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Research article

Fractional orthotriple fuzzy Choquet-Frank aggregation operators and their application in optimal selection for EEG of depression patients

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Abstract: The fractional orthotriple fuzzy sets (FOFSs) are a generalized fuzzy set model that is more accurate, practical, and realistic. It is a more advanced version of the present fuzzy set models that can be used to identify false data in real-world scenarios. Compared to the picture fuzzy set and Spherical fuzzy set, the fractional orthotriple fuzzy set (FOFS) is a powerful tool. Additionally, aggregation operators are effective mathematical tools for condensing a set of finite values into one value that assist us in decision making (DM) challenges. Due to the generality of FOFS and the benefits of aggregation operators, we established two new aggregation operators in this article using the Frank t-norm and conorm operation, which we have renamed the fractional orthotriple fuzzy Choquet-Frank averaging (FOFCFA) and fractional orthotriple fuzzy Choquet-Frank geometric (FOFCFG) operators. A few of these aggregation operators' characteristics are also discussed. To demonstrate the efficacy of the introduced work, the multi-attribute decision making (MADM) algorithm is discussed along with applications. To demonstrate the validity and value of the suggested work, a comparison of the proposed work has also been provided.

Keywords: fractional orthotriple fuzzy set; Frank operations; Choquet-Frank aggregation operators; decision making
Mathematics Subject Classification: 03E72, 47S40

1. Introduction

We apply several sentences with attributes that are not considered to be propositioned in our lifestyle problems because it is difficult to say whether or not these sentences are absolutely correct or incorrect. We encounter various forms of these sentences in our day-to-day issues. In order to deal with these kinds of situations, researchers in the fields of engineering, business, and social sciences need to use various modelling techniques. Zadeh [40] developed the notion of fuzzy set (FS) theory in 1965 as a solution to this problem by taking into account the membership degree (MD) that belongs to the range $[0, 1]$. Therefore, using the fuzzy set allows for the incorporation of incomplete and partial information into the decision version [10]. Numerous theories have been created in the context of fuzzy sets, and [6] presents a hybrid MCDM technique built on fuzzy DEMATEL, fuzzy TOPSIS, and fuzzy ANP to calculate green suppliers. One cannot dispute the fact that FS is the foundation of fuzzy set theory. Lin et al. [18] determine OWA operator weights using kernel density estimation. Researchers in this field have noted some short comings with fuzzy set. The intuitionistic fuzzy set (IFS), which takes into account the hesitation of experts, is how Atanassov [1] generalizes the concept of fuzzy set. Entropy is defined in [31] for intuitionistic fuzzy sets. For a moment, consider how IFS can only describe situations when the MD and non-membership degree (NMD) sum does not exceed 1, rendering it essentially useless in those situations. Hani et al. [16] developed linear Diophantine fuzzy graphs with new decision-making approach. Riaz et al. [28] defined an innovative bipolar fuzzy sine trigonometric aggregation operators and SIR method for medical tourism supply chain.

Yager [37] developed the notion of the Pythagorean fuzzy set (PyFS) to handle this complexity, which can also be used to solve the problem mentioned above. Since many researchers attempt to use this recently developed idea in their research, Akram et al. [2] established a group decision making (GDM) approach based on the Pythagorean fuzzy TOPSIS process. Additionally, based on the Pythagorean fuzzy TOPSIS method, a novel correlation measure and its application are provided in [17]. A Pythagorean fuzzy VIKOR technique was presented by Zhou and Chen [42] who also established a generalized distance measurement. The TODIM technique for MADM with linguistic 2-tuple PyF data was presented by Deng and Gao [13]. Yager [38] introduced a novel concept known in the literature as q-ROFS, which is the generalization of IFS and PyFS, as a result of the ongoing complexity of real-life issues and the development of FS theory. The concept of q-ROFS is also more broad. Because the sum of the qth powers of MD and NMD must fall within the range $[0, 1]$, it has a void range. Be aware that q-ROFS gives decision-makers access to a wider range of options. Garg et al. [14] presented some power aggregation operator (AO) and VIKOR method for complex q-ROFS keeping in mind that AOs are the fundamental tools to transform the total data into a unique value. The q-ROFS pointed weighted aggregation operators for MCDM were developed by Xing et al. [35]. Lin et al. [19] proposed linguistic q-rung orthopair fuzzy sets and their interaction partitioned Heronian mean aggregation operators.

But in many decision-making (DM) issues, like voting, human opinions consist of the following categories: Yes, no, abstinence, and refusal, which are not characterized in IFSs. It should be noted that all of the aforementioned existing literature only uses either MD or both MD and NMD. Therefore, neither the FSs nor the IFSs can treat this issue. Cuong [11] defined the notion of a picture fuzzy set (PFS) and added the neutral membership degree (NuMD) to deal with those types of situation when someone's opinions include more choice types, such as yes, abstention, no, and rejection, it is observed that PFS approaches are more applicable in these situations. Some methods and techniques have been introduced based on PFS, and it is a more general apparatus that is attracting more attention from researchers working in FS theory. In order to use the linguistic picture fuzzy TOPSIS method in enterprise resource systems, Zang et al. [41] provide an expanded version of it. Additionally, [22] describes an expanded picture fuzzy VIKOR technique and its applications in the beef industry. Lin et al. [21] defined picture fuzzy interactional partitioned Heronian mean aggregation operators: An application of MADM process. After that, Mahmood et al. [23] extended the idea of the picture fuzzy set to spherical fuzzy set. Due to the fact that SFS gives decision-makers more room, many researchers have focused on this idea and some new techniques have been created using SFS. Given that the TOPSIS method is well known for handling fuzzy information (FI), Barukab et al. [7] presented a novel technique for the TOPSIS method that is focused on entropy measurement and uses the Spherical fuzzy set. Additionally, Ashraf et al. [4] invented some novel aggregation operator, such as the Spherical fuzzy Dombi aggregation operator. Alaoui et al. [5] defined a novel analysis of fuzzy physical models by generalized fractional fuzzy operators.

The Frank t-norm and t-conorm are important generalizations of the probabilistic and Lukasiewicz t-norm and t-conorm. They are a large and adjustable family of continuous triangular norms. The Frank t-norm and t-conorm are more adaptable in the information fusion process and are better suited to modelling DMs problems because they both have a parameter. The functional equations of Frank and Alsina [8] were examined in two groups of commutative, associative, and increasing binary operators. Yager [36] defined Frank t-norms' additive generating function and used it to create a model of approximation reasoning. Frank t-norm scalar cardinality was the subject of an axiomatic method proposed by Casasnovas and Torrens [9], who also showed that the properties hold for other t-common and t-norms. In order to build a full algebraic lattice structure, Deschrijver [12] presented Frank t-norms-based extending operations to the lattice of closed interval-valued fuzzy sets (IVFS). Yahya et al. [39] developed Frank aggregation operators and their application to probabilistic hesitant fuzzy MADM. Tang et al. [32] proposed a MADM approaches based on dual hesitant fuzzy Frank aggregation operators. Mahnaz et al. [24] defined the idea of T-spherical fuzzy Frank aggregation operators and their application to decision making with unknown weight information. Riaz et al. [29] defines an interval-valued linear Diophantine fuzzy Frank aggregation operators with MCDM.

The picture fuzzy set and spherical fuzzy set frameworks have many applications in a variety of real-world contexts. We developed a novel expanded idea of a fractional orthotriple fuzzy set (FOFS) to get free of these restrictions. The proposed framework of FOFS has three membership degree $\mu_\Omega(\hbar), \eta_\Omega(\hbar), \chi_\Omega(\hbar) \in [0,1]$ in the universal set M with the condition $\mu_\Omega^f(\hbar) + \eta_\Omega^f(\hbar) + \chi_\Omega^f(\hbar) \leq 1, f \in Q^+$ (positive rational numbers) with $f \geq 1$, i.e., $f = p/q, f \neq 0, p, q \in N$, for each $\hbar \in M$. We observe that as rung f rises, the fractional orthotriple fuzzy space expands, giving observers more latitude to express their support for membership. Be aware that in order to deal with uncertainty and inaccurate information, we were able to obtain a fractional orthotriple fuzzy set that produced more precise and accurate rung fuzzy numbers. It is obvious that FOFSs are the general forms of picture fuzzy set and spherical fuzzy set, and if we put $p = q$ (for PFSs) and $p = 2n; q = n$ for all $n \in N$ (for SFSs), the corresponding set becomes too PFSs and SFSs, respectively. Due to their greater flexibility and suitability for dealing with uncertain information, FOFSs express more extensive fuzzy information.

There are two main advantages to the Frank t-conorm and t-norm. They begin by sharing some advantages with the Algebraic, Hamacher, and Einstein t-norm and t-conorm. Second, they have an additional boundary that results in a flexible and robust total cycle, making them stand out from other general t-norm and t-conorm. Hence, inspired by the previously mentioned investigation, the objective of this study as:

(1) To initiate the principle of some frank operational laws by using Frank t-norm and Frank t-conorm.

(2) To proposed fractional orthotriple fuzzy Choquet-Frank averaging operator, fractional orthotriple

fuzzy Choquet-Frank geometric operator, and their important properties are also elaborated.

(3) A multi-attribute decision making (MADM) approach is defined utilizing proposed operator to investigate the supremacy and consistency of the explored work.

(4) Finally, we discuss the benefits, sensitive analysis, and geometric exposes of the explored works with the aid of numerous examples in order to demonstrate the efficacy and the dominance of the initiated works.

Additionally, this article is structured as: In Section 2, we have overviews the concept of Frank t-norm and conorm, Choquet integral (CI), and fractional orthotriple fuzzy set. Several operational laws for fractional orthotriple fuzzy numbers are covered in Section 3 by Frank t-norm and t-conorm. The development of new aggregation operators for FOFSs, including the fractional orthotriple fuzzy Choquet-Frank averaging (FOFCFA) and fractional orthopair fuzzy Choquet-Frank geometric (FOFCFG) operators are discussed in detail in Section 4. Utilizing the aforementioned operators, we created the MADM approach for the fractional orthotriple fuzzy environment in Section 5. To verify the effectiveness of our method, numerical examples are discussed in Section 6. Comparative analysis of the proposed work with other techniques is covered in Section 7 to show the validity and impact of the suggested work. The conclusion was discussed in Section 8.

2. Preliminaries

In this section, we recall basic concept of fuzzy sets (FSs), Frank t-norm and co-norm, and the concept of Choquet integral.

Definition 2.1. [40] Suppose that M is an arbitrary nonempty set. A fuzzy set L is defined as

$$L = \{(\hbar, \mu_L(\hbar))|\hbar \in M\}, \tag{2.1}$$

where the function μ_L is a mapping from $M \to [0, 1]$, and for every $\hbar \in M, 0 \leq \mu_L(\hbar) \leq 1$, and function $\mu_L(\hbar)$ is called the membership degree of $\hbar$ in M.

