}_*\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right) + \mathfrak{G}_*\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right),$$

$$\frac{1}{\mathfrak{I}\left(\frac{1}{2}\right)} \mathfrak{G}^*\left(\left[\frac{\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}}{2}+\frac{(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}}{2}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right)$$

$$\leq \mathfrak{G}^*\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right) + \mathfrak{G}^*\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right).$$

In consequence, we obtain

$$\frac{1}{4\,\mathfrak{I}\left(\frac{1}{2}\right)} \mathfrak{G}_*\left(\left[\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right) \leq \frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}} \int_{\mathfrak{a}}^{\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}} \mathfrak{x}^{\mathfrak{p}-1} \mathfrak{G}_*(\mathfrak{x}, \jmath)\, d\mathfrak{x},$$

$$\frac{1}{4\mathfrak{I}\left(\frac{1}{2}\right)} \mathfrak{G}^*\left(\left[\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right) \leq \frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}} \int_{\mathfrak{a}}^{\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}} \mathfrak{x}^{\mathfrak{p}-1} \mathfrak{G}^*(\mathfrak{x}, \jmath)\, d\mathfrak{x}.$$

That is,

$$\frac{1}{4\mathfrak{I}\left(\frac{1}{2}\right)}\left[\mathfrak{G}_*\left(\left[\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right), \mathfrak{G}^*\left(\left[\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}}, \jmath\right)\right] \leq_I \frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\left[\int_{\mathfrak{a}}^{\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}} \mathfrak{x}^{\mathfrak{p}-1} \mathfrak{G}_*(\mathfrak{x}, \jmath)\, d\mathfrak{x}, \int_{\mathfrak{a}}^{\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}} \mathfrak{x}^{\mathfrak{p}-1} \mathfrak{G}^*(\mathfrak{x}, \jmath)\, d\mathfrak{x}\right].$$

It follows that

$$\frac{1}{4\mathfrak{I}\left(\frac{1}{2}\right)} \mathfrak{G}\left(\left[\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}}\right) \preccurlyeq \frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}} \int_{\mathfrak{a}}^{\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}} \mathfrak{x}^{\mathfrak{p}-1} \mathfrak{G}(\mathfrak{x})\, d\mathfrak{x}. \tag{49}$$

In a similar way as above, we have

$$\frac{1}{4\,\mathfrak{I}\left(\frac{1}{2}\right)} \mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+3\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}}\right) \preccurlyeq \frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}} \int_{\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}}^{\mathfrak{b}} \mathfrak{x}^{\mathfrak{p}-1} \mathfrak{G}(\mathfrak{x})\, d\mathfrak{x}. \tag{50}$$

Combining (49) and (50), we have

$$\frac{1}{4\mathfrak{I}\left(\frac{1}{2}\right)}\left[\mathfrak{G}\left(\left[\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}}\right) \tilde{+} \mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+3\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}}\right)\right] \preccurlyeq \frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}} \int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{x}^{\mathfrak{p}-1} \mathfrak{G}(\mathfrak{x})\, d\mathfrak{x}.$$

By using Theorem 4.1, we have

$$\frac{1}{4\left[\mathfrak{J}\left(\frac{1}{2}\right)\right]^2}\ \mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)=\frac{1}{4\left[\mathfrak{J}\left(\frac{1}{2}\right)\right]^2}\ \mathfrak{G}\left(\left[\frac{1}{2}.\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}+\frac{1}{2}.\frac{\mathfrak{a}^{\mathfrak{p}}+3\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}}\right).$$

Therefore, for each $\jmath\in[0,1]$, we have

$$\frac{1}{4\left[\mathfrak{J}\left(\frac{1}{2}\right)\right]^2}\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)=\frac{1}{4\left[\mathfrak{J}\left(\frac{1}{2}\right)\right]^2}\ \mathfrak{G}_*\left(\left[\frac{1}{2}.\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}+\frac{1}{2}.\frac{\mathfrak{a}^{\mathfrak{p}}+3\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right),$$

$$\frac{1}{4\left[\mathfrak{J}\left(\frac{1}{2}\right)\right]^2}\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)=\frac{1}{4\left[\mathfrak{J}\left(\frac{1}{2}\right)\right]^2}\mathfrak{G}^*\left(\left[\frac{1}{2}.\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}+\frac{1}{2}.\frac{\mathfrak{a}^{\mathfrak{p}}+3\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right),$$

$$\leq\frac{1}{4\left[\mathfrak{J}\left(\frac{1}{2}\right)\right]^2}\left[\mathfrak{J}\left(\frac{1}{2}\right)\mathfrak{G}_*\left(\left[\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)+\mathfrak{J}\left(\frac{1}{2}\right)\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+3\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\right],$$

$$\leq\frac{1}{4\left[\mathfrak{J}\left(\frac{1}{2}\right)\right]^2}\left[\mathfrak{J}\left(\frac{1}{2}\right)\mathfrak{G}^*\left(\left[\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)+\mathfrak{J}\left(\frac{1}{2}\right)\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+3\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\right],$$

$$=\rhd_{2*},$$

$$=\rhd_2{}^*,$$

$$\leq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)d\mathfrak{x},$$

$$\leq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)d\mathfrak{x},$$

$$\leq\left[\frac{\mathfrak{G}_*(\mathfrak{a},\jmath)+\mathfrak{G}_*(\mathfrak{b},\jmath)}{2}+\ \mathfrak{G}_*\left(\frac{\mathfrak{a}^{p}+\mathfrak{b}^{p}}{2},\jmath\right)\right]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s},$$

$$\leq\left[\frac{\mathfrak{G}^*(\mathfrak{a},\jmath)+\mathfrak{G}^*(\mathfrak{b},\jmath)}{2}+\ \mathfrak{G}^*\left(\frac{\mathfrak{a}^{p}+\mathfrak{b}^{p}}{2},\jmath\right)\right]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s},$$

$$=\rhd_{1*},$$

$$=\rhd_1{}^*,$$

$$\leq\left[\frac{\mathfrak{G}_*(\mathfrak{a},\jmath)+\mathfrak{G}_*(\mathfrak{b},\jmath)}{2}+\mathfrak{J}\left(\frac{1}{2}\right)\left(\mathfrak{G}_*(\mathfrak{a},\jmath)+\ \mathfrak{G}_*(\mathfrak{b},\jmath)\right)\right]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s},$$

$$\leq\left[\frac{\mathfrak{G}^*(\mathfrak{a},\jmath)+\mathfrak{G}^*(\mathfrak{b},\jmath)}{2}+\mathfrak{J}\left(\frac{1}{2}\right)\left(\mathfrak{G}^*(\mathfrak{a},\jmath)\ +\ \mathfrak{G}^*(\mathfrak{b},\gamma)\right)\right]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s},$$

$$=[\mathfrak{G}_*(\mathfrak{a},\gamma)+\ \mathfrak{G}_*(\mathfrak{b},\gamma)]\left[\frac{1}{2}+\mathfrak{J}\left(\frac{1}{2}\right)\right]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s},$$

$$=[\mathfrak{G}^*(\mathfrak{a},\gamma)+\ \mathfrak{G}^*(\mathfrak{b},\gamma)]\left[\frac{1}{2}+\mathfrak{J}\left(\frac{1}{2}\right)\right]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s},$$

that is,

$$\frac{1}{4\left[\mathfrak{J}\left(\frac{1}{2}\right)\right]^2}\ \mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)\preccurlyeq\rhd_2\preccurlyeq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\ (FR)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x})d\mathfrak{x}$$

$$\preccurlyeq\rhd_1\preccurlyeq[\mathfrak{G}(\mathfrak{a})\tilde{+}\mathfrak{G}(\mathfrak{b})]\left[\frac{1}{2}+\mathfrak{J}\left(\frac{1}{2}\right)\right]\int_0^1\mathfrak{J}(\mathfrak{s})\,d\mathfrak{s},$$

and hence, the result follows.

