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Research article

New fractional results for Langevin equations through extensive fractional operators

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Abstract: Fractional Langevin equations play an important role in describing a wide range of physical processes. For instance, they have been used to describe single-file predominance and the behavior of unshackled particles propelled by internal sounds. This article investigates fractional Langevin equations incorporating recent extensive fractional operators of different orders. Nonperiodic and nonlocal integral boundary conditions are assumed for the model. The Hyres-Ulam stability, existence, and uniqueness of the solution are defined and analyzed for the suggested equations. Also, we utilize Banach contraction principle and Krasnoselskii fixed point theorem to accomplish our results. Moreover, it will be apparent that the findings of this study include various previously obtained results as exceptional cases.

Keywords: extensive fractional integral operator; fixed point theorems; fractional Langevin equation; Hyres-Ulam stability; nonlocal conditions
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1. Introduction

Dynamical demeanor of natural phenomena are commonly represented by fractional differential mathematical models (FDMMs). Such these mathematical model have a memory and genetic properties, for example, viscoelastic deformation [1], bacterial chemotaxis [2], anomalous diffusion [3], stock market [4], unwinding and response energy and Behavior of Biomedical Materials etc [5]. Also, the FDMMs is an important subject of research that has wide applications in various fields such as fluid dynamics, biology, physics, problems of groundwaters, aerodynamics and hydrodynamics, image processing [6–9].

Recently, an effective fractional mathematical model for the transition dynamics of COVID-19 is investigated for diverse compartments [10]. A computational study for the dynamics of some fractional systems was offered with interesting applications [11]. Also, the Haar wavelet collocation method provides an efficient and sustainable methodology for approximate solutions of certain systems of fractional mathematical models [12].

Recent articles using a range of techniques (contraction mapping and Krasnoselskii's fixed point, common fixed point theorems, variational approaches, etc.) have examined the existence of nonlinear Langevin equation solutions for a variety of boundary conditions. Ahmad and Nieto [13, 14], discussed the solvability of Langevin nonlinear equation including two fractional orders in different intervals with boundary conditions of Dirichlet. The initial value problems for the fractional Langevin nonlinear equation were studied by Yu et al. [15] and Baghani [16]. Torres [17], offered a variational approach to examine the existence of solution for Langevin fractional equation. Barakat et al. [18], employed the general Hadamard-Caputo fractional operators to examine the the Hyres-Ulam stability, existence, and uniqueness for the solution of Langevin fractional equation.

Because fractional calculus has many qualities that classical calculus does not, there is growing interest in utilizing it to describe real-world occurrences. Most fractional-order derivatives, in contrast to integer-order derivatives, are non-local and include memory effects. This makes them more advantageous since, in many cases, the future state of the model depends not only on the present state but also on the model's past behavior. In a variety of scientific and technical disciplines, the use of fractional-order systems to simulate the behavior of real systems is growing in popularity due to this realistic quality.

The equation of Langevin has been vastly used to characterize the development of physical processes in fluctuating milieus. But the Langevin equation in ordinary derivative form fails to provide an accurate description of some complex dynamical systems. Many extensions of Langevin equations have been introduced to characterize dynamical phenomena in a fractal environments. One such extension is the fractional Langevin equation which is given by a fractional derivative instead of the ordinary derivative. The solutions of fractional version of Langevin equation gives a more adaptable model to fractal processes as contrasted with the standard one described by a fractional index [19, 20].

This motivates us to introduce a new fractional version of Langevin equation by using recent extensive fractional operators. According to Banach contraction principle and Krasnoselskii fixed point theorem, we study the existence and uniqueness of the solution to the suggested fractional Langevin equation. The boundary conditions are assumed as nonperiodic and nonlocal integral conditions. Moreover, Hyres-Ulam stability for this solution is defined and analyzed in a detailed manner.

In an easy way, our outcomes can be reduced into many results obtained for Langevin equation in previously research works.

2. Auxiliary fractional concepts

This section involves some fractional concepts that help us to comprehend the main results of the present work. Let $\rho \in \mathbb{C}$, $\text{Re}(\rho) > 0$, and $\pi \in (0, 1]$. Consider the continuous function σ: $[0, \infty) \times (0, 1] \to \mathbb{R}$ with the advantages: $\sigma(\tau, 1) = 1 \ \forall \ \tau \in [0, \infty)$, $\sigma(\tau, \pi) \neq 0 \ \forall \ (\tau, \pi) \in [0, \infty) \times (0, 1]$, and $\sigma(\tau, \pi_1) \neq \sigma(\tau, \pi_2)$ whenever $\pi_1, \pi_2 \in (0, 1]$ and $\pi_1 \neq \pi_2$.

Therefore, the definitions and essential features for the extensive fractional operators are offered as follows.