Definition 2.2. [1] Suppose that M is an arbitrary nonempty set. An IFS Ω in M is defined as

$$\Omega = \{(\hbar, \mu_\Omega(\hbar), \eta_\Omega(\hbar))|\hbar \in M\}, \tag{2.2}$$

where $\mu_\Omega :\to [0, 1]$ and $\eta_\Omega \to [0, 1]$ are called respectively MD and NMD, such as $\forall\hbar \in M : 0 \leq \mu_\Omega + \eta_\Omega \leq 1$. Hesitancy degree is defined as

$$\pi_\Omega = 1 - \mu_\Omega - \eta_\Omega.$$

Definition 2.3. [23] Suppose that M is an arbitrary non-empty set. A SFS is denoted by Ω and defined as

$$\Omega = \{(\hbar, \mu_\Omega(\hbar), \eta_\Omega(\hbar), \chi_\Omega(\hbar))|\hbar \in M\}, \tag{2.3}$$

where μ_Ω, η_Ω and χ_Ω are MD, NuMD and NMD functions $M \to [0, 1]$ with respect to $(\mu_\Omega)^2 + (\eta_\Omega)^2 + (\chi_\Omega)^2 \leq 1$. The hesitancy degree is defined as

$$\pi_\Omega = \sqrt{1 - (\mu_\Omega)^2 - (\eta_\Omega)^2 - (\chi_\Omega)^2}.$$

Definition 2.4. [27] Suppose that M is an arbitrary non-empty set. A FOFS is denoted by Ω and defined as

$$\Omega = \{(\hbar, \mu_\Omega(\hbar), \eta_\Omega(\hbar), \chi_\Omega(\hbar)) | \hbar \in M\}, \tag{2.4}$$

where μ_Ω, η_Ω and χ_Ω are MD, NuMD and NMD functions $M \to [0,1]$ with respect to $(\mu_\Omega)^f + (\eta_\Omega)^f + (\chi_\Omega)^f \leq 1$. The hesitancy degree is defined as

$$\pi_\Omega = \sqrt[f]{1 - (\mu_\Omega)^f - (\eta_\Omega)^f - (\chi_\Omega)^f}.$$

Definition 2.5. [26] Let $\Omega_1 = (\mu_1, \eta_1, \chi_1)$ be any FOFNs. Then, the score and accuracy function is defined as

$$S(\Omega_1) = (\mu_\Omega)^f - (\eta_\Omega)^f - (\chi_\Omega)^f, \tag{2.5}$$

and accuracy as

$$E(\Omega_1) = (\mu_\Omega)^f + (\eta_\Omega)^f + (\chi_\Omega)^f. \tag{2.6}$$

2.1. Frank operations

Frank t-norm and t-conormare initiated by Frank [33] and are defined as

$$T_i(\phi, \varphi) = \log_\theta\left(1 + \frac{(\theta^\phi - 1)(\theta^\varphi - 1)}{\theta - 1}\right), \text{ for all } \phi, \varphi \in [0,1], \theta \in (0, +\infty), \tag{2.7}$$

$$S_i(\phi, \varphi) = 1 - \log_\theta\left(1 + \frac{(\theta^\phi - 1)(\theta^\varphi - 1)}{\theta - 1}\right), \text{ for all } \phi, \varphi \in [0,1], \theta \in (0, +\infty). \tag{2.8}$$

Where $T_i(\phi, \varphi)$ satisfying;

(1) $T_i(1,1) = 1, T_i(\phi, 0) = T_i(0, \phi) = \phi$. (Boundary)

(2) If $\phi_1 \leq \phi_2, \varphi_1 \leq \varphi_2$ then $T_i(\phi_1, \varphi_2) \leq T_i(\phi_2, \varphi_2)$. (Monotonicity)

(3) $T_i(\phi_1, \phi_2) \leq T_i(\phi_2, \phi_1)$. (Commutativity)

(4) $T_i(\phi_1, T_i(\phi_2, \phi_3)) = T_i(T_i(\phi_2, \phi_2), \phi_3)$. (Associativity)

Also, $S_i(\phi, \varphi)$ satisfying

(1) $S_i(0,0) = 0, S_i(\phi, 1) = \phi$. (Boundary)

(2) If $\phi_1 \leq \phi_2, \varphi_1 \leq \varphi_2$ then $S_i(\phi_1, \varphi_2) \leq S_i(\phi_2, \varphi_2)$. (Monotonicity)

(3) $S_i(\phi_1, \phi_2) \leq S_i(\phi_2, \phi_1)$. (Commutativity)

(4) $S_i(\phi_1, S_i(\phi_2, \phi_3)) = S_i(S_i(\phi_2, \phi_2), \phi_3)$. (Associativity)

Two special cases of Frank t-norm and t-conorm [30] are the following:

(1) If $\theta \to 0$, then $T_i(\phi, \varphi) \to \phi.\varphi, S_i(\phi, \varphi) \to \phi + \varphi - \phi\varphi$ become to the algebraic t-norm and t-conorm.

(2) If $\theta \to \infty$, then $T_i(\phi, \varphi) \to \max(0, \phi + \varphi - 1), S_i(\phi, \varphi) \to \min(0, \phi + \varphi - 1)$ become to Lukasiewicz product and sum.

2.2. Choquet integral (CI) operator

Sugeno [30] introduced the idea of a fuzzy measure (FM), which is used to discuss how much weight should be given to each set of criteria in the Choquet integral model. We then talk about fuzzy measure and Choquet integral.

Definition 2.6. A fuzzy measure on universal set M is a mapping $\Xi : \Omega(\hbar) \to [0, 1]$, with the following condition:

(1) $\Xi(0) = 0, \Xi(M) = 1,$

(2) $uv \in M$ and $u \subseteq v$ then $\Xi(u) \leq \Xi(v)$.

However, it is typically exceedingly difficult because we have to choose $2^n - 2$ values for n criterion. Therefore, it is difficult to provide such a fuzzy measure by Definition 2.6. Thus, the following $\wp$ fuzzy measure is defined by Sugeno [30].

$$\Xi(u \cup v) = \Xi(u) + \Xi(v) + \wp\Xi(u)\Xi(v), \tag{2.9}$$

where $(u \cup v) = \phi$ and $\wp \in [-1, +\infty]$.

(1) If $\wp = 0$, then the $\wp$ fuzzy measure reduces to the $\Xi(u \cup v) = \Xi(u) + \Xi(v), u \cup v = \phi$, specified as an additive measure.

If all $\hbar \in M$ are independent, then

$$\Xi(u) = \sum_{\hbar_i \in u} \Xi(\hbar_i). \tag{2.10}$$

If all $\hbar \in M$ is finite, then

$$\Xi(\mu) = \Xi\left(\bigcup_{i=1}^{n} \hbar_i\right) = \begin{cases} \frac{1}{\wp}\left(\prod_{i=1}^{n} (1 + \lambda\Xi(u)) - 1\right), \wp \neq 0 \\ \sum_{\hbar_i \in u} \Xi(\hbar_i), \wp = 0. \end{cases} \tag{2.11}$$

Where $\hbar_i \cap \hbar_j = \phi$, for all $i, j = 1, ..., n$ and $i \neq j$.

(2) If $\wp > 0$, then $\wp$ fuzzy measure reduces to $\Xi(u \cup v) > \Xi(u) + \Xi(v)$, is called a super-additive measure.

(3) If $-1 \leq \wp < 0$, so $\wp$ fuzzy measure reduces to $u \cup v < \Xi(u) + \Xi(v)$, known as a sub-additive measure.

Definition 2.7. [34] Let $\wp$ is a positive real (PR)-valued mapping on M and Ξ is a fuzzy measure on M. Then, discrete CI of $\wp$ with respect to Ξ is defined as

$$\int \wp d\Xi = \sum_{i=1}^{n} \left(\Xi(u_{\xi(i)}) - \Xi(u_{\xi(i-1)})\right) \wp_{\xi(i)}, \tag{2.12}$$

where $\xi(i)$ is a permutation of $(1, ..., n)$ such as $\wp_{\xi(1)} \geq \wp_{\xi(2)} \geq ..., \wp_{\xi(n)}$, and $u_{\xi(0)} = \phi$, $u_{\xi(i)} = \left(G_{\xi(1)}, ..., G_{\xi(i)}\right)$.

3. Frank operational laws for fractional orthotriple fuzzy numbers

In this section of the article, we defined operational laws using the concept of Frank t-norm, t-conorm and fractional orthotriple fuzzy numbers.

Definition 3.1. Let $\Omega_1 = (\mu_1, \eta_1, \chi_1)$ and $\Omega_2 = (\mu_2, \eta_2, \chi_2)$ be any two FOFNs. Then, the following addition and multiplication laws are proposed using Frank t-norm and Frank t-conorm as

$$(1)\ \Omega_1 \oplus \Omega_2 = \left\{ \begin{array}{c} \sqrt[f]{1 - \log_\theta \left(1 + \frac{\left(\theta^{1-\mu_1^f} - 1\right)\left(\theta^{1-\mu_2^f} - 1\right)}{\theta - 1} \right)}, \\ \sqrt[f]{\log_\theta \left(1 + \frac{\left(\theta^{\eta_1^f} - 1\right)\left(\theta^{\eta_2^f} - 1\right)}{\theta - 1} \right)}, \sqrt[f]{\log_\theta \left(1 + \frac{\left(\theta^{\chi_1^f} - 1\right)\left(\theta^{\chi_2^f} - 1\right)}{\theta - 1} \right)} \end{array} \right\};$$

$$(2)\ \Omega_1 \otimes \Omega_2 = \left\{ \begin{array}{c} \sqrt[f]{\log_\theta \left(1 + \frac{\left(\theta^{\mu_1^f} - 1\right)\left(\theta^{\mu_2^f} - 1\right)}{\theta - 1} \right)}, \sqrt[f]{1 - \log_\theta \left(1 + \frac{\left(\theta^{1-\eta_1^f} - 1\right)\left(\theta^{1-\eta_2^f} - 1\right)}{\theta - 1} \right)}, \\ \sqrt[f]{1 - \log_\theta \left(1 + \frac{\left(\theta^{1-\chi_1^f} - 1\right)\left(\theta^{1-\chi_2^f} - 1\right)}{\theta - 1} \right)} \end{array} \right\};$$

$$(3)\ \lambda\Omega_1 = \left\{ \begin{array}{c} \sqrt[f]{1 - \log_\theta \left(1 + \frac{\left(\theta^{1-\mu_1^f} - 1\right)^\lambda}{(\theta - 1)^{\lambda-1}} \right)}, \sqrt[f]{\log_\theta \left(1 + \frac{\left(\theta^{\eta_1^f} - 1\right)^\lambda}{(\theta - 1)^{\lambda-1}} \right)}, \\ \sqrt[f]{\log_\theta \left(1 + \frac{\left(\theta^{\chi_1^f} - 1\right)^\lambda}{(\theta - 1)^{\lambda-1}} \right)} \end{array} \right\};$$

$$(4)\ \Omega_1^\lambda = \left\{ \begin{array}{c} \sqrt[f]{\log_\theta \left(1 + \frac{\left(\theta^{\mu_1^f} - 1\right)^\lambda}{(\theta - 1)^{\lambda}} \right)}, \sqrt[f]{1 - \log_\theta \left(1 + \frac{\left(\theta^{1-\eta_1^f} - 1\right)^\lambda}{(\theta - 1)^{\lambda-1}} \right)}, \\ \sqrt[f]{1 - \log_\theta \left(1 + \frac{\left(\theta^{1-\chi_1^f} - 1\right)^\lambda}{(\theta - 1)^{\lambda-1}} \right)} \end{array} \right\}.$$

4. Fractional orthotriple fuzzy Choquet-Frank aggregation operators

Using Frank operational laws of FOFNs and by Choquet integral, defined in the previous section, we will introduce the fractional orthotriple fuzzy Choquet-Frank aggregation operators, the first one is fractional orthotriple fuzzy Choquet-Frank average (FOFCFA) operator and the second one is fractional orthotriple fuzzy Choquet-Frank geometric (FOFCFG) operator.

