AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(3): 6278–6287.
DOI:10.3934/math.2023317
Received: 30 October 2022
Revised: 12 December 2022
Accepted: 21 December 2022
Published: 03 January 2023

Research article

Two theorems on direct products of gyrogroups

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Abstract: We extend two standard theorems on groups to gyrogroups: the direct product theorem and the cancellation theorem for direct products. Firstly, we prove that under a certain condition a gyrogroup G can be decomposed as the direct product of two subgyrogroups. Secondly, we prove that finite gyrogroups can be cancelled in direct products: if $A \cong B$, then $A \times H \cong B \times K$ or $H \times A \cong K \times B$ implies $H \cong K$, where A, B, H, and K are finite gyrogroups.

Keywords: direct product theorem; cancellation theorem; decomposition of gyrogroup; gyrogroup; homomorphism
Mathematics Subject Classification: Primary 20N05; Secondary 20A05, 20E34

1. Introduction

Let us mention two standard theorems involving direct products of groups in abstract algebra as follows.

Theorem 1.1. *If H and K are normal subgroups of a group (written multiplicatively) that have trivial intersection, then the internal direct product HK is isomorphic to the external direct product $H \times K$ as groups.*

Theorem 1.2. *Let G, H, and K be finite groups. If $G \times H$ and $G \times K$ are isomorphic as groups, then H and K are isomorphic as groups.*

Theorem 1.1 is sometimes referred to as the *direct product theorem* for groups (see, for instance, Theorem 9 in p. 171 of [2]). Theorem 1.2 is sometimes referred to as the *cancellation theorem* for direct products of finite groups (see, for instance, [5]). The latter theorem may be used for comparing external direct products of finite groups such as the uniqueness part of the fundamental theorem of finite abelian groups (see, for instance, p. 213 of [4]).

The notion of a gyrogroup is introduced as a suitable generalization of groups. For a detailed discussion of the formation of gyrogroups, the reader is referred to [14], for instance. For algebraic aspects of gyrogroups, the reader is referred to [1, 3, 6, 7, 13], for instance. Roughly speaking, a gyrogroup is a non-associative algebraic structure that shares several properties with groups. In fact, every group may be viewed as a gyrogroup with gyroautomorphisms being the identity automorphism. Important theorems on groups can be naturally extended to gyrogroups. This motivates us to continue studying algebraic aspects of gyrogroups. In the present article, we prove the two aforementioned theorems in the case of gyrogroups.

2. Preliminaries

For the basic theory of gyrogroups, the reader is referred to [11, 14]. The formal definition of a gyrogroup can be found in p. 17 of [14]. In this section, we summarize basic terminology, notation, and results in gyrogroup theory for reference.

In the case when $\oplus$ is a binary operation on a non-empty set G, let $\mathrm{Aut}\,(G)$ be the set of all automorphisms of $(G, \oplus)$. Let G be a gyrogroup. Recall that G satisfies the *left gyroassociative law*:

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}\,[a, b](c) \tag{2.1}$$

for all $a, b, c \in G$, where $\mathrm{gyr}\,[a, b]$ is an automorphism in $\mathrm{Aut}\,(G)$. We remark that G has the unique two-sided identity, denoted by e. Moreover, any element a in G has the unique two-sided inverse, denoted by $\ominus a$. The automorphism $\mathrm{gyr}\,[a, b]$ in (2.1) is called the *gyroautomorphism* generated by a and b. It can be proved that G also satisfies the *right gyroassociative law*:

$$(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}\,[b, a](c)). \tag{2.2}$$

for all $a, b, c \in G$. The *cooperation* of G, denoted by $\boxplus$, is defined as $a \boxplus b = a \oplus \mathrm{gyr}\,[a, \ominus b](b)$ for all $a, b \in G$. In addition, we define $a \ominus b = a \oplus (\ominus b)$ and $a \boxminus b = a \boxplus (\ominus b)$. Throughout the article, if X is a non-empty set, then I_X denotes the identity map on X; that is, $\mathrm{I}_X(x) = x$ for all $x \in X$.