Definition 2.1. *[21] Let Z:* $[0, \infty) \to \mathbb{R}$ *be a continuous function. The extensive fractional integral operator of Z is defined as:*

$$I_{\sigma}^{\rho,\pi}(Z(\tau)) = \frac{1}{\Gamma(\rho)} \int_0^{\tau} \Delta^{\rho-1}(\tau, u, \pi) \frac{Z(u)}{\sigma(u, \pi)} du, \tag{2.1}$$

where

$$\Delta(\tau, u, \pi) = \int_u^{\tau} \frac{dv}{\sigma(v, \pi)}. \tag{2.2}$$

Definition 2.2. *[21] In the Riemann–Liouville concept, the extensive fractional differential operator for the function Z is given as:*

$$D_{\sigma}^{\rho,\pi}(Z(\tau)) = \frac{1}{\Gamma(m-\rho)} D_{\sigma}^{m,\pi} \left(\int_0^{\tau} \Delta^{m-\rho-1}(\tau, u, \pi) \frac{Z(u)}{\sigma(u, \pi)} du \right), \tag{2.3}$$

where $m = [\text{Re}(\rho)] + 1$, $D_{\sigma}^{\pi} Z(t) = \sigma(t, \pi) \frac{dZ}{dt}$ and $D_{\sigma}^{m,\pi} Z(t) = D_{\sigma}^{\pi}.D_{\sigma}^{\pi}...D_{\sigma}^{\pi}$ (m-times).

Definition 2.3. *[21] Suppose that the set of all absolutely continuous functions on* $[0, a]$ *and whose derivative of order* $(m-1)$ *is absolutely continuous are denoted by* $\mathcal{W}_{\sigma}^{m,\pi}([0, a])$*, where* $m \in N$*. In the Caputo sense, the extensive fractional differential operator for the function* $Z \in \mathcal{W}_{\sigma}^{m,\pi}([0, a])$ *is defined as:*

$${}^{C}D_{\sigma}^{\rho,\pi}(Z(\tau)) = \frac{1}{\Gamma(m-\rho)} \int_0^{\tau} \Delta^{m-\rho-1}(\tau, u, \pi) \frac{D_{\sigma}^{m,\pi}(Z(u))du}{\sigma(u, \pi)}, \tag{2.4}$$

where $\pi > 0$*,* $Re[\rho] > 0$*, and* $m = Re[\rho] + 1$*.*

The following lemmas were recently presented by Hyder and Barakat in [21].

Lemma 2.1. *[21] Let* $\rho \in \mathbb{C}$*,* $Re(\rho) > 0$*, and* $\pi \in (0, 1]$*. Consider the continuous function* σ*:* $[0, \infty) \times (0, 1] \to \mathbb{R}$*. Then, for any* $\pi \in (0, 1]$ *we have*

- $|\left(I_{\sigma}^{\rho,\pi}(\Delta^{q-1}(\tau, 0, \pi)) \right)|(Z(\tau)) = \dfrac{\Gamma(q)}{\Gamma(\rho+q)} \Delta^{\rho+q-1}(\tau, 0, \pi).$
- $|\left({}^{C}D_{\sigma}^{\rho,\pi}(\Delta^{q-1}(\tau, 0, \pi)) \right)|(Z(\tau)) == \dfrac{\Gamma(q)}{\Gamma(q-\rho)} \Delta^{q-\rho-1}(\tau, 0, \pi).$

Lemma 2.2. *[21] Assume that* $\rho \in \mathbb{C}$ *with* $Re(\rho) > 0$*,* $m = [Re(\rho)] + 1$ *and* $Z \in \mathcal{W}_{\sigma}^{m,\pi}([0, a])$*. Then*

$$I_{\sigma}^{\rho,\pi} \ ({}^{C}D_{\sigma}^{\rho,\pi} \ Z(t)) = Z(t) - \sum_{r=1}^{m} \frac{D_{\sigma}^{\rho-r,\pi} Z(0)}{\Gamma(\rho - r + 1)} \Delta^{\rho-r}(t, 0, \pi).$$

Furthermore, a solution exists for the next fractional differential equation

$$^{C}D_{\sigma}^{\rho,\pi}Z(t) = 0, \tag{2.5}$$

and it takes the form

$$Z(t) = \sum_{l=0}^{m-1} S_l \Delta^l(t,0,\pi), \tag{2.6}$$

where $S_l = \frac{D_{\sigma}^{l,\pi}Z(0)}{\Gamma(l+1)}$.

Theorem 2.1. *[22] Assume that the Banach space $\mathfrak{C}[0,a]$ has a closed, bounded, convex, and nonempty subset called ∇. If $\mathcal{H}_1, \mathcal{H}_2$ are two operators from ∇ to $\mathfrak{C}[0,a]$ such that:*

- $\mathcal{H}_1 x + \mathcal{H}_2 y \in \nabla$ *for each* $x, y \in \nabla$.
- $\mathcal{H}_1$ *is compact and continuous.*
- $\mathcal{H}_2$ *fulfills a contraction condition.*

Then $\exists\ Z \in \nabla$ *such that* $Z = \mathcal{H}_1 Z + \mathcal{H}_2 Z$.

3. Main results

The following lemma is a crucial tool for our research. Consequently, we should demonstrate it before disclosing our results.