4.1. Fractional orthotriple fuzzy Choquet-Frank average operator

Definition 4.1. Let $\Omega_i = (\mu_i, \eta_i, \chi_i)\,(i = 1, ..., n)$ is a family of FOFNs, and $\xi(i)$ is a permutation of $(i = 1, ..., n)$ such as $\Omega_{\xi(i)} \geq \Omega_{\xi(2)} \geq ..., \geq \Omega_{\xi(n)}$. Then, a FOFCFA operator is defined as;

$$FOFCFA(\Omega_1, ..., \Omega_n) = \sum_{i=1}^{n} \left(\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)\right)\Omega_{\xi(i)}. \tag{4.1}$$

Where $u_{\xi(i)} = \left(G_{\xi(1)}, ..., G_{\xi(i)}\right), u_{\xi(i)} = \phi$, also $G_{\xi(i)}$ is the attribute related to $\Omega_{\xi(i)}$.

Theorem 4.1. Let $\Omega_i = (\mu_i, \eta_i, \chi_i)\,(i = 1, ..., n)$ is a family of FOFNs, and $\xi(i)$ is a permutation of $(i = 1, ..., n)$ such that $\Omega_{\xi(i)} \geq \Omega_{\xi(2)} \geq ..., \geq \Omega_{\xi(n)}$. $G_{\xi(i)}$ is the attribute related to $\Omega_{\xi(i)}, u_{\xi(i)} = \phi$, $u_{\xi(i)} = \left(G_{\xi(1)}, ..., G_{\xi(i)}\right)$. Then, using FOFCFA operator the aggregated value is also a FOFN and,

$$\begin{aligned} &FOFCFA(\Omega_1, ..., \Omega_n) \\ = &\left\{ \begin{array}{c} \sqrt[f]{1 - \log_\theta\left(1 + \prod_{i=1}^{n}\left(\theta^{1-\mu_{\xi(i)}^f} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{\log_\theta\left(1 + \prod_{i=1}^{n}\left(\theta^{\eta_{\xi(i)}^f} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{\log_\theta\left(1 + \prod_{i=1}^{n}\left(\theta^{\chi_{\xi(i)}^f} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)} \end{array} \right\}. \end{aligned} \tag{4.2}$$

Proof. It is very easy to show FOFCFA is a FOFN, so we ignore here. Next, we prove Eq (4.2) by MI on n. If $n = 2$, then by operational law (3).

$$\begin{aligned} &\left(\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)\right).\Omega_{\xi(i)} \\ = &\left\{ \begin{array}{c} \sqrt[f]{1 - \log_\theta\left(1 + \dfrac{\left(\theta^{1-\mu_{\xi(i)}^f} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}}{(\theta-1)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right) - 1}}\right)}, \\ \sqrt[f]{\log_\theta\left(1 + \dfrac{\left(\theta^{\eta_{\xi(i)}^f} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}}{(\theta-1)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right) - 1}}\right)}, \\ \sqrt[f]{\log_\theta\left(1 + \dfrac{\left(\theta^{\chi_{\Omega_i}^f} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}}{(\theta-1)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right) - 1}}\right)} \end{array} \right\}. \end{aligned}$$

Then,

$$\begin{aligned} &FOFCFA(\Omega_1, \Omega_2) \\ = &\left(\Xi\left(u_{\xi(1)}\right) - \Xi\left(u_{\xi(0)}\right)\right).\Omega_{\xi(1)} \oplus \left(\Xi\left(u_{\xi(2)}\right) - \Xi\left(u_{\xi(1)}\right)\right).\Omega_{\xi(2)} \end{aligned}$$

$$= \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\frac{\prod_{i=1}^{2}\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{(\theta-1)^{\sum_{i=1}^{2}\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)-1}}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^{2}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{(\theta-1)^{\sum_{i=1}^{2}\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)-1}}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^{2}\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{(\theta-1)^{\sum_{i=1}^{2}\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)-1}}\right)} \end{array} \right\}.$$

Since, $\sum_{i=1}^{2}\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)=1$. So, we get

$$FOFCFA(\Omega_1,\Omega_2) = \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\prod_{i=1}^{2}\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{2}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{2}\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \end{array} \right\}.$$

So Eq (4.2) is true for $n=2$.
Now, if Eq (4.2) is true for $n=k$,

$$\begin{aligned} & FOFCFA(\Omega_1,...,\Omega_k) \\ = & \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\prod_{i=1}^{k}\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{k}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{k}\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \end{array} \right\}. \end{aligned}$$

For $n=k+1$

$$\begin{aligned} & FOFCFA(\Omega_1,...,\Omega_k,\Omega_{k+1}) \\ = & FOFCFA(\Omega_1,...,\Omega_k)\oplus\left(\Xi\left(u_{\xi(k)}\right)-\Xi\left(u_{\xi(k-1)}\right)\right).\Omega_{\xi(k+1)} \\ = & \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\prod_{i=1}^{k}\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{k}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{k}\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \end{array} \right\} \end{aligned}$$

$$\oplus\left\{\begin{array}{c}\sqrt[f]{1-\log_\theta\left(1+\frac{\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{(\theta-1)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)-1}}\right)},\\ \sqrt[f]{\log_\theta\left(1+\frac{\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{(\theta-1)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)-1}}\right)},\\ \sqrt[f]{\log_\theta\left(1+\frac{\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{(\theta-1)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)-1}}\right)}\end{array}\right\}$$

$$=\left\{\begin{array}{c}\sqrt[f]{1-\log_\theta\left(1+\frac{\prod_{i=1}^{k+1}\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{(\theta-1)^{\sum_{i=1}^{k+1}\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)-1}}\right)},\\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^{k+1}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{(\theta-1)^{\sum_{i=1}^{k+1}\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)-1}}\right)},\\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^{k+1}\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\sum_{i=1}^{n}\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{(\theta-1)^{\sum_{i=1}^{k+1}\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)-1}}\right)}\end{array}\right\}.$$

Since, $\sum_{i=1}^{k+1}\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)=1$. So, we get

$$\begin{aligned}&FOFCFA(\Omega_1,...,\Omega_{k+1})\\ =&\left\{\begin{array}{c}\sqrt[f]{1-\log_\theta\left(1+\prod_{i=1}^{k+1}\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)},\\ \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{k+1}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)},\\ \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{k+1}\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}\end{array}\right\}.\end{aligned}$$

This implies that Eq (4.2) true for $n=k+1$. This shows that Eq (4.2) true for all values of n.

Theorem 4.2. Let $\Omega_i=(\mu_i,\eta_i,\chi_i)\,(i=1,...,n)$ is a family of FOFNs, and $\xi(i)$ is a permutation of $(i=1,...,n)$ such that $\Omega_{\xi(i)}\geq\Omega_{\xi(2)}\geq...,\geq\Omega_{\xi(n)}$. Then,

$$FOFCFA(\lambda.\Omega_1,...,\lambda.\Omega_n)=\lambda.FOFCFA(\Omega_1,...,\Omega_n). \tag{4.3}$$

Proof. By operational law (3) and Theorem 4.1 we have

$$\lambda.\Omega_i = \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\frac{\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^\lambda}{(\theta-1)^{\lambda-1}}\right)}, \sqrt[f]{\log_\theta\left(1+\frac{\left(\theta^{\eta^f_{\xi(i)}}-1\right)^\lambda}{(\theta-1)^{\lambda-1}}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\left(\theta^{\chi^f_{\xi(i)}}-1\right)^\lambda}{(\theta-1)^{\lambda-1}}\right)} \end{array} \right\}.$$

Then,

$$\begin{aligned} & FOFCFA(\lambda.\Omega_1,...,\lambda.\Omega_n) \\ = & \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\frac{\prod_{i=1}^n\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\lambda\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)}}{(\theta-1)^{(\lambda-1)}}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^n\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\lambda\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)}}{(\theta-1)^{(\lambda-1)}}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^n\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\lambda\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)}}{(\theta-1)^{(\lambda-1)}}\right)} \end{array} \right\}. \end{aligned}$$

Now,

$$\begin{aligned} & \lambda.FOFCFA(\Omega_1,...,\Omega_n) \\ = & \lambda.\left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\prod_{i=1}^n\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^n\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^n\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \end{array} \right\} \\ = & \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\frac{\prod_{i=1}^n\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\lambda\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)}}{(\theta-1)^{(\lambda-1)}}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^n\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\lambda\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)}}{(\theta-1)^{(\lambda-1)}}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^n\left(\theta^{\chi^f_{\Omega_i}}-1\right)^{\lambda\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)}}{(\theta-1)^{(\lambda-1)}}\right)} \end{array} \right\} \end{aligned}$$

$$= \quad FOFCFA(\lambda.\Omega_1, ..., \lambda.\Omega_n).$$

Therefore, Eq (4.3) holds, so proof completed.