Example 4.5. Let $\mathfrak{p}$ be an odd number and $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}$, for $\mathfrak{s}\in[0,1]$, and the $FIVF$ $\mathfrak{G}:[\mathfrak{a},\mathfrak{b}]=[2,3]\to\mathbb{F}_C(\mathbb{R})$ defined by, $\mathfrak{G}_\jmath(\mathfrak{x})=\left[\jmath\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right),(2-\jmath)\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)\right]$, as in Example 4.3, then $\mathfrak{G}(\mathfrak{x})$ is $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$ and satisfying (38). We have $\mathfrak{G}_*(\mathfrak{x},\jmath)=\jmath\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)$ and $\mathfrak{G}^*(\mathfrak{x},\jmath)=(2-\jmath)\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)$. We now compute the following

$$[\mathfrak{G}_*(\mathfrak{a},\jmath) + \mathfrak{G}_*(\mathfrak{b},\jmath)]\left[\frac{1}{2}+\mathfrak{J}\left(\frac{1}{2}\right)\right]\int_0^1 \mathfrak{J}(\mathfrak{s})\,d\mathfrak{s} = \frac{4-\sqrt{2}-\sqrt{3}}{2}\jmath,$$
$$[\mathfrak{G}^*(\mathfrak{a},\jmath) + \mathfrak{G}^*(\mathfrak{b},\jmath)]\left[\frac{1}{2}+\mathfrak{J}\left(\frac{1}{2}\right)\right]\int_0^1 \mathfrak{J}(\mathfrak{s})\,d\mathfrak{s} = \frac{4-\sqrt{2}-\sqrt{3}}{2}(2-\jmath),$$

$$\rhd_{1_*} = \left[\frac{\mathfrak{G}_*(\mathfrak{a},\jmath)+\mathfrak{G}_*(\mathfrak{b},\jmath)}{2} + \mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\right]\int_0^1 \mathfrak{J}(\mathfrak{s})\,d\mathfrak{s} = \frac{8-\sqrt{2}-\sqrt{3}-\sqrt{10}}{4}\jmath,$$
$$\rhd_1{}^* = \left[\frac{\mathfrak{G}^*(\mathfrak{a},\jmath)+\mathfrak{G}^*(\mathfrak{b},\jmath)}{2} + \mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\right]\int_0^1 \mathfrak{J}(\mathfrak{s})\,d\mathfrak{s} = \frac{8-\sqrt{2}-\sqrt{3}-\sqrt{10}}{4}(2-\jmath),$$

$$\rhd_{2_*} = \frac{1}{4\mathfrak{J}\left(\frac{1}{2}\right)}\left[\mathfrak{G}_*\left(\left[\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)+\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+3\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\right] = \frac{5-\sqrt{11}}{4}\jmath,$$
$$\rhd_2{}^* = \frac{1}{4\mathfrak{J}\left(\frac{1}{2}\right)}\left[\mathfrak{G}^*\left(\left[\frac{3\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)+\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+3\mathfrak{b}^{\mathfrak{p}}}{4}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\right] = \frac{5-\sqrt{11}}{4}(2-\jmath),$$

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right) = \frac{4-\sqrt{10}}{2}\jmath,$$
$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right) = \frac{4-\sqrt{10}}{2}(2-\jmath).$$

Then, we obtain that

$$\frac{4-\sqrt{10}}{2}\jmath \le \frac{5-\sqrt{11}}{4}\jmath \le \frac{21}{50}\jmath \le \frac{8-\sqrt{2}-\sqrt{3}-\sqrt{10}}{4}\jmath \le \frac{4-\sqrt{2}-\sqrt{3}}{2}\jmath,$$
$$\frac{4-\sqrt{10}}{2}(2-\jmath) \le \frac{5-\sqrt{11}}{4}(2-\jmath) \le \frac{21}{50}(2-\jmath) \le \frac{8-\sqrt{2}-\sqrt{3}-\sqrt{10}}{4}(2-\jmath) \le \frac{4-\sqrt{2}-\sqrt{3}}{2}(2-\jmath).$$

Hence, Theorem 4.4 is verified.

Next, Theorems 4.6 and 4.7 obtain the fuzzy interval integral inequalities for the product of $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$s

Theorem 4.6. Let $\mathfrak{G},\mathcal{J} : [\mathfrak{a},\mathfrak{b}] \to \mathbb{F}_C(\mathbb{R})$ be two $(\mathfrak{p},\mathfrak{J}_1)$-convex and $(\mathfrak{p},\mathfrak{J}_2)$-convex $FIVF$s with non-negative real valued functions $\mathfrak{J}_1,\mathfrak{J}_2 : [0,1] \to \mathbb{R}$, respectively, whose $\jmath$-levels define the family of IVFs $\mathfrak{G}_\jmath$, $\mathcal{J}_\jmath : [\mathfrak{a},\mathfrak{b}] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are, respectively, given by $\mathfrak{G}_\jmath(\mathfrak{x}) = [\mathfrak{G}_*(\mathfrak{x},\jmath),\mathfrak{G}^*(\mathfrak{x},\jmath)]$ and $\mathcal{J}_\jmath(\mathfrak{x}) = [\mathcal{J}_*(\mathfrak{x},\jmath),\mathcal{J}^*(\mathfrak{x},\jmath)]$ for all $\mathfrak{x} \in [\mathfrak{a},\mathfrak{b}]$ and for all $\jmath \in [0,1]$. If $\mathfrak{G} \tilde{\times} \mathcal{J} \in \mathcal{FR}_{([\mathfrak{a},\mathfrak{b}],\jmath)}$, then

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(FR)\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x}) \tilde{\times} \mathcal{J}(\mathfrak{x})d\mathfrak{x} \preccurlyeq \mathcal{M}(\mathfrak{a},\mathfrak{b})\int_0^1 \mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(\mathfrak{s})\,d\mathfrak{s} \tilde{+} \mathcal{N}(\mathfrak{a},\mathfrak{b})\int_0^1 \mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(1-\mathfrak{s})\,d\mathfrak{s}, \quad (51)$$

where $\mathcal{M}(\mathfrak{a},\mathfrak{b}) = \mathfrak{G}(\mathfrak{a}) \tilde{\times} \mathcal{J}(\mathfrak{a}) \tilde{+} \mathfrak{G}(\mathfrak{b}) \tilde{\times} \mathcal{J}(\mathfrak{b})$, $\mathcal{N}(\mathfrak{a},\mathfrak{b}) = \mathfrak{G}(\mathfrak{a}) \tilde{\times} \mathcal{J}(\mathfrak{b}) \tilde{+} \mathfrak{G}(\mathfrak{b}) \tilde{\times} \mathcal{J}(\mathfrak{a})$, and $\mathcal{M}(\mathfrak{a},\mathfrak{b}) = [\mathcal{M}_*((\mathfrak{a},\mathfrak{b}),\jmath),\mathcal{M}^*((\mathfrak{a},\mathfrak{b}),\jmath)]$ and $\mathcal{N}(\mathfrak{a},\mathfrak{b}) = [\mathcal{N}_*((\mathfrak{a},\mathfrak{b}),\jmath),\mathcal{N}^*((\mathfrak{a},\mathfrak{b}),\jmath)]$.

