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Research article

Qualitative behavior of a higher-order fuzzy difference equation

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Abstract: In this paper, we investigate the qualitative behavior of the fuzzy difference equation

$$z_{n+1} = \frac{Az_{n-s}}{B + C\prod_{i=0}^{s} z_{n-i}}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, (z_n) is a sequence of positive fuzzy numbers, A, B, C and the initial conditions z_{-j}, $j = 0, 1, ..., s$ are positive fuzzy numbers and s is a positive integer. Moreover, two examples are given to verify the effectiveness of the results obtained.

Keywords: fuzzy difference equations; existence of positive solutions; boundedness; convergence
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1. Introduction

Difference equations play an important role in modeling problems that arise in biology, physics, engineering, finance and many other areas [1, 2]. In many cases, the obtained models are restricted in their ability to describe phenomena due to the incomplete or uncertain information about the variables, parameters and initial conditions available. To take into account these uncertainties or lack of precision, we may use a fuzzy environment in variables, parameters and initial conditions, by turning general difference equations into fuzzy difference equations. The most striking applications of the fuzzy notion can be seen in some branches of engineering. See [3, 4] for a few applications from mechanical engineering. The fuzzy notion also has applications in differential equations, which are closely related to difference equations. See [5]. Most numerical methods convert differential equations to difference equations, and their relationship is discussed in [6].

In [7], Bajo and Liz investigated the behavior of the ordinary difference equation

$$x_{n+1} = \frac{x_{n-1}}{a + bx_{n-1}x_n}, \tag{1.1}$$

where the parameters a, b and the initial conditions x_{-1}, x_0 are real numbers.

Moreover, in [8], Rahman et al. investigated the behavior of the fuzzy difference equation

$$x_{n+1} = \frac{x_{n-1}}{A + Bx_{n-1}x_n}, \tag{1.2}$$

where A, B and the initial conditions x_{-1}, x_0 are positive fuzzy numbers.

In [9], Shojaei et al. investigated the behavior of the ordinary difference equation

$$x_{n+1} = \frac{\alpha x_{n-2}}{\beta + \gamma x_{n-2}x_{n-1}x_n}, \tag{1.3}$$

where the parameters α, β, γ and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.

Also, in [10], Atak et al. investigated the behavior of the fuzzy difference equation

$$z_{n+1} = \frac{z_{n-2}}{C + z_{n-2}z_{n-1}z_n}, \tag{1.4}$$

where (z_n) is a sequence of positive fuzzy numbers, C and the initial conditions z_{-2}, z_{-1}, z_0 are positive fuzzy numbers.

Motivated by above mentioned studies, in this paper, we investigate the qualitative behavior of the fuzzy difference equation

$$z_{n+1} = \frac{Az_{n-s}}{B + C\prod_{i=0}^{s} z_{n-i}}, \tag{1.5}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, (z_n) is a sequence of positive fuzzy numbers, A, B, C and the initial conditions z_{-j} $(j = 0, 1, ..., s)$ are positive fuzzy numbers and s is a positive integer. Moreover, some examples are given to verify the effectiveness of the results obtained.

Recently some researchers have studied fuzzy difference equations and properties of their solutions in different approaches. See, for example, [11–20]. We also refer to [21, 22] for some fundamentals of ordinary difference equations.

2. Preliminaries

In this section, we give some definitions which will be used in this paper. For more details see [23–25].

Definition 2.1. *Consider a fuzzy set A which is a function from the set of real numbers* $\mathbb{R}$ *into the interval* $[0, 1]$. *We say that A is a fuzzy number if it satisfies the following properties*

(a) A is normal, i.e., $\exists\, x_0 \in \mathbb{R}$ *with* $A(x_0) = 1$,
(b) A is fuzzy convex, i.e., $A(tx_1 + (1-t)x_2) \geq \min\{A(x_1), A(x_2)\}$, $\forall t \in [0,1]$ *and* $x_1, x_2 \in \mathbb{R}$,
(c) A is upper semicontinuous on $\mathbb{R}$,
(d) A is compactly supported, i.e., $\overline{\{x \in \mathbb{R} : A(x) > 0\}}$ *is compact.*

Let us denote by $\mathbb{R}_F$ the space of all fuzzy numbers. For $0 < \alpha \leq 1$ and $A \in \mathbb{R}_F$, we denote α-cuts of fuzzy number A by $[A]^\alpha = \{x \in \mathbb{R}, A(x) \geq \alpha\}$ and $[A]^0 = \overline{\{x \in \mathbb{R}, A(x) \geq 0\}}$. We call $[A]^0$, the support of fuzzy number A and denote it by $supp\,(A)$.

The fuzzy number A is called positive if $supp\,(A) \subset (0, \infty)$. We denote by $\mathbb{R}_F^+$, the space of all positive fuzzy numbers.

Definition 2.2. *(a) Let* $A, B \in \mathbb{R}_F$ *with* $[A]^\alpha = [A_l^\alpha, A_r^\alpha]$ *and* $[B]^\alpha = [B_l^\alpha, B_r^\alpha]$ *for* $\alpha \in [0,1]$. *We define* $\|A\|$ *on the space of fuzzy numbers as follow;*

$$\|A\| = \sup\max\left\{\left|A_l^\alpha\right|, \left|A_r^\alpha\right|\right\},$$

where sup *is taken for all* $\alpha \in [0,1]$. *We recall the following metric*

$$D(A,B) = \sup\left\{\max\left\{\left|A_l^\alpha - B_l^\alpha\right|, \left|A_r^\alpha - B_r^\alpha\right|\right\}\right\},$$

where sup *is taken for all* $\alpha \in [0,1]$.