Theorem 4.3. Let $\Omega_i = (\mu_i, \eta_i, \chi_i)\,(i = 1, ..., n)$ is a family of FOFNs, and $\xi(i)$ is a permutation of $(i = 1, ..., n)$ such that $\Omega_{\xi(i)} \geq \Omega_{\xi(2)} \geq ..., \geq \Omega_{\xi(n)}$. Then,

$$FOFCFA(\Omega_1 \oplus \Omega, ..., \Omega_n \oplus \Omega) = FOFCFA(\Omega_1, ..., \Omega_n) \oplus \Omega. \tag{4.4}$$

Proof. Using operational law (1) and Theorem 4.1, we get

$$(\Omega_i \oplus \Omega) = \left\{ \begin{array}{c} \sqrt[f]{1 - \log_\theta \left(1 + \frac{\left(\theta^{1-\mu_i^f} - 1\right)\left(\theta^{1-\mu^f} - 1\right)}{\theta - 1} \right)}, \\ \sqrt[f]{\log_\theta \left(1 + \frac{\left(\theta^{\eta_i^f} - 1\right)\left(\theta^{\eta^f} - 1\right)}{\theta - 1} \right)}, \\ \sqrt[f]{\log_\theta \left(1 + \frac{\left(\theta^{\chi_i^f} - 1\right)\left(\theta^{\chi^f} - 1\right)}{\theta - 1} \right)} \end{array} \right\}.$$

Thus,

$$\begin{aligned} & FOFCFA(\Omega_1 \oplus \Omega, ..., \Omega_n \oplus \Omega) \\ = & \left\{ \begin{array}{c} \sqrt[f]{1 - \log_\theta \left(1 + \frac{\left(\theta^{1-\mu^f} - 1\right) \prod_{i=1}^{n} \left(\theta^{1-\mu_i^f} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}}{\theta - 1} \right)}, \\ \sqrt[f]{\log_\theta \left(1 + \frac{\left(\theta^{\eta^f} - 1\right) \prod_{i=1}^{n} \left(\theta^{\eta_i^f} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}}{\theta - 1} \right)}, \\ \sqrt[f]{\log_\theta \left(1 + \frac{\left(\theta^{\chi^f} - 1\right) \prod_{i=1}^{n} \left(\theta^{\chi_i^f} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}}{\theta - 1} \right)} \end{array} \right\}. \end{aligned}$$

Now,

$$\begin{aligned} & FOFCFA(\Omega_1, ..., \Omega_n) \oplus \Omega \\ = & \left\{ \begin{array}{c} \sqrt[f]{1 - \log_\theta \left(1 + \prod_{i=1}^{n} \left(\theta^{1-\mu_{\xi(i)}^f} - 1 \right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)} \right)}, \\ \sqrt[f]{\log_\theta \left(1 + \prod_{i=1}^{n} \left(\theta^{\eta_{\xi(i)}^f} - 1 \right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)} \right)}, \\ \sqrt[f]{\log_\theta \left(1 + \prod_{i=1}^{n} \left(\theta^{\chi_{\xi(i)}^f} - 1 \right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)} \right)} \end{array} \right\} \oplus (\mu, \eta, \chi) \end{aligned}$$

$$= \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\frac{\left(\theta^{1-\mu^f}-1\right)\prod_{i=1}^{n}\left(\theta^{1-\mu_i^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\left(\theta^{\eta^f}-1\right)\prod_{i=1}^{n}\left(\theta^{\eta_i^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\left(\theta^{\chi^f}-1\right)\prod_{i=1}^{n}\left(\theta^{\chi_i^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)} \end{array} \right\}$$

$$= FOFCFA(\Omega_1 \oplus \Omega, ..., \Omega_n \oplus \Omega).$$

Theorem 4.4. Let $\Omega_i = (\mu_i, \eta_i, \chi_i)\,(i = 1, ..., n)$ is a family of FOFNs, and $\xi(i)$ is a permutation of $(i = 1, ..., n)$ such that $\Omega_{\xi(i)} \geq \Omega_{\xi(2)} \geq ..., \geq \Omega_{\xi(n)}$. Then,

$$\begin{aligned} & FOFCFA(\Omega_{u_1} \oplus \Omega_{v_1}, ..., \Omega_{u_n} \oplus \Omega_{v_n}) \\ = \; & FOFCFA(\Omega_{u_1}, ..., \Omega_{u_n}) \oplus FOF\mathbb{C}FA(\Omega_{v_1}, ..., \Omega_{v_n}). \end{aligned} \tag{4.5}$$

Proof. Using operational law (3) and Theorem 4.1, we get

$$(\Omega_{u_i} \oplus \Omega_{v_i}) = \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\frac{\left(\theta^{1-\mu_{u_i}^f}-1\right)\left(\theta^{1-\mu_{v_i}^f}-1\right)}{\theta-1}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\left(\theta^{\eta_{u_i}^f}-1\right)\left(\theta^{\eta_{v_i}^f}-1\right)}{\theta-1}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\left(\theta^{\chi_{u_i}^f}-1\right)\left(\theta^{\chi_{v_i}^f}-1\right)}{\theta-1}\right)} \end{array} \right\}.$$

Thus,

$$\begin{aligned} & FOFCFA(\Omega_{u_1} \oplus \Omega_{v_1}, ..., \Omega_{u_n} \oplus \Omega_{v_n}) \\ = & \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\frac{\prod_{i=1}^{n}\left(\theta^{1-\mu_{u_i}^f}-1\right)\left(\theta^{1-\mu_{v_i}^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^{n}\left(\theta^{\eta_{u_i}^f}-1\right)\left(\theta^{\eta_{v_i}^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^{n}\left(\theta^{\chi_{u_i}^f}-1\right)\left(\theta^{\chi_{v_i}^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)} \end{array} \right\}. \end{aligned}$$

As

$$FOFCFA(\Omega_{u_1}, ..., \Omega_{u_n})$$

$$
= \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\frac{\prod_{i=1}^{n}\left(\theta^{1-\mu_{u_i}^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^{n}\left(\theta^{\eta_{u_i}^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^{n}\left(\theta^{\chi_{u_i}^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)} \end{array} \right\}
$$

and

$$
\begin{aligned}
& FOFCFA(\Omega_{v_1},...,\Omega_{v_n}) \\
= & \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\frac{\prod_{i=1}^{n}\left(\theta^{1-\mu_{v_i}^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^{n}\left(\theta^{\eta_{v_i}^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^{n}\left(\theta^{\chi_{v_i}^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)} \end{array} \right\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& FOFCFA(\Omega_{u_1},...,\Omega_{u_n}) \oplus FOFCFA(\Omega_{v_1},...,\Omega_{v_n}) \\
= & \left\{ \begin{array}{c} \sqrt[f]{1-\log_\theta\left(1+\frac{\prod_{i=1}^{n}\left(\theta^{1-\mu_{u_i}^f}-1\right)\left(\theta^{1-\mu_{v_i}^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^{n}\left(\theta^{\eta_{u_i}^f}-1\right)\left(\theta^{\eta_{v_i}^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)}, \\ \sqrt[f]{\log_\theta\left(1+\frac{\prod_{i=1}^{n}\left(\theta^{\chi_{u_i}^f}-1\right)\left(\theta^{\chi_{v_i}^f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\theta-1}\right)} \end{array} \right\} \\
= & FOFCFA(\Omega_{u_1} \oplus \Omega_{v_1},...,\Omega_{u_n} \oplus \Omega_{v_n}).
\end{aligned}
$$

Equation (4.5) holds, so proof completed.

The following properties of the FOFCFA operator can be simply demonstrated.

Theorem 4.5. (Idempotency) Let $\Omega_i = (\mu_i, \eta_i, \chi_i)\,(i = 1, ..., n)$ is a family of FOFNs, if all $\Omega_i (i = 1, ..., n)$ are equal i.e., $\Omega_i = \Omega = (\mu, \eta, \chi)$ for all i. Then,

$$
FOFCFA(\Omega_1, ..., \Omega_n) = FOFCFA(\Omega, ..., \Omega) = \Omega. \tag{4.6}
$$

Proof. The proof is straight forward.

Theorem 4.6. (Monotonicity) Let $\Omega_{u_i} = (\mu_{u_i}, \eta_{u_i}, \chi_{u_i})$ and $\Omega_{v_i} = (\mu_{v_i}, \eta_{v_i}, \chi_{v_i})$ $(i = 1, ..., n)$ are two family of FOFNs if $\mu_{u_i} \leq \mu_{v_i}, \eta_{u_i} \leq \eta_{v_i}$ and $\chi_{u_i} \leq \chi_{v_i}$ for all i. Then,

$$FOFCFA(\Omega_{u_1}, ..., \Omega_{u_n}) \leq FOFCFA(\Omega_{v_1}, ..., \Omega_{v_n}). \tag{4.7}$$

Proof. The proof is straight forward. □

Theorem 4.7. (Boundedness) Let $\Omega_i = (\mu_i, \eta_i, \chi_i)$ $(i = 1, ..., n)$ is a family of FOFNs, if $\Omega^+ = (\max(\mu_i), \min(\eta_i), \min(\chi_i))$ and $\Omega^- = (\min(\mu_i), \max(\eta_i), \max(\chi_i))$. Then,

$$\Omega^- \leq FOFCFA(\Omega_1, ..., \Omega_n) \leq \Omega^+. \tag{4.8}$$

Proof. The proof is straight forward. □

Next, we discussed some special cases of parameters for different values.

Theorem 4.8. Let $\Omega_i = (\mu_i, \eta_i, \chi_i)$ $(i = 1, ..., n)$ is a family of FOFNs. Then,

Case 1. If $\theta \to 1$, then FOFCFA operator becomes a FOFCA operator using the algebraic t-norm and t-conorm and represented as;

$$\lim_{\theta \to 1} FOFCFA = \left\{ \begin{array}{c} \sqrt[f]{1 - \prod_{i=1}^{n} \left(1 - \mu_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}}, \\ \sqrt[f]{\left(\prod_{i=1}^{n} \left(\eta_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{\left(\prod_{i=1}^{n} \left(\chi_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)} \end{array} \right\}. \tag{4.9}$$

Case 2. If $\theta \to 1, \Xi_i = \Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)$, then FOFCFA operator becomes fractional orthotriple fuzzy weighted averaging (FOFWA) operator.

Case 3. If $\theta \to 1, \Xi_i = \Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)$ and $\Xi(u) = \sum_{i=1}^{|u|} w_i$ for all $u \in X$, where $w = (w_1, ..., w_n), w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Also, $|u|$ is the ordered set of u, then FOFCFA operator becomes fractional orthotriple fuzzy ordered weighted averaging (FOFOWA) operator.

Case 4. If $\theta \to \infty$, then FOFCFA operator becomes traditional arithmetic weighted averaging operator, defined as;

$$\lim_{\theta \to \infty} FOFCFA = \left\{ \begin{array}{c} \sqrt[f]{1 - \sum_{i=1}^{n} \left(\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)\right) \mu_{\xi(i)}^{f}}, \\ \sqrt[f]{\sum_{i=1}^{n} \left(\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)\right) \eta_{\xi(i)}^{f}}, \\ \sqrt[f]{\sum_{i=1}^{n} \left(\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)\right) \chi_{\xi(i)}^{f}}, \end{array} \right\}. \tag{4.10}$$

Proof. We have only proved Cases 1 and 4, Cases 2 and 3 can be easily obtained from Cases 1 and 4.