Proof. Since $\mathfrak{G}$ is a $(\mathfrak{p},\mathfrak{J}_1)$-convex $FIVF$ and $\mathcal{J}$ is a $(\mathfrak{p},\mathfrak{J}_2)$-convex $FIVF$ then, for each $\jmath \in [0,1]$, we have

$$\mathfrak{G}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\leq\mathfrak{I}_1(\mathfrak{s})\mathfrak{G}_*(\mathfrak{a},\mathscr{j})+\mathfrak{I}_1(1-\mathfrak{s})\mathfrak{G}_*(\mathfrak{b},\mathscr{j}),$$
$$\mathfrak{G}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\leq\mathfrak{I}_1(\mathfrak{s})\mathfrak{G}^*(\mathfrak{a},\mathscr{j})+\mathfrak{I}_1(1-\mathfrak{s})\mathfrak{G}^*(\mathfrak{b},\mathscr{j}),$$

and

$$\mathcal{J}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\leq\mathfrak{I}_2(\mathfrak{s})\mathcal{J}_*(\mathfrak{a},\mathscr{j})+\mathfrak{I}_2(1-\mathfrak{s})\mathcal{J}_*(\mathfrak{b},\mathscr{j}),$$
$$\mathcal{J}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\leq\mathfrak{I}_2(\mathfrak{s})\mathcal{J}^*(\mathfrak{a},\mathscr{j})+\mathfrak{I}_2(1-\mathfrak{s})\mathcal{J}^*(\mathfrak{b},\mathscr{j}).$$

From the definition of $(\mathfrak{p},\mathfrak{I})$-convex $FIVF$s it follows that $\mathfrak{G}(\mathfrak{x})\succcurlyeq\tilde{0}$ and $\mathcal{J}(\mathfrak{x})\succcurlyeq\tilde{0}$, so

$$\begin{aligned}
&\mathfrak{G}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\times\mathcal{J}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\\
&\leq\begin{pmatrix}\mathfrak{I}_1(\mathfrak{s})\mathfrak{G}_*(\mathfrak{a},\mathscr{j})\\+\mathfrak{I}_1(1-\mathfrak{s})\mathfrak{G}_*(\mathfrak{b},\mathscr{j})\end{pmatrix}\times\begin{pmatrix}\mathfrak{I}_2(\mathfrak{s})\mathcal{J}_*(\mathfrak{a},\mathscr{j})\\+\mathfrak{I}_2(1-\mathfrak{s})\mathcal{J}_*(\mathfrak{b},\mathscr{j})\end{pmatrix}\\
&=\mathfrak{G}_*(\mathfrak{a},\mathscr{j})\times\mathcal{J}_*(\mathfrak{a},\mathscr{j})[\mathfrak{I}_1(\mathfrak{s})\,\mathfrak{I}_2(\mathfrak{s})]\\
&+\mathfrak{G}_*(\mathfrak{b},\mathscr{j})\times\mathcal{J}_*(\mathfrak{b},\mathscr{j})[\mathfrak{I}_1(1-\mathfrak{s})\,\mathfrak{I}_2(1-\mathfrak{s})]\\
&+\mathfrak{G}_*(\mathfrak{a},\mathscr{j})\times\mathcal{J}_*(\mathfrak{b},\mathscr{j})\,\mathfrak{I}_1(\mathfrak{s})\,\mathfrak{I}_2(1-\mathfrak{s})\\
&+\mathfrak{G}_*(\mathfrak{b},\mathscr{j})\times\mathcal{J}_*(\mathfrak{a},\mathscr{j})\,\mathfrak{I}_1(1-\mathfrak{s})\,\mathfrak{I}_2(\mathfrak{s}),\\
&\mathfrak{G}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\times\mathcal{J}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\\
&\leq\begin{pmatrix}\mathfrak{I}_1(\mathfrak{s})\mathfrak{G}^*(\mathfrak{a},\mathscr{j})\\+\mathfrak{I}_1(1-\mathfrak{s})\mathfrak{G}^*(\mathfrak{b},\mathscr{j})\end{pmatrix}\times\begin{pmatrix}\mathfrak{I}_2(\mathfrak{s})\,\mathcal{J}^*(\mathfrak{a},\mathscr{j})\\+\mathfrak{I}_2(1-\mathfrak{s})\,\mathcal{J}^*(\mathfrak{b},\mathscr{j})\end{pmatrix}\\
&=\mathfrak{G}^*(\mathfrak{a},\mathscr{j})\times\mathcal{J}^*(\mathfrak{a},\mathscr{j})[\mathfrak{I}_1(\mathfrak{s})\,\mathfrak{I}_2(\mathfrak{s})]\\
&+\mathfrak{G}^*(\mathfrak{b},\mathscr{j})\times\mathcal{J}^*(\mathfrak{b},\mathscr{j})[\mathfrak{I}_1(1-\mathfrak{s})\,\mathfrak{I}_2(1-\mathfrak{s})]\\
&+\mathfrak{G}^*(\mathfrak{a},\mathscr{j})\times\mathcal{J}^*(\mathfrak{b},\mathscr{j})\,\mathfrak{I}_1(\mathfrak{s})\,\mathfrak{I}_2(1-\mathfrak{s})\\
&+\mathfrak{G}^*(\mathfrak{b},\mathscr{j})\times\mathcal{J}^*(\mathfrak{a},\mathscr{j})\,\mathfrak{I}_1(1-\mathfrak{s})\,\mathfrak{I}_2(\mathfrak{s}),
\end{aligned}$$

Integrating both sides of above inequality over [0, 1] we get

$$\begin{aligned}
&\int_0^1\mathfrak{G}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\times\mathcal{J}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\\
&=\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\mathscr{j})\times\mathcal{J}_*(\mathfrak{x},\mathscr{j})\,d\mathfrak{x}\\
&\leq\left(\mathfrak{G}_*(\mathfrak{a},\mathscr{j})\times\mathcal{J}_*(\mathfrak{a},\mathscr{j})+\mathfrak{G}_*(\mathfrak{b},\mathscr{j})\times\mathcal{J}_*(\mathfrak{b},\mathscr{j})\right)\int_0^1\mathfrak{I}_1(\mathfrak{s})\,\mathfrak{I}_2(\mathfrak{s})\,d\mathfrak{s}\\
&+\left(\mathfrak{G}_*(\mathfrak{a},\mathscr{j})\times\mathcal{J}_*(\mathfrak{b},\mathscr{j})+\mathfrak{G}_*(\mathfrak{b},\mathscr{j})\times\mathcal{J}_*(\mathfrak{a},\mathscr{j})\right)\int_0^1\mathfrak{I}_1(\mathfrak{s})\,\mathfrak{I}_2(1-\mathfrak{s})\,d\mathfrak{s},\\
&\int_0^1\mathfrak{G}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\times\mathcal{J}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\mathscr{j}\right)\\
&=\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\mathscr{j})\times\mathcal{J}^*(\mathfrak{x},\mathscr{j})\,d\mathfrak{x}\\
&\leq\left(\mathfrak{G}^*(\mathfrak{a},\mathscr{j})\times\mathcal{J}^*(\mathfrak{a},\mathscr{j})+\mathfrak{G}^*(\mathfrak{b},\mathscr{j})\times\mathcal{J}^*(\mathfrak{b},\mathscr{j})\right)\int_0^1\mathfrak{I}_1(\mathfrak{s})\,\mathfrak{I}_2(\mathfrak{s})\,d\mathfrak{s}\\
&+\left(\mathfrak{G}^*(\mathfrak{a},\mathscr{j})\times\mathcal{J}^*(\mathfrak{b},\mathscr{j})+\mathfrak{G}^*(\mathfrak{b},\mathscr{j})\times\mathcal{J}^*(\mathfrak{a},\mathscr{j})\right)\int_0^1\mathfrak{I}_1(\mathfrak{s})\,\mathfrak{I}_2(1-\mathfrak{s})\,d\mathfrak{s}.
\end{aligned}$$

