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Research article

A robust study of the transmission dynamics of syphilis infection through non-integer derivative

Rashid Jan[1], Adil Khurshaid[1], Hammad Alotaibi[2] and Mustafa Inc[3,4,*]

[1] Department of Mathematics, University of Swabi, Swabi 23561, KPK Pakistan

[2] Department of Mathematics and Statistics, Faculty of Science, Taif University, P.O. Box 1109, Taif 21944, Saudi Arabia

[3] Department of Mathematics, Firat University, Elazig 23119, Turkiye

[4] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

* **Correspondence:** Email: minc@firat.edu.tr.

Abstract: One of the most harmful and widespread sexually transmitted diseases is syphilis. This infection is caused by the Treponema Palladum bacterium that spreads through sexual intercourse and is projected to affect 12 million people annually worldwide. In order to thoroughly examine the complex and all-encompassing dynamics of syphilis infection. In this article, we constructed the dynamics of syphilis using the fractional derivative of the Atangana-Baleanu for more accurate outcomes. The basic theory of non-integer derivative is illustrated for the examination of the recommended model. We determined the steady-states of the system and calculated the $\mathcal{R}_0$ for the intended fractional model with the help of the next-generation method. The infection-free steady-state of the system is locally stable if $\mathcal{R}_0 < 1$ through jacobian matrix method. The existence and uniqueness of the fractional order system are investigate by applying the fixed-point theory. The iterative solution of our model with fractional order was then carried out by utilising a newly generated numerical approach. Finally, numerical results are computed for various values of the factor Φ and other parameters of the system. The solution pathways and chaotic phenomena of the system are highlighted. Our findings show that fractional order derivatives provide more precise and realistic information regarding the dynamics of syphilis infection.

Keywords: syphilis dynamics; Atangana-Baleanu; stability analysis; fixed-point theorem; numerical method; dynamical behavior
Mathematics Subject Classification: 4C05, 92D25

1. Introduction

Treponema pallidum is a spirochete that has a morphology for different body organs and tissues, causes the infectious illness named as syphilis infection, which has complicated clinical symptoms [1–3]. Syphilis is one of the most contagious illnesses that is frequently spread through sexual contact [4]. There are two basic ideas that are most popular, however the exact cause of syphilis is uncertain. One hypothesis argues that the illness originated in America and was brought to Europe after Columbus exploration in 1492. One more contends that syphilis was widespread in central Africa before Columbus expedition and spread to Europe [5]. In a secluded Guyanan community, researchers have uncovered a DNA-related strain of *T. pallidum*, that causes an illness that resembles both syphilis and yaws. This creature may have served as the ancestor from whom *T. pallidum* originated many millions of years ago, according to certain theories [6]. Extragenital inoculations are conceivable but are less prevalent because most syphilis transmission occurs when infected individuals share sex toys or engage in sexual activity without using condoms [7]. The Syphilis illness has three major stages: the primary, secondary, and latent. The primary stage symptom may manifest as a singular, painless chancre at the infection area. Without treatment, this illness advances to the secondary stage with indications resembling lymphnodes, skin rashes, and muscous membrane lesions [8]. The signs of the infection are more visible at this stage. The disease progresses to a latent phase if untreated along with secondary stage symptoms and has the potential to be fatal or harmful to internal organs [8]. It has been challenging to eradicate this disease entirely regardless the discovery of penicillin in the middle of the 20^{th} century. It continues to be a significant concern among human illnesses, and its frequency is rising quickly in many regions of the globe [9–11].

In order to fully comprehend the dynamics of infectious diseases, epidemiological models are crucial to create efficient control measures [12, 13]. To be more precise, study of these models forecasts important variables that are crucial to the transmission and treatment of the infection [14, 15]. An equation to forecast a disease was one of the early successes of mathematical epidemiology. In order to determine variables for diverse contagious illnesses and utilise those variables to examine the impacts of potential therapies, mathematical models make use of several fundamental mathematical assumptions and principles [16–18]. These models have been constructed in large numbers to analyze the mechanics of the propagation of syphilis infection. In [19, 20], the authors structured the propagation of the infection with the effect of treatment. Various categories were created throughout the population based on factors including age, sexual intercourse, and gender [21], but the latter stages of syphilis were integrated. Milner and Zhao [22], offered an ODE model based on partial immunization and vaccination (presuming a 30 effective vaccine is made) in more recent study, and they demonstrated that there exists backward bifurcation for specific input variables. Mostly of the researchers illustrated the dynamics of this infection through integer derivative, therefore non-integer framework is a best option to demonstrate the dynamics of syphilis infection.

Fractional-calculus has been quite popular in recent years due to its wide range of applications in different research areas [23–25]. It has the capability to consider the memory impact [26, 27], which occurs frequently in biological models. In recent years, a novel FO derivative has been developed in [28, 29]. These novel concepts have been successfully applied to simulating real-world issues in

several disciplines, such as physics, engineering, biology and several other areas [30–32]. A single kernel fractional derivative was initially proposed by Riemann and Louiville. In [28], Caputo and Fabrizio offered a fresh definition of fractional derivative without single kernel that many scholars found to be accurate and useful. A Few years ago, Atangana and Baleanu developed a fractional operator in the Caputo sense based on the generalised Mittag-Leffler function with non-singular and non-local kernel named as ABC derivative [33]. In the sense of CF and ABC, numerous non-integer order models have been constructed [34–37]. To be more specific, the dynamics of biological processes are more accurately explored through non-integer derivative . The authors in [37, 38], numerically illustrated the fractional dynamics of HIV infection while the numerical examination of bovine babesiosis infection has been presented in [39]. These findings made fractional-calculus more attractive for the researchers and scientists. Therefore, the dynamics of syphilis infection is structured in fractional framework for more precise findings. The main objective of this work is to formulate the intricate transmission of syphilis infection between men and women with primary and secondary infections to obtain more realistic results. The effect of the antibiotic on the transmission route of syphilis infection will also be a part of this study.

A brief introduction to the infectious syphilis model is described in Section 1, along with basic definitions and terminologies related to the FO derivative are involved. The formation of syphilis model is presented in Section 2. In Section 3, we examine the model of infectious syphilis while also looking at equilibrium points and the basic reproduction number $\mathcal{R}_0$. In Section 4, the fixed-point theorem is utilized to demonstrate the existence and uniqueness of the solution to the specified FO derivative model of Syphilis. In Section 5, we presented a numerical scheme for the solution of our model while the model syphilis infection is illustrated numerically in Section 6. We finish up by presenting the results and conclusions of our study analysis in Section 7.

Preliminaries

Here, we present a couple of mathematical theorems and definition of Caputo [40] and Atangana-Baleanu fractional derivatives [33], which will be utilized in the upcoming sections.

Definition 1.1. *Let $g : [p, q] \to \mathcal{R}$ be a given function, then the derivative of Caputo for g stated in [40] is given by*

$$ {}^{C}_{p}D^{\phi}_{a}(g(a)) = \frac{1}{\Gamma(\mathrm{m}-\phi)} \int_{p}^{a} \mathrm{g}^{\mathrm{m}}(\varepsilon)(a-\varepsilon)^{\mathrm{m}-\phi-1} d\varepsilon, $$

for $\phi \in (\mathrm{m}-1, \mathrm{m})$, where $\mathrm{m} \in \mathbf{Z}$.

Definition 1.2. *Let g be a given function such that $g \in H^1(p, q)$, $q > p$, and $\Phi \in [0, 1]$, the ABC is then described as follows*

$$ {}^{ABC}_{p}D^{\Phi}_{a}g(a) = \frac{B(\Phi)}{1-\Phi} \int_{p}^{a} g'(\varepsilon) E_{\Phi}\Big[-\Phi\frac{(a-\varepsilon)^{\Phi}}{1-\Phi}\Big] d\varepsilon. $$

Definition 1.3. *The ${}^{ABC}_{p}I^{\Phi}_{a}g(a)$ denotes the FO integral associated with the AB derivative and is defined as*

$$ {}^{ABC}_{p}I^{\Phi}_{a}g(a) = \frac{1-\Phi}{B(\Phi)}g(a) + \frac{\Phi}{B(\Phi)\Gamma(\Phi)} \int_{p}^{a} g(\varepsilon)(a-\varepsilon)^{\Phi-1} d\varepsilon. $$

It is clear that the original function can be achieved when Φ approaches zero.

Theorem 1.1. *If g is a function such that $g \in C[p, q]$, then the following conclusion is acceptable [33]:*

$$\|{}_{p}^{ABC}D_{a}^{\Phi}(g(a))\| < \frac{B(\Phi)}{1-\Phi}\|g(a)\|, \text{ where } \|g(a)\| = max_{p\le a\le q}|g(a)|.$$

Moreover, the Lipschitz condition fulfilled by ABC derivative [33] is as

$$\|{}_{p}^{ABC}D_{a}^{\Phi}g_1(a) - {}_{p}^{ABC}D_{a}^{\Phi}g_2(a)\| < \Omega_1\|g_1(a) - g_2(a)\|.$$

Theorem 1.2. *[33]. The FO differential equation of the following form has one and only one solution*

$${}_{p}^{ABC}D_{a}^{\Phi}g(a) = \mathbb{X}(a),$$

given by

$$g(a) = \frac{1-\Phi}{B(\Phi)}\mathbb{X}(a) + \frac{\Phi}{B(\Phi)\Gamma(\Phi)}\int_{p}^{a}\mathbb{X}(\varepsilon)(a-\varepsilon)^{\Phi-1}d\varepsilon.$$

2. Evaluation of fractional dynamics

In formulation of the model, the humans strength $N(t)$ is categorized into the following inclusive compartments: susceptible men $S_m(t)$, susceptible women $S_w(t)$, men with primary infection $I_{mp}(t)$, women with primary infection $I_{wp}(t)$, men with secondary infection $I_{ms}(t)$, women with secondary infection $I_{ws}(t)$, men with latent infection $L_m(t)$, women with latent infection $L_w(t)$, recovered men $R_m(t)$, and recovered women $R_w(t)$. Then, we have

$$N(t) = S_m(t) + S_w(t) + I_{mp}(t) + I_{wp}(t) + I_{ms}(t) + I_{ws}(t) + L_m(t) + L_w(t) + R_m(t) + R_w(t).$$

The recruitment rate to S_m is indicated by π_m at time t. The population has grown due to the rate φ_m of males who survived syphilis illness after losing their natural resistance. The number of vulnerable men decreases due to the growth of newly infected men with syphilis who proceed to the men with primary stage by $\alpha_w\psi\frac{I_{wp}+I_{ws}+L_w}{N}S_m$, where ψ denote the average number of sexual partners per hour and α_w stands for the possibility that syphilis illness will be transmitted by a woman. The number of vulnerable males decreases at a proportion of μ due to the natural causes of mortality. Thus the equation seems to be

$$\frac{dS_m}{dt} = \pi_m + \varphi_m R_m - \alpha_w\psi\left(\frac{I_{wp} + I_{ws} + L_w}{N}\right)S_m - \mu S_m.$$

The recruitment rate to S_w is indicated by π_w at time t. Syphilis illness recovery and subsequent immune system degradation also lead to an increase in vulnerable women at rate φ_w. The population will be diminished by becoming affected with syphilis illness at the amount $\alpha_m\psi\left(\frac{I_{mp}+I_{ms}+L_m}{N}\right)S_w$ and migrating to women who are already afflicted with the disease, where α_m is the likelihood that these males will transmit the disease to women. Deaths brought on by natural causes lower the population at a rate of μ. Therefore

$$\frac{dS_w}{dt} = \pi_w + \varphi_w R_w - \alpha_m\psi\left(\frac{I_{mp} + I_{ms} + L_m}{N}\right)S_w - \mu S_w.$$