(b) Let (x_n) *be a sequence of positive fuzzy numbers and* $x \in \mathbb{R}_F$. *Then, we say that*

$$\lim_{n\to\infty} x_n = x \quad \textit{iff} \quad \lim_{n\to\infty} D(x_n, x) = 0.$$

The following lemma and definition are given in [24].

Lemma 2.1. *Let* $X, Y \in \mathbb{R}_F$ *and* $[X]^\alpha = [X_l^\alpha, X_r^\alpha]$, $[Y]^\alpha = [Y_l^\alpha, Y_r^\alpha]$ *for* $\alpha \in [0,1]$ *be the* α*-cuts of* X, Y, *respectively. Let Z be a fuzzy number such that* $[Z]^\alpha = [Z_l^\alpha, Z_r^\alpha]$ *for* $\alpha \in [0,1]$. *Then,* $MIN\{X, Y\} = Z$ *(resp.* $MAX\{X, Y\} = Z$*) if and only if* $\min\left\{X_l^\alpha, Y_l^\alpha\right\} = Z_l^\alpha$ *and* $\min\{X_r^\alpha, Y_r^\alpha\} = Z_r^\alpha$ *(resp.* $\max\left\{X_l^\alpha, Y_l^\alpha\right\} = Z_l^\alpha$ *and* $\max\{X_r^\alpha, Y_r^\alpha\} = Z_r^\alpha$*).*

Definition 2.3. *(a) We say that a sequence of positive fuzzy numbers* (x_n) *is bounded and persistent if there exist* $n_0 \in \mathbb{N}$ *and* $C, D \in \mathbb{R}_F^+$ *such that* $MIN\{x_n, C\} = C$ *and* $MAX\{x_n, D\} = D$ *for* $n \geq n_0$.

(b) We say that (x_n) *for* $n \in \mathbb{N}_0$ *is an unbounded sequence if the* $\|x_n\|$ *for* $n \in \mathbb{N}_0$ *is an unbounded sequence.*

We need the following lemma which is given in [18] in the next section.

Lemma 2.2. *Let* f *be a continuous function from* $\mathbb{R}^+ \times \mathbb{R}^+ \times ... \times \mathbb{R}^+$ *into* $\mathbb{R}^+$ *and* $B_0, B_1, ..., B_k \in \mathbb{R}_F$. *Then,*

$$[f(B_0, B_1, ..., B_k)]^\alpha = f([B_0]^\alpha, [B_1]^\alpha, ..., [B_k]^\alpha) \textit{ for } \alpha \in (0,1].$$

3. Main results

In this section, we study the existence and properties of the positive solutions to (1.5). If (z_n) is a sequence of positive fuzzy numbers which satisfies (1.5), we say (z_n) is a positive solution of (1.5).

Theorem 3.1. *Consider (1.5) where* $A, B, C \in \mathbb{R}_F^+$. *Then, for any positive fuzzy numbers* z_{-j} $(j = 0, 1, ..., s)$ *there exists a unique positive solution* (z_n) *of (1.5) with the initial conditions* z_{-j} $(j = 0, 1, ..., s)$.

Proof. Suppose that there exists a sequence of fuzzy numbers (z_n) satisfying (1.5) with the initial conditions z_{-j} $(j = 0, 1, ..., s)$. Consider the α-cuts

$$\begin{aligned} [z_n]^\alpha &= [L_n^\alpha, R_n^\alpha], \\ [A]^\alpha &= [A_l^\alpha, A_r^\alpha], \\ [B]^\alpha &= [B_l^\alpha, B_r^\alpha], \\ [C]^\alpha &= [C_l^\alpha, C_r^\alpha] \end{aligned} \tag{3.1}$$

for $n = -s, -s+1, ...$ and all $\alpha \in (0, 1]$. Then, from (1.5)-(3.1) and Lemma 2.2, it follows that

$$\begin{aligned} [z_{n+1}]^\alpha &= \left[\frac{A z_{n-s}}{B + C \prod_{i=0}^{s} z_{n-i}} \right]^\alpha = \frac{[A]^\alpha [z_{n-s}]^\alpha}{[B]^\alpha + [C]^\alpha \prod_{i=0}^{s} [z_{n-i}]^\alpha} \\ &= \frac{[A_l^\alpha, A_r^\alpha][L_{n-s}^\alpha, R_{n-s}^\alpha]}{[B_l^\alpha, B_r^\alpha] + [C_l^\alpha, C_r^\alpha] \prod_{i=0}^{s} [L_{n-i}^\alpha, R_{n-i}^\alpha]} \\ &= \left[\frac{A_l^\alpha L_{n-s}^\alpha}{B_r^\alpha + C_r^\alpha \prod_{i=0}^{s} R_{n-i}^\alpha}, \frac{A_r^\alpha R_{n-s}^\alpha}{B_l^\alpha + C_l^\alpha \prod_{i=0}^{s} L_{n-i}^\alpha} \right] \end{aligned}$$

from which we have

$$L_{n+1}^\alpha = \frac{A_l^\alpha L_{n-s}^\alpha}{B_r^\alpha + C_r^\alpha \prod_{i=0}^{s} R_{n-i}^\alpha}, \; R_{n+1}^\alpha = \frac{A_r^\alpha R_{n-s}^\alpha}{B_l^\alpha + C_l^\alpha \prod_{i=0}^{s} L_{n-i}^\alpha}, \tag{3.2}$$

for $n \in \mathbb{N}_0$ and $\alpha \in (0, 1]$. Then, it is clear that for any (L_j^α, R_j^α), $j = -s, -s+1, ..., 0$, there exists a unique solution (L_n^α, R_n^α) with the initial conditions (L_j^α, R_j^α), $j = -s, -s+1, ..., 0$ for all $\alpha \in (0, 1]$.