It follows that

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath) \times \mathcal{J}_*(\mathfrak{x},\jmath)\, d\mathfrak{x}$$
$$\leq \mathcal{M}_*\big((\mathfrak{a},\mathfrak{b}),\jmath\big)\int_0^1 \mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(\mathfrak{s})\, d\mathfrak{s} + \mathcal{N}_*\big((\mathfrak{a},\mathfrak{b}),\jmath\big)\int_0^1 \mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(1-\mathfrak{s})\, d\mathfrak{s},$$
$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath) \times \mathcal{J}^*(\mathfrak{x},\jmath)\, d\mathfrak{x}$$
$$\leq \mathcal{M}^*\big((\mathfrak{a},\mathfrak{b}),\jmath\big)\int_0^1 \mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(\mathfrak{s})\, d\mathfrak{s} + \mathcal{N}^*\big((\mathfrak{a},\mathfrak{b}),\jmath\big)\int_0^1 \mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(1-\mathfrak{s})\, d\mathfrak{s},$$

that is,

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\left[\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath) \times \mathcal{J}_*(\mathfrak{x},\jmath) d\mathfrak{x}\,, \int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath) \times \mathcal{J}^*(\mathfrak{x},\jmath)\, d\mathfrak{x}\right]$$
$$\leq_I \left[\mathcal{M}_*\big((\mathfrak{a},\mathfrak{b}),\jmath\big), \mathcal{M}^*\big((\mathfrak{a},\mathfrak{b}),\jmath\big)\right]\int_0^1 \mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(\mathfrak{s})\, d\mathfrak{s}$$
$$+\left[\mathcal{N}_*\big((\mathfrak{a},\mathfrak{b}),\jmath\big), \mathcal{N}^*\big((\mathfrak{a},\mathfrak{b}),\jmath\big)\right]\int_0^1 \mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(1-\mathfrak{s})\, d\mathfrak{s}.$$

Thus,

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(FR)\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x}) \tilde{\times} \mathcal{J}(\mathfrak{x}) d\mathfrak{x} \preccurlyeq \mathcal{M}(\mathfrak{a},\mathfrak{b})\int_0^1 \mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(\mathfrak{s})\, d\mathfrak{s}\ \tilde{+}\mathcal{N}(\mathfrak{a},\mathfrak{b})\int_0^1 \mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(1-\mathfrak{s})\, d\mathfrak{s},$$

and the theorem has been established.

Theorem 4.7. Let $\mathfrak{G}, \mathcal{J} : [\mathfrak{a},\mathfrak{b}] \to \mathbb{F}_C(\mathbb{R})$ be two $(\mathfrak{p},\mathfrak{J}_1)$-convex and $(\mathfrak{p},\mathfrak{J}_2)$-convex *FIVF*s with non-negative real valued functions $\mathfrak{J}_1, \mathfrak{J}_2 : [0,1] \to \mathbb{R}^+$ such that $\mathfrak{J}_1, \mathfrak{J}_2 \not\equiv 0$ and $\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right) \neq 0$, respectively, whose $\jmath$-levels define the family of *IVF*s $\mathfrak{G}_\jmath$, $\mathcal{J}_\jmath : [\mathfrak{a},\mathfrak{b}] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ are, respectively, given by $\mathfrak{G}_\jmath(\mathfrak{x}) = [\mathfrak{G}_*(\mathfrak{x},\jmath), \mathfrak{G}^*(\mathfrak{x},\jmath)]$ and $\mathcal{J}_\jmath(\mathfrak{x}) = [\mathcal{J}_*(\mathfrak{x},\jmath), \mathcal{J}^*(\mathfrak{x},\jmath)]$ for all $\mathfrak{x} \in [\mathfrak{a},\mathfrak{b}]$ and for all $\jmath \in [0,1]$. If $\mathfrak{G} \tilde{\times} \mathcal{J} \in \mathcal{FR}_{([\mathfrak{a},\mathfrak{b}],\jmath)}$, then

$$\frac{1}{2\,\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)}\,\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right) \tilde{\times} \mathcal{J}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right) \preccurlyeq \frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(FR)\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x}) \tilde{\times} \mathcal{J}(\mathfrak{x}) d\mathfrak{x}$$
$$\tilde{+}\mathcal{M}(\mathfrak{a},\mathfrak{b})\int_0^1 \mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(1-\mathfrak{s})\, d\mathfrak{s}\,\tilde{+}\mathcal{N}(\mathfrak{a},\mathfrak{b})\int_0^1 \mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(\mathfrak{s})\, d\mathfrak{s}, \tag{52}$$

where $\mathcal{M}(\mathfrak{a},\mathfrak{b}) = \mathfrak{G}(\mathfrak{a}) \tilde{\times} \mathcal{J}(\mathfrak{a}) \tilde{+} \mathfrak{G}(\mathfrak{b}) \tilde{\times} \mathcal{J}(\mathfrak{b})$, $\mathcal{N}(\mathfrak{a},\mathfrak{b}) = \mathfrak{G}(\mathfrak{a}) \tilde{\times} \mathcal{J}(\mathfrak{b}) \tilde{+} \mathfrak{G}(\mathfrak{b}) \tilde{\times} \mathcal{J}(\mathfrak{a})$, and $\mathcal{M}(\mathfrak{a},\mathfrak{b}) = \left[\mathcal{M}_*\big((\mathfrak{a},\mathfrak{b}),\jmath\big), \mathcal{M}^*\big((\mathfrak{a},\mathfrak{b}),\jmath\big)\right]$ and $\mathcal{N}(\mathfrak{a},\mathfrak{b}) = \left[\mathcal{N}_*\big((\mathfrak{a},\mathfrak{b}),\jmath\big), \mathcal{N}^*\big((\mathfrak{a},\mathfrak{b}),\jmath\big)\right]$.

Proof. By hypothesis, for each $\jmath \in [0,1]$, we have

$$\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right) \times \mathcal{J}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)$$
$$\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right) \times \mathcal{J}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)$$

$$\leq \mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix} \mathfrak{G}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right) \\ +\mathfrak{G}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}_*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}},\jmath]^{\frac{1}{\mathfrak{p}}}\right) \end{bmatrix}$$

$$+\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix} \mathfrak{G}_*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right) \\ +\mathfrak{G}_*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}_*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right) \end{bmatrix},$$

$$\leq \mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix} \mathfrak{G}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right) \\ +\mathfrak{G}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}^*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right) \end{bmatrix}$$

$$+\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix} \mathfrak{G}^*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right) \\ +\mathfrak{G}^*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}^*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}},\jmath]^{\frac{1}{\mathfrak{p}}}\right) \end{bmatrix},$$

$$\leq \mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix} \mathfrak{G}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right) \\ +\mathfrak{G}_*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}_*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right) \end{bmatrix}$$

$$+\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix} \left(\mathfrak{J}_1(\mathfrak{s})\,\mathfrak{G}_*(\mathfrak{a},\jmath)+\mathfrak{J}_1(1-\mathfrak{s})\,\mathfrak{G}_*(\mathfrak{b},\jmath)\right) \\ \times\left(\mathfrak{J}_2(1-\mathfrak{s})\,\mathcal{J}_*(\mathfrak{a},\jmath)+\mathfrak{J}_2(\mathfrak{s})\,\mathcal{J}_*(\mathfrak{b},\jmath)\right) \\ +\left(\mathfrak{J}_1(1-\mathfrak{s})\,\mathfrak{G}_*(\mathfrak{a},\jmath)+\mathfrak{J}_1(\mathfrak{s})\,\mathfrak{G}_*(\mathfrak{b},\jmath)\right) \\ \times\left(\mathfrak{J}_2(\mathfrak{s})\,\mathcal{J}_*(\mathfrak{a},\jmath)+\mathfrak{J}_2(1-\mathfrak{s})\,\mathcal{J}_*(\mathfrak{b},\jmath)\right) \end{bmatrix},$$