(1) As

$$\lim_{\theta\to 1} FOFCFA = \lim_{\theta\to 1}\left\{\begin{array}{c}\sqrt[f]{1-\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)},\\ \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)},\\ \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}\end{array}\right\}.$$

We only need to prove that

$$\lim_{\theta\to 1}\sqrt[f]{1-\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{1-\prod_{i=1}^{n}\left(1-\mu^f_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}$$
$$\lim_{\theta\to 1}\sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\eta^f_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{\left(\prod_{i=1}^{n}\left(\eta^f_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}$$
$$\lim_{\theta\to 1}\sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\chi^f_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{\left(\prod_{i=1}^{n}\left(\chi^f_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}.$$

We first prove that

$$\lim_{\theta\to 1}\left(\sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\eta^f_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}\right) = \sqrt[f]{\left(\prod_{i=1}^{n}\left(\eta^f_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}.$$

As $\theta \to 1$, then $\left(\eta^f_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)} \to 0$, thus by equivalent infinitesimal replacement law we have that $\ln(1+x) \sim x(x>0)$ and logarithmic transform, we have

$$\begin{aligned}&\lim_{\theta\to 1}\sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\eta^f_{\xi(i)}-1-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}\\ =\ & \frac{\sqrt[f]{\ln\left(1+\prod_{i=1}^{n}\left(\eta^f_{\xi(i)}-1-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}}{\ln\theta}\\ \sim\ & \frac{\sqrt[f]{\left(1+\prod_{i=1}^{n}\left(\eta^f_{\xi(i)}-1-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}}{\ln\theta}.\end{aligned}$$

By Taylor series and $\ln\theta > 0$, we get

$$\begin{aligned}\eta^f_{\xi(i)}-1 &= 1+\eta^f_{\xi(i)}\ln\theta+\frac{\eta^f_{\xi(i)}}{2}(\ln\theta)^2+... = 1+\eta^f_{\xi(i)}\ln\theta+O(\ln\theta)\\ \eta^f_{\xi(i)}-1 &= \eta^f_{\xi(i)}\ln\theta+O(\ln\theta).\end{aligned}$$

Thus, $\eta^{f}_{\xi(i)} - 1 \to \eta^{f}_{\xi(i)} \ln\theta$. Then,

$$\lim_{\theta\to 1}\sqrt[f]{\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\eta^{f}_{\xi(i)}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \frac{\sqrt[f]{\ln\left(1+\prod_{i=1}^{n}\left(\eta^{f}_{\xi(i)}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}}{\ln\theta}$$

$$\lim_{\theta\to 1}\sqrt[f]{\frac{\left(\prod_{i=1}^{n}\left(\eta^{f}_{\xi(i)}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}{\ln\theta}} = \lim_{\theta\to 1}\sqrt[f]{\frac{\left(\prod_{i=1}^{n}\left(\eta^{f}_{\xi(i)}\ln\theta\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}{\ln\theta}}$$

$$\sqrt[f]{\frac{\left(\prod_{i=1}^{n}\left(\eta^{f}_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)\ln\theta}{\ln\theta}} = \sqrt[f]{\left(\prod_{i=1}^{n}\left(\eta^{f}_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}.$$

So,

$$\lim_{\theta\to 1}\sqrt[f]{\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\eta^{f}_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{\left(\prod_{i=1}^{n}\left(\eta^{f}_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}.$$

Similarly, we can prove that

$$\lim_{\theta\to 1}\sqrt[f]{1-\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\theta^{1-\mu^{f}_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{1-\prod_{i=1}^{n}\left(1-\mu^{f}_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}$$

$$\lim_{\theta\to 1}\sqrt[f]{\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\chi^{f}_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{\left(\prod_{i=1}^{n}\left(\chi^{f}_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}.$$

Thus, we can write

$$\lim_{\theta\to 1} FOFCFA = \left\{\begin{array}{c} \sqrt[f]{1-\prod_{i=1}^{n}\left(1-\mu^{f}_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}, \\ \sqrt[f]{\left(\prod_{i=1}^{n}\left(\eta^{f}_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{\left(\prod_{i=1}^{n}\left(\chi^{f}_{\xi(i)}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \end{array}\right\}.$$

Which is the required proof.

(2) We simply need to demonstrate that for Proof (4).

$$\left\{\begin{array}{c} \lim_{\theta\to\infty}\sqrt[f]{1-\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \\ =\sqrt[f]{1-\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\mu^f_{\xi(i)}}, \\ \lim_{\theta\to\infty}\sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \\ =\sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\eta^f_{\xi(i)}}, \\ \lim_{\theta\to\infty}\sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \\ =\sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\chi^f_{\xi(i)}}, \end{array}\right\}$$

We first prove that

$$\lim_{\theta\to\infty}\sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}=\sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\eta^f_{\xi(i)}},$$

As, $\sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}$ is continuous, so we get

$$\lim_{\theta\to\infty}\sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}=\sqrt[f]{\lim_{\theta\to\infty}\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}.$$

Thus by logarithmic transform and by L-Hospital rule, we get

$$\lim_{\theta\to\infty}\sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}=\sqrt[f]{\lim_{\theta\to\infty}\frac{\ln\left(1+\prod_{i=1}^{n}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}{\ln\theta}}$$

$$=\sqrt[f]{\lim_{\theta\to\infty}\frac{\dfrac{\prod_{i=1}^{n}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\left(1+\prod_{i=1}^{n}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\frac{\eta^f_{\xi(i)}\theta^{\eta^f_{\xi(i)}-1}}{\eta^f_{\xi(i)}-1}}}{\frac{1}{\theta}}}$$

$$=\sqrt[f]{\lim_{\theta\to\infty}\frac{1}{\frac{1+1}{\prod_{i=1}^{n}\left(\theta^{\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}}\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\frac{\eta^f_{\xi(i)}\theta^{\eta^f_{\xi(i)}}-1}{\eta^f_{\xi(i)}-1}}$$

$$=\sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\eta^f_{\xi(i)}}.$$

Similarly, we can prove

$$\left\{\begin{array}{c}\lim_{\theta\to\infty}\sqrt[f]{1-\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{1-\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \\ =\sqrt[f]{1-\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\mu^f_{\xi(i)}}, \\ \lim_{\theta\to\infty}\sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \\ =\sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\chi^f_{\xi(i)}},\end{array}\right\}.$$

So,

$$\lim_{\theta\to\infty}FOFCFA=\left\{\begin{array}{c}\sqrt[f]{1-\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\mu^f_{\xi(i)}}, \\ \sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\eta^f_{\xi(i)}}, \\ \sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\chi^f_{\xi(i)}},\end{array}\right\}.$$

Which is the required proof.

4.2. Fractional orthotriple fuzzy Choquet-Frank geometric operator

Definition 4.2. Let $\Omega_i=(\mu_i,\eta_i,\chi_i)\,(i=1,...,n)$ is a family of FOFNs, and $\xi(i)$ is a permutation of $(i=1,...,n)$, such that $\Omega_{\xi(i)}\geq\Omega_{\xi(2)}\geq...,\geq\Omega_{\xi(n)}$. Then, a FOFCFA operator is defined as

$$FOFCFG(\Omega_1,...,\Omega_n)=\prod_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\Omega_{\xi(i)}. \tag{4.11}$$

Where $u_{\xi(i)}=\left(G_{\xi(1)},...,G_{\xi(i)}\right),u_{\xi(i)}=\phi$, also $G_{\xi(i)}$ is the attribute related to $\Omega_{\xi(i)}$.

Theorem 4.9. Let $\Omega_i=(\mu_i,\eta_i,\chi_i)\,(i=1,...,n)$ is a family of FOFNs, and $\xi(i)$ is a permutation of $(i=1,...,n)$, such that $\Omega_{\xi(i)}\geq\Omega_{\xi(2)}\geq...,\geq\Omega_{\xi(n)}$. $G_{\xi(i)}$ is the attribute related to $\Omega_{\xi(i)},u_{\xi(i)}=\phi$, $u_{\xi(i)}=\left(G_{\xi(1)},...,G_{\xi(i)}\right)$. Then, using FOFCFG operator the aggregated value is also a FOFN and

$$\begin{aligned}&FOFCFG(\Omega_1,...,\Omega_n) \\ &=\left\{\begin{array}{c}\sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{1-\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{1-\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{1-\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{1-\chi^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}\end{array}\right\}.\end{aligned} \tag{4.12}$$

Proof. Proof is same as Theorem 4.1. □

Theorem 4.10. Let $\Omega_i = (\mu_i, \eta_i, \chi_i)\,(i = 1, ..., n))$ is a family of FOFNs, and $\xi(i)$ is a permutation of $(i = 1, ..., n)$ such that $\Omega_{\xi(i)} \geq \Omega_{\xi(2)} \geq ..., \geq \Omega_{\xi(n)}$. Then,

$$FOFCFG(\lambda.\Omega_1, ..., \lambda.\Omega_n) = \lambda.FOFCFG(\Omega_1, ..., \Omega_n). \tag{4.13}$$

Proof. Proof is same as Theorem 4.1.

Theorem 4.11. Let $\Omega_i = (\mu_i, \eta_i, \chi_i)\,(i = 1, ..., n)$ is a family of FOFNs, and $\xi(i)$ is a permutation of $(i = 1, ..., n)$ such as $\Omega_{\xi(i)} \geq \Omega_{\xi(2)} \geq ..., \geq \Omega_{\xi(n)}$. Then,

$$FOFCFG(\Omega_1 \oplus \Omega, ..., \Omega_n \oplus \Omega) = FOFCFG(\Omega_1, ..., \Omega_n) \oplus \Omega. \tag{4.14}$$

Proof. Proof is same as Theorem 4.1.

Theorem 4.12. Let $\Omega_i = (\mu_i, \eta_i, \chi_i)\,(i = 1, ..., n)$ is a family of FOFNs, and $\xi(i)$ is a permutation of $(i = 1, ..., n)$ such as $\Omega_{\xi(i)} \geq \Omega_{\xi(2)} \geq ..., \geq \Omega_{\xi(n)}$. Then,

$$\begin{aligned} & FOFCFG(\Omega_{u_1} \oplus \Omega_{v_1}, ..., \Omega_{u_n} \oplus \Omega_{v_n}) \\ = \; & FOFCFG(\Omega_{u_1}, ..., \Omega_{u_n}) \oplus FOFCFG(\Omega_{v_1}, ..., \Omega_{v_n}). \end{aligned} \tag{4.15}$$

Proof. Proof is same as Theorem 4.1.

The following properties of the FOFCFG operator can then be simply demonstrated.