Now, we prove that $[L_n^\alpha, R_n^\alpha]$ for all $\alpha \in (0, 1]$, where (L_n^α, R_n^α) is the solution of the system (3.2) with the initial conditions (L_j^α, R_j^α), $j = -s, -s+1, ..., 0$, determines the solution (z_n) of (1.5) with the initial conditions z_{-j} $(j = 0, 1, ..., s)$ such that

$$[z_n]^\alpha = [L_n^\alpha, R_n^\alpha], \; \alpha \in (0, 1], \; n = -s, -s+1, ... \tag{3.3}$$

Since $A, B, C, z_{-j} (j = 0, 1, ..., s) \in \mathbb{R}_F^+$ for any $\alpha_1, \alpha_2 \in (0, 1]$ and $\alpha_1 \leq \alpha_2$, we have

$$\begin{aligned} 0 < A_l^{\alpha_1} \leq A_l^{\alpha_2} \leq A_r^{\alpha_2} \leq A_r^{\alpha_1}, \\ 0 < B_l^{\alpha_1} \leq B_l^{\alpha_2} \leq B_r^{\alpha_2} \leq B_r^{\alpha_1}, \\ 0 < C_l^{\alpha_1} \leq C_l^{\alpha_2} \leq C_r^{\alpha_2} \leq C_r^{\alpha_1}, \\ 0 < L_j^{\alpha_1} \leq L_j^{\alpha_2} \leq R_j^{\alpha_2} \leq R_j^{\alpha_1}, \end{aligned} \tag{3.4}$$

for $j=-s,-s+1,...,0$. We prove by the induction that

$$0<L_n^{\alpha_1}\le L_n^{\alpha_2}\le R_n^{\alpha_2}\le R_n^{\alpha_1},\ n\in\mathbb{N}_0. \tag{3.5}$$

From (3.4), we see that (3.5) holds for $n=-s,-s+1,...,0$. Suppose that (3.5) is valid for $n\le k$, $k\in\{1,2,...\}$. Then, from (3.2), (3.4) and (3.5) for $n\le k$, it follows that

$$\begin{aligned} L_{k+1}^{\alpha_1} &= \frac{A_l^{\alpha_1}L_{k-s}^{\alpha_1}}{B_r^{\alpha_1}+C_r^{\alpha_1}\prod_{i=0}^{s}R_{k-i}^{\alpha_1}} \le \frac{A_l^{\alpha_2}L_{k-s}^{\alpha_2}}{B_r^{\alpha_2}+C_r^{\alpha_2}\prod_{i=0}^{s}R_{k-i}^{\alpha_2}} = L_{k+1}^{\alpha_2} \\ &\le \frac{A_r^{\alpha_2}R_{k-s}^{\alpha_2}}{B_l^{\alpha_2}+C_l^{\alpha_2}\prod_{i=0}^{s}L_{k-i}^{\alpha_2}} = R_{k+1}^{\alpha_2} \le \frac{A_r^{\alpha_1}R_{k-s}^{\alpha_1}}{B_l^{\alpha_1}+C_l^{\alpha_1}\prod_{i=0}^{s}L_{k-i}^{\alpha 1}} = R_{k+1}^{\alpha_1}. \end{aligned}$$

Therefore, (3.5) is satisfied. Moreover, from (3.2), we have

$$L_1^{\alpha}=\frac{A_l^{\alpha}L_{-s}^{\alpha}}{B_r^{\alpha}+C_r^{\alpha}\prod_{i=0}^{s}R_{-i}^{\alpha}},\ R_1^{\alpha}=\frac{A_r^{\alpha}R_{-s}^{\alpha}}{B_l^{\alpha}+C_l^{\alpha}\prod_{i=0}^{s}L_{-i}^{\alpha}},\ \text{for all } \alpha\in(0,1]. \tag{3.6}$$

Then, since A,B,C,z_{-j} ($j=0,1,...,s$) $\in\mathbb{R}_F^+$, we have that A_l^{α}, A_r^{α}, B_l^{α}, B_r^{α}, C_l^{α}, C_r^{α}, L_{-j}^{α}, R_{-j}^{α} ($j=0,1,...,s$) are left continuous. So, from (3.6) we see that L_1^{α} and R_1^{α} are also left continuous. Working inductively we can easily obtain that L_n^{α} and R_n^{α} are left continuous for $n\in\mathbb{N}$.