$$\leq \mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix} \mathfrak{G}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right) \\ +\mathfrak{G}^*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}^*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right) \end{bmatrix}$$

$$+\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix} \left(\mathfrak{J}_1(\mathfrak{s})\,\mathfrak{G}^*(\mathfrak{a},\jmath)+\mathfrak{J}_1(1-\mathfrak{s})\,\mathfrak{G}^*(\mathfrak{b},\jmath)\right) \\ \times\left(\mathfrak{J}_2(1-\mathfrak{s})\,\mathcal{J}^*(\mathfrak{a},\jmath)+\mathfrak{J}_2(\mathfrak{s})\,\mathcal{J}^*(\mathfrak{b},\jmath)\right) \\ +\left(\mathfrak{J}_1(1-\mathfrak{s})\,\mathfrak{G}^*(\mathfrak{a},\jmath)+\mathfrak{J}_1(\mathfrak{s})\,\mathfrak{G}^*(\mathfrak{b},\jmath)\right) \\ \times\left(\mathfrak{J}_2(\mathfrak{s})\,\mathcal{J}^*(\mathfrak{a},\jmath)+\mathfrak{J}_2(1-\mathfrak{s})\,\mathcal{J}^*(\mathfrak{b},\jmath)\right) \end{bmatrix},$$

$$=\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix} \mathfrak{G}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right) \\ +\mathfrak{G}_*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}_*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right) \end{bmatrix}$$

$$+2\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\begin{bmatrix} \{\mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(\mathfrak{s})+\mathfrak{J}_1(1-\mathfrak{s})\,\mathfrak{J}_2(1-\mathfrak{s})\}\mathcal{N}_*\left((\mathfrak{a},\mathfrak{b}),\jmath\right) \\ +\{\mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(1-\mathfrak{s})+\mathfrak{J}_1(1-\mathfrak{s})\,\mathfrak{J}_2(\mathfrak{s})\}\mathcal{M}_*\left((\mathfrak{a},\mathfrak{b}),\jmath\right) \end{bmatrix},$$

$$= \mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\mathfrak{G}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\\+\mathfrak{G}^*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}^*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\end{array}\right]$$
$$+2\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)\left[\begin{array}{c}\{\mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(\mathfrak{s})+\mathfrak{J}_1(1-\mathfrak{s})\mathfrak{J}_2(1-\mathfrak{s})\}\mathcal{N}^*((\mathfrak{a},\mathfrak{b}),\jmath)\\+\{\mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(1-\mathfrak{s})+\mathfrak{J}_1(1-\mathfrak{s})\mathfrak{J}_2(\mathfrak{s})\}\mathcal{M}^*((\mathfrak{a},\mathfrak{b}),\jmath)\end{array}\right].$$

Integrating over $[0,1]$, we have

$$\frac{1}{2\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)}\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\leq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(R)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\times\mathcal{J}_*(\mathfrak{x},\jmath)d\mathfrak{x}$$
$$+\mathcal{M}_*((\mathfrak{a},\mathfrak{b}),\jmath)\int_0^1\mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(1-\mathfrak{s})\,d\mathfrak{s}+\mathcal{N}_*((\mathfrak{a},\mathfrak{b}),\jmath)\int_0^1\mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(\mathfrak{s})\,d\mathfrak{s},$$
$$\frac{1}{2\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)}\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\leq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(R)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\times\mathcal{J}^*(\mathfrak{x},\jmath)d\mathfrak{x}$$
$$+\mathcal{M}^*((\mathfrak{a},\mathfrak{b}),\jmath)\int_0^1\mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(1-\mathfrak{s})\,d\mathfrak{s}+\mathcal{N}^*((\mathfrak{a},\mathfrak{b}),\jmath)\int_0^1\mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(\mathfrak{s})\,d\mathfrak{s},$$

that is,

$$\frac{1}{2\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)}\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)\tilde{\times}\mathcal{J}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)\preccurlyeq\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(FR)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x})\tilde{\times}\mathcal{J}(\mathfrak{x})d\mathfrak{x}$$

$$\tilde{+}\mathcal{M}(\mathfrak{a},\mathfrak{b})\int_0^1\mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(1-\mathfrak{s})\,d\mathfrak{s}\,\tilde{+}\mathcal{N}(\mathfrak{a},\mathfrak{b})\int_0^1\mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(\mathfrak{s})\,d\mathfrak{s}.$$

Hence, we have the required result.

Example 4.8. Let $\mathfrak{p}$ be an odd number and $\mathfrak{J}_1(\mathfrak{s})=\mathfrak{s}$, $\mathfrak{J}_2(\mathfrak{s})=1$, for $\mathfrak{s}\in[0,1]$, and the $(\mathfrak{p},\mathfrak{J}_1)$-convex and $(\mathfrak{p},\mathfrak{J}_2)$-convex $FIVF$s $\mathfrak{G},\mathcal{J}:[\mathfrak{a},\mathfrak{b}]=[2,3]\to\mathbb{F}_C(\mathbb{R})$ are, respectively, defined by $\mathfrak{G}_{\jmath}(\mathfrak{x})=\left[\jmath\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right),(2-\jmath)\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)\right]$, as in Example 4.3, and $\mathcal{J}_{\jmath}(\mathfrak{x})=[\jmath\mathfrak{x}^{\mathfrak{p}},(2-\jmath)\mathfrak{x}^{\mathfrak{p}}]$. Since $\mathfrak{G}(x)$ and $\mathcal{J}(x)$ both are $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$s, $\mathfrak{G}_*(\mathfrak{x},\jmath)=\jmath\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)$, $\mathfrak{G}^*(\mathfrak{x},\jmath)=(2-\jmath)\left(2-\mathfrak{x}^{\frac{\mathfrak{p}}{2}}\right)$, and $\mathcal{J}_*(\mathfrak{x},\jmath)=\jmath\mathfrak{x}^{\mathfrak{p}}$, $\mathcal{J}^*(\mathfrak{x},\jmath)=(2-\jmath)\mathfrak{x}^{\mathfrak{p}}$, we compute the following.

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\times\mathcal{J}_*(\mathfrak{x},\jmath)d\mathfrak{x}=\jmath^2,$$
$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\times\mathcal{J}^*(\mathfrak{x},\jmath)d\mathfrak{x}=(2-\jmath)^2,$$

$$\mathcal{M}_*((\mathfrak{a},\mathfrak{b}),\jmath)\int_0^1\mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(\mathfrak{s})\,d\mathfrak{s}=\left(10-2\sqrt{2}-3\sqrt{3}\right)\frac{\jmath^2}{2},$$
$$\mathcal{M}^*((\mathfrak{a},\mathfrak{b}),\jmath)\int_0^1\mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(\mathfrak{s})\,d\mathfrak{s}=\left(10-2\sqrt{2}-3\sqrt{3}\right)\frac{(2-\jmath)^2}{2},$$

$$\mathcal{N}_*((\mathfrak{a},\mathfrak{b}),\jmath)\int_0^1\mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(1-\mathfrak{s})\,d\mathfrak{s}=\left(10-3\sqrt{2}-2\sqrt{3}\right)\frac{\jmath^2}{2}$$
$$\mathcal{N}^*((\mathfrak{a},\mathfrak{b}),\jmath)\int_0^1\mathfrak{J}_1(\mathfrak{s})\mathfrak{J}_2(1-\mathfrak{s})\,d\mathfrak{s}=\left(10-3\sqrt{2}-2\sqrt{3}\right)\frac{(2-\jmath)^2}{2},$$

for each $\jmath\in[0,1]$, which means

$$\jmath^2 \le \left(20 - 5\sqrt{2} - 5\sqrt{3}\right)\frac{\jmath^2}{2},$$
$$(2-\jmath)^2 \le \left(20 - 5\sqrt{2} - 5\sqrt{3}\right)\frac{(2-\jmath)^2}{2}.$$