Let G and H be gyrogroups. A map φ from G to H is called a *gyrogroup homomorphism* if $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$ for all $a, b \in G$. A bijective gyrogroup homomorphism is called a *gyrogroup isomorphism*. If there exists a gyrogroup isomorphism from G to H, then we say that G is *isomorphic* to H, denoted by $G \cong H$. Let $\varphi : G \to H$ be a gyrogroup homomorphism. The *kernel* of φ, denoted by $\ker \varphi$, is defined by $\ker \varphi = \{a \in G : \varphi(a) = e\}$. The *image* of φ, denoted by $\mathrm{im}\,\varphi$, is defined by $\mathrm{im}\,\varphi = \{b \in H : b = \varphi(a) \text{ for some } a \in G\}$.

Let G be a gyrogroup. Recall that a non-empty subset H of G is a *subgyrogroup* of G if H forms a gyrogroup under the operation inherited from G. If H and K are subgyrogroups of G, define $H \oplus K = \{h \oplus k : h \in H, k \in K\}$. A subgyrogroup H of G is called an *L-subgyrogroup* if $\mathrm{gyr}\,[a, h](H) = H$ for all $a \in G, h \in H$. If H is an L-subgyrogroup of G and if G is finite, then the *index formula* holds:

$$|G| = [G : H]|H|, \tag{2.3}$$

where $[G : H] = |\{a \oplus H : a \in G\}|$ (cf. Corollary 22 of [12]). A subgyrogroup N of G is *normal*, denoted by $N \trianglelefteq G$, provided there is a gyrogroup homomorphism φ from G to a gyrogroup such that $N = \ker \varphi$. If $N \trianglelefteq G$, then $\mathrm{gyr}\,[a, b](N) = N$ for all $a, b \in G$ (see the proof of Proposition 35 of [11]).

Furthermore, the set of left cosets of N in G, $G/N = \{a \oplus N : a \in G\}$, forms a gyrogroup under the operation defined by $(a \oplus N) \oplus (b \oplus N) = (a \oplus b) \oplus N$ for all $a, b \in G$, called the *quotient gyrogroup* of G by N. In this case, the map $a \mapsto a \oplus N$, $a \in G$, defines a gyrogroup homomorphism, called the *canonical projection*.

Theorem 2.1. *(See Chapter 2 of [14]) Let G be a gyrogroup and let a, b, $c \in G$.*

1) $a \oplus b = a \oplus c$ *implies* $b = c$; (left cancellation law I)
2) $\ominus a \oplus (a \oplus b) = b$; (left cancellation law II)
3) $(b \ominus a) \boxplus a = b$; (right cancellation law I)
4) $(b \boxminus a) \oplus a = b$. (right cancellation law II)

Proposition 2.1. *([12, Part 3 of Proposition 23]) If $\varphi : G \to H$ is a gyrogroup homomorphism, then $\varphi(\mathrm{gyr}\,[a, b](c)) = \mathrm{gyr}\,[\varphi(a), \varphi(b)](\varphi(c))$ for all $a, b, c \in G$.*

Proposition 2.2. *([12, Proposition 26]) Let $\varphi\colon G \to H$ be a gyrogroup homomorphism and let $a, b \in G$. Then $\ominus a \oplus b \in \ker\varphi$ if and only if $\varphi(a) = \varphi(b)$.*

Let G be a gyrogroup. The set of permutations of G is denoted by $\mathrm{Sym}\,(G)$, which forms a group under composition of maps. For each $a \in G$, the *left gyrotranslation* L_a is defined by $L_a(x) = a \oplus x$, $x \in G$, which is a permutation of G (cf. Theorem 10 of [12]). Define $\widehat{G}$ to be the set of left gyrotranslations,

$$\widehat{G} = \{L_a : a \in G\}, \tag{2.4}$$

and define $\mathrm{Sym}_e(G)$ to be the set of permutations of G leaving the identity e fixed,

$$\mathrm{Sym}_e(G) = \{\rho \in \mathrm{Sym}\,(G) : \rho(e) = e\}. \tag{2.5}$$

Theorem 2.2. *([12, Theorem 11]) Let G be a gyrogroup. For each $\sigma \in \mathrm{Sym}\,(G)$, σ can be written uniquely as $\sigma = L_a \circ \rho$, where $a \in G$ and $\rho \in \mathrm{Sym}_e(G)$.*

Let G and H be gyrogroups. As defined in [10], the *direct product* of G and H, denoted by $G \times H$, is a gyrogroup with underlying set $\{(g, h) : g \in G, h \in H\}$ whose operation is given componentwise by

$$(a, b) \oplus (c, d) = (a \oplus c, b \oplus d) \tag{2.6}$$

for all $a, c \in G$ and for all $b, d \in H$.