Now, we prove that $\overline{\cup_{\alpha\in(0,1]}[L_n^{\alpha},R_n^{\alpha}]}$ is compact. It is sufficient to prove that $\cup_{\alpha\in(0,1]}[L_n^{\alpha},R_n^{\alpha}]$ is bounded. Let $n=1$, since A,B,C,z_{-j} ($j=0,1,...,s$) $\in\mathbb{R}_F^+$, there exist constants

$$M_A,N_A,M_B,N_B,M_C,N_C,M_{-j},N_{-j}>0$$

for $j=0,1,...,s$ such that

$$\begin{aligned} [A_l^{\alpha},A_r^{\alpha}]&\subset[M_A,N_A],\\ [B_l^{\alpha},B_r^{\alpha}]&\subset[M_B,N_B],\\ [C_l^{\alpha},C_r^{\alpha}]&\subset[M_C,N_C],\\ [L_{-j}^{\alpha},R_{-j}^{\alpha}]&\subset[M_{-j},N_{-j}] \end{aligned} \tag{3.7}$$

for $j=0,1,...,s$. Therefore, from (3.6) and (3.7) we can easily prove that

$$[L_1^{\alpha},R_1^{\alpha}]\subset\left[\frac{M_AM_{-s}}{N_B+N_c\prod_{i=0}^{s}N_{-i}},\frac{N_AN_{-s}}{M_B+M_c\prod_{i=0}^{s}M_{-i}}\right] \tag{3.8}$$

from which it is clear that

$$\cup_{\alpha\in(0,1]}\,[L_1^{\alpha},R_1^{\alpha}]\subset\left[\frac{M_AM_{-s}}{N_B+N_c\prod_{i=0}^{s}N_{-i}},\frac{N_AN_{-s}}{M_B+M_c\prod_{i=0}^{s}M_{-i}}\right]\ \text{for all } \alpha\in(0,1]. \tag{3.9}$$

Also, (3.9) implies that $\overline{\cup_{\alpha\in(0,1]}[L_1^\alpha, R_1^\alpha]}$ is compact and $\overline{\cup_{\alpha\in(0,1]}[L_1^\alpha, R_1^\alpha]} \subset (0,\infty)$. Working inductively, we can easily see that $\overline{\cup_{\alpha\in(0,1]}[L_n^\alpha, R_n^\alpha]}$ is compact and

$$\overline{\cup_{\alpha\in(0,1]}[L_n^\alpha, R_n^\alpha]} \subset (0,\infty), \text{ for } n \in \mathbb{N}. \tag{3.10}$$

Therefore, using (3.5), (3.10) and since L_n^α, R_n^α are left continuous, we see that $[L_n^\alpha, R_n^\alpha]$ determines a sequence of positive fuzzy numbers (z_n) such that (3.3) holds.

We prove now that (z_n) is the solution of (1.5) with the initial conditions z_{-j} $(j = 0, 1, ..., s)$. Since

$$[z_{n+1}]^\alpha = [L_{n+1}^\alpha, R_{n+1}^\alpha] = \left[\frac{A_l^\alpha L_{-s}^\alpha}{B_r^\alpha + C_r^\alpha \prod_{i=0}^{s} R_{-i}^\alpha}, \frac{A_r^\alpha R_{n-s}^\alpha}{B_l^\alpha + C_l^\alpha \prod_{i=0}^{s} L_{n-i}^\alpha}\right] = \left[\frac{Az_{n-s}}{B + C\prod_{i=0}^{s} z_{n-i}}\right]^\alpha$$

for all $\alpha \in (0, 1]$, we have that (z_n) is the solution of (1.5) with the initial conditions z_{-j} $(j = 0, 1, ..., s)$.

Suppose that there exists another solution $(\widetilde{z}_n)$ of (1.5) with the initial conditions z_{-j} $(j = 0, 1, ..., s)$. Then, we can easily show that

$$[\widetilde{z}_n]^\alpha = [L_n^\alpha, R_n^\alpha] \text{ for } \alpha \in (0, 1] \text{ and } n \in \mathbb{N}_0. \tag{3.11}$$

Then, from (3.3) and (3.11), $[z_n]^\alpha = [\widetilde{z}_n]^\alpha$ for $\alpha \in (0, 1]$ and $n = -s, -s + 1, ...$ from which we get $z_n = \widetilde{z}_n$ for $n = -s, -s + 1, ...$ Thus, the proof is completed. □

In order to study the further dynamics of (1.5), we apply the results concerning the following system of ordinary difference equations

$$u_{n+1} = \frac{a_1 u_{n-s}}{b_1 + c_1 \prod_{i=0}^{s} v_{n-i}}, \; v_{n+1} = \frac{a_2 v_{n-s}}{b_2 + c_2 \prod_{i=0}^{s} u_{n-i}}, \; n \in \mathbb{N}_0, \tag{3.12}$$

where the parameters $a_1, b_1, c_1, a_2, b_2, c_2$ and initial conditions u_{-j}, v_{-j} $(j = 0, 1, ..., s)$ are positive real numbers and s is a positive integer. Note that system (3.12) can be written as

$$x_{n+1} = \frac{x_{n-s}}{q + \prod_{i=0}^{s} y_{n-i}}, \; y_{n+1} = \frac{y_{n-s}}{p + \prod_{i=0}^{s} x_{n-i}}, \; n \in \mathbb{N}_0, \tag{3.13}$$

by the change of variables $u_n = \left(\frac{a_2}{c_2}\right)^{\frac{1}{s+1}} x_n$, $v_n = \left(\frac{a_1}{c_1}\right)^{\frac{1}{s+1}} y_n$ with $q = \frac{b_1}{a_1}$ and $p = \frac{b_2}{a_2}$. The equilibrium points of (3.13) are the solutions of the equations

$$\overline{x} = \frac{\overline{x}}{q + \prod_{i=0}^{s} \overline{y}}, \; \overline{y} = \frac{\overline{y}}{p + \prod_{i=0}^{s} \overline{x}}.$$

We need the following results concerning the behavior of the solutions of the system (3.13) which has been presented in [26].