Lemma 3.1. *The integral version of the following fractional Langevin equation*

$$\begin{cases} ^{C}D_{\sigma}^{\theta,\pi}(^{C}D_{\sigma}^{\rho,\pi} + \nu)Z(t) = \phi(t, Z(t));\ t \in [0,1],\ \nu > 0,\ 2 < \theta < 3,\ 0 < \rho < 2, \\ Z(0) = 0, Z'(0) = 0,\ \ Z''(0) = 0, \\ ^{C}D_{\sigma}^{\rho,\pi}Z(0) = I_{\sigma}^{\delta,\pi}Z(\eta),\ \ 0 < \eta < 1,\ \delta > 0, \\ ^{C}D_{\sigma}^{\rho,\pi}Z(a) + kZ(a) = 0,\ \ a > 0,\ k \in \mathbb{R},\ k \neq \nu, \end{cases} \tag{3.1}$$

has the following solution

$$\begin{aligned} Z(t) &= \frac{1}{\Gamma(\theta+\rho)} \int_0^t \Delta^{\theta+\rho-1}(t,u,\pi) \frac{\phi(u, Z(u))}{\sigma(u,\pi)} du - \frac{\nu}{\Gamma(\rho)} \int_0^t \Delta^{\rho-1}(t,u,\pi) \frac{Z(u)}{\sigma(u,\pi)} du \\ &+ \frac{\Delta^{\rho+1}(t,0,\pi)}{(k-\nu)\Delta^{\rho+1}(a,0,\pi) + \Delta(a,0,\pi)\Gamma(\rho+2)} \Big(\frac{(\nu-k)}{\Gamma(\theta+\rho)} \int_0^a \Delta^{\theta+\rho-1}(a,u,\pi) \frac{\phi(u,Z(u))}{\sigma(u,\pi)} du \\ &- \frac{1}{\Gamma(\theta)} \int_0^a \Delta^{\theta-1}(a,u,\pi) \frac{\phi(u,Z(u))}{\sigma(u,\pi)} du - \frac{\nu(\nu-k)}{\Gamma(\rho)} \int_0^a \Delta^{\rho-1}(a,u,\pi) \frac{Z(u)}{\sigma(u,\pi)} du \\ &+ \frac{(\nu-k)\Delta^{\rho}(a,0,\pi) - \Gamma(\rho+1)}{\Gamma(\rho+1)\Gamma(\delta)} \int_0^{\eta} \Delta^{\delta-1}(\eta,u,\pi) \frac{Z(u)}{\sigma(u,\pi)} du \Big) \\ &+ \frac{\Delta^{\rho}(t,0,\pi)}{\Gamma(\rho+1)\Gamma(\delta)} \int_0^{\eta} \Delta^{\delta-1}(\eta,u,\pi) \frac{Z(u)}{\sigma(u,\pi)} du. \end{aligned}$$

Proof. When we solve Eq (3.1) using the extensive fractional integral operator $I_{\sigma}^{\theta,\pi}$, we get

$$({}^{C}D_{\sigma}^{\rho,\pi}+\nu)Z(t)=I_{\sigma}^{\theta,\pi}\phi(t,Z(t))+\Big[A_0+A_1\Delta(t,0,\pi)\Big]. \tag{3.2}$$

Where $A_0, A_1 \in$ R. Once more by employing the extensive fractional integral operator of order ρ to Eq (3.2), we acquire

$$Z(t)=I_{\sigma}^{\theta+\rho,\pi}\phi(t,Z(t))-\nu I_{\sigma}^{\rho,\pi}Z(t)+I_{\sigma}^{\rho,\pi}(A_0)+I_{\sigma}^{\rho,\pi}(A_1\Delta(t,0,\pi)). \tag{3.3}$$

Since

$$I_{\sigma}^{\rho,\pi}(A_0)=\frac{A_0(\Delta(t,0,\pi))^{\rho}}{\Gamma(\rho+1)}, \tag{3.4}$$

and

$$I_{\sigma}^{\rho,\pi}(A_1\Delta(t,0,\pi))=\frac{A_1(\Delta(t,0,\pi))^{\rho+1}}{\Gamma(\rho+2)}. \tag{3.5}$$

By utilizing (3.4) and (3.5). Then Eq (3.3) appears to take on the following form

$$Z(t)=I_{\sigma}^{\theta+\rho,\pi}\phi(t,Z(t))-\nu I_{\sigma}^{\rho,\pi}Z(t)+\frac{A_0(\Delta(t,0,\pi))^{\rho}}{\Gamma(\rho+1)}+\frac{A_1(\Delta(t,0,\pi))^{\rho+1}}{\Gamma(\rho+2)}$$
$$+A_2(\Delta(t,0,\pi))^2+A_3\Delta(t,0,\pi)+A_4,$$

where the real constants A_0, A_1, A_2, A_3 and A_4 can be determined by using the boundary conditions from (3.2) as follows:

$$Z(0)=0, Z'(0)=0 \text{ and } Z''(0)=0 \text{ implies } A_4=0, A_3=0, \text{ and } A_2=0.$$

Therefore,

$$Z(t)=I_{\sigma}^{\theta+\rho,\pi}\phi(t,Z(t))-\nu I_{\sigma}^{\rho,\pi}Z(t)+\frac{A_0(\Delta(t,0,\pi))^{\rho}}{\Gamma(\rho+1)}+\frac{A_1(\Delta(t,0,\pi))^{\rho+1}}{\Gamma(\rho+2)}. \tag{3.6}$$