Proposition 2.3. *Let A, B, C, and D be gyrogroups.*

1) *Then $A \times B \cong B \times A$.*
2) *If $A \cong B$ and $B \cong C$, then $A \cong C$.*
3) *If $A \cong C$ and $B \cong D$, then $A \times B \cong C \times D$.*

Proof. The map Φ defined by $\Phi(a, b) = (b, a)$, $a \in A, b \in B$, is a gyrogroup isomorphism from $A \times B$ to $B \times A$. If $\phi : A \to B$ and $\psi : B \to C$ are gyrogroup isomorphisms, then $\psi \circ \phi$ is a gyrogroup isomorphism from A to C. If $\phi : A \to C$ and $\psi : B \to D$ are gyrogroup isomorphisms, then the map θ defined by $\theta(a, b) = (\phi(a), \psi(b))$, $a \in A, b \in B$, is a gyrogroup isomorphism from $A \times B$ to $C \times D$. □

3. Main results

The main purpose of this article is to generalize Theorems 1.1 and 1.2 from groups to gyrogroups. The direct product theorem for gyrogroups proves useful in understanding the gyrogroup that arises in Cayley's theorem for gyrogroups, as we will see shortly. The cancellation theorem for gyrogroups may be used to compare direct products of finite gyrogroups, as in the case of groups.

3.1. Direct product decomposition

To prove the direct product theorem for gyrogroups, we need two preliminary lemmas involving gyroautomorphisms, which are important in their own right.

Lemma 3.1. *Let H and K be normal subgyrogroups of a gyrogroup G such that $H \cap K = \{e\}$.*

1) *Then $h \oplus k = k \oplus h$ for all $h \in H, k \in K$.*
2) *If $h \in H$ and $b \in G$, then $\mathrm{gyr}\,[h, b](k) = k$ for all $k \in K$.*
3) *If $k \in K$ and $a \in G$, then $\mathrm{gyr}\,[a, k](h) = h$ for all $h \in H$.*

Proof. Since $H \trianglelefteq G$ and $K \trianglelefteq G$, we obtain $H = \ker \varphi$ and $K = \ker \psi$, where φ and ψ are homomorphisms of G to some gyrogroups. Let $h \in H$ and $k \in K$. Since φ preserves the operations and $\varphi(h) = e$, it follows that

$$\varphi(\ominus h \oplus (\ominus k \oplus (h \oplus k))) = \ominus\varphi(h) \oplus (\ominus\varphi(k) \oplus (\varphi(h) \oplus \varphi(k))) = e.$$

Hence, $\ominus h \oplus (\ominus k \oplus (h \oplus k)) \in H$. Similarly, $\psi(\ominus h \oplus (\ominus k \oplus (h \oplus k))) = e$ and then $\ominus h \oplus (\ominus k \oplus (h \oplus k)) \in K$. Since $H \cap K = \{e\}$, we obtain $\ominus h \oplus (\ominus k \oplus (h \oplus k)) = e$, which implies $h \oplus k = k \oplus h$ by the left cancellation law II. This proves Part 1.

Let $h \in H$, $b \in G$, and $k \in K$. Since $\mathrm{gyr}\,[h, b](K) = K$, it follows that $\mathrm{gyr}\,[h, b](k) = k'$ for some $k' \in K$. By Proposition 2.1, $\varphi(k') = \varphi(\mathrm{gyr}\,[h, b](k)) = \mathrm{gyr}\,[\varphi(h), \varphi(b)](\varphi(k)) = \varphi(k)$ since $\varphi(h) = e$. By Proposition 2.2, $\ominus k \oplus k' \in H$. Hence, $\ominus k \oplus k' \in H \cap K$. By assumption, $\ominus k \oplus k' = e$, which implies $k' = k$ by the left cancellation law II. This proves Part 2. Part 3 is proved in a similar fashion to Part 2. □

Lemma 3.2. *([8, Lemma 1]) If H and K are subgyrogroups of a gyrogroup G with the property that $\mathrm{gyr}\,[a, b](K) \subseteq K$ for all $a, b \in H$ and $H \cap K = \{e\}$, then $h_1 \oplus k_1 = h_2 \oplus k_2$, where $h_1, h_2 \in H, k_1, k_2 \in K$, implies $h_1 = h_2$ and $k_1 = k_2$.*

Proof. See the proof of Lemma 1 of [8]. □

The following theorem gives a criterion to decompose a gyrogroup into the direct product of its subgyrogroups. This generalizes the familiar direct product theorem for groups.