If $p < 1$, $q < 1$, then (3.13) has equilibrium points $(0, 0)$ and $(\sqrt[s+1]{1-p}, \sqrt[s+1]{1-q})$. In addition, if $p < 1$, $q = 1$, then system (3.13) has an equilibrium $(\sqrt[s+1]{1-p}, 0)$ and if $p = 1$, $q < 1$, then system (3.13) has an equilibrium $(0, \sqrt[s+1]{1-q})$. Finally, if $p > 1$, $q > 1$, then system (3.13) has the unique equilibrium $(0, 0)$.

Theorem 3.2. *Let (x_n, y_n) be any positive solution of system (3.13), then the following statements are true:*

(**1**) *For every $m \geq 0$, the following results hold:*

$$0 \leq x_n \leq \begin{cases} \left(\frac{1}{q}\right)^{m+1} x_{-s}, & n = (s+1)m + 1, \\ \left(\frac{1}{q}\right)^{m+1} x_{-s+1}, & n = (s+1)m + 2, \\ \vdots & \vdots \\ \left(\frac{1}{q}\right)^{m+1} x_0, & n = (s+1)m + s + 1 \end{cases} \tag{3.14}$$

and

$$0 \leq y_n \leq \begin{cases} \left(\frac{1}{p}\right)^{m+1} y_{-s}, & n = (s+1)m + 1, \\ \left(\frac{1}{p}\right)^{m+1} y_{-s+1}, & n = (s+1)m + 2, \\ \vdots & \vdots \\ \left(\frac{1}{p}\right)^{m+1} y_0, & n = (s+1)m + p + 1. \end{cases} \tag{3.15}$$

(**2**) *If $p < 1$ and $q < 1$, then the following statements are true for $j = -s, -s+1, ..., 0$.*

(**i**) *If $(x_j, y_j) \in (0, \sqrt[s+1]{1-p}) \times (\sqrt[s+1]{1-q}, \infty)$, then $(x_n, y_n) \in (0, \sqrt[s+1]{1-p}) \times (\sqrt[s+1]{1-q}, \infty)$.*

(**ii**) *If $(x_j, y_j) \in (\sqrt[s+1]{1-p}, \infty) \times (0, \sqrt[s+1]{1-q})$, then $(\sqrt[s+1]{1-p}, \infty) \times (0, \sqrt[s+1]{1-q})$.*

(**3**) *If $p > 1$ and $q > 1$, then every positive solution (x_n, y_n) of system (3.13) converges to $(0, 0)$ as $n \to \infty$.*

The following corollary can be obtained from Theorem 3.2.

Corollary 3.1. *Let (x_n, y_n) be positive solution to system (3.13), then the following statements are true.*

(**1**) *If $p > 1$ and $q > 1$, then every positive solution (x_n, y_n) of system (3.13) is bounded and persistent.*

(**2**) *If $p < 1$ and $q < 1$, then the system (3.13) has unbounded solutions:*

(**i**) *If $x_j \in \left(0, \sqrt[s+1]{1-p}\right)$ and $y_j \in \left(\sqrt[s+1]{1-q}, \infty\right)$ for $j = -s, -s+1, ..., 0$, then $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = \infty$.*

(**ii**) *If $x_j \in \left(\sqrt[s+1]{1-p}, \infty\right)$ and $y_j \in \left(0, \sqrt[s+1]{1-q}\right)$ for $j = -s, -s+1, ..., 0$, then $\lim_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} x_n = \infty$.*

Proof. (**1**) Let $\beta = \max\{x_{-s}, x_{-s+1}, ..., x_0\}$ and $\gamma = \max\{y_{-s}, y_{-s+1}, ..., y_0\}$, then it is easy to see from system (3.13) that

$$0 \leq x_n \leq \beta \text{ and } 0 \leq y_n \leq \gamma$$

for all $n \in \mathbb{N}_0$. Hence, every positive solution (x_n, y_n) of system (3.13) is bounded and persistent.

(**2**) We only prove the part (i), since the part (ii) can be proved similarly.

(**i**) Assume that (x_n, y_n) is a positive solution of system (3.13) such that $x_j \in \left(0, \sqrt[s+1]{1-p}\right)$ and $y_j \in \left(\sqrt[s+1]{1-q}, \infty\right)$ for $j = -s, -s+1, ..., 0$. Then, from system (3.13), we obtain the following inequalities

$$
\begin{aligned}
x_1 &= \frac{x_{-s}}{q+\prod_{i=0}^{s} y_{-i}} < \frac{x_{-s}}{q+\prod_{i=0}^{s} \sqrt[s+1]{1-q}} = x_{-s},\\
x_2 &= \frac{x_{-s+1}}{q+\prod_{i=0}^{s} y_{-i+1}} < \frac{x_{-s+1}}{q+\prod_{i=0}^{s} \sqrt[s+1]{1-q}} = x_{-s+1},\\
x_3 &= \frac{x_{-s+2}}{q+\prod_{i=0}^{s} y_{-i+2}} < \frac{x_{-s+2}}{q+\prod_{i=0}^{s} \sqrt[s+1]{1-q}} = x_{-s+2},\\
&\vdots
\end{aligned}
$$