On the other hand using the boundary conditions

$${}^{C}D_{\sigma}{}^{\rho,\pi}Z(0)=I_{\sigma}^{\delta,\pi}Z(\eta), \quad 0<\eta<1,\ \delta>0,$$

and

$${}^{C}D_{\sigma}^{\rho,\pi}Z(a)+kZ(a)=0, \quad k\in\mathbb{R},\ k\neq\nu.$$

Setting $t=0$ in (3.2) we obtain

$$A_0=I_{\sigma}^{\delta,\pi}Z(\eta). \tag{3.7}$$

And, using $t=a$ in (3.2), (3.6) and from Eq (3.7), we have

$$A_1=\frac{\Gamma(\rho+2)}{(k-\nu)\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2)}\Big(\frac{(\nu-k)}{\Gamma(\theta+\rho)}\int_0^a\Delta^{\theta+\rho-1}(a,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du$$
$$-\frac{1}{\Gamma(\theta)}\int_0^a\Delta^{\theta-1}(a,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du-\frac{\nu(\nu-k)}{\Gamma(\rho)}\int_0^a\Delta^{\rho-1}(a,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du$$
$$+\frac{(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^{\eta}\Delta^{\delta-1}(\eta,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du\Big). \tag{3.8}$$

Hence, the following equation is obtained by substituting the values from the Eqs (3.7) and (3.8) in Eq (3.6)

$$\begin{aligned}
Z(t) &= \frac{1}{\Gamma(\theta+\rho)}\int_0^t \Delta^{\theta+\rho-1}(t,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du - \frac{\nu}{\Gamma(\rho)}\int_0^t \Delta^{\rho-1}(t,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du \\
&+ \frac{\Delta^{\rho+1}(t,0,\pi)}{(k-\nu)\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2)}\Big(\frac{(\nu-k)}{\Gamma(\theta+\rho)}\int_0^a \Delta^{\theta+\rho-1}(a,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du \\
&- \frac{1}{\Gamma(\theta)}\int_0^a \Delta^{\theta-1}(a,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du - \frac{\nu(\nu-k)}{\Gamma(\rho)}\int_0^a \Delta^{\rho-1}(a,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du \\
&+ \frac{(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du\Big) \\
&+ \frac{\Delta^{\rho}(t,0,\pi)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du.
\end{aligned} \tag{3.9}$$

□

The Bananch space of all continuous functions $\mathfrak{g}$ such that $\mathfrak{g}$: $[0,a] \to \mathbb{R}$ is now taken into consideration, and it is denoted by $(\mathfrak{C}[0,a], \|.\|)$ and its norm is indicated by $\|\mathfrak{g}\| = \sup_{t\in[0,a]} \mathfrak{g}(t)$.

To start analyzing our results First, let's define the operator Let's start by defining an operator $\mathcal{H}$: $\mathfrak{C}[0,a] \to \mathfrak{C}[0,a]$ as shown below

$$\begin{aligned}
\mathcal{H}Z(t) &= \frac{1}{\Gamma(\theta+\rho)}\int_0^t \Delta^{\theta+\rho-1}(t,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du - \frac{\nu}{\Gamma(\rho)}\int_0^t \Delta^{\rho-1}(t,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du \\
&+ \frac{\Delta^{\rho+1}(t,0,\pi)}{(k-\nu)\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2)}\Big(\frac{(\nu-k)}{\Gamma(\theta+\rho)}\int_0^a \Delta^{\theta+\rho-1}(a,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du \\
&- \frac{1}{\Gamma(\theta)}\int_0^a \Delta^{\theta-1}(a,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du - \frac{\nu(\nu-k)}{\Gamma(\rho)}\int_0^a \Delta^{\rho-1}(a,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du \\
&+ \frac{(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du\Big) \\
&+ \frac{\Delta^{\rho}(t,0,\pi)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du.
\end{aligned} \tag{3.10}$$

It should be emphasized that the Langevin Eq (3.1) has a solution if the operator $\mathcal{H}$ has a fixed point.

3.1. Fixed point result's existence

The following theorem is now presented in order to validate our existence result using the fixed point approach.

Theorem 3.1. *Assume that the continuous function* ϕ: $[0,a]\times\mathbb{R} \to \mathbb{R}$ *is satisfied with the following conditions:*

- *(I)* $|\phi(t,Z(t))| \leq M(t)\ \forall(t,Z(t)) \in [0,a]\times\mathbb{R}$ *and* $M(t) \in (\mathfrak{C}[0,a],\mathbb{R}^+)$, *with* $\sup_t |M(t)| = \|M\|$.
- *(II)* $|\phi(t,Z(t)) - \phi(t,Z_1(t))| \leq \lambda|Z(t)-Z_1(t)|\ \forall t\in[0,a],\ \ Z,Z_1 \in \mathfrak{C}[0,a]$ *and* $\lambda > 0$.