Theorem 4.13. (Idempotency) Let $\Omega_i = (\mu_i, \eta_i, \chi_i)\,(i = 1, ..., n)$ is a family of FOFNs, if all $\Omega_i (i = 1, ..., n)$ are equal i.e., $\Omega_i = \Omega = (\mu, \eta, \chi)$ for all i. Then,

$$FOFCFG(\Omega_1, ..., \Omega_n) = FOFCFG(\Omega, ..., \Omega) = \Omega. \tag{4.16}$$

Proof. The proof is straight forward.

Theorem 4.14. (Monotonicity) Let $\Omega_{u_i} = (\mu_{u_i}, \eta_{u_i}, \chi_{u_i})$ and $\Omega_{v_i} = (\mu_{v_i}, \eta_{v_i}, \chi_{v_i})\,(i = 1, ..., n)$ are two family of FOFNs if $\mu_{u_i} \leq \mu_{v_i}, \eta_{u_i} \leq \eta_{v_i}$and $\chi_{u_i} \leq \chi_{v_i}$ for all i. Then,

$$FOFCFG(\Omega_{u_1}, ..., \Omega_{u_n}) \leq FOFCFG(\Omega_{v_1}, ..., \Omega_{v_n}). \tag{4.17}$$

Theorem 4.15. (Boundedness) Let $\Omega_i = (\mu_i, \eta_i, \chi_i)\,(i = 1, ..., n)$ is a family of FOFNs, if $\Omega^+ = (\max(\mu_i), \min(\eta_i), \min(\chi_i))$ and $\Omega^- = (\min(\mu_i), \max(\eta_i), \max(\chi_i))$. Then,

$$\Omega^- \leq FOFCFG(\Omega_1, ..., \Omega_n) \leq \Omega^+. \tag{4.18}$$

Next, we have different special cases of parameters for different values.

Theorem 4.16. Let $\Omega_i = (\mu_i, \eta_i, \chi_i)\,(i = 1, ..., n)$ is a family of FOFNs. Then,

Case 5. If $\theta \to 1$, then FOFCFG operator becomes a FOFCG operator using the algebraic t-norm and t-conorm and represented as

$$\lim_{\theta \to 1} FOFCFG = \left\{ \begin{array}{c} \sqrt[f]{\left(\prod_{i=1}^{n} \left(\mu_{\xi(i)}^{f} \right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)} \right)}, \\ \sqrt[f]{1 - \prod_{i=1}^{n} \left(1 - \eta_{\xi(i)}^{f} \right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}}, \\ \sqrt[f]{1 - \prod_{i=1}^{n} \left(1 - \chi_{\xi(i)}^{f} \right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}} \end{array} \right\}. \tag{4.19}$$

Case 6. If $\theta \to 1, \Xi_i = \Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)$, then FOFCFG operator becomes fractional orthotriple fuzzy weighted geometric (FOFWG) operator.

Case 7. If $\theta \to 1, \Xi_i = \Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)$ and $\Xi(u) = \sum_{i=1}^{|u|} w_i$ for all $u \in X$, where $w = (w_1, ..., w_n), w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Also, $|u|$ is the ordered set of u, then FOFCFG operator becomes fractional orthotriple fuzzy ordered weighted geometric (FOFOWG) operator.

Case 8. If $\theta \to \infty$, then FOFCFG operator becomes traditional geometric weighted averaging operator defined as;

$$\lim_{\theta \to \infty} FOFCFG = \left\{ \begin{array}{c} \sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)\right)\mu_{\xi(i)}^{f}}, \\ \sqrt[f]{1 - \sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)\right)\eta_{\xi(i)}^{f}}, \\ \sqrt[f]{1 - \sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)\right)\chi_{\xi(i)}^{f}} \end{array} \right\}. \tag{4.20}$$

Proof. We have only proved Cases 1 and 4, Cases 2 and 3 can be easily obtained from Cases 1 and 4.

(1) As

$$\lim_{\theta \to 1} FOFCFG = \lim_{\theta \to 1} \left\{ \begin{array}{c} \sqrt[f]{\log_\theta\left(1 + \prod_{i=1}^{n}\left(\theta^{\mu_{\xi(i)}^{f}} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)} \\ \sqrt[f]{1 - \log_\theta\left(1 + \prod_{i=1}^{n}\left(\theta^{1-\eta_{\xi(i)}^{f}} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{1 - \log_\theta\left(1 + \prod_{i=1}^{n}\left(\theta^{1-\chi_{\xi(i)}^{f}} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)} \end{array} \right\}.$$

We only need to prove that

$$\lim_{\theta \to 1} \sqrt[f]{\log_\theta\left(1 + \prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{\left(\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)}$$

$$\lim_{\theta \to 1} \sqrt[f]{1 - \log_\theta\left(1 + \prod_{i=1}^{n}\left(\theta^{1-\eta_{\xi(i)}^{f}} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{1 - \prod_{i=1}^{n}\left(1 - \eta_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}}$$

$$\lim_{\theta \to 1} \sqrt[f]{1 - \log_\theta\left(1 + \prod_{i=1}^{n}\left(\theta^{1-\chi_{\xi(i)}^{f}} - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{1 - \prod_{i=1}^{n}\left(1 - \chi_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}}.$$

We first prove that

$$\lim_{\theta \to 1}\left(\sqrt[f]{\log_\theta\left(1 + \prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)}\right) = \sqrt[f]{\left(\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)}.$$

As $\theta \to 1$, then $\left(\mu_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)} \to 0$, thus by equivalent infinitesimal replacement law we have that $\ln(1 + x) \sim x (x > 0)$ and logarithmic transform, we have

$$\lim_{\theta \to 1} \sqrt[f]{\log_\theta\left(1 + \prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f} - 1 - 1\right)^{\Xi\left(u_{\xi(i)}\right) - \Xi\left(u_{\xi(i-1)}\right)}\right)}$$

$$
\begin{aligned}
&= \frac{\sqrt[f]{\ln\left(1+\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}-1-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}}{\ln\theta} \\
&\sim \frac{\sqrt[f]{\left(1+\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}-1-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}}{\ln\theta}.
\end{aligned}
$$

By Taylor series and $\ln\theta > 0$, we get

$$
\begin{aligned}
\mu_{\xi(i)}^{f}-1 &= 1+\mu_{\xi(i)}^{f}\ln\theta+\frac{\mu_{\xi(i)}^{f}}{2}(\ln\theta)^{2}+...=1+\mu_{\xi(i)}^{f}\ln\theta+O(\ln\theta) \\
\mu_{\xi(i)}^{f}-1 &= \mu_{\xi(i)}^{f}\ln\theta+O(\ln\theta).
\end{aligned}
$$

Thus, $\mu_{\xi(i)}^{f}-1 \to \mu_{\xi(i)}^{f}\ln\theta$. Then,

$$
\begin{aligned}
&\lim_{\theta\to1}\sqrt[f]{\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \frac{\sqrt[f]{\ln\left(1+\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}}{\ln\theta} \\
&\lim_{\theta\to1}\sqrt[f]{\frac{\left(\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}{\ln\theta}} = \lim_{\theta\to1}\sqrt[f]{\frac{\left(\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}\ln\theta\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}{\ln\theta}} \\
&\sqrt[f]{\frac{\left(\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)\ln\theta}{\ln\theta}} = \sqrt[f]{\left(\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}.
\end{aligned}
$$

So,

$$
\lim_{\theta\to1}\sqrt[f]{\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{\left(\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}.
$$

Similarly, we can prove that

$$
\begin{aligned}
&\lim_{\theta\to1}\sqrt[f]{1-\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\theta^{1-\eta_{\xi(i)}^{f}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{1-\prod_{i=1}^{n}\left(1-\eta_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}} \\
&\lim_{\theta\to1}\sqrt[f]{\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\chi_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{\left(\prod_{i=1}^{n}\left(\chi_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}.
\end{aligned}
$$

Thus, we can write

$$\lim_{\theta\to 1} FOFCFG = \left\{ \begin{array}{c} \sqrt[f]{\left(\prod_{i=1}^{n}\left(\mu_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}, \\ \sqrt[f]{1-\prod_{i=1}^{n}\left(1-\eta_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}, \\ \sqrt[f]{1-\prod_{i=1}^{n}\left(1-\chi_{\xi(i)}^{f}\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}} \end{array} \right\}.$$

Which is the required proof.

(2) We simply need to demonstrate that for Proof (4).

$$\left\{ \begin{array}{c} \lim_{\theta\to\infty} \sqrt[f]{\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\theta^{\mu_{\xi(i)}^{f}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \\ = \sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\mu_{\xi(i)}^{f}}, \\ \lim_{\theta\to\infty} \sqrt[f]{1-\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\theta^{1-\eta_{\xi(i)}^{f}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \\ = \sqrt[f]{1-\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\eta_{\xi(i)}^{f}}, \\ \lim_{\theta\to\infty} \sqrt[f]{1-\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\theta^{1-\chi_{\xi(i)}^{f}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \\ = \sqrt[f]{1-\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\chi_{\xi(i)}^{f}} \end{array} \right\}.$$

We first prove that

$$\lim_{\theta\to\infty} \sqrt[f]{\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\theta^{\mu_{\xi(i)}^{f}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\mu_{\xi(i)}^{f}}.$$

As, $\sqrt[f]{\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\theta^{\mu_{\xi(i)}^{f}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}$ is continuous, so we get

$$\lim_{\theta\to\infty} \sqrt[f]{\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\theta^{\mu_{\xi(i)}^{f}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} = \sqrt[f]{\lim_{\theta\to\infty}\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\theta^{\mu_{\xi(i)}^{f}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}.$$

Thus by logarithmic transform and by L-Hospital rule, we get

$$\begin{aligned} &\lim_{\theta\to\infty} \sqrt[f]{\log_{\theta}\left(1+\prod_{i=1}^{n}\left(\theta^{\mu_{\xi(i)}^{f}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \\ = &\sqrt[f]{\lim_{\theta\to\infty}\frac{\ln\left(1+\prod_{i=1}^{n}\left(\theta^{\mu_{\xi(i)}^{f}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)}{\ln\theta}} \end{aligned}$$

$$
\begin{aligned}
&= \sqrt[f]{\lim_{\theta\to\infty} \frac{\frac{\prod_{i=1}^{n}\left(\theta^{\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}{\left(1+\prod_{i=1}^{n}\left(\theta^{\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\frac{\mu^f_{\xi(i)}\theta^{\mu^f_{\xi(i)}}-1}{\mu^f_{\xi(i)}-1}}}{\frac{1}{\theta}}} \\
&= \sqrt[f]{\lim_{\theta\to\infty} \frac{1}{\frac{1+1}{\prod_{i=1}^{n}\left(\theta^{\mu^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}}} \sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\frac{\mu^f_{\xi(i)}\theta^{\mu^f_{\xi(i)}}-1}{\mu^f_{\xi(i)}-1}} \\
&= \sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\mu^f_{\xi(i)}}.
\end{aligned}
$$

Similarly, we can prove

$$
\left\{
\begin{array}{c}
\lim_{\theta\to\infty} \sqrt[f]{1-\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{1-\eta^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \\
= \sqrt[f]{1-\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\eta^f_{\xi(i)}}, \\
\lim_{\theta\to\infty} \sqrt[f]{\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\chi^f_{\xi(i)}}-1\right)^{\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)}\right)} \\
= \sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\chi^f_{\xi(i)}},
\end{array}
\right\}.
$$

So,

$$
\lim_{\theta\to\infty} FOFCFG = \left\{
\begin{array}{c}
\sqrt[f]{\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\mu^f_{\xi(i)}}, \\
\sqrt[f]{1-\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\eta^f_{\xi(i)}}, \\
\sqrt[f]{1-\sum_{i=1}^{n}\left(\Xi\left(u_{\xi(i)}\right)-\Xi\left(u_{\xi(i-1)}\right)\right)\chi^f_{\xi(i)}}
\end{array}
\right\}.
$$

Which is the required proof.