and

$$
\begin{aligned}
y_1 &= \frac{y_{-s}}{p+\prod_{i=0}^{s} x_{-i}} > \frac{y_{-s}}{p+\prod_{i=0}^{s} \sqrt[s+1]{1-p}} = y_{-s},\\
y_2 &= \frac{y_{-s+1}}{p+\prod_{i=0}^{s} x_{-i+1}} > \frac{y_{-s+1}}{p+\prod_{i=0}^{s} \sqrt[s+1]{1-p}} = y_{-s+1},\\
y_3 &= \frac{y_{-s+2}}{p+\prod_{i=0}^{s} x_{-i+2}} > \frac{y_{-s+2}}{p+\prod_{i=0}^{s} \sqrt[s+1]{1-p}} = y_{-s+2},\\
&\vdots
\end{aligned}
$$

from which it follows that

$$\lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} y_n = \infty.$$

So, the proof is completed. □

Theorem 3.3. *Consider the fuzzy difference Eq (1.5). The the following statements are true.*
(**1**) *If $A_r^\alpha < B_l^\alpha$ for all $\alpha \in (0,1]$, then every positive solution of (1.5) is bounded and persistent.*
(**2**) *If there exists an $\overline{\alpha} \in (0,1]$ such that $B_r^{\overline{\alpha}} < A_l^{\overline{\alpha}}$, then the Eq (1.5) has unbounded solutions.*

Proof. (**1**) Consider the following system of ordinary difference equations

$$s_{n+1} = \frac{\gamma_A s_{n-p}}{\beta_B + \beta_C \prod_{i=0}^{p} t_{n-i}},\ t_{n+1} = \frac{\beta_A t_{n-p}}{\gamma_B + \gamma_C \prod_{i=0}^{p} s_{n-i}},\ n \in \mathbb{N}_0 \tag{3.16}$$

where

$$
\begin{aligned}
[A]^\alpha &= [A_l^\alpha, A_r^\alpha] \subset \overline{\cup_{\alpha\in(0,1]}[A_l^\alpha, A_r^\alpha]} \subset [\gamma_A, \beta_A],\\
[B]^\alpha &= [B_l^\alpha, B_r^\alpha] \subset \overline{\cup_{\alpha\in(0,1]}[B_l^\alpha, B_r^\alpha]} \subset [\gamma_B, \beta_B],\\
[C]^\alpha &= [C_l^\alpha, C_r^\alpha] \subset \overline{\cup_{\alpha\in(0,1]}[C_l^\alpha, C_r^\alpha]} \subset [\gamma_C, \beta_C].
\end{aligned} \tag{3.17}
$$

Let (s_n, t_n) be a solution of system (3.16) with the initial conditions $(s_{-j}, t_{-j}) = (\gamma_{-j}, \beta_{-j})$ for $j = 0, 1, ..., p$ where γ_{-j} and β_{-j} are given

$$[L_{-j}^\alpha, R_{-j}^\alpha] \subset \overline{\cup_{\alpha\in(0,1]}[L_{-j}^\alpha, R_{-j}^\alpha]} \subset [\gamma_{-j}, \beta_{-j}] \text{ for } j = -p, -p+1, ..., 0. \tag{3.18}$$

Then, from (3.16) and (3.17) it folows that

$$s_1 = \frac{\gamma_A s_{-p}}{\beta_B + \beta_C \prod_{i=0}^{p} t_{-i}} \le \frac{A_l^\alpha L_{-p}^\alpha}{B_r^\alpha + C_r^\alpha \prod_{i=0}^{p} R_{-i}^\alpha} = L_1^\alpha \tag{3.19}$$

and

$$t_1 = \frac{\beta_A t_{-p}}{\gamma_B + \gamma_C \prod_{i=0}^{p} s_{-i}} \geq \frac{A_r^\alpha R_{-p,\alpha}}{B_l^\alpha + C_l^\alpha \prod_{i=0}^{p} L_{-i}^\alpha} = R_1^\alpha. \tag{3.20}$$

Hence, by induction, we get $s_n \leq L_n^\alpha$ and $R_n^\alpha \leq t_n$ for $n \in \mathbb{N}$. Assume that $A_r^\alpha < B_l^\alpha$ for all $\alpha \in (0,1]$, then it follows that $\gamma_A < \beta_B$ and $\beta_A < \gamma_B$. From (2) of Theorem 3.2, the solution (s_n, t_n) of system (3.15) is bounded and persistent, which is the solution (z_n) of (1.5). This completes the proof of (1).

(**2**) Suppose that there exists an $\overline{\alpha} \in (0,1]$ such that $B_r^{\overline{\alpha}} < A_l^{\overline{\alpha}}$. If $A_l^{\overline{\alpha}} = a_1$, $A_r^{\overline{\alpha}} = a_2$, $B_l^{\overline{\alpha}} = b_2$, $B_r^{\overline{\alpha}} = b_1$, $L_n^{\overline{\alpha}} = x_n$ and $R_n^{\overline{\alpha}} = y_n$ for $n = -s, -s+1, ...$, then we can apply (i) of (2) in Corollary 3.1 to system (3.2) (We can use (ii) of (2) in Corollary 3.1, too). If there exists an $\overline{\alpha} \in (0,1]$ such that $B_r^{\overline{\alpha}} < A_l^{\overline{\alpha}}$ and $x_j \in \left(0, \sqrt[s+1]{1-p}\right)$ and $y_j \in \left(\sqrt[s+1]{1-q}, \infty\right)$ for $j = -s, -s+1, ..., 0$, then there exist solutions (x_n, y_n) of system (3.13) where $\overline{\alpha} = \alpha$ with initial conditions (x_{-j}, y_{-j}) for $j = 0, 1, ..., s$ such that

$$\lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} y_n = \infty. \tag{3.21}$$