Because of the spread of newly infectious syphilis people from the susceptible man at the amount $\alpha_w\psi\left(\frac{I_{wp}+I_{ws}+L_w}{N}\right)S_m$, the number of men with primary stage $I_{mp}(t)$ grows and is lowered as a result of a

rate γ_m migration of males with later stage syphilis I_{ms}. Natural mortality is indicated by μ and the use of antibiotics in therapy at rate ξ_{m_1} substantially reduces this population, resulting in

$$\frac{dI_{mp}}{dt} = \alpha_w \psi \left(\frac{I_{wp} + I_{ws} + L_w}{N} \right) S_m - \gamma_m I_{mp} - \mu I_{mp} - \xi_{m_1} I_{mp}.$$

The incidence rate from susceptible class of women is $\alpha_m \psi \left(\frac{I_{mp}+I_{ms}+L_m}{N} \right) S_w$ which moves to $I_{wp}(t)$ and is decreased when syphilis progresses to the secondary stage in infected women (I_{ws}) at rate γ_w. The population is declining because of natural mortality at rate μ and the use of antibiotic at the rate ρ_{w_1}. Thus

$$\frac{dI_{wp}}{dt} = \alpha_m \psi \left(\frac{I_{mp} + I_{ms} + L_m}{N} \right) S_w - \gamma_w I_{wp} - \mu I_{wp} - \rho_{w_1} I_{wp}.$$

The transition of men from the class (I_{mp}) to the class (I_{ms}) occurs at rate γ_m causes a rise in the amount of infected men $I_{ms}(t)$. Males with secondary stage of the infection (I_{ms}) proceed at a rate β_m to men with latent stage syphilis (L_m), which results in a reduction in the population. Natural death rate is indicated by μ and ξ_{m_2} is the rate of antibiotics which move from the current class to the next class. The equation is given by

$$\frac{dI_{ms}}{dt} = \gamma_m I_{mp} - \beta_m I_{ms} - \xi_{m_2} I_{ms} - \mu I_{ms}.$$

The transition of women from the class I_{wp} to the class (I_{ws}) occurs at the rate γ_w causes a rise in the amount of infected women $I_{ws}(t)$. Women with secondary stage syphilis (I_{ws}) proceed at a rate β_w to women with latent stage syphilis (L_w), which results in a reduction in the population. By dying from a natural cause at the rate μ and through the use of antibiotics at rate ρ_{w_2}, the population is further diminished and the equation becomes

$$\frac{dI_{ws}}{dt} = \gamma_w I_{wp} - \beta_w I_{ws} - \rho_{w_2} I_{ws} - \mu I_{ws}.$$

The population of men transfer from I_{ms} at rate β_m to the class (L_m) while the reduction of natural death at rate μ decreases them. The addition of therapy (antibiotics) at the rate ξ_{m_3} progressively reduces this population, resulting in the equation being as follows

$$\frac{dL_m}{dt} = \beta_m I_{ms} - \mu L_m - \xi_{m_3} L_m.$$

The population women moves from the class (I_{ws}) at rate β_w to the class (L_w) while the reduction of natural death at rate μ decreases them. The addition of therapy (antibiotics) at the rate ρ_{w_3} progressively reduces this population, resulting in the equation being as follows

$$\frac{dL_w}{dt} = \beta_w I_{ws} - \mu L_w - \rho_{w_3} L_w.$$

The advancement of cured men from primary, secondary, and latent phases of syphilis (I_{mp}, I_{ms}, L_m) respectively, at rates ξ_{m_1}, ξ_{m_2}, and ξ_{m_3}, which are the recovery rates of syphilis illness in the men community, results in an increase in $R_m(t)$. This population is diminished by natural mortality at a rate of μ and by loss of immunity brought on by medication and migration of such persons to the vulnerable man population at a rate of φ_m, resulting in

$$\frac{dR_m}{dt} = \xi_{m_1} I_{mp} + \xi_{m_2} I_{ms} + \xi_{m_3} L_m - \mu R_m - \varphi_m R_m.$$

The advancement of cured women from primary, secondary, and latent phases of syphilis (I_{wp}, I_{ws}, L_w) respectively, at rates ρ_{w_1}, ρ_{w_2} , and ρ_{w_3}, which are the recovery rates of syphilis illness in the women community, results in an increase in $R_w(t)$. This population is diminished by natural mortality at a rate of μ and by loss of immunity brought on by medication and migration of such people to the vulnerable women population at a rate of φ_w, resulting in

$$\frac{dR_w}{dt} = \rho_{w_1} I_{wp} + \rho_{w_2} I_{ws} + \rho_{w_3} L_w - \mu R_w - \varphi_w R_w.$$

Since there are three stages of infection, so the syphilis model it described in detail as follows:

$$\begin{cases} \frac{dS_m}{dt} = \pi_{\mathrm{m}} + \varphi_m R_m - \alpha_w \psi(\frac{I_{wp}+I_{ws}+L_w}{N}) S_m - \mu S_m, \\ \frac{dI_{mp}}{dt} = \alpha_w \psi(\frac{I_{wp}+I_{ws}+L_w}{N}) S_m - \gamma_m I_{mp} - \mu I_{mp} - \xi_{m_1} I_{mp}, \\ \frac{dI_{ms}}{dt} = \gamma_m I_{mp} - \beta_m I_{ms} - \xi_{m_2} I_{ms} - \mu I_{ms}, \\ \frac{dL_m}{dt} = \beta_m I_{ms} - \mu L_m - \xi_{m_3} L_m, \\ \frac{dR_m}{dt} = \xi_{m_1} I_{mp} + \xi_{m_2} I_{ms} + \xi_{m_3} L_m - \mu R_m - \varphi_m R_m, \\ \frac{dS_w}{dt} = \pi_w + \varphi_w R_w - \alpha_m \psi(\frac{I_{mp}+I_{ms}+L_m}{N}) S_w - \mu S_w, \\ \frac{dI_{wp}}{dt} = \alpha_m \psi(\frac{I_{mp}+I_{ms}+L_m}{N}) S_f - \gamma_w I_{wp} - \mu I_{wp} - \rho_{w_1} I_{wp}, \\ \frac{dI_{ws}}{dt} = \gamma_w I_{wp} - \beta_w I_{ws} - \rho_{w_2} I_{ws} - \mu I_{ws}, \\ \frac{dL_w}{dt} = \beta_w I_{ws} - \mu L_w - \rho_{w_3} L_w, \\ \frac{dR_w}{dt} = \rho_{w_1} I_{wp} + \rho_{w_2} I_{ws} + \rho_{w_3} L_w - \mu R_w - \varphi_w R_w, \end{cases} \tag{2.1}$$

subject to the initial conditions:

$$\begin{cases} S_m(0) = S_{m0}(0) \geq 0, & I_{mp}(0) = I_{mp0} \geq 0, \\ I_{ms}(0) = I_{ms0}(0) \geq 0, & L_m(0) = L_{m0}(0) \geq 0, \\ S_w(0) = S_{w0}(0) \geq 0, & I_{wp}(0) = I_{wp0} \geq 0, \\ I_{ws}(0) = I_{ws0}(0) \geq 0, & L_w(0) = L_{w0}(0) \geq 0. \end{cases} \tag{2.2}$$

To obtain the dynamical behaviors of the overall population of system (2.1), the related subcategories are combined, which results in

$$\frac{dN}{dt} = \pi_{\mathrm{m}} + \pi_w - \mu N.$$

It is well-known that the results of fractional-calculus are more precise and accurate. Fractional systems possess hereditary property and can capture the nonlocal behavior of biological systems. Therefore, we represent our system through ABC derivative as

$$\begin{cases} {}^{ABC}_{0}D^{\Phi}_{t} S_m = \pi_{\mathrm{m}} + \varphi_m R_m - \alpha_w \psi(\frac{I_{wp}+I_{ws}+L_w}{N}) S_m - \mu S_m, \\ {}^{ABC}_{0}D^{\Phi}_{t} I_{mp} = \alpha_w \psi(\frac{I_{wp}+I_{ws}+L_w}{N}) S_m - \gamma_m I_{mp} - \mu I_{mp} - \xi_{m_1} I_{mp}, \\ {}^{ABC}_{0}D^{\Phi}_{t} I_{ms} = \gamma_m I_{mp} - \beta_m I_{ms} - \xi_{m_2} I_{ms} - \mu I_{ms}, \\ {}^{ABC}_{0}D^{\Phi}_{t} L_m = \beta_m I_{ms} - \mu L_m - \xi_{m_3} L_m, \\ {}^{ABC}_{0}D^{\Phi}_{t} R_m = \xi_{m_1} I_{mp} + \xi_{m_2} I_{ms} + \xi_{m_3} L_m - \mu R_m - \varphi_m R_m, \\ {}^{ABC}_{0}D^{\Phi}_{t} S_w = \pi_w + \varphi_w R_w - \alpha_m \psi(\frac{I_{mp}+I_{ms}+L_m}{N}) S_w - \mu S_w, \\ {}^{ABC}_{0}D^{\Phi}_{t} I_{wp} = \alpha_m \psi(\frac{I_{mp}+I_{ms}+L_m}{N}) S_w - \gamma_w I_{wp} - \mu I_{wp} - \rho_{w_1} I_{wp}, \\ {}^{ABC}_{0}D^{\Phi}_{t} I_{ws} = \gamma_w I_{wp} - \beta_w I_{ws} - \rho_{w_2} I_{ws} - \mu I_{ws}, \\ {}^{ABC}_{0}D^{\Phi}_{t} L_w = \beta_w I_{ws} - \mu L_w - \rho_{w_3} L_w, \\ {}^{ABC}_{0}D^{\Phi}_{t} R_w = \rho_{w_1} I_{wp} + \rho_{w_2} I_{ws} + \rho_{w_3} L_w - \mu R_w - \varphi_w R_w. \end{cases} \tag{2.3}$$

The fractional derivative ${}_0^{ABC}D_t^{\Phi}$ utilised in the recommended system (2.3) has often been referred to as Atangana-Baleanu derivative in the Caputo interpretation.

3. Investigation of the dynamics

In this section of the paper, we will analyze the recommended fractional model of the infection. The stead-states, basic reproduction number, stability of steady-states will be investigated.

3.1. Steady-states

To determine the steady-states of (2.3), we set all the fractional derivatives to zero as

$$\begin{cases} 0 = \pi_{\mathrm{m}} + \varphi_m R_m - \alpha_w \psi(\frac{I_{wp}+I_{ws}+L_w}{N}) S_m - \mu S_m, \\ 0 = \alpha_w \psi(\frac{I_{wp}+I_{ws}+L_w}{N}) S_m - \gamma_m I_{mp} - \mu I_{mp} - \xi_{m_1} I_{mp}, \\ 0 = \gamma_m I_{mp} - \beta_m I_{ms} - \xi_{m_2} I_{ms} - \mu I_{ms}, \\ 0 = \beta_m I_{ms} - \mu L_m - \xi_{m_3} L_m, \\ 0 = \xi_{m_1} I_{mp} + \xi_{m_2} I_{ms} + \xi_{m_3} L_m - \mu R_m - \varphi_m R_m, \\ 0 = \pi_w + \varphi_w R_w - \alpha_m \psi(\frac{I_{mp}+I_{ms}+L_m}{N}) S_w - \mu S_w, \\ 0 = \alpha_m \psi(\frac{I_{mp}+I_{ms}+L_m}{N}) S_w - \gamma_w I_{wp} - \mu I_{wp} - \rho_{w_1} I_{wp}, \\ 0 = \gamma_w I_{wp} - \beta_w I_{ws} - \rho_{w_2} I_{ws} - \mu I_{ws}, \\ 0 = \beta_w I_{ws} - \mu L_w - \rho_{w_3} L_w, \\ 0 = \rho_{w_1} I_{wp} + \rho_{w_2} I_{ws} + \rho_{w_3} L_w - \mu R_w - \varphi_w R_w. \end{cases} \tag{3.1}$$