Hence, Theorem 4.6 is demonstrated.
For Theorem 4.7, we have

$$\frac{1}{2\,\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)}\,\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)=\frac{20-5\sqrt{10}}{4}\jmath^2,$$
$$\frac{1}{2\,\mathfrak{J}_1\left(\frac{1}{2}\right)\mathfrak{J}_2\left(\frac{1}{2}\right)}\,\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\times\mathcal{J}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)=\frac{20-5\sqrt{10}}{4}(2-\jmath)^2,$$
$$\mathcal{M}_*\big((\mathfrak{a},\mathfrak{b}),\jmath\big)\int_0^1\mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(1-\mathfrak{s})\,d\mathfrak{s} = \left(10-2\sqrt{2}-3\sqrt{3}\right)\frac{\jmath^2}{2},$$
$$\mathcal{M}^*\big((\mathfrak{a},\mathfrak{b}),\jmath\big)\int_0^1\mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(1-\mathfrak{s})\,d\mathfrak{s} = \left(10-2\sqrt{2}-3\sqrt{3}\right)\frac{(2-\jmath)^2}{2},$$
$$\mathcal{N}_*\big((\mathfrak{a},\mathfrak{b}),\jmath\big)\int_0^1\mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(\mathfrak{s})\,d\mathfrak{s} = \left(10-3\sqrt{2}-2\sqrt{3}\right)\frac{\jmath^2}{2},$$
$$\mathcal{N}^*\big((\mathfrak{a},\mathfrak{b}),\jmath\big)\int_0^1\mathfrak{J}_1(\mathfrak{s})\,\mathfrak{J}_2(\mathfrak{s})\,d\mathfrak{s} = \left(10-3\sqrt{2}-2\sqrt{3}\right)\frac{(2-\jmath)^2}{2},$$

for each $\jmath \in [0,1]$, which means

$$\frac{20-5\sqrt{10}}{4}\jmath^2 \le \left(1+\frac{20-5\sqrt{2}-5\sqrt{3}}{2}\right)\jmath^2,$$
$$\frac{20-5\sqrt{10}}{4}(2-\jmath)^2 \le \left(1+\frac{20-5\sqrt{2}-5\sqrt{3}}{2}\right)(2-\jmath)^2.$$

Hence, Theorem 4.7 is verified.
Next, Theorems 4.9 and 4.10 give the second HH-Fejér inequality and the first HH-Fejér inequality for $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$, respectively.

Theorem 4.9. (Second HH-Fejér inequality for $\mathfrak{J}$-convex $FIVF$) Let $\mathfrak{G}:[\mathfrak{a},\mathfrak{b}]\to\mathbb{F}_C(\mathbb{R})$ be a $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$ with $\mathfrak{a}<\mathfrak{b}$ and $\mathfrak{J}:[0,1]\to\mathbb{R}^+$, whose $\jmath$-levels define the family of IVFs $\mathfrak{G}_\jmath:[\mathfrak{a},\mathfrak{b}]\subset\mathbb{R}\to{\mathcal{K}_C}^+$ are given by $\mathfrak{G}_\jmath(\mathfrak{x})=[\mathfrak{G}_*(\mathfrak{x},\jmath),\mathfrak{G}^*(\mathfrak{x},\jmath)]$ for all $\mathfrak{x}\in[\mathfrak{a},\mathfrak{b}]$ and for all $\jmath\in[0,1]$. If $\mathfrak{G}\in\mathcal{FR}_{([\mathfrak{a},\mathfrak{b}],\jmath)}$ and $\Omega:[\mathfrak{a},\mathfrak{b}]\to\mathbb{R},\Omega(\mathfrak{x})\ge 0$, $\mathfrak{p}$-symmetric with respect to $\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}$, then

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\,(FR)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x})\Omega(\mathfrak{x})d\mathfrak{x}\preccurlyeq[\mathfrak{G}(\mathfrak{a})\tilde{+}\mathfrak{G}(\mathfrak{b})]\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}. \tag{53}$$

If $\mathfrak{G}$ is $(\mathfrak{p},\mathfrak{J})$-concave $FIVF$, then inequality (53) is reversed.

Proof. Let $\mathfrak{G}$ be a $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$. Then, for each $\jmath\in[0,1]$, we have

$$\begin{aligned}&\mathfrak{G}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)\\&\quad\le\big(\mathfrak{J}(\mathfrak{s})\mathfrak{G}_*(\mathfrak{a},\jmath)+\mathfrak{J}(1-\mathfrak{s})\mathfrak{G}_*(\mathfrak{b},\jmath)\big)\Omega\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right),\\&\mathfrak{G}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)\\&\quad\le\big(\mathfrak{J}(\mathfrak{s})\mathfrak{G}^*(\mathfrak{a},\jmath)+\mathfrak{J}(1-\mathfrak{s})\mathfrak{G}^*(\mathfrak{b},\jmath)\big)\Omega\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right).\end{aligned}\tag{54}$$

Also,

$$
\begin{aligned}
&\mathfrak{G}_*\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)\\
&\quad\leq\left(\mathfrak{J}(1-\mathfrak{s})\mathfrak{G}_*(\mathfrak{a},\jmath)+\mathfrak{J}(\mathfrak{s})\mathfrak{G}_*(\mathfrak{b},\jmath)\right)\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right),\\
&\mathfrak{G}^*\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)\\
&\quad\leq\left(\mathfrak{J}(1-\mathfrak{s})\mathfrak{G}^*(\mathfrak{a},\jmath)+\mathfrak{J}(\mathfrak{s})\mathfrak{G}^*(\mathfrak{b},\jmath)\right)\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right).
\end{aligned}
\tag{55}
$$

After adding (54) and (55) and integrating over $[0,1]$, we get

$$
\begin{aligned}
&\int_0^1\mathfrak{G}_*\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\
&\qquad+\int_0^1\mathfrak{G}_*\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\
&\leq\int_0^1\left[\begin{array}{l}\mathfrak{G}_*(\mathfrak{a},\jmath)\left\{\mathfrak{J}(\mathfrak{s})\Omega\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)+\mathfrak{J}(1-\mathfrak{s})\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)\right\}\\+\mathfrak{G}_*(\mathfrak{b},\jmath)\left\{\mathfrak{J}(1-\mathfrak{s})\Omega\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)+\mathfrak{J}(\mathfrak{s})\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)\right\}\end{array}\right]d\mathfrak{s}\\
&=2\mathfrak{G}_*(\mathfrak{a},\jmath)\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}+2\mathfrak{G}_*(\mathfrak{b},\jmath)\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s},\\
&\int_0^1\mathfrak{G}^*\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\
&\qquad+\int_0^1\mathfrak{G}^*\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\
&\leq\int_0^1\left[\begin{array}{l}\mathfrak{G}^*(\mathfrak{a},\jmath)\left\{\mathfrak{J}(\mathfrak{s})\Omega\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)+\mathfrak{J}(1-\mathfrak{s})\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)\right\}\\+\mathfrak{G}^*(\mathfrak{b},\jmath)\left\{\mathfrak{J}(1-\mathfrak{s})\Omega\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)+\mathfrak{J}(\mathfrak{s})\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)\right\}\end{array}\right]d\mathfrak{s}.\\
&=2\mathfrak{G}^*(\mathfrak{a},\jmath)\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}+2\mathfrak{G}^*(\mathfrak{b},\jmath)\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}.
\end{aligned}
$$