Therefore, there is at least one solution on $[0, a]$ *for the fractional Langevin Eq (3.1) if*

$$\mathfrak{L}_3 < 1,$$

where

$$\begin{aligned}\mathfrak{L}_3 = &\frac{\lambda|\nu-k|\Lambda^{2\rho+\theta}}{(|k-\nu|\Lambda^{\rho}+\Gamma(\rho+2))\Gamma(\theta+\rho+1)} + \frac{\lambda\Lambda^{\rho+\theta}}{(|k-\nu|\Lambda^{\rho}+\Gamma(\rho+2))\Gamma(\theta+1)}\\ &+ \frac{\nu|\nu-k|\Lambda^{2\rho}}{(|k-\nu|\Lambda^{\rho}+\Gamma(\rho+2))\Gamma(\rho+1)} + \frac{(|\nu-k|\Lambda^{\rho}+\Gamma(\rho+1))\Lambda^{\delta+\rho}}{(|k-\nu|\Lambda^{\rho}+\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)}\\ &+ \frac{\Lambda^{\rho+\delta}}{\Gamma(\rho+1)\Gamma(\delta+1)}.\end{aligned}$$

Proof. Choose $\Lambda = \sup_{t\in[0,a} \Delta(t, 0, \pi)$ and $\epsilon = \frac{\mathfrak{L}_1\|M\|}{1-\mathfrak{L}_2}$, where

$$\mathfrak{L}_1 = \Big\{\frac{\Lambda^{\theta+\rho}}{\Gamma(\theta+\rho+1)} + \frac{|\nu-k|\Lambda^{\theta+2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+\rho+1)} + \frac{\Lambda^{\theta+\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+1)}\Big\},$$

$$\begin{aligned}\mathfrak{L}_2 = &\Big\{\frac{|\nu|\Lambda^{\rho}}{\Gamma(\rho+1)} + \frac{\nu|\nu-k|\Lambda^{2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)}\\ &+ \frac{(|\nu-k|\Lambda^{\rho}+\Gamma(\rho+1))\Lambda^{\delta+\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)}\\ &+ \frac{\Lambda^{\delta+2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)}\Big\}.\end{aligned}$$

The closed ball $\overline{\mathcal{B}}_\epsilon$ is described as $\overline{\mathcal{B}}_\epsilon = \{u \in \mathfrak{C}[0, a]\colon \|u\| \leq \epsilon\}$.
The following is a description of the operators $\mathcal{H}_1, \mathcal{H}_2 \in \overline{\mathcal{B}}_\epsilon$, when combined:

$$\mathcal{H}_1 Z(t) = \frac{1}{\Gamma(\theta+\rho)}\int_0^t \Delta^{\theta+\rho-1}(t,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du - \frac{\nu}{\Gamma(\rho)}\int_0^t \Delta^{\rho-1}(t,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du,$$

and

$$\begin{aligned}\mathcal{H}_2 Z(t) = &\frac{\Delta^{\rho+1}(t,0,\pi)}{(k-\nu)\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2)}\Big(\frac{(\nu-k)}{\Gamma(\theta+\rho)}\int_0^a \Delta^{\theta+\rho-1}(a,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du\\ &- \frac{1}{\Gamma(\theta)}\int_0^a \Delta^{\theta-1}(a,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du - \frac{\nu(\nu-k)}{\Gamma(\rho)}\int_0^a \Delta^{\rho-1}(a,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du\\ &+ \frac{(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du\Big)\\ &+ \frac{\Delta^{\rho}(t,0,\pi)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du.\end{aligned}$$

From the linearity of Banach space, we have $\mathcal{H}z(t) = \mathcal{H}_1 z(t) + \mathcal{H}_2 z(t) \in \overline{\mathcal{B}}_\epsilon$ on $[0, a]$.
To complete our proof firstly, we'll demonstrate that

$$\mathcal{H}Z(t) = \mathcal{H}_1 Z_1(t) + \mathcal{H}_2 Z_2(t) \in \overline{\mathcal{B}}_\epsilon; \forall Z_1, Z_2 \in \overline{\mathcal{B}}_\epsilon.$$