The meaningful steady-states of the system (2.3) are infection-free and endemic steady-state. We calculated and assessed the Jacobian of system (2.3) at infection-free steady-state for the stability analysis of the system. The indicators of the Jacobian's eigenvalues are used to evaluate the local stability of E_0. For infection-free steady-state, we take the first equation of system (3.1) without infection as

$$0 = \pi_{\mathrm{m}} + \varphi_m 0 - \alpha_w \psi(0) S_m - \mu S_m,$$

which implies that $S_m^0 = \frac{\pi_{\mathrm{m}}}{\mu}$. Similarly, from the sixth equation of (3.1), we have

$$0 = \pi_w + \varphi_w 0 - \alpha_m \psi(0) S_w - \mu S_w,$$

which implies that $S_w^0 = \frac{\pi_{\mathrm{w}}}{\mu}$. Thus, the infection-free steady-state is

$$\left(S_m^0, I_{mp}^0, I_{ms}^0, L_m^0, R_m^0, S_w^0, I_{wp}^0, I_{ws}^0, L_w^0, R_w^0\right) = \left(\frac{\pi_{\mathrm{m}}}{\mu}, 0, 0, 0, 0, \frac{\pi_{\mathrm{w}}}{\mu}, 0, 0, 0, 0\right). \tag{3.2}$$

Let us assume that endemic steady-state is indicated by

$$E^{**} = \left(S_m^{**}, I_{mp}^{**}, I_{ms}^{**}, L_m^{**}, R_m^{**}, S_w^{**}, I_{wp}^{**}, I_{ws}^{**}, L_w^{**}, R_w^{**}\right),$$

of the recommended fractional system of the infection. Let

$$\lambda_m^{**} = \frac{\alpha_m \psi(I_{mp}^{**} + I_{ms}^{**} + L_m^{**})}{N^{**}}, \ \lambda_w^{**} = \frac{\alpha_w \psi(I_{wp}^{**} + I_{ws}^{**} + L_w^{**})}{N^{**}},$$

indicate the force of infection, and let

$$N^{**} = S_m^{**} + I_{mp}^{**} + I_{ms}^{**} + L_m^{**} + R_m^{**} + S_w^{**} + I_{wp}^{**} + I_{ws}^{**} + L_w^{**} + R_w^{**}.$$

Then the above system (3.1) become as

$$\begin{cases} 0 = \pi_{\mathrm{m}} + \varphi_m R_m^{**} - \alpha_w \psi \lambda_m^{**} S_m^{**} - \mu S_m^{**}, \\ 0 = \alpha_w \psi \lambda_m^{**} S_m^{**} - \gamma_m I_{mp}^{**} - \mu I_{mp}^{**} - \xi_{m_1} I_{mp}^{**}, \\ 0 = \gamma_m I_{mp}^{**} - \beta_m I_{ms}^{**} - \xi_{m_2} I_{ms}^{**} - \mu I_{ms}^{**}, \\ 0 = \beta_m I_{ms}^{**} - \mu L_m^{**} - \xi_{m_3} L_m^{**}, \\ 0 = \xi_{m_1} I_{mp}^{**} + \xi_{m_2} I_{ms}^{**} + \xi_{m_3} L_m^{**} - \mu R_m^{**} - \varphi_m R_m^{**}, \\ 0 = \pi_w + \varphi_w R_w^{**} - \alpha_m \psi \lambda_w^{**} S_w^{**} - \mu S_w^{**}, \\ 0 = \alpha_m \psi \lambda_w^{**} S_w - \gamma_w I_{wp}^{**} - \mu I_{wp}^{**} - \rho_{w_1} I_{wp}^{**}, \\ 0 = \gamma_w I_{wp}^{**} - \beta_w I_{ws}^{**} - \rho_{w_2} I_{ws}^{**} - \mu I_{ws}^{**}, \\ 0 = \beta_w I_{ws}^{**} - \mu L_w^{**} - \rho_{w_3} L_w^{**}, \\ 0 = \rho_{w_1} I_{wp}^{**} + \rho_{w_2} I_{ws}^{**} + \rho_{w_3} L_w^{**} - \mu R_w^{**} - \varphi_w R_w^{**}. \end{cases} \tag{3.3}$$

Solving the above system (3.3) through mathematical skills, we have

$$\begin{cases} S_m^{**} = \frac{b_1 b_2 I_{ms}^*}{\gamma_m (\lambda_w^{**} + \mu)}, \\ I_{mp}^{**} = \frac{b_2 I_{ms}^*}{\gamma_m}, \\ I_{ms}^{**} = \frac{\pi_{\mathrm{m}} \gamma_m b_3 b_4}{b_1 b_2 b_3 b_4 - (\xi_{m_3} \beta_m \gamma_m \varphi_m + k_3 \xi_{m_2} \gamma_m \psi_m + \varphi_m \xi_{m_1} b_2 b_3)}, \\ L_m^{**} = \frac{\beta_m I_{ms}^*}{b_3}, \\ R_m^{**} = \frac{b_1 b_2 I_{ms}^{**} - \gamma_m \pi_m}{\gamma_m \varphi_m}, \\ S_w^{**} = \frac{h_1 h_2 I_{ws}^{**}}{\gamma_w (\lambda_m^* + \mu)}, \\ I_{wp}^{**} = \frac{h_2 I_{ws}^*}{\gamma_w}, \\ I_{ws}^{**} = \frac{\pi_w \gamma_w h_3 h_4}{h_1 h_2 h_3 h_4 - (\rho_{w_3} \beta_w \gamma_w \varphi_w + h_3 \rho_{w_2} \gamma_w \psi_w + \varphi_w \rho_{w_1} h_2 h_3)}, \\ L_w^{**} = \frac{\beta_w I_{ws}^{**}}{h_3}, \\ R_w^{**} = \frac{(\gamma_w + \mu + \rho_{w_1})(\beta_w + \rho_{w_2} + \mu) I_{ws}^{**} - \gamma_w \pi_w}{\gamma_w \varphi_2}, \end{cases} \tag{3.4}$$

where $b_1 = \gamma_m + \mu + \xi_{m_1}$, $b_2 = \beta_m + \mu + \xi_{m_2}$, $b_3 = \mu + \xi_{m_3}$, $b_4 = \mu + \varphi_m$, and $h_1 = \gamma_w + \mu + \rho_{w_1}$, $h_2 = \beta_w + \rho_{w_2} + \mu$, $h_3 = \mu + \rho_{w_3}$, $h_4 = \mu + \varphi_w$.

3.2. Reproduction parameter

The basic reproduction number $\mathcal{R}_0$ is the average number of secondary illnesses brought on by one infected individual when the entire community is vulnerable. $\mathcal{R}_0 = \rho(FV^{-1})$ is used to represent the epidemiological threshold for syphilis illness, where ρ is the predominate eigenvalue. We also utilised the methods in [41] to obtain the basic reproduction number for system (2.3), we have

$$F = \begin{pmatrix} \alpha_w \psi \left(\frac{I_{wp} + I_{w_s} + L_w}{N} \right) S_m \\ 0 \\ 0 \\ \alpha_m \psi \left(\frac{I_{mp} + I_{ms} + L_m}{N} \right) S_w \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} (\gamma_m + \mu + \xi_{m_1}) I_{mp} \\ -\gamma_m I_{mp} + (\beta_m + \mu + \xi_{m_2}) I_{ms} \\ -\beta_m I_{ms} + (\mu + \xi_{m_3}) L_m \\ (\gamma_w + \mu + \rho_{w_1}) I_{wp} \\ -\gamma_w I_{wp} + (\beta_w + \mu + \rho_{w_2}) I_{ws} \\ -\beta_w I_{ws} + (\mu + \rho_{w_3}) L_w \end{pmatrix}.$$

Here, the transition terms are present in the matrices F and V of the recommended model (2.3). At syphilis-free equilibrium, the jacobian matrices of F and V are evaluated, we obtain the following

$$F = \begin{pmatrix} 0 & 0 & 0 & \frac{\alpha_w \psi \pi_{\mathrm{m}}}{\pi_{\mathrm{m}}+\pi_w} & \frac{\alpha_w \psi \pi_{\mathrm{m}}}{\pi_{\mathrm{m}}+\pi_w} & \frac{\alpha_w \psi \pi_{\mathrm{m}}}{\pi_{\mathrm{m}}+\pi_w} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\alpha_m \psi \pi_w}{\pi_m+\pi_w} & \frac{\alpha_m \psi \pi_w}{\pi_m+\pi_w} & \frac{\alpha_m \psi \pi_w}{\pi_m+\pi_w} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$V = \begin{pmatrix} \gamma_m + \mu + \xi_{m_1} & 0 & 0 & 0 & 0 & 0 \\ -\gamma_m & \beta_m + \mu + \xi_{m_2} & 0 & 0 & 0 & 0 \\ 0 & -\beta_m & \mu + \xi_{m_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_w + \mu + \rho_{w_1} & 0 & 0 \\ 0 & 0 & 0 & -\gamma_w & \beta_w + \mu + \rho_{w_2} & 0 \\ 0 & 0 & 0 & 0 & -\beta_w & \mu + \rho_{w_3} \end{pmatrix}.$$

Therefore, model (2.3) has the basic reproduction number $\mathcal{R}_0$, given by

$$\mathcal{R}_0 = \sqrt{\frac{\psi^2 \alpha_w \alpha_m \pi_{\mathrm{m}} \pi_w \left(\beta_m \gamma_m + \gamma_m q_2 + q_2 q_3\right)\left(\beta_w \gamma_w + \gamma_w q_6 + q_5 q_6\right)}{\left(\pi_{\mathrm{m}} + \pi_w\right)^2 q_1 q_2 q_3 q_4 q_5 q_6}}, \tag{3.5}$$

where $q_1 = (\gamma_m + \mu + \xi_{m_1})$, $q_2 = (\beta_m + \mu + \xi_{m_2})$, $q_3 = (\mu + \xi_{m_3})$, $q_4 = (\gamma_w + \mu + \rho_{w_1})$, $q_5 = (\beta_w + \mu + \rho_{w_2})$, and $q_6 = (\mu + \rho_{w_3})$. The reproduction parameter of a system is a significant value which predict about the status of the infection.

3.3. Stability analysis

Theorem 3.1. *If $\mathcal{R}_0 < 1$, the syphilis-free equilibrium of system (2.3) is locally asymptotically stable otherwise it is unstable.*

Proof. At syphilis-free equilibrium, the Jacobian matrix of system (2.3) is as follows

$$\mathcal{J}\left(E^0\right) = \begin{pmatrix} -g_1 & 0 & 0 & 0 & \varphi_1 & 0 & -\frac{\alpha_w \psi \pi_{\mathrm{m}}}{\pi_{\mathrm{m}}+\pi_w} & -\frac{\alpha_w \psi \pi_{\mathrm{m}}}{\pi_{\mathrm{m}}+\pi_w} & -\frac{\alpha_w \psi \pi_{\mathrm{m}}}{\pi_{\mathrm{m}}+\pi_w} & 0 \\ 0 & -g_2 & 0 & 0 & 0 & 0 & \frac{\alpha_w \psi \pi_{\mathrm{m}}}{\pi_{\mathrm{m}}+\pi_w} & \frac{\alpha_w \psi \pi_{\mathrm{m}}}{\pi_{\mathrm{m}}+\pi_w} & \frac{\alpha_w \psi \pi_{\mathrm{m}}}{\pi_{\mathrm{m}}+\pi_w} & 0 \\ 0 & \gamma_m & -g_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_m & -g_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_{m_1} & \xi_{m_2} & \xi_{m_3} & -g_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\alpha_w \psi \pi_w}{\pi_{\mathrm{m}}+\pi_w} & -\frac{\alpha_w \psi \pi_w}{\pi_{\mathrm{m}}+\pi_w} & -\frac{\alpha_w \psi \pi_w}{\pi_{\mathrm{m}}+\pi_w} & 0 & -g_6 & 0 & 0 & 0 & \varphi_w \\ 0 & \frac{\alpha_m \psi \pi_w}{\pi_{\mathrm{m}}+\pi_w} & \frac{\alpha_m \psi \pi_w}{\pi_{\mathrm{m}}+\pi_w} & \frac{\alpha_m \psi \pi_w}{\pi_{\mathrm{m}}+\pi_w} & 0 & 0 & -g_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma_w & -g_8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_w & -g_9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_{w_1} & \rho_{w_2} & \rho_{w_3} & -g_{10} \end{pmatrix},$$

where

$g_1 = \mu,\ g_2 = (\gamma_m + \mu + \xi_{m_1}),\ g_3 = (\beta_m + \mu + \xi_{m_2}),\ g_4 = (\mu + \xi_{m_3}),\ g_5 = (\mu + \varphi_m),$
$g_6 = \mu,\ g_7 = (\gamma_w + \mu + \rho_{w_1}),\ g_8 = (\gamma_w + \mu + \rho_{w_2}),\ g_9 = (\mu + \rho_{w_3}),$ and $g_{10} = (\mu + \varphi_w).$