Theorem 3.1. *(Direct product theorem for gyrogroups) Let G be a gyrogroup. If H and K are subgyrogroups of G such that*

(i) *$G = H \oplus K$;*
(ii) *$H \trianglelefteq G$ and $K \trianglelefteq G$;*
(iii) *$H \cap K = \{e\}$,*

then $G \cong H \times K$ as gyrogroups.

Proof. Define a map ϕ by $\phi(h,k) = h \oplus k$ for all $h \in H, k \in K$. Then ϕ sends $H \times K$ to G. We show that ϕ is an isomorphism. By condition (i), ϕ is surjective. Let $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Suppose that $\phi(h_1,k_1) = \phi(h_2,k_2)$. Then $h_1 \oplus k_1 = h_2 \oplus k_2$. By Lemma 3.2, $h_1 = h_2$ and $k_1 = k_2$. This proves that ϕ is injective. Using the left and right gyroassociative laws and Lemma 3.1, we obtain

$$\begin{aligned}
\phi((h_1,k_1) \oplus (h_2,k_2)) &= \phi(h_1 \oplus h_2, k_1 \oplus k_2) \\
&= (h_1 \oplus h_2) \oplus (k_1 \oplus k_2) \\
&= h_1 \oplus (h_2 \oplus \operatorname{gyr}[h_2,h_1](k_1 \oplus k_2)) \\
&= h_1 \oplus (h_2 \oplus (k_1 \oplus k_2)) \\
&= h_1 \oplus ((h_2 \oplus k_1) \oplus \operatorname{gyr}[h_2,k_1](k_2)) \\
&= h_1 \oplus ((h_2 \oplus k_1) \oplus k_2) \\
&= h_1 \oplus ((k_1 \oplus h_2) \oplus k_2) \\
&= h_1 \oplus (k_1 \oplus (h_2 \oplus \operatorname{gyr}[h_2,k_1](k_2))) \\
&= h_1 \oplus (k_1 \oplus (h_2 \oplus k_2)) \\
&= (h_1 \oplus k_1) \oplus \operatorname{gyr}[h_1,k_1](h_2 \oplus k_2) \\
&= (h_1 \oplus k_1) \oplus (\operatorname{gyr}[h_1,k_1](h_2) \oplus \operatorname{gyr}[h_1,k_1](k_2)) \\
&= (h_1 \oplus k_1) \oplus (h_2 \oplus k_2) \\
&= \phi(h_1,k_1) \oplus \phi(h_2,k_2).
\end{aligned}$$

This proves that ϕ preserves the gyrogroup operations. □

The converse of the above theorem also holds, in the sense of the following theorem.

Theorem 3.2. *Let A and B be gyrogroups and let $G = A \times B$. Then G contains subgyrogroups H and K such that*

(i) $G = H \oplus K$;
(ii) $H \trianglelefteq G$ *and* $K \trianglelefteq G$;
(iii) $H \cap K = \{(e,e)\}$.

Proof. Set $H = \{(a,e) : a \in A\}$ and $K = \{(e,b) : b \in B\}$. By definition, $(a,b) = (a,e) \oplus (e,b)$ for all $a \in A, b \in B$. Hence, $G = H \oplus K$. Let $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ be the projection maps defined by $\pi_1(a,b) = a$ and $\pi_2(a,b) = b$ for all $(a,b) \in A \times B$, respectively. Then π_1 and π_2 are gyrogroup homomorphisms such that $H = \ker \pi_2$ and $K = \ker \pi_1$. This shows that $H \trianglelefteq G$ and $K \trianglelefteq G$. It is clear that $H \cap K = \{(e,e)\}$. □

We give a few concrete examples to illustrate Theorem 3.1 below.