Since Ω is symmetric,

$$
\begin{aligned}
&=2[\mathfrak{G}_*(\mathfrak{a},\jmath)+\mathfrak{G}_*(\mathfrak{b},\jmath)]\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s},\\
&=2[\mathfrak{G}^*(\mathfrak{a},\jmath)+\mathfrak{G}^*(\mathfrak{b},\jmath)]\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}.
\end{aligned}
\tag{56}
$$

Since

$$
\begin{aligned}
&\int_0^1\mathfrak{G}_*\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\
&\qquad=\int_0^1\mathfrak{G}_*\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\
&\qquad\qquad=\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x},\\
&\int_0^1\mathfrak{G}^*\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\
&\qquad=\int_0^1\mathfrak{G}^*\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\
&\qquad\qquad=\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}.
\end{aligned}
\tag{57}
$$

From (57) and integrating with respect to $\mathfrak{s}$ over $[0,1]$, we have

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}\le[\mathfrak{G}_*(\mathfrak{a},\jmath)+\mathfrak{G}_*(\mathfrak{b},\jmath)]\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s},$$

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}\le[\mathfrak{G}^*(\mathfrak{a},\jmath)+\mathfrak{G}^*(\mathfrak{b},\jmath)]\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s},$$

that is,

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\left[\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x},\ \int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}\right]$$

$$\le_I[\mathfrak{G}_*(\mathfrak{a},\jmath)+\mathfrak{G}_*(\mathfrak{b},\jmath),\mathfrak{G}^*(\mathfrak{a},\jmath)+\mathfrak{G}^*(\mathfrak{b},\jmath)]\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s},$$

and hence,

$$\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}(FR)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x})\Omega(\mathfrak{x})d\mathfrak{x}\preccurlyeq[\mathfrak{G}(\mathfrak{a})\tilde{+}\mathfrak{G}(\mathfrak{b})]\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}.$$

Theorem 4.10. (First HH-Fejér inequality for $\mathfrak{J}$-convex $FIVF$) Let $\mathfrak{G}:[\mathfrak{a},\mathfrak{b}]\to\mathbb{F}_C(\mathbb{R})$ be a $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$ with $\mathfrak{a}<\mathfrak{b}$ and $\mathfrak{J}:[0,1]\to\mathbb{R}^+$such that $\mathfrak{J}\left(\frac{1}{2}\right)\not\equiv 0$, whose $\jmath$-levels define the family of IVFs $\mathfrak{G}_\jmath:[\mathfrak{a},\mathfrak{b}]\subset\mathbb{R}\to\mathcal{K}_C^{+}$ are given by $\mathfrak{G}_\jmath(\mathfrak{x})=[\mathfrak{G}_*(\mathfrak{x},\jmath),\mathfrak{G}^*(\mathfrak{x},\jmath)]$ for all $\mathfrak{x}\in[\mathfrak{a},\mathfrak{b}]$ and for all $\jmath\in[0,1]$. If $\mathfrak{G}\in\mathcal{FR}_{([\mathfrak{a},\mathfrak{b}],\jmath)}$ and $\Omega:[\mathfrak{a},\mathfrak{b}]\to\mathbb{R},\Omega(\mathfrak{x})\ge 0$, $\mathfrak{p}$-symmetric with respect to $\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}$, and $\int_{\mathfrak{a}}^{\mathfrak{b}}\Omega(\mathfrak{x})d\mathfrak{x}>0$, then

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{p}+\mathfrak{b}^{p}}{2}\right]^{\frac{1}{p}}\right)\preccurlyeq\frac{1}{\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{p-1}\Omega(\mathfrak{x})d\mathfrak{x}}(FR)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{p-1}\mathfrak{G}(\mathfrak{x})\Omega(\mathfrak{x})d\mathfrak{x}. \tag{58}$$

If $\mathfrak{G}$ is $(\mathfrak{p},\mathfrak{J})$-concave $FIVF$, then inequality (58) is reversed.

Proof. Since $\mathfrak{G}$ is a $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$, for each $\jmath\in[0,1]$ we have

$$\begin{aligned}\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)&\le\mathfrak{J}\left(\frac{1}{2}\right)\left(\mathfrak{G}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)+\mathfrak{G}_*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\right),\\ \mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)&\le\mathfrak{J}\left(\frac{1}{2}\right)\left(\mathfrak{G}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)+\mathfrak{G}^*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\right).\end{aligned} \tag{59}$$

By multiplying (59) by $\Omega\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)=\Omega\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)$ and integrating it by $\mathfrak{s}$ over $[0,1]$, we obtain

$$\begin{aligned}&\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\int_0^1\Omega\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)d\mathfrak{s}\\ &\le\mathfrak{J}\left(\frac{1}{2}\right)\begin{pmatrix}\int_0^1\mathfrak{G}_*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\+\int_0^1\mathfrak{G}_*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\end{pmatrix},\\ &\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\int_0^1\Omega\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\ &\le\mathfrak{J}\left(\frac{1}{2}\right)\begin{pmatrix}\int_0^1\mathfrak{G}^*\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left([\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\+\int_0^1\mathfrak{G}^*\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\end{pmatrix}.\end{aligned} \tag{60}$$

Since

$$\int_0^1 \mathfrak{G}_*\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}$$
$$=\int_0^1 \mathfrak{G}_*\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}$$
$$=\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x},$$
$$\int_0^1 \mathfrak{G}^*\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[\mathfrak{s}\mathfrak{a}^{\mathfrak{p}}+(1-\mathfrak{s})\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s} \tag{61}$$
$$=\int_0^1 \mathfrak{G}^*\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\Omega\left(\left[(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}\right]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}$$
$$=\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x},$$

from (61), (60) we have

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\le\frac{1}{\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\Omega(\mathfrak{x})d\mathfrak{x}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x},$$
$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\le\frac{1}{\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\Omega(\mathfrak{x})d\mathfrak{x}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}.$$

From that, we have

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\left[\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right),\ \mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\right]$$
$$\le_I\frac{1}{\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\Omega(\mathfrak{x})d\mathfrak{x}}\left[\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x},\ \int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}\right],$$

that is,

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}}\right)\preccurlyeq\frac{1}{\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\Omega(\mathfrak{x})d\mathfrak{x}}\ (FR)\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}.$$

This completes the proof.

Remark 4.11. If in the Theorems 4.9 and 4.10 $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}^{s}$, then we obtain the appropriate theorems for $(\mathfrak{p},s)$-convex $FIVF$s on the second sense (see [85]):

If in the Theorems 4.9 and 10 $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}$, then we obtain the appropriate theorems for $\mathfrak{p}$-convex $FIVF$s (see [85]).

If $\mathfrak{G}_*(\mathfrak{a},\jmath)=\mathfrak{G}^*(\mathfrak{a},\jmath)$ with $\jmath=1$, then Theorems 4.9 and 4.10 reduce to classical first and second HH-Fejér inequality for $(\mathfrak{p},\mathfrak{J})$-convex function.

If in the Theorems 4.9 and 4.10, $\mathfrak{G}_*(\mathfrak{a},\jmath)=\mathfrak{G}^*(\mathfrak{a},\jmath)$ with $\jmath=1$ and $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}$, then we obtain the appropriate theorems for $\mathfrak{p}$-convex function (see [53]).

If in the Theorems 4.9 and 4.10, $\mathfrak{G}_*(\mathfrak{a},\jmath)=\mathfrak{G}^*(\mathfrak{a},\jmath)$ with $\jmath=1$, $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}$ and $\mathfrak{p}=1$, then we obtain the appropriate theorems for convex function, [42].