Moreover, if $x_{-j} < y_{-j}$ ($j = 0, 1, ..., s$), we can find $z_{-j} \in \mathbb{R}_F^+$ such that

$$[z_j]^\alpha = [L_j^\alpha, R_j^\alpha] \text{ for } \alpha \in (0,1] \tag{3.22}$$

and

$$[z_j]^{\overline{\alpha}} = [L_j^{\overline{\alpha}}, R_j^{\overline{\alpha}}] = [x_j, y_j] \tag{3.23}$$

for $j = -s, -s+1, ..., 0$. Let (z_n) be a positive solution of (1.5) with the initial conditions z_{-j} $(j = 0, 1, ..., s)$ and $[z_n]^\alpha = [L_n^\alpha, R_n^\alpha]$ for $\alpha \in (0,1]$. Since (3.22) and (3.23) hold and (L_n^α, R_n^α) satisfies system (3.2), we have

$$[z_n]^{\overline{\alpha}} = [L_n^{\overline{\alpha}}, R_n^{\overline{\alpha}}] = [x_n, y_n]. \tag{3.24}$$

Therefore, from (3.21), (3.24) and since

$$\|z_n\| = \sup_{\alpha\in(0,1]} \max\left\{\left|L_n^\alpha\right|, \left|R_n^\alpha\right|\right\} \geq \max\left\{\left|L_n^{\overline{\alpha}}\right|, \left|R_n^{\overline{\alpha}}\right|\right\} = R_n^{\overline{\alpha}}$$

where sup is taken for all $\alpha \in (0,1]$, it is clear that solution (z_n) is unbounded. This completes the proof of (2). □

Theorem 3.4. *If* $A_r^\alpha < B_l^\alpha$ *for all* $\alpha \in (0,1]$, *then every positive solution* (z_n) *of (1.5) converges to* 0 *as* $n \to \infty$.

Proof. Let (z_n) be a positive solution of (1.5) such that (3.3) holds with $A_r^\alpha < B_l^\alpha$ for all $\alpha \in (0,1]$. Then, we can apply (3) of Theorem 3.2 to system (3.2). So, we get

$$\lim_{n\to\infty} L_n^\alpha = \lim_{n\to\infty} R_n^\alpha = 0. \tag{3.25}$$

Therefore, from (3.25), we get

$$\lim_{n\to\infty} D(z_n, 0) = \lim_{n\to\infty}\left(\sup_{\alpha\in(0,1]}\left\{\max\left\{\left|L_n^\alpha - 0\right|, \left|R_n^\alpha - 0\right|\right\}\right\}\right) = 0.$$

This completes the proof. □

4. Numerical examples

In this section, to verify obtained results, we give two numerical examples for $s = 3$ with different values of A, B, C where the initial conditions $z_{-3}, z_{-2}, z_{-1}, z_0$ are satisfied

$$z_{-3}(x) = \begin{cases} \frac{4x-0.4}{2}, & 0.1 \le x \le 0.6, \\ \frac{4.4-4x}{2}, & 0.6 \le x \le 1.1, \end{cases}$$

$$z_{-2}(x) = \begin{cases} \frac{5x-1}{2}, & 0.2 \le x \le 0.6, \\ \frac{5-5x}{2}, & 0.6 \le x \le 1, \end{cases}$$

$$z_{-1}(x) = \begin{cases} \frac{4x-1}{2}, & 0.25 \le x \le 0.75, \\ \frac{5-4x}{2}, & 0.75 \le x \le 1.25, \end{cases} \tag{4.1}$$

$$z_0(x) = \begin{cases} \frac{5x-2.5}{2}, & 0.5 \le x \le 0.9, \\ \frac{6.5-5x}{2}, & 0.9 \le x \le 1.3. \end{cases}$$

From (4.1), we get

$$[z_{-3}]^\alpha = \left[\frac{2\alpha+0.4}{4}, \frac{4.4-2\alpha}{4}\right],$$

$$[z_{-2}]^\alpha = \left[\frac{2\alpha+1}{5}, \frac{5-2\alpha}{5}\right],$$

$$[z_{-1}]^\alpha = \left[\frac{2\alpha+1}{4}, \frac{5-2\alpha}{4}\right],$$

$$[z_0]^\alpha = \left[\frac{2\alpha+2.50}{5}, \frac{6.50-2\alpha}{5}\right]$$

for all $\alpha \in [0, 1]$.