Here, $\mathcal{J}(E^0)$ implies that $\lambda_1 = -\mu < 0,\ \lambda_2 = -(\mu + \varphi_{\rm m}) < 0,\ \lambda_3 = -\mu < 0,$ and $\lambda_4 = -(\mu + \varphi_w) < 0,$ and also give the following

$$\begin{pmatrix} -g_2 & 0 & 0 & \frac{\alpha_w\psi\pi_{\rm m}}{\pi_{\rm m}+\pi_w} & \frac{\alpha_w\psi\pi_{\rm m}}{\pi_{\rm m}+\pi_w} & \frac{\alpha_w\psi\pi_{\rm m}}{\pi_{\rm m}+\pi_w} \\ \gamma_m & -g_3 & 0 & 0 & 0 & 0 \\ 0 & \beta_m & -g_4 & 0 & 0 & 0 \\ \frac{\alpha_m\psi\pi_w}{\pi_{\rm m}+\pi_w} & \frac{\alpha_m\psi\pi_w}{\pi_{\rm m}+\pi_w} & \frac{\alpha_m\psi\pi_w}{\pi_{\rm m}+\pi_w} & -g_7 & 0 & 0 \\ 0 & 0 & 0 & \gamma_w & -g_8 & 0 \\ 0 & 0 & 0 & 0 & \beta_w & -g_9 \end{pmatrix}. \tag{3.6}$$

The following is a definition of the characteristic equation for (3.6) is given by

$$\lambda^6 + B_1\lambda^5 + B_2\lambda^4 + B_3\lambda^3 + B_4\lambda^2 + B_5\lambda + B_6 = 0, \tag{3.7}$$

where the coefficients of (3.7) are given as

$$\begin{aligned}
B_1 &= (g_2 + g_3 + g_4 + g_7 + g_8 + g_9),\\
B_2 &= g_9(g_2 + g_3 + g_4 + g_7 + g_8) + g_3(g_2 + g_3) + g_2g_3 + g_7(g_2 + g_3 + g_4) + g_8(g_2 + g_3 + g_4 + g_7) - b_1c_1,\\
B_3 &= (g_9(g_4(g_2 + g_3) + g_2g_3 + g_7(g_2 + g_3 + g_4) + g_8(g_2 + g_3 + g_4 + g_7)) + g_7(g_4(g_2 + g_3) + g_2g_3)\\
&\quad + g_8(g_4(g_2 + g_3) + g_2g_3 + g_7(g_2 + g_3 + g_4)) + b_1c_1g_2 + g_2g_3g_4 - b_1c_1(2 + \gamma_w + \gamma_m) + (g_2 + g_3 + g_4)),\\
B_4 &= (c_1(\gamma_m b_1 g_2 + \gamma_m b_1 g_3) + g_8(g_7(g_4(g_2 + g_3) + g_2g_3) - \gamma_m b_1c_1 + b_1c_1g_2 + g_2g_3g_4 - b_1c_1(g_2 + g_3 + g_4))\\
&\quad - (\gamma_m b_1c_1 - b_1c_1g - 2)(g_2 + g_3 + g_4)\gamma_w(b_1c_1g_2 - \gamma_m b_1c_1 + b_1c_1g_7)\\
&\quad + g_9(g_7(g_4(g_2 + g_3) + g_2g_3) + g_8(g_4(g_2 + g_3) - b_1c_1 + g_2g_3 + g_7(g_2 + g_3\\
&\quad + g_4)) - \gamma_w b_1c_1 - \gamma_m b_1c_1 + b_1c_1g_2 + g_2g_3g_4 - b_1c_1(g_2 + g_3 + g_4)) - b_1c_1g_2^2 - b_1c_1(g_4(g_2 + g_3) + g_2g_3)\\
&\quad + g_2g_3g_4g_7 - \gamma_w b_1c_1(g_2 + g_3 + g_4 + g_7) - \beta_w\gamma_w b_1c_1 - \beta_m\gamma_m b_1c_1),\\
B_5 &= (\beta_w(\gamma_w(b_1c_1g_2 - \gamma_m b_1c_1 + b_1c_1g_7) + \gamma_w b_1c_1g_8)\\
&\quad - \gamma_w\left(g_7(b_1c_1g_2 - \gamma_m b_1c_1 + b_1c_1g_7) - c_1(\gamma_m b_1g_2 + \gamma_m b_1g_3) + c_1\left(c_1b_1^2 + b_1g_2^2\right)\right.\\
&\quad + \beta_m\gamma_m b_1c - 1) - g_9((\gamma_m b_1c_1 - b_1c_1g_2)(g_2 + g_3 + g_4) - g_8(g_7(g_4(g_2 + g_3) + g_2g_3) - \gamma_m b_1c_1 + b_1c_1g_2\\
&\quad + g_2g_3g_4 - b_1c_1(g_2 + g_3 + g_4)) - c_1(\gamma_m b_1g_2 + \gamma_m b_1g_3) - \gamma_w(b_1c_1g_2 - \gamma_m b_1c_1 + b_1c_1g_7) + b_1c_1g_2^2\\
&\quad + b_1c_1(g_4(g_2 + g_3) + g_2g - 3) - g_2g_3g_4g_7 + \gamma_w b_1c - 1(g_2 + g_3 + g_4 + g_7) + \beta_m\gamma_m b_1c - 1) - g_8((\gamma_m b_1c_1\\
&\quad - b_1c_1g_2)(g_2 + g_3 + g_4) - c_1(\gamma_m b_1g_2 + \gamma_m b_1g_3) + b_1c_1g_2^2 + b_1c_1(g_4(g_2 + g_3) + g_2g_3) - g_2g - 3g - 4g_7\\
&\quad + b_1c_1g_2)(g_2 + g_3 + g_4) - c_1(\gamma_m b_1g_2 + \gamma_m b_1g_3) + b_1c_1g_2^2 + b_1c_1(g_4(g_2 + g_3) + g_2g_3) - g_2g - 3g - 4g_7\\
&\quad + \beta_m\gamma_m b_1c_1) + \gamma_w(b_1c_1g_2 - \gamma_m b_1c_1 + b_1c_1g_7)(g_2 + g_3 + g_4 + g_7) - \gamma_w b_1c_1(g_4(g_2 + g_3) - b_1c_1 + g_2g_3\\
&\quad + g_7(g_2 + g_3 + g_4)) - \beta_w\gamma_w b_1c_1(g_2 + g_3 + g_4 + g_7 + g_8)),\\
B_6 &= (\beta_w(\gamma_w(b_1c_1g_2 - \gamma_m b_1c_1 + b_1c_1g_7) + \gamma_w b_1c_1g_8)(g_2 + g_3 + g_4 + g_7 + g_8)\\
&\quad - g_9(\gamma_w(g_7(b_1c_1g_2 - \gamma_m b_1c_1 + b_1c_1g_7) - c_1(\gamma_m b_1g_2 + \gamma_m b_1g_3) + c_1\left(c_1b_1^2 + b_1g_2^2\right) + \beta_m\gamma_m b_1c_1)\\
&\quad + g_8\left((\gamma_m b_1c_1 - b_1c_1g_2)(g_2 + g_3 + g_4) - c_1(\gamma_m b_1g_2 + \gamma_m b_1g_3) + b_1c_1g_2^2\right.
\end{aligned}$$

$+\ b_1c_1\left(g_4\left(g_2+g_3\right)+g_2g_3\right)-g_2g_3g_4g_7+\beta_m\gamma_mb_1c_1)-\gamma_w\left(b_1c_1g_2-\gamma_mb_1c_1+b_1c_1g_7\right)\left(g_2+g_3+g_4\right.$
$\left.+\ g_7\right)+\gamma_wb_1c_1\left(g_4\left(g_2+g_3\right)-b_1c_1+g_2g_3+g_4\left(g_2+g_3+g_4\right)\right))$
$-\beta_w\left(\gamma_w\left(g_7\left(b_1c_1g_2-\gamma_mb_1c_1+b_1c_1g_7\right)\right.\right.$
$\left.\left.-\ g_7\right)+\gamma_wb_1c_1\left(g_4\left(g_2+g_3\right)-b_1c_1+g_2g_3+g_4\left(g_2+g_3+g_4\right)\right)\right)-\beta_w\left(\gamma_w\left(g_7\left(b_1c_1g_2-\gamma_mb_1c_1+b_1c_1g_7\right)\right.\right.$
$\left.\left.-\ c_1\left(\gamma_mb_1g_2+\gamma_mb_1g_1\right)+c_1\left(c_1b_1^2+b_1g_2^2\right)+\beta_m\gamma_mb_1c-1\right)+g_8\left(\gamma_w\left(b_1c_1g_2-\gamma_mb_1c_1+b_1c_1g_7\right)\right.\right.$
$\left.\left.+\ \gamma_wb_1c_1g_8\right)\right)-\beta_w\gamma_wb_1c_1\left(g_4\left(g_2+g_3\right)-b_1c_1+g_2g_3+g_7\left(g_2+g_3+g_4\right)+g_8\left(g_2+g_3+g_4+g_7\right)\right)\Big).$

By utilizing the Routh-Hurwitz argument, one can determine that the above Eq (3.7) fulfills all the conditions of Routh-Hurwitz. Hence, the infection-free steady-state of the proposed model is locally asymptotically stable. □

4. Fractional order model solution

In this section, the solution of the recommended system (2.3) will be investigated with the help of fixed-point theory. Through the use of fixed-point theory we had also demonstrated the existence and uniqueness of the FO model (2.3) solution. The following form can be used to quickly describe the system of Eq (2.3) as

$$\begin{cases} {}_0^{ABC}D_t^{\Phi}y(t)=\mathbb{X}(a,y(a)), \\ y(0)=y_0,\quad 0<a<\mathrm{A}<\infty. \end{cases} \tag{4.1}$$

In system (4.1), the vector function $\mathbb{X}$ is continuous and $y(a)=\left(S_m,I_{mp},I_{ms},L_m,R_m,S_w,I_{wp},I_{ws},L_w,R_w\right)$ represents the state variables as vectors, while $\mathbb{X}$ is defined as

$$\mathbb{X}=\begin{pmatrix}\mathbb{X}_1\\ \mathbb{X}_2\\ \mathbb{X}_3\\ \mathbb{X}_4\\ \mathbb{X}_5\\ \mathbb{X}_6\\ \mathbb{X}_7\\ \mathbb{X}_8\\ \mathbb{X}_9\\ \mathbb{X}_{10}\end{pmatrix}=\begin{pmatrix}\pi_{\mathrm{m}}+\varphi_mR_m-\alpha_w\psi(\frac{I_{wp}+I_{ws}+L_w}{N})S_m-\mu S_m\\ \alpha_w\psi(\frac{I_{wp}+I_{ws}+L_w}{N})S_m-\gamma_mI_{mp}-\mu I_{mp}-\xi_{m_1}I_{mp}\\ \gamma_mI_{mp}-\beta_mI_{ms}-\xi_{m_2}I_{ms}-\mu I_{ms}\\ \beta_mI_{ms}-\mu L_m-\xi_{m_3}L_m\\ \xi_{m_1}I_{mp}+\xi_{m_2}I_{ms}+\xi_{m_3}L_m-\mu R_m-\varphi_mR_m\\ \pi_w+\varphi_wR_w-\alpha_m\psi(\frac{I_{mp}+I_{ms}+L_m}{N})S_w-\mu S_w\\ \alpha_m\psi(\frac{I_{mp}+I_{ms}+L_m}{N})S_w-\gamma_wI_wp-\mu I_{wp}-\rho_{w_1}I_{wp}\\ \gamma_wI_{wp}-\beta_wI_{ws}-\rho_{w_2}I_{ws}-\mu I_{ws}\\ \beta_wI_{ws}-\mu L_w-\rho_{w_3}L_w\\ \rho_{w_1}I_{wp}+\rho_{w_2}I_{ws}+\rho_{w_3}L_w-\mu R_w-\varphi_wR_w\end{pmatrix},$$

and $y_0(a)=\left(S_m(0),I_{mp}(0),I_{ms}(0),L_m(0),R_m(0),S_w(0),I_{wp}(0),I_{ws}(0),L_w(0),R_w(0)\right)$, is a suitable starting condition vector for state variables. Additionally, the Lipschitz function satisfies the criterion. Also,$\mathbb{X}$ is declared as

$$\|\mathbb{X}(a,y_1(a))-\mathbb{X}(a,y_2(a))\| \quad\leq\quad \mathbb{N}\|y_1(a)-y_2(a)\|. \tag{4.2}$$

Next, we assert and demonstrate the subsequent theorem regarding the existence and originality of the solution of FO dynamical model (2.3).