Example 3.1. *Let A be the gyrogroup $\operatorname{Dih}(G_8)$ given in Example 5 of [8], called the dihedralized gyrogroup of G_8. Let B be the gyrogroup Q_{16}^{gyr} given in Example 5.2 of [9]. Then $G = A \times B$ is a finite gyrogroup of order 256. Set $H = \{(a,(0,0)) : a \in A\}$ and $K = \{((0,0),b) : b \in B\}$. As in the proof of Theorem 3.2, H and K are distinct normal subgyrogroups of G such that $H \cap K$ contains precisely the identity of G.*

Example 3.2. *Referring to the Einstein gyrogroup* $(\mathbb{B}, \oplus_E)$ *in Section 3.8 of [14], we know by Theorem 12 of [12] that* $\mathrm{Sym}(\mathbb{B})$ *forms a gyrogroup under the operation given by*

$$\sigma \oplus \tau = L_{\mathbf{u}\oplus_E \mathbf{v}} \circ (\alpha \circ \beta) \tag{3.1}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{B}, \alpha, \beta \in \mathrm{Sym}_{\mathbf{0}}(\mathbb{B}) = \{\rho \in \mathrm{Sym}(\mathbb{B}) : \rho(\mathbf{0}) = \mathbf{0}\}$. *Using* (3.1)*, one can check that* $H = \{L_{\mathbf{u}} : \mathbf{u} \in \mathbb{B}\}$ *and* $K = \mathrm{Sym}_{\mathbf{0}}(\mathbb{B})$ *are distinct normal subgyrogroups of* $\mathrm{Sym}(\mathbb{B})$ *such that* $H \cap K$ *contains precisely the identity of* $\mathrm{Sym}(\mathbb{B})$.

The previous example can be generalized to arbitrary gyrogroups as follows. Recall that *Cayley's theorem* for gyrogroups states that every gyrogroup is isomorphic to a gyrogroup of permutations. Let G be a gyrogroup. Let σ and τ be arbitrary permutations of G. By Theorem 2.2, σ and τ have unique factorizations $\sigma = L_a \circ \alpha$ and $\tau = L_b \circ \beta$, where $a, b \in G$ and $\alpha, \beta \in \mathrm{Sym}_e(G)$. This allows us to introduce a gyrogroup operation on $\mathrm{Sym}(G)$, given by

$$\sigma \oplus \tau = L_{a\oplus b} \circ (\alpha \circ \beta). \tag{3.2}$$

Then $(\mathrm{Sym}(G), \oplus)$ becomes a gyrogroup containing $\widehat{G}$ as an isomorphic copy of G. Moreover, by the direct product theorem, $(\mathrm{Sym}(G), \oplus) \cong \widehat{G} \times \mathrm{Sym}_e(G)$ as gyrogroups. This leads to new insight into the gyrogroup version of Cayley's theorem.

3.2. Cancellation in direct products of finite gyrogroups

To prove the cancellation theorem for direct products of finite gyrogroups, we need preliminary results, which are important in their own right. We remark that this method of proof follows the same steps as in the case of finite groups. Let G and H be gyrogroups. Define $\mathrm{Hom}(G, H)$ to be the set of all homomorphisms from G to H,

$$\mathrm{Hom}(G, H) = \{\varphi : \varphi \text{ is a homomorphism from } G \text{ to } H\}, \tag{3.3}$$

and define $\mathrm{IHom}(G, H)$ to be the set of all injective homomorphisms from G to H,

$$\mathrm{IHom}(G, H) = \{\varphi : \varphi \text{ is an injective homomorphism from } G \text{ to } H\}. \tag{3.4}$$

Proposition 3.1. *Let* $G, H,$ *and* K *be gyrogroups.*

1) *If* $H \cong K$*, then there is a bijection from* $\mathrm{Hom}(G, H)$ *to* $\mathrm{Hom}(G, K)$.
2) *There is a bijection from* $\mathrm{Hom}(G, H) \times \mathrm{Hom}(G, K)$ *to* $\mathrm{Hom}(G, H \times K)$.
3) *There is a bijection from* $\mathrm{IHom}(G, H)$ *to* $\mathrm{IHom}(G/\{e\}, H)$.