Example 4.1. *Consider Eq (1.5) where the initial conditions are satisfied (4.1) and A, B, C are satisfied*

$$A = \begin{cases} 4x-1, & 0.25 \le x \le 0.5, \\ 3-4x, & 0.5 \le x \le 0.75, \end{cases}$$

$$B = \begin{cases} x-1, & 1 \le x \le 2, \\ 3-x, & 2 \le x \le 3, \end{cases} \tag{4.2}$$

$$C = \begin{cases} 2x-1, & 0.5 \le x \le 1, \\ 3-2x, & 1 \le x \le 1.5. \end{cases}$$

Then, from (4.2), we get $[A]^\alpha = \left[\frac{\alpha+1}{4}, \frac{3-\alpha}{4}\right]$, $[B]^\alpha = [\alpha+1, 3-\alpha]$ *and* $[C]^\alpha = \left[\frac{\alpha+1}{2}, \frac{3-\alpha}{2}\right]$ *for all* $\alpha \in (0, 1]$. *By Theorem 3.1, there exists a unique solution. Since* $A_r^\alpha < B_l^\alpha$ *for all* $\alpha \in [0, 1]$, *then by case (1) in Theorem 3.3, the positive solution* (z_n) *of fuzzy difference Eq (1.5) is bounded and persistent and by Theorem 3.4, it converges to* 0 *as* $n \to \infty$. *For* $\alpha_1 = 0.2$ *and* $\alpha_2 = 0.8$, *the* $\alpha-$*cuts of the solution* L_n^α *and* R_n^α *are depicted in Figures 1 and 2, respectively.*

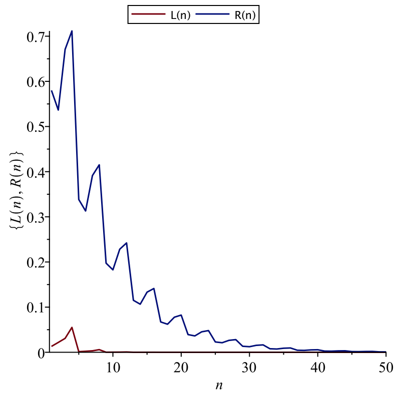

Figure 1. α-cuts of the solution for $\alpha = 0.2$, in Example 4.1.

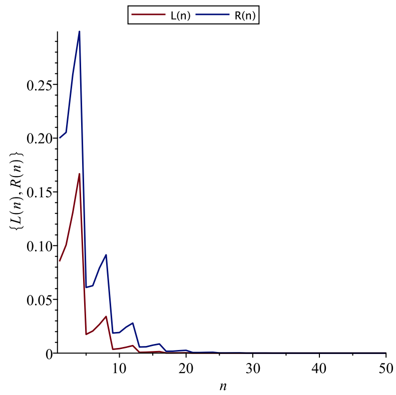

Figure 2. α-cuts of the solution for $\alpha = 0.8$, in Example 4.1.

Example 4.2. *Consider Eq (1.5) where the initial conditions are satisfied (4.1) and A, B, C are satisfied*

$$A = \begin{cases} x-2, & 2 \leq x \leq 3, \\ 4-x, & 3 \leq x \leq 4, \end{cases}$$

$$B = \begin{cases} 4x-1, & 0.25 \leq x \leq 0.5, \\ 3-4x, & 0.5 \leq x \leq 0.75, \end{cases} \tag{4.3}$$

$$C = \begin{cases} x-1, & 1 \leq x \leq 2, \\ 3-x, & 2 \leq x \leq 3. \end{cases}$$

Then, from (4.3), we get $[A]^\alpha = [\alpha+2, 4-\alpha]$, $[B]^\alpha = \left[\frac{\alpha+1}{4}, \frac{3-\alpha}{4}\right]$ *and* $[C]^\alpha = [\alpha+1, 3-\alpha]$ *for all* $\alpha \in (0,1]$. *By Theorem 3.1 there exists a unique positive solution. For any* $\alpha \in [0,1]$, *we have* $A_l^\alpha > B_r^\alpha$. *So, by case (2) in Theorem 3.3, the corresponding fuzzy difference equation has unbounded solutions. For* $\alpha_1 = 0.2$ *and* $\alpha_2 = 0.8$, *the* α*-cuts of the solution* L_n^α *and* R_n^α *are depicted in Figures 3 and 4, respectively.*

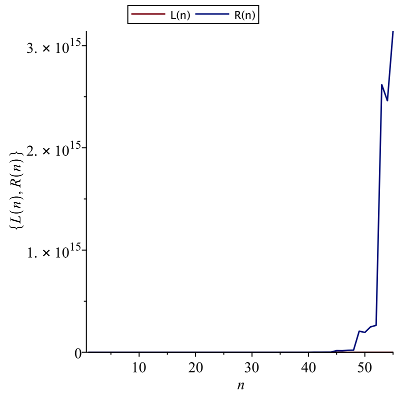

Figure 3. α-cuts of the solution for $\alpha = 0.2$, in Example 4.2.

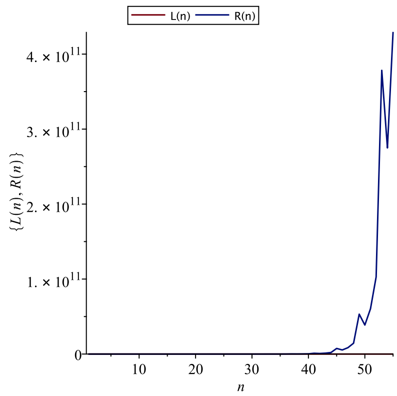

Figure 4. α-cuts of the solution for $\alpha = 0.8$, in Example 4.2.

5. Conclusions

In this study, we investigated behavior of the fuzzy difference equation $z_{n+1} = Az_{n-s}/(B + C \prod_{i=0}^{s} z_{n-i})$. We have shown that, under certain conditions, the positive solutions of this equation converge to zero. We have also considered the case where the solutions are unbounded. Finally, we have supported our theoretical results via two numerical examples. This study extends the results in the references [8, 10].

Conflict of interest

The authors declare that they have no conflicts of interest.

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