Theorem 4.1. *If the following condition is met, then the given system of Eq (2.3), will have a unique solution*

$$\frac{(1-\Phi)}{ABC(\Phi)}\mathbb{N}+\frac{\Phi}{ABC(\Phi)\Gamma(\Phi)}\mathrm{A}_{max}^{\Phi}\mathbb{N}<1. \tag{4.3}$$

Proof. The non-linear Volterra integral equation shown below is obtained by applying the Atangana-Beleanu (AB) fractional integral shown in Definition 1.3 to system (4.1), in order to obtain the desired result

$$y(a)=y_0+\frac{1-\Phi}{ABC(\Phi)}\mathbb{X}(a,y(a))+\frac{\Phi}{ABC(\Phi)\Gamma(\Phi)}\int_0^a(a-\varepsilon)^{\Phi-1}\mathbb{X}(\varepsilon,y(\varepsilon))d\varepsilon. \tag{4.4}$$

Assume that $\mathrm{I}=(0,\mathrm{A})$, and the operator $\Omega:\mathrm{C}(\mathrm{I},\mathrm{R}^{10})\to\mathrm{C}(\mathrm{I},\mathrm{R}^{10})$ defined by

$$\Omega\left[y(a)\right]=y_0+\frac{1-\Phi}{ABC(\Phi)}\mathbb{X}(a,y(a))+\frac{\Phi}{ABC(\Phi)\Gamma(\Phi)}\int_0^a(a-\varepsilon)^{\Phi-1}\mathbb{X}(\varepsilon,y(\varepsilon))d\varepsilon. \tag{4.5}$$

Equation (4.4) can be expressed as the following

$$y(a)=\Omega\left[y(a)\right], \tag{4.6}$$

$\|.\|_I$, is used to indicate the supremum norm on I, which is described by

$$\|y(a)\|_I=\sup_{a\in I}\|y(a)\|,\quad y(a)\in\mathrm{C}. \tag{4.7}$$

Obviously, $\mathrm{C}(\mathrm{I},\mathrm{R}^{10})$ with norm $\|.\|_I$ create a Banach space, in addition to which it is readily apparent that

$$\left\|\int_0^a\mathcal{L}(a,\varepsilon)y(\varepsilon)d\varepsilon\right\|\leq A\|\mathcal{L}(a,\varepsilon)\|_I\|y(a)\|_I, \tag{4.8}$$

with $y(a)\in\mathrm{C}(\mathrm{I},\mathrm{R}^{10})$, $\mathcal{L}(a,\varepsilon)\in\mathrm{C}(\mathrm{I},\mathrm{R})$ such that

$$\|\mathcal{L}(a,\varepsilon)\|_I=\sup_{a,\varepsilon\in I}|\mathcal{L}(a,\varepsilon)|. \tag{4.9}$$

We get the following by utilising the definition of Ω provided in (4.6) as

$$\begin{aligned}\|\Omega[y_1(a)]-\Omega[y_2(a)]\|_I &\leq \left\|\frac{(1-\Phi)}{ABC(\Phi)}(\mathbb{X}(a,y_1(a))-\mathbb{X}(a,y_2(a))+\frac{\Phi}{ABC(\Phi)\Gamma(\Phi)}\right.\\ &\quad\left.\times\int_0^a(a-\varepsilon)^{\Phi-1}(\mathbb{X}(\varepsilon,y_1(\varepsilon))-\mathbb{X}(\varepsilon,y_2(\varepsilon)))d\varepsilon\right\|_I.\end{aligned} \tag{4.10}$$

Additionally, after simplification, we obtain the following by implementing the Lipschitz condition (4.2) and the triangle inequality to the solution in (4.8)

$$\|\Omega[y_1(a)]-\Omega[y_1(a)]\|_I \leq \left(\frac{(1-\Phi)\mathbb{N}}{ABC(\Phi)}+\frac{\Phi}{ABC(\Phi)\Gamma(\Phi)}\mathbb{N}\mathrm{A}_{max}^{\Phi}\right)\|y_1(a)-y_2(a)\|_I. \tag{4.11}$$

As a consequence, we obtain

$$\|\Omega[y_1(a)]-\Omega[y_1(a)]\|_I \leq \mathcal{B}\|y_1(a)-y_2(a)\|_I, \tag{4.12}$$

where

$$\mathcal{B}=\frac{(1-\Phi)\mathbb{N}}{ABC(\Phi)}+\frac{\Phi}{ABC(\Phi)\Gamma(\Phi)}\mathbb{N}\mathrm{A}_{max}^{\Phi}.$$

It is obvious that Ω will be a contraction if condition (4.3) is met. This demonstrates that the fractional order dynamical system (4.1) has a singular solution. □

5. Numerical scheme

In this section, we emphasise the suggested model's numerical solution (2.3). In order to numerically solve our system with the non-singular and non-local kernel, we first design an iterative approach. Then, simulations are run to show the suggested FO dynamical system. We use the newly created numerical technique described in [42] to approximate the AB integral operator. We briefly examine the aforementioned approach and apply it to our dynamical system in order to get an iterative scheme using the newly established numerical techniques followed in [42] for the system (2.3). System (4.1) is rewritten into the fractional integral equation form shown below using the fundamental theorem of fractional calculus

$$y(a)-y(0) = \frac{(1-\Phi)}{ABC(\Phi)}\mathbb{X}(a,y(a))+\frac{\Phi}{ABC(\Phi)\times\Gamma(\Phi)}\int_0^a \mathbb{X}(\varepsilon,x(\varepsilon))(a-\varepsilon)^{\Phi-1}d\varepsilon. \tag{5.1}$$

At $a=a_{j+1},\ j=0,1,2,...,$ we have

$$\begin{aligned} y(a_{j+1})-y(0) &= \frac{1-\Phi}{ABC(\Phi)}\mathbb{X}(a_j,y(a_j))+ \\ &\quad \frac{\Phi}{ABC(\Phi)\times\Gamma(\Phi)}\int_0^{a_{j+1}} \mathbb{X}(\varepsilon,y(\varepsilon))(a_{j+1}-\varepsilon)^{\Phi-1}d\Phi, \\ &= \frac{1-\Phi}{ABC(\Phi)}\mathbb{X}(a_j,y(a_j))+ \\ &\quad \frac{\Phi}{ABC(\Phi)\times\Gamma(\Phi)}\sum_{\kappa=0}^{j}\int_{a_\kappa}^{a_{\kappa+1}} \mathbb{X}(\varepsilon,y(\varepsilon))(a_{j+1}-\varepsilon)^{\Phi-1}d\Phi. \end{aligned} \tag{5.2}$$

The function $\mathbb{X}(\varepsilon,y(\varepsilon))$ can be estimated over the interval $[a_\kappa,a_{\kappa+1}]$, we apply the interpolation polynomial

$$\mathbb{X}(\varepsilon,y(\varepsilon)) \cong \frac{\mathbb{X}(a_\kappa,y(a_\kappa))}{\mathrm{h}}(a-a_{\kappa-1})-\frac{\mathbb{X}a_{\kappa-1},y(a_{\kappa-1}))}{\mathrm{h}}(a-a_\kappa), \tag{5.3}$$

substituting in (5.2) we get

$$\begin{aligned} y(a_{j+1}) &= y(0)+\frac{1-\Phi}{ABC(\Phi)}\mathbb{X}(a_j,y(a_j))+\frac{\Phi}{ABC(\Phi)\times\Gamma(\Phi)} \\ &\quad \sum_{\kappa=0}^{j}\Big(\frac{\mathbb{X}(a_\kappa,y(a_\kappa))}{\mathrm{h}}\int_{a_\kappa}^{a_{\kappa+1}}(a-a_{\kappa-1})(a_{j+1}-a)^{\Phi-1}dt \\ &\quad -\frac{\mathbb{X}(a_{\kappa-1},y(a_{\kappa-1}))}{\mathrm{h}}\int_{a_\kappa}^{a_{\kappa+1}}(a-a_\kappa)(a_{j+1}-a)^{\Phi-1}dt\Big), \end{aligned} \tag{5.4}$$

the approximate solution after the computation of these integrals are obtained as:

$$
\begin{aligned}
y(a_{j+1}) &= y(a_0) + \frac{1-\Phi}{ABC(\Phi)}\mathbb{X}(a_j, y(a_j)) + \frac{\Phi}{ABC(\Phi)}\sum_{\kappa=0}^{j} \\
&\Big(\frac{\mathrm{h}^{\Phi}\mathbb{X}(a_\kappa, y(a_\kappa))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi}(j-\kappa+2+\Phi) - (j-\kappa)^{\Phi}(j-\kappa+2+2\Phi)) \\
&-\frac{\mathrm{h}^{\Phi}\mathbb{X}(a_{\kappa-1}, y(a_{\kappa-1}))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi+1} - (j-\kappa)^{\Phi}(j-\kappa+1+\Phi))\Big). \qquad (5.5)
\end{aligned}
$$

Finally, we obtained the recurrent formulas shown below for the proposed model:

$$
\begin{aligned}
S_m(a_{j+1}) &= S_m(a_0) + \frac{1-\Phi}{ABC(\Phi)}\mathbb{X}_1(a_j, y(a_j)) + \frac{\Phi}{ABC(\Phi)}\sum_{\kappa=0}^{j} \\
&\Big(\frac{\mathrm{h}^{\Phi}\mathbb{X}_1(a_\kappa, y(a_\kappa))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi}(j-\kappa+2+\Phi) - (j-\kappa)^{\Phi}(j-\kappa+2+2\Phi)) \\
&-\frac{\mathrm{h}^{\Phi}\mathbb{X}_1(a_{\kappa-1}, y(a_{\kappa-1}))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi+1} - (j-\kappa)^{\Phi}(j-\kappa+1+\Phi))\Big), \\
I_{mp}(a_{j+1}) &= I_{mp}(a_0) + \frac{1-\Phi}{ABC(\Phi)}\mathbb{X}_2(a_j, y(a_j)) + \frac{\Phi}{ABC(\Phi)}\sum_{\kappa=0}^{j} \\
&\Big(\frac{\mathrm{g}^{\Phi}\mathbb{X}_2(a_\kappa, y(a_\kappa))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi}(j-\kappa+2+\Phi) - (j-\kappa)^{\Phi}(j-\kappa+2+2\Phi)) \\
&-\frac{\mathrm{g}^{\Phi}\mathbb{X}_2(a_{\kappa-1}, y(a_{\kappa-1}))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi+1} - (j-\kappa)^{\Phi}(j-\kappa+1+\Phi))\Big), \\
I_{ms}(a_{j+1}) &= I_{ms}(a_0) + \frac{1-\Phi}{ABC(\Phi)}\mathbb{X}_3(a_j, y(a_j)) + \frac{\Phi}{ABC(\Phi)}\sum_{\kappa=0}^{j} \\
&\Big(\frac{\mathrm{h}^{\Phi}\mathbb{X}_3(a_\kappa, y(a_\kappa))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi}(j-\kappa+2+\Phi) - (j-\kappa)^{\Phi}(j-\kappa+2+2\Phi)) \\
&-\frac{\mathrm{h}^{\Phi}\mathbb{X}_3(a_{\kappa-1}, y(a_{\kappa-1}))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi+1} - (j-\kappa)^{\Phi}(j-\kappa+1+\Phi))\Big), \\
L_m(a_{j+1}) &= L_m(a_0) + \frac{1-\Phi}{ABC(\Phi)}\mathbb{X}_4(a_j, y(a_j)) + \frac{\Phi}{ABC(\Phi)}\sum_{\kappa=0}^{j} \\
&\Big(\frac{\mathrm{g}^{\Phi}\mathbb{X}_4(a_\kappa, y(a_\kappa))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi}(j-\kappa+2+\Phi) - (j-\kappa)^{\Phi}(j-\kappa+2+2\Phi)) \\
&-\frac{\mathrm{g}^{\Phi}\mathbb{X}_4(a_{\kappa-1}, y(a_{\kappa-1}))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi+1} - (j-\kappa)^{\Phi}(j-\kappa+1+\Phi))\Big), \\
R_m(a_{j+1}) &= R_m(a_0) + \frac{1-\Phi}{ABC(\Phi)}\mathbb{X}_5(a_j, y(a_j)) + \frac{\Phi}{ABC(\Phi)}\sum_{\kappa=0}^{j} \\
&\Big(\frac{\mathrm{h}^{\Phi}\mathbb{X}_5(a_\kappa, y(a_\kappa))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi}(j-\kappa+2+\Phi) - (j-\kappa)^{\Phi}(j-\kappa+2+2\Phi))
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\mathrm{h}^{\Phi}\mathbb{X}_5(a_{\kappa-1},y(a_{\kappa-1}))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi+1}-(j-\kappa)^{\Phi}(j-\kappa+1+\Phi))\Big),\\
S_w(a_{j+1}) &= S_w(a_0)+\frac{1-\Phi}{ABC(\Phi)}\mathbb{X}_6(a_j,y(a_j))+\frac{\Phi}{ABC(\Phi)}\sum_{\kappa=0}^{j}\\
&\Big(\frac{\mathrm{g}^{\Phi}\mathbb{X}_6(a_\kappa,y(a_\kappa))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi}(j-\kappa+2+\Phi)-(j-\kappa)^{\Phi}(j-\kappa+2+2\Phi))\\
&-\frac{\mathrm{g}^{\Phi}\mathbb{X}_6(a_{\kappa-1},y(a_{\kappa-1}))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi+1}-(j-\kappa)^{\Phi}(j-\kappa+1+\Phi))\Big),\\
I_{wp}(a_{j+1}) &= I_{wp}(a_0)+\frac{1-\Phi}{ABC(\Phi)}\mathbb{X}_7(a_j,y(a_j))+\frac{\Phi}{ABC(\Phi)}\sum_{\kappa=0}^{j}\\
&\Big(\frac{\mathrm{h}^{\Phi}\mathbb{X}_7(a_\kappa,y(a_\kappa))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi}(j-\kappa+2+\Phi)-(j-\kappa)^{\Phi}(j-\kappa+2+2\Phi))\\
&-\frac{\mathrm{h}^{\Phi}\mathbb{X}_7(a_{\kappa-1},y(a_{\kappa-1}))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi+1}-(j-\kappa)^{\Phi}(j-\kappa+1+\Phi))\Big),\\
I_{ws}(a_{j+1}) &= I_{ws}(a_0)+\frac{1-\Phi}{ABC(\Phi)}\mathbb{X}_8(a_j,y(a_j))+\frac{\Phi}{ABC(\Phi)}\sum_{\kappa=0}^{j}\\
&\Big(\frac{\mathrm{g}^{\Phi}\mathbb{X}_8(a_\kappa,y(a_\kappa))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi}(j-\kappa+2+\Phi)-(j-\kappa)^{\Phi}(j-\kappa+2+2\Phi))\\
&-\frac{\mathrm{g}^{\Phi}\mathbb{X}_8(a_{\kappa-1},y(a_{\kappa-1}))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi+1}-(j-\kappa)^{\Phi}(j-\kappa+1+\Phi))\Big),\\
L_w(a_{j+1}) &= L_w(a_0)+\frac{1-\Phi}{ABC(\Phi)}\mathbb{X}_9(a_j,y(a_j))+\frac{\Phi}{ABC(\Phi)}\sum_{\kappa=0}^{j}\\
&\Big(\frac{\mathrm{h}^{\Phi}\mathbb{X}_9(a_\kappa,y(a_\kappa))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi}(j-\kappa+2+\Phi)-(j-\kappa)^{\Phi}(j-\kappa+2+2\Phi))\\
&-\frac{\mathrm{h}^{\Phi}\mathbb{X}_9(a_{\kappa-1},y(a_{\kappa-1}))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi+1}-(j-\kappa)^{\Phi}(j-\kappa+1+\Phi))\Big),\\
R_w(a_{j+1}) &= R_w(a_0)+\frac{1-\Phi}{ABC(\Phi)}\mathbb{X}_{10}(a_j,y(a_j))+\frac{\Phi}{ABC(\Phi)}\sum_{\kappa=0}^{j}\\
&\Big(\frac{\mathrm{g}^{\Phi}\mathbb{X}_{10}(a_\kappa,y(a_\kappa))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi}(j-\kappa+2+\Phi)-(j-\kappa)^{\Phi}(j-\kappa+2+2\Phi))\\
&-\frac{\mathrm{g}^{\Phi}\mathbb{X}_{10}(a_{\kappa-1},y(a_{\kappa-1}))}{\Gamma(\Phi+2)}((j+1-\kappa)^{\Phi+1}-(j-\kappa)^{\Phi}(j-\kappa+1+\Phi))\Big). \qquad (5.6)
\end{aligned}
$$

This scheme will be utilized to illustrate the solution pathways of the recommended system. There are numerous numerical schemes in the literature for fractional system. Here, we mainly focussed to show the chaotic and dynamical behaviour of the system through the above technique. However, accuracy, stability and other properties of the numerical scheme will be investigated in the future work. In [43–45], the authors considered an epidemic model of covid-19 infection through Caputo derivative and we considered the dynamics of syphilis infection through ABC derivative. These models are

different, the analysis is different and the results of these models are different from each other.

6. Results and discussion

In this section, we demonstrated numerical findings obtained during different scenarios. Applying the above recently proposed iterative approach (5.6), we will graphical represent the outcomes of the fractional order model (2.3) of syphilis infection. The dynamical behavior and chaotic behavior will be demonstrated numerically with the help of different input values. The parameter values are taken from Table 1 for simulations and the state-variables are taken to be $S_m(0) = 1200, I_{mp}(0) = 80, I_{ms}(0) = 130, L_m(0) = 100, R_m(0) = 80, S_w(0) = 1000, I_{wp}(0) = 100, I_{ws}(0) = 300, L_w(0) = 200$ and $R_w(0) = 130$.

Here, we perform different simulations to understand the dynamics of the recommended model of syphilis infection. We visualized the solution pathways and chaotic behavior of the system. In the first simulation presented in Figures 1 and 2, we assumed different values of fractional order and illustrate the solution pathways of the proposed system. In Figure 1, we have shown the time series of infected individuals of men and women with $\Phi = 0.4, 0.5, 0.6, 0.7$ while the value of fractional order Φ are assumed to be $0.7, 0.8, 0.9, 1.0$ in Figure 2. It can be seen that smaller value of Φ reduce the infection level in primary and secondary individuals of both the classes. Figure 2 also represents comparison of integer and non-integer cases which shows that fractional framework is more flexible and suitable for data fitting. We noticed that the order of the fractional operator has a positive impact on the infected individuals of both the class and the control of fractional order can control the infection level.

In the second simulation illustrated in Figure 3, we variate the input parameter ψ and assume it to be $0.40, 0.43, 0.46$ and 0.49. The rise of this parameter increase the infection level of primary infection of men and women and is recommended to be a critical factor. This parameter is predicted to be critical which increase the risk of syphilis in the community We noticed the effect of ψ on the infected individuals of men and women in this simulation. In the third simulation presented in Figures 4–6, we have shown the chaotic phenomena of the system with different input parameters. In Figure 4, the chaotic phenomena is demonstrated with $\gamma_m = 0.435, \gamma_w = 0.23$ and $\Psi = 0.8$ while in Figure 5, we represent the chaotic plots with $\gamma_m = 0.0435, \gamma_w = 0.023$ and $\Psi = 1.0$. In Figure 6, we demonstrated the chaos of the system with the input parameter $\psi_m = 0.35, \psi_w = 0.16$ and $\Psi = 1.0$. It has been noticed that the chaos of the system is closely related to the initial values of state variables and input parameters. The chaotic phenomena bring the system to the unstable situation, therefore, further investigation is needed to find out the most flexible values of the parameters to control the chaos of the system.

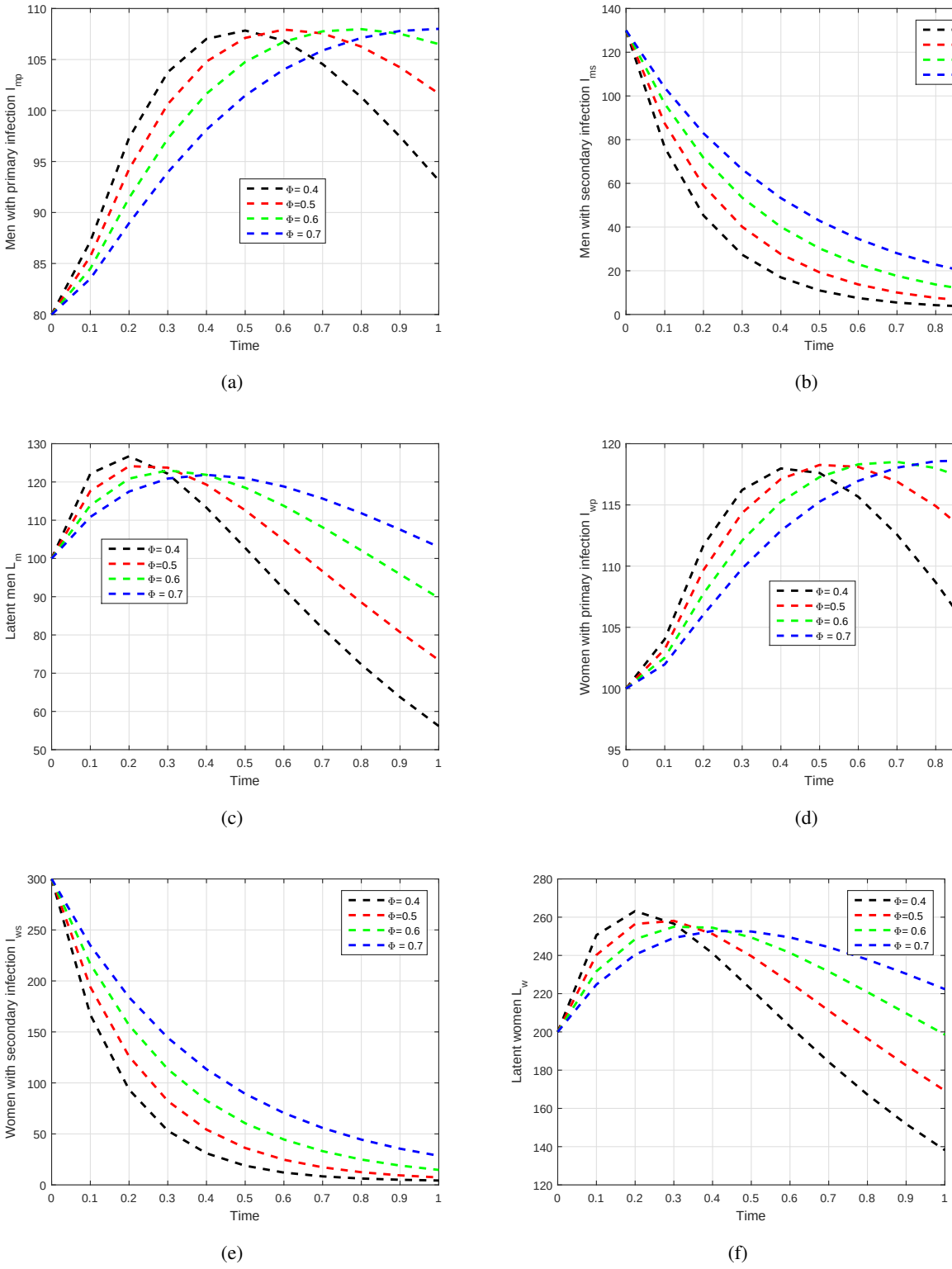
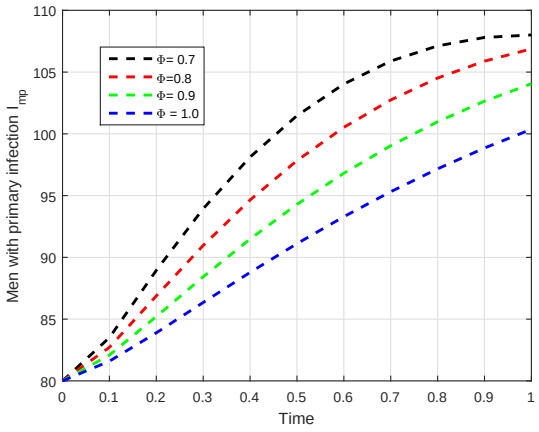
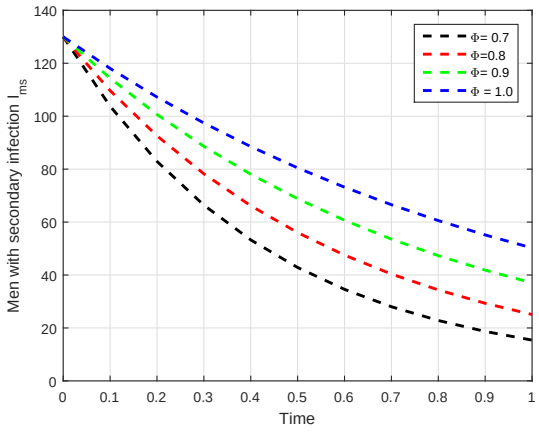