Proof. To prove Part 1, suppose that $\Phi : H \to K$ is an isomorphism. Define a map σ by $\sigma(\varphi) = \Phi \circ \varphi$ for all $\varphi \in \mathrm{Hom}(G, H)$. As the composite of gyrogroup homomorphisms is again a gyrogroup homomorphism, $\sigma(\varphi)$ lies in $\mathrm{Hom}(G, K)$. Hence, σ sends $\mathrm{Hom}(G, H)$ to $\mathrm{Hom}(G, K)$. Note that Φ^{-1} is an isomorphism from K to H. We obtain similarly that the map τ defined by $\tau(\psi) = \Phi^{-1} \circ \psi$ for all $\psi \in \mathrm{Hom}(G, K)$ sends $\mathrm{Hom}(G, K)$ to $\mathrm{Hom}(G, H)$. Direct computation shows that $\sigma \circ \tau = \mathrm{I}_{\mathrm{Hom}(G,K)}$ and $\tau \circ \sigma = \mathrm{I}_{\mathrm{Hom}(G,H)}$. Hence, σ is bijective and $\sigma^{-1} = \tau$.

For $\alpha \in \mathrm{Hom}(G, H)$ and $\beta \in \mathrm{Hom}(G, K)$, define a map $\alpha \times \beta$ by

$$\alpha \times \beta(g) = (\alpha(g), \beta(g)) \quad \text{for all } g \in G.$$

It is not difficult to see that $\alpha\times\beta$ is a homomorphism from G to $H\times K$. Define a map σ by $\sigma(\alpha,\beta)=\alpha\times\beta$ for all $\alpha\in \mathrm{Hom}\,(G,H),\beta\in \mathrm{Hom}\,(G,K)$. Then σ maps $\mathrm{Hom}\,(G,H)\times \mathrm{Hom}\,(G,K)$ to $\mathrm{Hom}\,(G,H\times K)$. It is easy to see that σ is injective. Suppose that $\varphi\in \mathrm{Hom}\,(G,H\times K)$. Let $\pi_1 : H\times K\to H$ and $\pi_2 : H\times K\to K$ be the projection maps defined by $\pi_1(h,k)=h$ and $\pi_2(h,k)=k$ for all $(h,k)\in H\times K$, respectively. Then $\pi_1\circ\varphi\in \mathrm{Hom}\,(G,H)$ and $\pi_2\circ\varphi\in \mathrm{Hom}\,(G,K)$. Let $g\in G$ and assume that $\varphi(g)=(h,k)$ with $h\in H, k\in K$. Then $\sigma(\pi_1\circ\varphi,\pi_2\circ\varphi)(g)=(\pi_1(\varphi(g)),\pi_2(\varphi(g)))=(h,k)=\varphi(g)$ and so $\sigma(\pi_1\circ\varphi,\pi_2\circ\varphi)=\varphi$. This proves that σ is surjective, which completes the proof of Part 2. The proof of Part 3 is straightforward. □

Proposition 3.2. *Let G and H be gyrogroups. Then there is a bijection from* $\mathrm{Hom}\,(G,H)$ *to* $\bigcup_{N\trianglelefteq G}\mathrm{IHom}\,(G/N,H)$.

Proof. First, note that if $N\trianglelefteq G$ and $K\trianglelefteq G$, then $G/N=G/K$ if and only if $N=K$. In fact, $G/N=G/K$ implies $e\oplus N$ is the identity of G/K. Hence, $e\oplus N=e\oplus K$ by the uniqueness of the identity and so $N=K$. It follows that $\mathrm{IHom}\,(G/N,H)$ and $\mathrm{IHom}\,(G/K,H)$ have empty intersection whenever $N\neq K$.