$$
\begin{aligned}
\|\mathcal{H}_1Z_1+\mathcal{H}_2Z_2\| &= \sup_{t\in[0,a]} |\mathcal{H}_1Z_1(t)+\mathcal{H}_2Z_2(t)|\\
&= \sup_{t\in[0,a]} \Big\{\Big|\frac{1}{\Gamma(\theta+\rho)}\int_0^t \Delta^{\theta+\rho-1}(t,u,\pi)\frac{\phi(u,Z_1(u))}{\sigma(u,\pi)}du - \frac{\nu}{\Gamma(\rho)}\int_0^t \Delta^{\rho-1}(t,u,\pi)\frac{Z_1(u)}{\sigma(u,\pi)}du\\
&+ \frac{\Delta^{\rho+1}(t,0,\pi)}{(k-\nu)\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2)}\Big(\frac{(\nu-k)}{\Gamma(\theta+\rho)}\int_0^a \Delta^{\theta+\rho-1}(a,u,\pi)\frac{\phi(u,Z_2(u))}{\sigma(u,\pi)}du\\
&- \frac{1}{\Gamma(\theta)}\int_0^a \Delta^{\theta-1}(a,u,\pi)\frac{\phi(u,Z_2(u))}{\sigma(u,\pi)}du - \frac{\nu(\nu-k)}{\Gamma(\rho)}\int_0^a \Delta^{\rho-1}(a,u,\pi)\frac{Z_2(u)}{\sigma(u,\pi)}du\\
&+ \frac{(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{Z_2(u)}{\sigma(u,\pi)}du\Big)\\
&+ \frac{\Delta^{\rho}(t,0,\pi)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{Z_2(u)}{\sigma(u,\pi)}du\Big|\Big\}\\
&\le \sup_{t\in[0,a]} \Big\{\frac{1}{\Gamma(\theta+\rho)}\int_0^t \Delta^{\theta+\rho-1}(t,u,\pi)\frac{|\phi(u,Z_1(u))|}{\sigma(u,\pi)}du + \frac{|\nu|}{\Gamma(\rho)}\int_0^t \Delta^{\rho-1}(t,u,\pi)\frac{|Z_1(u)|}{\sigma(u,\pi)}du\\
&+ \frac{\Delta^{\rho+1}(t,0,\pi)}{|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2)}\Big(\frac{|\nu-k|}{\Gamma(\theta+\rho)}\int_0^a \Delta^{\theta+\rho-1}(a,u,\pi)\frac{|\phi(u,Z_2(u))|}{\sigma(u,\pi)}du\\
&+ \frac{1}{\Gamma(\theta)}\int_0^a \Delta^{\theta-1}(a,u,\pi)\frac{|\phi(u,Z_2(u))|}{\sigma(u,\pi)}du + \frac{\nu|\nu-k|}{\Gamma(\rho)}\int_0^a \Delta^{\rho-1}(a,u,\pi)\frac{|Z_2(u)|}{\sigma(u,\pi)}du\\
&+ \frac{|(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)|}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{|Z_2(u)|}{\sigma(u,\pi)}du\Big)\\
&+ \frac{\Delta^{\rho}(t,0,\pi)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{|Z_2(u)|}{\sigma(u,\pi)}du|\Big\}.
\end{aligned}
$$

Using condition *I*, we have

$$
\begin{aligned}
\|\mathcal{H}_1Z_1+\mathcal{H}_2Z_2\| \le \|M\| &\sup_{t\in[0,a]} \Big\{\frac{1}{\Gamma(\theta+\rho)}\int_0^t \Delta^{\theta+\rho-1}(t,u,\pi)\frac{du}{\sigma(u,\pi)}\\
&+ \frac{|\nu-k|\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+\rho)}\int_0^a \Delta^{\theta+\rho-1}(a,u,\pi)\frac{du}{\sigma(u,\pi)}\\
&+ \frac{\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta)}\int_0^a \Delta^{\theta-1}(a,u,\pi)\frac{du}{\sigma(u,\pi)}\Big\}\\
&+ \epsilon \sup_{t\in[0,a]} \Big\{\Big(\frac{|\nu|}{\Gamma(\rho)} + \frac{\nu|\nu-k|\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho)}\Big)\int_0^a \Delta^{\rho-1}(a,u,\pi)\frac{du}{\sigma(u,\pi)}\\
&+ \Big(\frac{(|(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)|)\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta)}\\
&+ \frac{\Delta^{2\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta)}\Big)\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{du}{\sigma(u,\pi)}\Big\}.
\end{aligned}
$$

Hence,

$$\begin{aligned}
&\|\mathcal{H}_1 Z_1 + \mathcal{H}_2 Z_2\| \\
&\le \|M\|\Big\{\frac{\sup_{t\in[0,a]}\Delta^{\theta+\rho}(t,0,\pi)}{\Gamma(\theta+\rho+1)} + \frac{|\nu-k|\sup_{t\in[0,a]}\Delta^{\rho+1}(t,0,\pi)\Delta^{\theta+\rho}(a,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+\rho+1)} \\
&+ \frac{\sup_{t\in[0,a]}\Delta^{\rho+1}(t,0,\pi)\Delta^{\theta}(a,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+1)}\Big\} \\
&+ \epsilon\Big\{\frac{|\nu|\Delta^{\rho}(a,0,\pi)}{\Gamma(\rho+1)} + \frac{\nu|\nu-k|\Delta^{\rho}(a,0,\pi)\sup_{t\in[0,a]}\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)} \\
&+ \frac{(|(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)|)\sup_{t\in[0,a]}\Delta^{\rho+1}(t,0,\pi)\Delta^{\delta}(\eta,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)} \\
&+ \frac{\sup_{t\in[0,a]}\Delta^{2\rho+1}(t,0,\pi)\Delta^{\delta}(\eta,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)}\Big\} \\
&\le \|M\|\Big\{\frac{\Lambda^{\theta+\rho}}{\Gamma(\theta+\rho+1)} + \frac{|\nu-k|\Lambda^{\theta+2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+\rho+1)} \\
&+ \frac{\Lambda^{\theta+\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+1)}\Big\} \\
&+ \epsilon\Big\{\frac{|\nu|\Lambda^{\rho}}{\Gamma(\rho+1)} + \frac{\nu|\nu-k|\Lambda^{2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)} \\
&+ \frac{(|\nu-k|\Lambda^{\rho}+\Gamma(\rho+1))\Lambda^{\delta+\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)} \\
&+ \frac{\Lambda^{\delta+2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)}\Big\} \\
&\le \|M\|\mathfrak{L}_1 + \epsilon\mathfrak{L}_2 \le \epsilon.
\end{aligned}$$