5. Approach for MADM based on fractional orthotriple fuzzy Choquet-Frank aggregation operators

In this section, we establish an approach to MADM difficulties with fractional orthotriple fuzzy information using the established aggregation operators. MADM problem can be explained as; let $H = (H_1, ..., H_m)$ be a set of m alternative, $\mathbb{C} = (\mathbb{C}_1, ..., \mathbb{C}_n)$ be a set of n attributes. Suppose a fractional orthotriple fuzzy number $\Omega_i = \left(\mu_{ij}, \eta_{ij}, \chi_{ij}\right)$ an attribute value is given by decision makers for the alternative h_i with respect to the attribute $\mathbb{C}_j$, satisfies the following condition $\mu_{ij}, \eta_{ij}, \chi_{ij} \in [0, 1]$ and $0 \leq \mu^f_{ij} + \eta^f_{ij} + \chi^f_{ij} \leq 1$. Then, for any solutions, we can use the proposed operators, with given steps:

Step 1: Normalize the given decision matrix, normally, attributes can be categorized into two kinds, the first one is the benefit attribute and the second one is the cast attribute. The point to be noted, no need to normalize the decision making matrix if attributes are of the same type (benefit type). If attributes have different values, then normalized decision matrix by;

$$\Omega_{ij} = \begin{cases} \left(\mu_{ij}, \eta_{ij}, \chi_{ij}\right), & \text{if attribute are benefit type} \\ \left(\chi_{ij}, \eta_{ij}, \mu_{ij}\right), & \text{if attribute are cost type} \end{cases}$$

Step 2: By score function in Definition 2.5, we determine $\Xi(\Omega_{ij})$ to rearrange Ω_{ij} for every alternative $H_i(i = 1, ..., m)$ in descending order. If $\Xi(\Omega_{ij}) = \Xi(\Omega_{ik})$, we relate $E(\Omega_{ij})$ and $E(\Omega_{ik})$, so partial evaluation Ω_{ij} of H_i the alternative is rearranged such as $\Omega_{i(j)} \geq \Omega_{i(j-1)}$.
Step 3: Find the fuzzy measure of n attributes of H_i. To prevent difficulty, we use Eq (2.9) for defining the fuzzy measure.
Step 4: Determine the whole preferences for every alternative. For the alternative $H_i(i = 1, ..., m)$, use the FOFCFA or FOFCFG operator to aggregate the values of the attributes.
Step 5: Determine the score values for each alternative.
Step 6: Choose the best option after ranking the alternatives.

6. Numerical example

Nowadays, a human mental disorder, such as depression, is a serious global health problem. One of the many characteristics of the psychological illness is a significant and ongoing depressed mood. Due to an extremely busy lifestyle, mental illness or depression is on the rise. Persistent depression, , and frustration are three major medical manifestations of the disorder that is affecting an increasing number of people. In some cases, suicidal thoughts also manifest in these conditions. The World Health Organization (WHO) estimated that by the end of 2020, depression affected 350 million people worldwide, making it the second most common disease after heart disease. The depression has been properly studied as a mental disorder disease since the middle of the 19th century. According to Beck's theory of cognitive breakdown, which was developed in the 1960s, depressed people have negative thoughts about both themselves and the people around them. Clinical diagnosis has been hampered, though, by the fact that the etiology of depression is unknown and the manifestations are challenging. The doctors use a variety of diagnostic techniques and routine tests to identify this illness, and each of these tests yields a unique set of diagnostic findings. This leads to the conclusion that patients who exhibit symptoms of depression rely on self-care to carry out a management plan of treatment and that there is no such physical signal that can be used as a measurable constraint. An initial examination, a diagnosis, and overcoming anticipation depression are very difficult. Due to the recent rapid advancements in cognitive science and sensor technology, researchers can now use electroencephalograms (EEGs) to physically record brain activity. Electroencephalograms are used to assess the clinical history of the disorder. EEG physiological signals can be used to measure and record depression's relationship to brain activity. Several clinical tests of brain function can be conducted using EEG recordings.

According to a large body of research, EEG signals are used to analyze brain activity. Recently, the sample entropy bi-spectrum entropy, approximate entropy and renyi entropy have been studied for classifier selection of EEG signals for depression and classify into different classes of depression

EEG signals. It's obvious that more knowledge is needed about the processes that occur in the brain before and during stress in order to monitor diagnosis and treatment. The statistical performance of the resulting features was evaluated, and they were classified as potential neural networks, k-nearest neighbor algorithm (k-NN), Gaussian mixture model (GMM), Decision tree (DT) and Probabilistic neural network (PNN). The proposed intelligent decision technique analyzes the k-nearest neighbour algorithm (k-NN), Gaussian mixture model (GMM), Decision tree (DT) and probabilistic neural network (PNN) of EEG signals of depression using the sample entropy bispectrum entropy, approximate entropy and renyi entropy under the FOF information.

The decision experts are finalized the following method for classifier selection H_1: K-nearest neighbor algorithm(k-NN), H_2: Gaussian mixture model (GMM), H_3: Decision tree (DT) and H_4: Probabilistic neural network (PNN). The classifier is further evaluated by decision experts based on attribute $\mathbb{C}_1$: Sample entropy, $\mathbb{C}_2$: Bispectrum entropy, $\mathbb{C}_3$: Approximate entropy $\mathbb{C}_4$: Renyi entropy. The experts give their assessment for the five classifiers of EEG based on five attribute in the Table 1.

Table 1. Fractional orthopair fuzzy decision matrix.

	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$
H_1	$(0.7, 0.5, 0.4)$	$(0.5, 0.8, 0.3)$	$(0.9, 0.7, 0.3)$	$(0.6, 0.8, 0.4)$
H_2	$(0.6, 0.8, 0.7)$	$(0.9, 0.6, 0.5)$	$(0.6, 0.5, 0.8)$	$(0.5, 0.4, 0.9)$
H_3	$(0.4, 0.9, 0.6)$	$(0.6, 0.8, 0.4)$	$(0.5, 0.8, 0.6)$	$(0.8, 0.3, 0.8)$
H_4	$(0.8, 0.4, 0.5)$	$(0.7, 0.5, 0.9)$	$(0.3, 0.4, 0.9)$	$(0.7, 0.5, 0.6)$

Step 1: The decision matrix does not require normalization because all attributes are of the benefit type.

Step 2: To demonstrate a fuzzy measure for the n attributes of $\mathbb{C}_i$. The fuzzy measure of $\mathbb{C}_i$ attributes should be as following:

$$\Xi(\mathbb{C}_1) = 0.3, \Xi(\mathbb{C}_2) = 0.4, \Xi(\mathbb{C}_3) = 0.2, \Xi(\mathbb{C}_4) = 0.3.$$

The fuzzy measure of attribute sets is calculated using the $\wp$-FM. First, the value of $\wp$ is determined by Eq (2.9). We get $\wp = (-0.3417)$, and then Eq (2.11) can be used to find the fuzzy measure of the attribute set $\mathbb{C} = (\mathbb{C}_1, ..., \mathbb{C}_4)$.

$$\begin{aligned}
\Xi(\mathbb{C}_1, \mathbb{C}_2) &= 0.271, \Xi(\mathbb{C}_1, \mathbb{C}_3) = 0.482, \Xi(\mathbb{C}_1, \mathbb{C}_4) = 0.162, \Xi(\mathbb{C}_2, \mathbb{C}_4) = 0.370, \\
\Xi(\mathbb{C}_3, \mathbb{C}_4) &= 0.517, \Xi(\mathbb{C}_2, \mathbb{C}_3) = 0.209, \Xi(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3) = 0.116, \Xi(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_4) = 0.464, \\
\Xi(\mathbb{C}_1, \mathbb{C}_3, \mathbb{C}_4) &= 0.487, \Xi(\mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4) = 0.541, \Xi(\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4) = 1.
\end{aligned}$$

Step 3: By using score functions, we reorder the FOFNs in the following order, according to Table 1:

$$\begin{aligned}
\Omega_{1\xi(1)} &= (0.9, 0.7, 0.3), \Omega_{1\xi(2)} = (0.7, 0.5, 0.4), \Omega_{1\xi(3)} = (0.6, 0.8, 0.4), \Omega_{1\xi(4)} = (0.5, 0.8, 0.3) \\
\Omega_{2\xi(1)} &= (0.9, 0.6, 0.5), \Omega_{2\xi(2)} = (0.6, 0.8, 0.7), \Omega_{2\xi(3)} = (0.6, 0.5, 0.8), \Omega_{2\xi(4)} = (0.5, 0.4, 0.9) \\
\Omega_{3\xi(1)} &= (0.8, 0.3, 0.8), \Omega_{3\xi(2)} = (0.6, 0.8, 0.4), \Omega_{3\xi(3)} = (0.5, 0.8, 0.6), \Omega_{3\xi(4)} = (0.4, 0.9, 0.6) \\
\Omega_{4\xi(1)} &= (0.8, 0.4, 0.5), \Omega_{4\xi(2)} = (0.7, 0.5, 0.6), \Omega_{4\xi(3)} = (0.7, 0.5, 0.9), \Omega_{4\xi(4)} = (0.3, 0.4, 0.9).
\end{aligned}$$