Set $\mathcal{U}=\bigcup_{N\trianglelefteq G}\mathrm{IHom}\,(G/N,H)$. For each $\varphi\in \mathrm{Hom}\,(G,H)$, let $\overline{\varphi}$ be the isomorphism from $G/\ker\varphi$ to $\mathrm{im}\,\varphi$ defined as in the proof of the first isomorphism theorem (cf. Theorem 28 of [12]) by the equation $\overline{\varphi}(a\oplus\ker\varphi)=\varphi(a)$ for all $a\in G$. Then $\overline{\varphi}$ is an injective homomorphism from $G/\ker\varphi$ to H; that is, $\overline{\varphi}\in \mathrm{IHom}\,(G/\ker\varphi,H)$. Define a map σ by $\sigma(\varphi)=\overline{\varphi}$ for all $\varphi\in \mathrm{Hom}\,(G,H)$. Then σ maps $\mathrm{Hom}\,(G,H)$ to $\mathcal{U}$. For each normal subgyrogroup N of G, let π_N be the canonical projection from G to G/N. For each $\psi\in\mathcal{U}$, as noted above, there is a unique normal subgyrogroup N of G such that $\psi\in \mathrm{IHom}\,(G/N,H)$ and we can define $\tau(\psi)=\psi\circ\pi_N$. Then τ defines a map from $\mathcal{U}$ to $\mathrm{Hom}\,(G,H)$. We show that σ and τ are inverses of each other. Let $\varphi\in \mathrm{Hom}\,(G,H)$. Note that $(\tau\circ\sigma)(\varphi)=\tau(\overline{\varphi})=\overline{\varphi}\circ\pi_{\ker\varphi}$ and that $(\overline{\varphi}\circ\pi_{\ker\varphi})(a)=\overline{\varphi}(a\oplus\ker\varphi)=\varphi(a)$ for all $a\in G$. Hence, $\overline{\varphi}\circ\pi_{\ker\varphi}=\varphi$. This proves that $\tau\circ\sigma=\mathrm{I}_{\mathrm{Hom}\,(G,H)}$. Let $\psi\in\mathcal{U}$. Then $\psi\in \mathrm{IHom}\,(G/N,H)$, where $N\trianglelefteq G$. Note that $(\sigma\circ\tau)(\psi)=\sigma(\psi\circ\pi_N)=\overline{\psi\circ\pi_N}$. Since $\ker\,(\psi\circ\pi_N)=N$, we obtain by definition that

$$\overline{\psi\circ\pi_N}(a\oplus N)=\overline{\psi\circ\pi_N}(a\oplus\ker\,(\psi\circ\pi_N))=(\psi\circ\pi_N)(a)=\psi(a\oplus N)$$

for all $a\in G$. Hence, $\overline{\psi\circ\pi_N}=\psi$. This proves that $\sigma\circ\tau=\mathrm{I}_{\mathcal{U}}$. Therefore, σ is bijective and $\sigma^{-1}=\tau$. □

Suppose that G and H are finite gyrogroups. Because the union in the previous proposition is disjoint, we derive a counting formula for homomorphisms from G to H in terms of injective homomorphisms from G/N to H, where N runs over all normal subgyrogroups of G.

Corollary 3.1. *If G and H are finite gyrogroups, then*

$$|\mathrm{Hom}\,(G,H)|=\sum_{N\trianglelefteq G}|\mathrm{IHom}\,(G/N,H)|. \tag{3.5}$$

Proof. Since G and H are finite, $\mathrm{Hom}\,(G,H)$ is a finite set. As noted in the proof of Proposition 3.2, if $N\neq K$, then $\mathrm{IHom}\,(G/N,H)\cap \mathrm{IHom}\,(G/K,H)=\emptyset$. Hence, $\mathrm{Hom}\,(G,H)$ is the disjoint union of the sets $\mathrm{IHom}\,(G/N,H)$ as N varies over all normal subgyrogroups of G. Thus, $|\mathrm{Hom}\,(G,H)|=\left|\bigcup_{N\trianglelefteq G}\mathrm{IHom}\,(G/N,H)\right|=\sum_{N\trianglelefteq G}|\mathrm{IHom}\,(G/N,H)|$. □

We are now in a position to prove a left cancellation law for direct products of finite gyrogroups: if G, H, and K are finite gyrogroups, then $G \times H \cong G \times K$ implies $H \cong K$.

Theorem 3.3. *(Left cancellation in direct products) Let G, H, and K be finite gyrogroups. If $G \times H \cong G \times K$, then $H \cong K$.*

Proof. Suppose that $G \times H \cong G \times K$. First, we show that

$$|\mathrm{Hom}\,(L, H)| = |\mathrm{Hom}\,(L, K)|$$

for all finite gyrogroups L. Let L be a finite gyrogroup. Note that $|\mathrm{Hom}\,(L, G)| \neq 0$ because the trivial homomorphism t defined by $t(a) = e$ for all $a \in L$ is an element in $\mathrm{Hom}\,(L, G)$. By Part 1 and Part 2 of Proposition 3.1,

$$\begin{aligned} |\mathrm{Hom}\,(L, G)||\mathrm{Hom}\,(L, H)| &= |\mathrm{Hom}\,(L, G \times H)| \\ &= |\mathrm{Hom}\,(L, G \times K)| \\ &= |\mathrm{Hom}\,(L, G)||\mathrm{Hom}\,(L, K)|, \end{aligned}$$

which implies $|\mathrm{Hom}\,(L, H)| = |\mathrm{Hom}\,(L, K)|$.