Thus,

$$\mathcal{H}Z(t) = \mathcal{H}_1 Z_1(t) + \mathcal{H}_2 Z_2(t) \in \overline{\mathcal{B}}_\epsilon.$$

The second step is to show that the contraction operator $\mathcal{H}_2$ exists. Let's say that there are two elements in the Banach space $(C[0,a],\|.\|)$ called w_1 and w_2 such that

$$\begin{aligned}
\|\mathcal{H}_2 w_1 - \mathcal{H}_2 w_2\| &= \sup_{t\in[0,a]}|\mathcal{H}_2 w_1(t) - \mathcal{H}_2 w_2(t)| \\
&\le \frac{\sup_{t\in[0,a]}\Delta^{\rho+1}(t,0,\pi)}{|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2)}\Big(\frac{|\nu-k|}{\Gamma(\theta+\rho)}\int_0^a \Delta^{\theta+\rho-1}(a,u,\pi)\frac{|\phi(u,w_1(u))-\phi(u,w_1(u))|}{\sigma(u,\pi)}du \\
&+ \frac{1}{\Gamma(\theta)}\int_0^a \Delta^{\theta-1}(a,u,\pi)\frac{|\phi(u,w_1(u))-\phi(u,w_1(u))|}{\sigma(u,\pi)}du + \frac{\nu|\nu-k|}{\Gamma(\rho)}\int_0^a \Delta^{\rho-1}(a,u,\pi)\frac{|w_1(u)-w_2(u)|}{\sigma(u,\pi)}du \\
&+ \frac{|\nu-k|\Delta^{\rho}(a,0,\pi)+\Gamma(\rho+1)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{|w_1(u)-w_2(u)|}{\sigma(u,\pi)}du\Big) \\
&+ \frac{\Delta^{\rho}(t,0,\pi)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \sup_{t\in[0,a]}\Delta^{\delta-1}(\eta,u,\pi)\frac{|w_1(u)-w_2(u)|}{\sigma(u,\pi)}du.
\end{aligned}$$

Using condition (II) we obtain

$$\begin{aligned}\|\mathcal{H}_2 w_1 - \mathcal{H}_2 w_2\| \leq \Big\{ & \frac{\lambda|\nu - k|\Lambda^{2\rho+\theta}}{(|k-\nu|\Lambda^{\rho} + \Gamma(\rho+2))\Gamma(\theta+\rho+1)} + \frac{\lambda\Lambda^{\rho+\theta}}{(|k-\nu|\Lambda^{\rho} + \Gamma(\rho+2))\Gamma(\theta+1)} \\ &+ \frac{\nu|\nu-k|\Lambda^{2\rho}}{(|k-\nu|\Lambda^{\rho} + \Gamma(\rho+2))\Gamma(\rho+1)} + \frac{(|\nu-k|\Lambda^{\rho} + \Gamma(\rho+1))\Lambda^{\delta+\rho}}{(|k-\nu|\Lambda^{\rho} + \Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)} \\ &+ \frac{\Lambda^{\rho+\delta}}{\Gamma(\rho+1)\Gamma(\delta+1)} \Big\} \|w_1 - w_2\| \leq \mathfrak{L}_3 \|w_1 - w_2\|.\end{aligned}$$

Given that $\mathfrak{L}_3 < 1$ is assumed to be greater than 1, $\mathcal{H}_2$ is therefore a contraction map.

Finally, we aim to demonstrate the continuity and compactness of $\mathcal{H}_1$. Starting by continuity, the operator $\mathcal{H}_1$ is continuous because ϕ is a continuous function on $t \in [0, a]$.

To be considered compact, $\mathcal{H}_1$ must be equicontinuous on $\overline{\mathcal{B}}_\epsilon$ and uniformly bounded.

$$\begin{aligned}\|\mathcal{H}_1 Z\| &= \sup_{t\in[0,a]} |\mathcal{H}_1 Z(t)| \\ &\leq \sup_{t\in[0,a]} \Big\{ \frac{1}{\Gamma(\theta+\rho)} \int_0^t \Delta^{\theta+\rho-1}(t,u,\pi) \frac{|\phi(u, Z(u))|}{\sigma(u,\pi)} du + \frac{\nu}{\Gamma(\rho)} \int_0^t \Delta^{\rho-1}(t,u,\pi) \frac{|Z(u)|}{\sigma(u,\pi)} du \\ &\leq \|M\| \sup_{t\in[0,a]} \Big\{ \frac{1}{\Gamma(\theta+\rho)} \int_0^t \Delta^{\theta+\rho-1}(t,u,\pi) \frac{du}{\sigma(u,\pi)} \Big\} + \epsilon \sup_{t\in[0,a]} \Big\{ \frac{\nu}{\Gamma(\rho)} \int_0^t \Delta^{\rho-1}(t,u,\pi) \frac{du}{\sigma(u,\pi)} \Big\} \\ &\leq \frac{\|M\|\Lambda^{\theta+\rho}}{\Gamma(\theta+\rho+1)} + \frac{\nu\epsilon\Lambda^{\rho}}{\Gamma(\rho+1)} < \infty.\end{aligned}$$

This demonstrates how uniformly bounded $\mathcal{H}_1$ is.