Then, we obtain

$$
\begin{aligned}
u_{1\xi(1)} &= (\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4), u_{1\xi(2)} = (\mathbb{C}_1, \mathbb{C}_3, \mathbb{C}_4), u_{1\xi(3)} = (\mathbb{C}_1, \mathbb{C}_4), u_{1\xi(4)} = \mathbb{C}_4 \\
u_{2\xi(1)} &= (\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4), u_{2\xi(2)} = (\mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4), u_{2\xi(3)} = (\mathbb{C}_2, \mathbb{C}_4), u_{2\xi(4)} = \mathbb{C}_2 \\
u_{3\xi(1)} &= (\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4), u_{3\xi(2)} = (\mathbb{C}_1, \mathbb{C}_3, \mathbb{C}_4), u_{3\xi(3)} = (\mathbb{C}_3, \mathbb{C}_4), u_{3\xi(4)} = \mathbb{C}_4 \\
u_{4\xi(1)} &= (\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4), u_{4\xi(2)} = (\mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4), u_{4\xi(3)} = (\mathbb{C}_2, \mathbb{C}_4), u_{4\xi(4)} = \mathbb{C}_2.
\end{aligned}
$$

Step 4: Using the FOFCFA operator and $f = 3$, to combine the attribute values for alternative $H = (H_1, ..., H_4)$. We have a set of overall values if $\theta = 2$.

$$
\begin{aligned}
\Omega_1 &= (0.4291, 0.6385, 0.3816), \Omega_2 = (0.5982, 0.3997, 0.4872) \\
\Omega_3 &= (0.6271, 0.4208, 0.2752), \Omega_4 = (0.5183, 0.3174, 0.5309).
\end{aligned}
$$

Step 5: Find the scores of $\Omega_i (i = 1, ..., 4)$, using Definition 2.5, we obtained.

$$S(\Omega_1) = 0.3525, S(\Omega_2) = 0.3702, S(\Omega_3) = 0.4492, S(\Omega_4) = 0.4137.$$

Thus, the ranking of the overall values is $\Omega_3 > \Omega_4 > \Omega_2 > \Omega_1$.

Step 6: Give ranking to the alternatives $H_i (i = 1, ..., 4)$ according to the rank values of $\Omega_i (i = 1, ..., 4)$, we get a result: $H_3 > H_4 > H_2 > H_1$. Thus, H_3 is the best alternative, indicating that the Decision Tree is the best choice.

6.1. Sensitivity analysis

Based on the DM's preferences, a variety of values may be applied to the parameter. To examine the variation in the ranking of the four options depending on the value of the parameter, we assigned θ values ranging from 1 to 50 and computed the scores for these four alternatives. Table 2 provides a summary of the scores assigned to the alternatives by the FOFCFA operator. It is clear that when the value of θ rises from 1 to 50, the scores assigned to each alternative rise as well. We can see that the four options are always in the same order when we use $\theta \in [1, 50]$, and Decision Tree is the best choice. When the parameter is between $(1, +\infty)$, we can observe that the FOFCFA operator increases monotonically with respect to that parameter. The values of the alternative increasing if we adjust the parameter. In Table 2, the fluctuations in the scores are readily seen in relation to the values of the parameter θ.

Table 2. Overall ranking of alternatives using different value of parameter.

Parameter θ	Scores values				Ranking
	$S(\Omega_1)$	$S(\Omega_2)$	$S(\Omega_3)$	$S(\Omega_4)$	
$\theta = 1$	0.2146	0.2372	0.2641	0.2277	$H_3 > H_4 > H_2 > H_1$
$\theta = 2$	0.3525	0.3702	0.4492	0.4136	$H_3 > H_4 > H_2 > H_1$
$\theta = 3$	0.4417	0.4602	0.4997	0.4736	$H_3 > H_4 > H_2 > H_1$
$\theta = 10$	0.5752	0.5891	0.6142	0.5975	$H_3 > H_4 > H_2 > H_1$
$\theta = 25$	0.7350	0.7424	0.7846	0.7570	$H_3 > H_4 > H_2 > H_1$
$\theta = 50$	0.7764	0.7982	0.8869	0.8473	$H_3 > H_4 > H_2 > H_1$

We can see from the analysis of Table 2 above that the parameter θ represents the decision makers' preferences, and the decision makers can select the appropriate values of θ in accordance with their preferences. We may generate several scoring functions, which in turn allow us to generate various ranks of the alternatives as well as many optimal alternatives, by selecting various values for the parameter θ. In other words, different choices of the parameter θ could result in different ultimate optimal decisions. Since we can choose different values for the parameter in light of various practical situations, using the developed aggregation operators.

Figure 1 shows graphically the ranking of the alternatives.

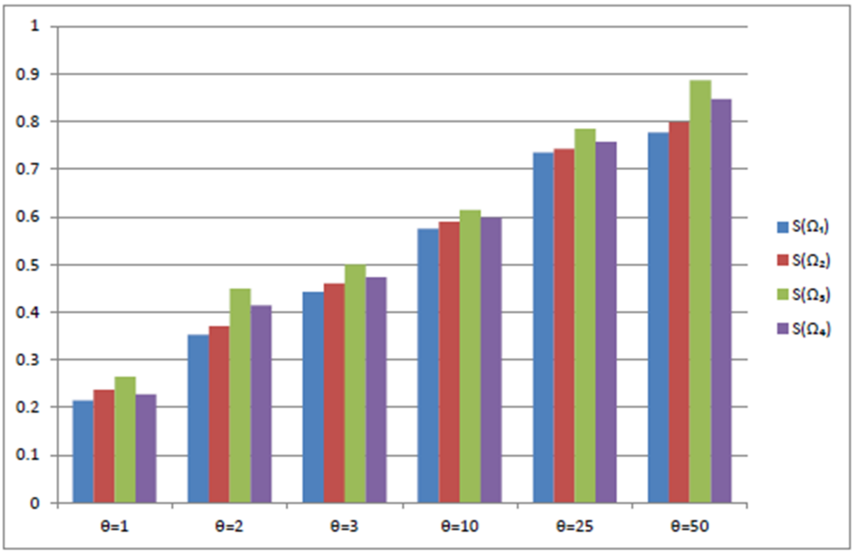

Figure 1. Ranking of the alternatives using proposed operator.

6.2. Comparative analysis

In this section, we'll lay the groundwork for a collaborative analysis of the previously presented work using a few tried-and-true techniques from the literature. The main points of discussion are listed below.

Here, we will compare our work with concepts; some new Pythagorean fuzzy Choquet-Frank aggregation operators for multi-attribute decision making [34]; an approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets [23]; T-spherical fuzzy Frank aggregation operators and their application to decision making with unknown weight information [24]; similarity measures for fractional orthotriple fuzzy sets using cosine and cotangent functions [25], Banzhaf-Choquet Copula based operators for managing fractional orthotriple fuzzy information [26] and Fractional orthotriple fuzzy rough Hamacher aggregation operators [27]. The final results are now presented in Table 3 for this discussion.

Table 3. Ranking and score values of alternatives using different operators.

Method	Scores values				Ranking
	$S(\Omega_1)$	$S(\Omega_2)$	$S(\Omega_3)$	$S(\Omega_4)$	
Xing et al. [34]	0.1526	0.1627	0.1905	0.1724	$H_3 > H_4 > H_2 > H_1$
Mahmood et al. [23]	0.3851	0.3725	0.3519	0.6472	$H_3 > H_1 > H_2 > H_3$
Mahnaz et al. [24]	0.5136	0.5472	0.5961	0.4620	$H_3 > H_2 > H_1 > H_4$
Naeem et al. [25]	0.6168	0.6291	0.6634	0.6472	$H_3 > H_4 > H_2 > H_1$
Qiyas et al. [26]	0.6097	0.5997	0.5821	0.6015	$H_3 > H_4 > H_1 > H_2$
Qiyas et al. [27]	0.4317	0.4421	0.4872	0.4543	$H_3 > H_4 > H_2 > H_1$

Figure 2 shows graphically the ranking of the alternatives.

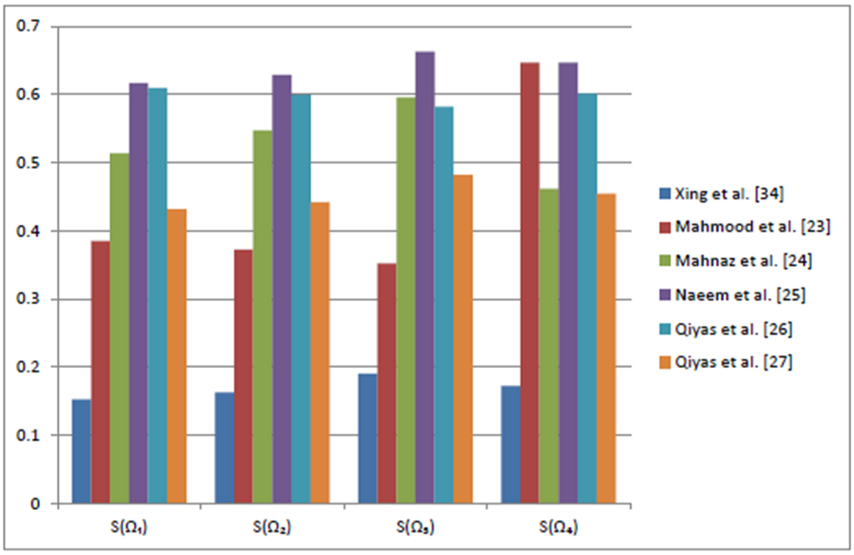

Figure 2. Ranking of the alternatives using different methods.

7. Conclusions

Fractional orthotriple set is more strong apparatus than picture fuzzy set and spherical fuzzy set, and it provides extra space to decision makers for decision making in several real-life problems. Also, aggregation operators are wont to reduce the set of finite values into one value, so motivated by the generality of fractional orthotriple fuzzy set and basic characteristics of aggregation operators, during this article. We've initiated some new aggregation operator supported Frank t-norm and t-conorm, called FOFCFA and FOFCFG operators. Moreover, some basic properties of those aggregation operators are elaborated intimately. Multi-attribute decision making algorithm supported these operators has introduced an alongside application to point out the effectiveness of introduced work. The proposed algorithm is considered for classifier selection for EEG under depression information based on given attribute. The proposed decision making method analyzes the k-nearest neighbor algorithm, Gaussian mixture model, decision tree and probabilistic neural network. The proposed method is attribute based for analyzing optimal classifier selection for EEG under depression patients. We used the classifier method to obtain depression patients data in normal situations and abnormal situation based on the given attributes. Furthermore, a comparative study of the proposed work has been given to point out the authenticity and the superiority of the proposed work. Although the focus of this paper is hybrid sets of fuzzy sets, it can also be applied to other types of structures.

How to determine the parameter θ in the proposed operators according to the practical situations has not been discussed in the current paper, which is an interesting topic and is worthy to be further studied in the future. Moreover, in the future we will apply the proposed operators and approach to some practical applications such as soft sets, Cubic fuzzy sets, Bipolar fuzzy sets, fractional orthotriple fuzzy rough sets, Linear Diophantine fuzzy sets, Linear Diophantine fuzzy graphs.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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