If $\Omega(\mathfrak{x})=1$, then combining Theorems 4.9 and 4.10, we get Theorem 4.1.

Example 4.12. We consider $\mathfrak{J}(\mathfrak{s})=\mathfrak{s}$, for $\mathfrak{s}\in[0,1]$, and the $FIVF$ $\mathfrak{G}:[1,4]\to\mathbb{F}_C(\mathbb{R})$ defined by,

$$\mathfrak{G}(\mathfrak{x})(\sigma)=\begin{cases}\frac{\sigma-e^{\mathfrak{x}^{\mathfrak{p}}}}{e^{\mathfrak{x}^{\mathfrak{p}}}}, & \sigma\in\left[e^{\mathfrak{x}^{\mathfrak{p}}},2e^{\mathfrak{x}^{\mathfrak{p}}}\right],\\ \frac{4e^{\mathfrak{x}^{\mathfrak{p}}}-\sigma}{2e^{\mathfrak{x}^{\mathfrak{p}}}}, & \sigma\in\left(2e^{\mathfrak{x}^{\mathfrak{p}}},4e^{\mathfrak{x}^{\mathfrak{p}}}\right],\\ 0, & otherwise,\end{cases} \tag{62}$$

Then, for each $\jmath\in[0,1]$, we have $\mathfrak{G}_{\jmath}(\mathfrak{x})=\left[(1+\jmath)e^{\mathfrak{x}^{\mathfrak{p}}},2(2-\jmath)e^{\mathfrak{x}^{\mathfrak{p}}}\right]$. Since end point functions $\mathfrak{G}_*(\mathfrak{x},\jmath)$, $\mathfrak{G}^*(\mathfrak{x},\jmath)$ are $(\mathfrak{p},\mathfrak{J})$-convex functions, for each $\jmath\in[0,1]$, $\mathfrak{G}(\mathfrak{x})$ is $(\mathfrak{p},\mathfrak{J})$-convex *FIVF*. If

$$\Omega(\mathfrak{x})=\begin{cases}\mathfrak{x}^{\mathfrak{p}}-1, & \sigma\in\left[1,\frac{5}{2}\right],\\ 4-\mathfrak{x}^{\mathfrak{p}}, & \sigma\in\left(\frac{5}{2},4\right],\end{cases} \tag{63}$$

where $\mathfrak{p}=1$, then we have

$$\begin{aligned}
&\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_1^4\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}=\frac{1}{3}\int_1^4\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}\\
&\quad=\frac{1}{3}\int_1^{\frac{5}{2}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}+\frac{1}{3}\int_{\frac{5}{2}}^4\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x},\\
&=\frac{1}{3}(1+\jmath)\int_1^{\frac{5}{2}}e^{\mathfrak{x}}(\mathfrak{x}-1)d\mathfrak{x}+\frac{1}{3}(1+\jmath)\int_{\frac{5}{2}}^4e^{\mathfrak{x}}(4-\mathfrak{x})d\mathfrak{x}\approx 11(1+\jmath),\\
&\frac{\mathfrak{p}}{\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\int_1^4\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}=\frac{1}{3}\int_1^4\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}\\
&\quad=\frac{1}{3}\int_1^{\frac{5}{2}}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}+\frac{1}{3}\int_{\frac{5}{2}}^4\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x},\\
&=\frac{2}{3}(2-\jmath)\int_1^{\frac{5}{2}}e^{\mathfrak{x}}(\mathfrak{x}-1)d\mathfrak{x}+\frac{2}{3}(2-\jmath)\int_{\frac{5}{2}}^4e^{\mathfrak{x}}(4-\mathfrak{x})d\mathfrak{x}\approx 22(2-\jmath),
\end{aligned} \tag{64}$$

and

$$\begin{aligned}
&[\mathfrak{G}_*(\mathfrak{a},\jmath)+\mathfrak{G}_*(\mathfrak{b},\jmath)]\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\
&[\mathfrak{G}^*(\mathfrak{a},\jmath)+\mathfrak{G}^*(\mathfrak{b},\jmath)]\int_0^1\mathfrak{J}(\mathfrak{s})\Omega\left([(1-\mathfrak{s})\mathfrak{a}^{\mathfrak{p}}+\mathfrak{s}\mathfrak{b}^{\mathfrak{p}}]^{\frac{1}{\mathfrak{p}}}\right)d\mathfrak{s}\\
&=(1+\jmath)[e+e^4]\left[\int_0^{\frac{1}{2}}3\mathfrak{s}^2d\mathfrak{x}+\int_{\frac{1}{2}}^1\mathfrak{s}(3-3\mathfrak{s})d\mathfrak{s}\right]\approx\frac{43}{2}(1+\jmath).\\
&=2(2-\jmath)[e+e^4]\left[\int_0^{\frac{1}{2}}3\mathfrak{s}^2d\mathfrak{x}+\int_{\frac{1}{2}}^1\mathfrak{s}(3-3\mathfrak{s})d\mathfrak{s}\right]\approx 43(2-\jmath).
\end{aligned} \tag{65}$$

From (64) and (65), we have

$$[11(1+\jmath),22(2-\jmath)]\leq_I\left[\frac{43}{2}(1+\jmath),43(2-\jmath)\right],\ \text{for each }\jmath\in[0,1].$$

Hence, Theorem 4.9 is verified.

For Theorem 4.10, we have

$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}_*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\approx\frac{61}{5}(1+\jmath),$$
$$\frac{1}{2\mathfrak{J}\left(\frac{1}{2}\right)}\mathfrak{G}^*\left(\left[\frac{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}{2}\right]^{\frac{1}{\mathfrak{p}}},\jmath\right)\approx\frac{122}{5}(2-\jmath)\,, \tag{66}$$

$$\int_{\mathfrak{a}}^{\mathfrak{b}}\Omega(\mathfrak{x})d\mathfrak{x}=\int_{1}^{\frac{5}{2}}(\mathfrak{x}-1)d\mathfrak{x}\int_{\frac{5}{2}}^{4}(4-\mathfrak{x})d\mathfrak{x}=\frac{9}{4},$$

$$\frac{1}{\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\Omega(\mathfrak{x})d\mathfrak{x}}\int_{1}^{4}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}_*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}\approx\frac{73}{5}(1+\jmath),$$
$$\frac{1}{\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{x}^{\mathfrak{p}-1}\Omega(\mathfrak{x})d\mathfrak{x}}\int_{1}^{4}\mathfrak{x}^{\mathfrak{p}-1}\mathfrak{G}^*(\mathfrak{x},\jmath)\Omega(\mathfrak{x})d\mathfrak{x}\approx\frac{293}{10}(2-\jmath). \tag{67}$$

From (66) and (67), we have

$$\left[\frac{61}{5}(1+\jmath),\frac{122}{5}(2-\jmath)\right]\leq_I\left[\frac{73}{5}(1+\jmath),\frac{293}{10}(2-\jmath)\right].$$

Hence, Theorem 4.10 is demonstrated.

5. Conclusions

The $(\mathfrak{p},\mathfrak{J})$-convex (concave, affine) class for $FIVF$s was established in this paper. For $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$, we created some brand-new discrete Jensen and Schur type inequalities. Additionally, using fuzzy Riemann integrals, we discovered several HH-inequalities for $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$s. Examples were used to demonstrate how our findings apply to a broad class of previously undiscovered and well-known inequalities for $(\mathfrak{p},\mathfrak{J})$-convex $FIVF$s and their variant forms. We will try to analyze Jensen and HH-inequalities for IVF and $FIVF$s on a temporal scale in the future as we explore these ideas. We hope that the theories and methods presented in this paper can serve as a springboard for additional study in this field.

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Conflict of interest

The authors declare that they have no competing interests.

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