Next, we show that $|\mathrm{IHom}\,(L, H)| = |\mathrm{IHom}\,(L, K)|$ for all finite gyrogroups L by induction on $|L|$. The case where $|L| = 1$ is clear. Suppose that $\{e\} \neq N \trianglelefteq L$. Since N is an L-subgyrogroup of G, we obtain that $|L/N| = \dfrac{|L|}{|N|} < |L|$. Hence, by the inductive hypothesis, $|\mathrm{IHom}\,(L/N, H)| = |\mathrm{IHom}\,(L/N, K)|$. By Part 3 of Proposition 3.1 and (3.5),

$$\begin{aligned} |\mathrm{IHom}\,(L, H)| &= |\mathrm{IHom}\,(L/\{e\}, H)| \\ &= |\mathrm{Hom}\,(L, H)| - \sum_{\{e\} \neq N \trianglelefteq L} |\mathrm{IHom}\,(L/N, H)| \\ &= |\mathrm{Hom}\,(L, K)| - \sum_{\{e\} \neq N \trianglelefteq L} |\mathrm{IHom}\,(L/N, K)| \\ &= |\mathrm{IHom}\,(L/\{e\}, K)| \\ &= |\mathrm{IHom}\,(L, K)|, \end{aligned}$$

which completes the induction.

In the particular case when $L = H$, we have $|\mathrm{IHom}\,(H, K)| = |\mathrm{IHom}\,(H, H)| \geq 1$ because the identity homomorphism on H is in $\mathrm{IHom}\,(H, H)$. Hence, there is a homomorphism $\phi \in \mathrm{IHom}\,(H, K)$. Since $|G||H| = |G \times H| = |G \times K| = |G||K|$, it follows that $|H| = |K|$. Since ϕ is injective, $|\phi(H)| = |H| = |K|$. Since $\phi(H) \subseteq K$ and K is finite, $\phi(H) = K$. This proves that ϕ is surjective and so ϕ becomes an isomorphism from H to K. Thus, $H \cong K$. □

Corollary 3.2. *(Right cancellation in direct products) Let G, H, and K be finite gyrogroups. If $H \times G \cong K \times G$, then $H \cong K$.*

Proof. This follows directly from Theorem 3.3 and Proposition 2.3. □

We may weaken the assumptions of Theorem 3.3 and Corollary 3.2, as shown in the following theorem.

Theorem 3.4. *(Cancellation in direct products) Let A, B, H, and K be finite gyrogroups such that $A \cong B$. If $A \times H \cong B \times K$ or $H \times A \cong K \times B$, then $H \cong K$.*

Proof. Suppose that $A \times H \cong B \times K$. Since $A \cong B$, it follows by Part 3 of Proposition 2.3 that $A \times H \cong B \times H$. By Part 2 of Proposition 2.3, $B \times H \cong B \times K$. By Theorem 3.3, $H \cong K$. The case where $H \times A \cong K \times B$ is proved similarly. □

We emphasize that the finiteness of gyrogroups in Theorem 3.3 is crucial. In fact, there are infinite gyrogroups (indeed, infinite groups) G, H, and K such that $G \times H \cong G \times K$ but H and K are not isomorphic. For example, let $G = \prod_{i=1}^{\infty} \mathbb{Z}$, let $H = \mathbb{Z}$, and let $K = \mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z}$ is the additive group of integers. Then $G \times H \cong G \times K$ but H and K are not isomorphic.

4. Conclusions

We prove a few theorems that involve direct products of gyrogroups, especially the direct product theorem for gyrogroups and some cancellation laws for direct products of finite gyrogroups. These results extend two well-known results in abstract algebra.

Acknowledgments

This work was supported by the Research Group in Mathematics and Applied Mathematics, Chiang Mai University.

Conflict of interest

The author declares no conflict of interest.

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