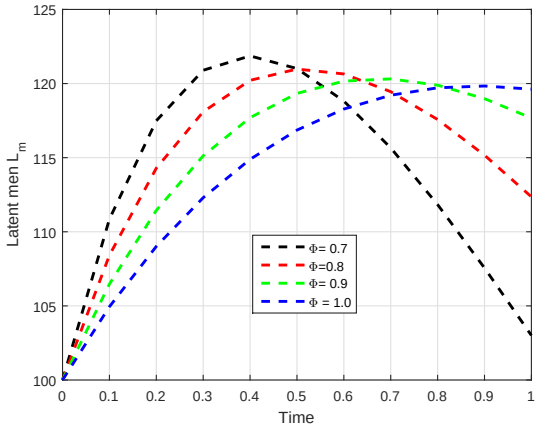
Figure 1. Plotting the solution pathways of the recommended model (2.3) of syphilis infection with variation of fractional order Φ, i.e., $\Phi = 0.4, 0.5, 0.6, 0.7$.



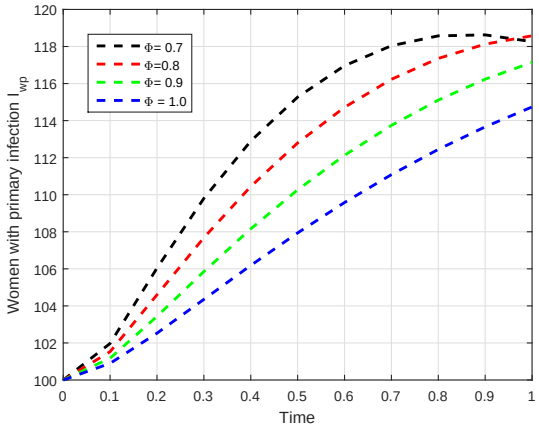
(a)



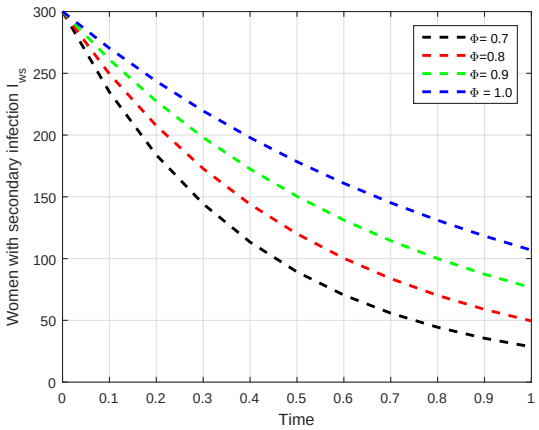
(b)



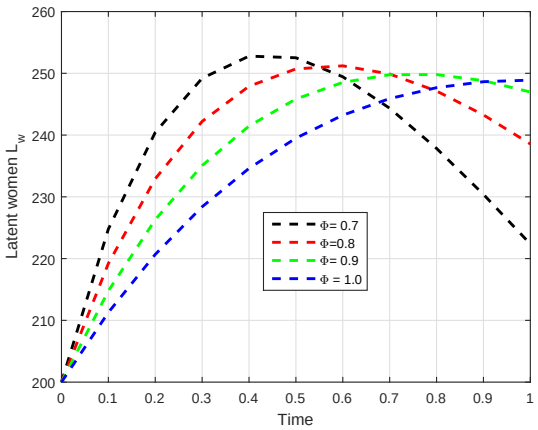
(c)



(d)

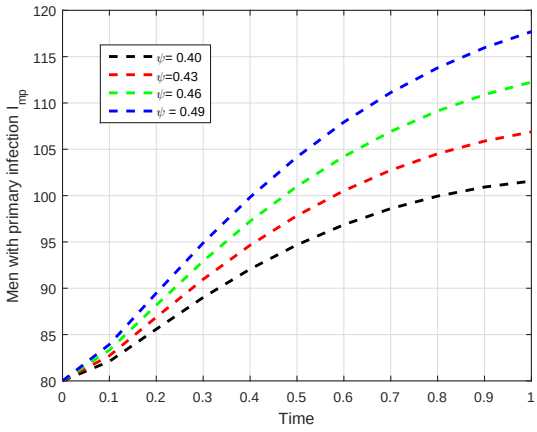


(e)

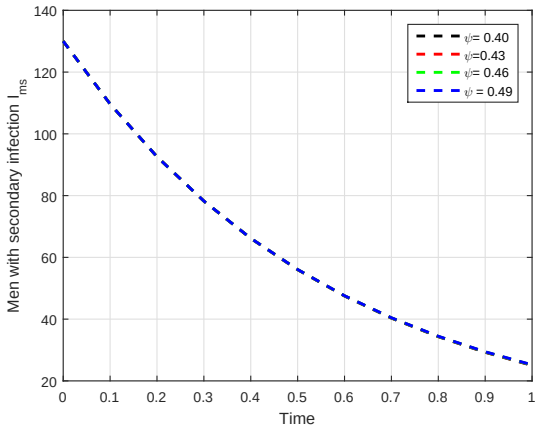


(f)

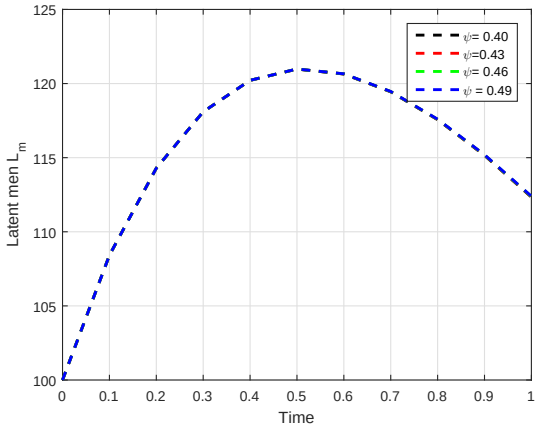
Figure 2. Illustration of the solution pathways of the recommended model (2.3) of syphilis infection with variation of fractional order Φ, i.e., $\Phi = 0.7, 0.8, 0.9, 1.0$.



(a)



(b)



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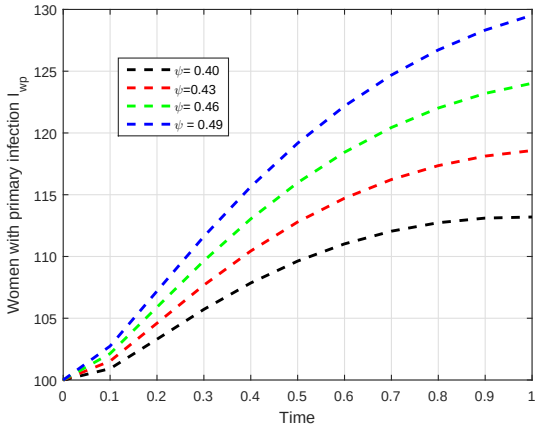

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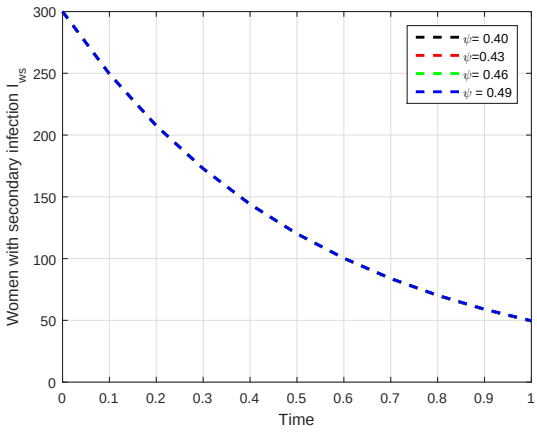
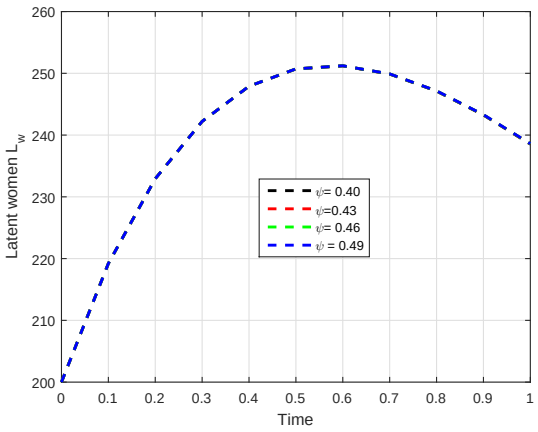
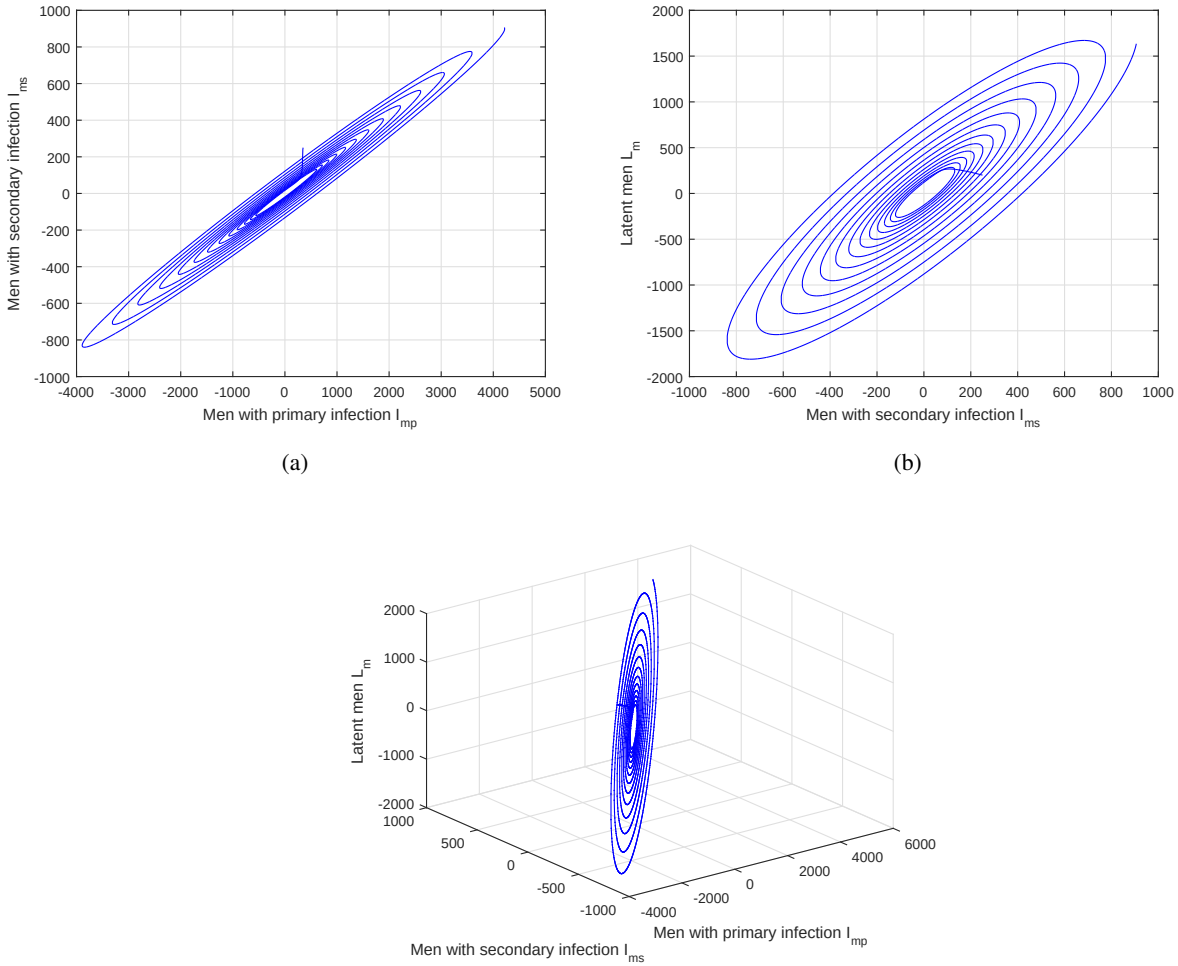


(e)



(f)

Figure 3. Graphical view analysis the solution pathways of the suggested model (2.3) of syphilis infection with variation of the parameter ψ.



(a) (b)

(c)

Figure 4. Illustration of the chaotic phenomena of the fractional system (2.3) of syphilis infection with $\gamma_m = 0.435$, $\gamma_w = 0.23$ and $\Phi = 0.8$.

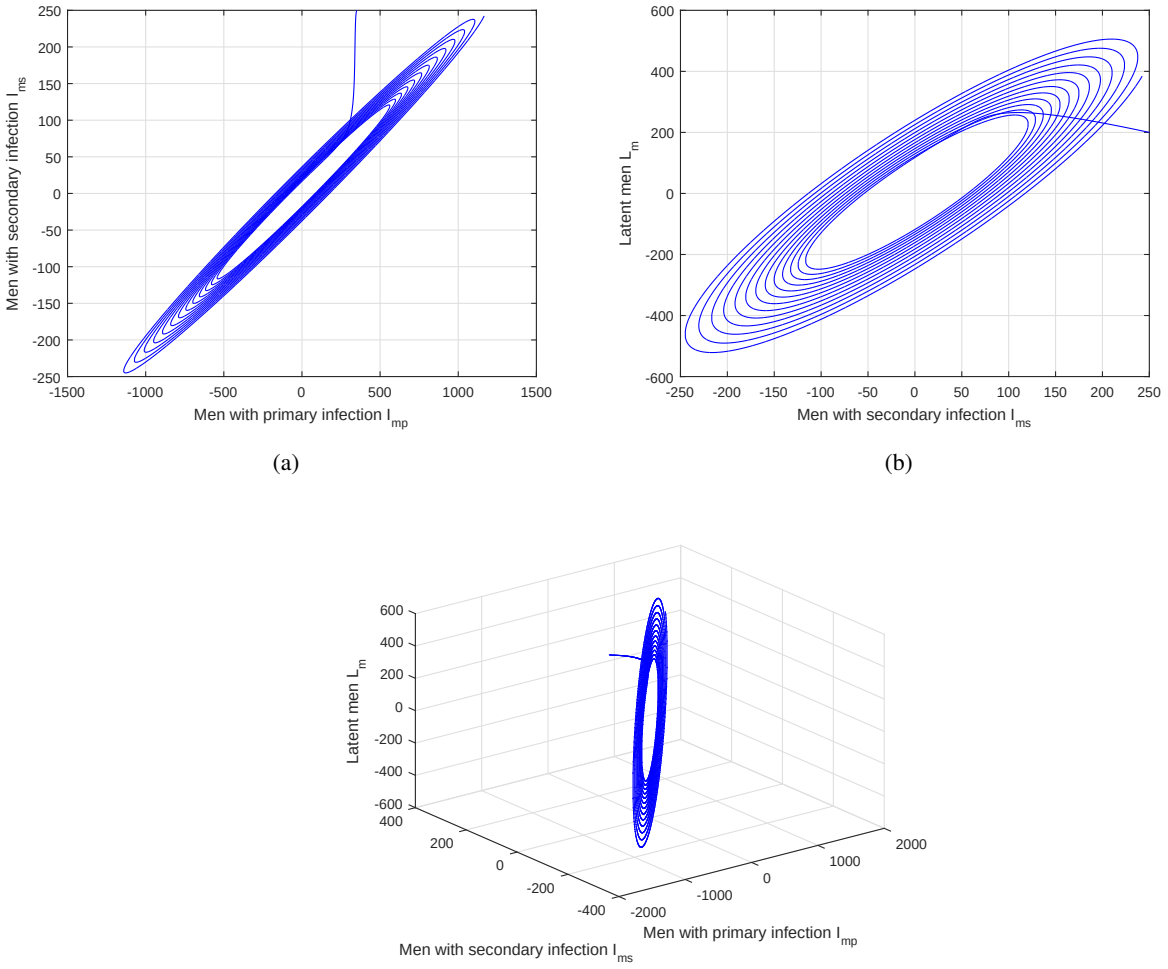
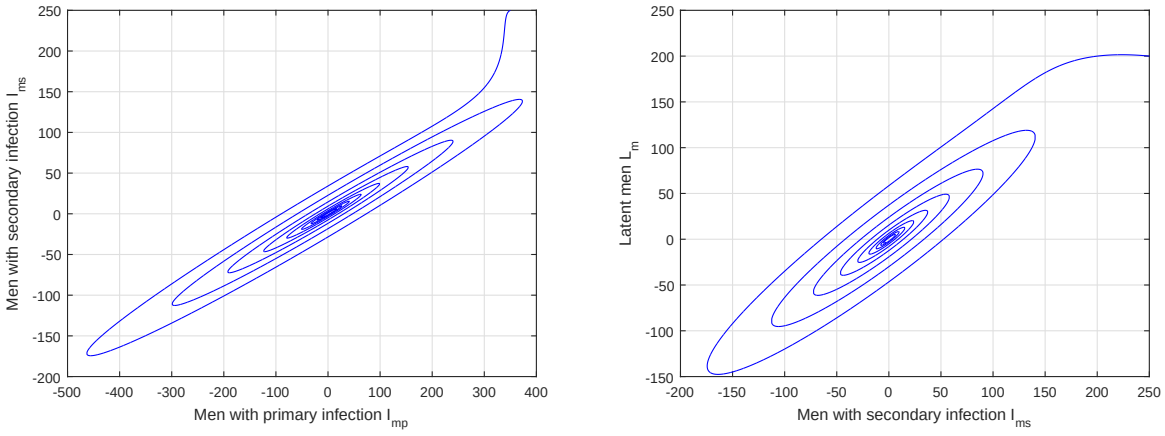
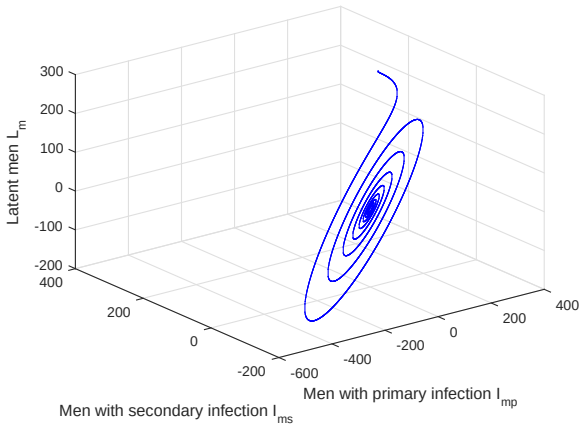


Figure 5. Illustration of the chaotic phenomena of the fractional system (2.3) of syphilis infection with $\gamma_m = 0.0435$, $\gamma_w = 0.023$ and $\Phi = 1.0$.



(a) (b)



(c)

Figure 6. Graphical view analysis of the chaotic phenomena of the fractional system (2.3) of syphilis infection with $\varphi_m = 0.35$, $\varphi_w = 0.16$ and $\Phi = 1.0$.

Table 1. Parameter values and description for the recommended fractional syphilis model.

Parameters	Description	Values	Source
β_m	Men to men transmission with secondary to latent class	Various	Assumed
π_{m}	Recruitment rate for men population by susceptible individuals	0.3	[43]
π_w	Recruitment rate for women population by susceptible individuals	0.45	[43]
α_w	Probability of women syphilis spreading infection	0.2	[19]
β_w	Woman to woman transmission with secondary to latent class	Various	Assumed
α_m	Probability of men syphilis spreading infection	0.5	[19]
γ_m	Rate at which a primary infected man develops secondary syphilis	0.01	[19,44]
γ_w	Movement rate from women with essential syphilis	0.627	[19,44]
ψ	Average number of partners for a woman with secondary syphilis per unit of time	Various	Assumed
φ_m	Syphilis recovery rate in infectious men	0.1	[43]
φ_w	Syphilis recovery rate in infectious women	0.1	[43]
ξ_{m_1}	Men with primary stage syphilis-treated rate	Various	Assumed
ξ_{m_2}	Men with secondary stage syphilis-treated rate	0.1	[43]
ξ_{m_3}	Men with latent stage syphilis-treated rate	0.2	[43]
ρ_{w_1}	Women with primary stage syphilis-treated rate	Various	Assumed
ρ_{w_2}	Women with secondary stage syphilis-treated rate	0.1	[43]
ρ_{w_3}	Women with latent stage syphilis-treated rate	0.2	[43]
μ	Rate of natural death	5.48×10^{-5}	[8]

7. Conclusions

In this research paper, a new approach in syphilis modeling via Atangana-Baleanu fractional derivative is suggested to deeply explore the intricate dynamics of syphilis infection. Instead of using the traditional order derivative, we had used fractional derivative to offer findings that were more realistic and trustworthy. We used the next-generation method in order to obtain $\mathcal{R}_0$ for the suggested fractional model. We have shown that the infection-free steady-state is locally asymptotically stable if $\mathcal{R}_0 < 1$ otherwise unstable. Existence and uniqueness of the fractional order system are investigated through fixed-point theory. A novel numerical method is being employed to iteratively solve the fractional order model. Eventually, the dynamical behaviour of the system is highlighted through the proposed numerical method. Moreover, the contribution of input parameters in the solution pathways of the system has been conceptualized through different scenarios. With particular input factor values, the system's chaotic behaviour is also demonstrated. We have shown the most significant factor of the system and illustrated that fractional derivatives provide more precise and realistic information regarding the dynamics of syphilis infection. In the future work, the recommended model of syphilis infection will be validated through real data and different control measure will be introduced in the system for the prevention of the infection.

Acknowledgments

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Conflict of interest

The authors disclosed no conflicts of interest in publishing this paper.

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