The operator $\mathcal{H}_1$'s compactness is then illustrated. We obtain for each $0 < t_1 < t_2 < a$,

$$\begin{aligned}|\mathcal{H}_1 Z(t_1) - \mathcal{H}_1 Z(t_2)| = \Big| & \frac{1}{\Gamma(\theta+\rho)} \int_0^{t_1} \Delta^{\theta+\rho-1}(t_1,u,\pi) \frac{\phi(u,Z(u))}{\sigma(u,\pi)} du - \frac{\nu}{\Gamma(\rho)} \int_0^{t_1} \Delta^{\rho-1}(t_1,u,\pi) \frac{|Z(u)|}{\sigma(u,\pi)} du \\ &- \frac{1}{\Gamma(\theta+\rho)} \int_0^{t_2} \Delta^{\theta+\rho-1}(t_2,u,\pi) \frac{|\phi(u,Z(u))|}{\sigma(u,\pi)} du + \frac{\nu}{\Gamma(\rho)} \int_0^{t_2} \Delta^{\rho-1}(t_2,u,\pi) \frac{|Z(u)|}{\sigma(u,\pi)} du \Big|,\end{aligned}$$

$$\begin{aligned}|\mathcal{H}_1 Z(t_1) - \mathcal{H}_1 Z(t_2)| = \Big| & \frac{1}{\Gamma(\theta+\rho)} \int_0^{t_1} \left[\Delta^{\theta+\rho-1}(t_1,u,\pi) - \Delta^{\theta+\rho-1}(t_2,u,\pi) \right] \frac{\phi(u,Z(u))}{\sigma(u,\pi)} du \\ &- \frac{1}{\Gamma(\theta+\rho)} \int_{t_1}^{t_2} \Delta^{\theta+\rho-1}(t_2,u,\pi) \frac{\phi(u,Z(u))}{\sigma(u,\pi)} du \\ &+ \frac{\nu}{\Gamma(\rho)} \int_0^{t_1} \left[\Delta^{\rho-1}(t_2,u,\pi) - \Delta^{\rho-1}(t_1,u,\pi) \right] \frac{Z(u)}{\sigma(u,\pi)} du \\ &+ \frac{\nu}{\Gamma(\rho)} \int_{t_1}^{t_2} \Delta^{\rho-1}(t_2,u,\pi) \frac{Z(u)}{\sigma(u,\pi)} du \Big| \\ \leq & \frac{\|M\|}{\Gamma(\theta+\rho)} \Big\{ \int_0^{t_1} \left| \Delta^{\theta+\rho-1}(t_1,u,\pi) - \Delta^{\theta+\rho-1}(t_2,u,\pi) \right| \frac{du}{\sigma(u,\pi)} \\ &+ \int_{t_1}^{t_2} \Delta^{\theta+\rho-1}(t_2,u,\pi) \frac{du}{\sigma(u,\pi)} \Big\}\end{aligned}$$

$$+\frac{\nu\epsilon}{\Gamma(\rho)}\Big\{\int_0^{t_1}\Big|\Delta^{\rho-1}(t_2,u,\pi)-\Delta^{\rho-1}(t_1,u,\pi)\Big|\frac{du}{\sigma(u,\pi)}+\int_{t_1}^{t_2}\Delta^{\rho-1}(t_2,u,\pi)\frac{du}{\sigma(u,\pi)}\Big\}.$$

When we consider the limit as t_2 approaches to t_1, we get $|\mathcal{H}_1 z(t_1)-\mathcal{H}_1 z(t_2)|$ tends to zero.

Hence, $\mathcal{H}_1$ is an equicontinuous. Based on the Arzela Ascoli theorem, we can say that $\mathcal{H}_1$ compacts on $\overline{\mathcal{B}}_\epsilon$. Thus, there is a point Z in $\overline{\mathcal{B}}_\epsilon$ such that $Z=\mathcal{H}Z$. Eq (3.10) therefore has at least one solution on $[0,a]$. □

3.2. Result of uniqueness

Theorem 3.2. *Consider the following continuous function $\phi:[0,a]\times\mathbb{R}\to\mathbb{R}$ satisfying the assumptions (I), (II), and (III) $\chi=\sup_{t\in[0,a]}|\phi(t,0)|<\infty$.*

Therefore, there is a unique solution on [0, a] for the Fractional Langevin Eq (3.1), if

$$\lambda\mathfrak{L}_1+\mathfrak{L}_2\le 1. \tag{3.11}$$

Proof. To identify the bounded, convex and closed ball $\overline{\mathcal{B}}_{\epsilon_1}=\{Z\in C[0,a]:\ \|Z\|\le\epsilon_1\}$, take $\epsilon_1>\frac{\mathfrak{L}_1\chi}{1-\lambda\mathfrak{L}_1-\mathfrak{L}_2}$.

So, we begin by showing that $\mathcal{H}$ is contractive and then demonstrate $\mathcal{H}\overline{\mathcal{B}}_{\epsilon_1}$ is bounded. Thus, for all $Z_1,Z_2\in C[0,a]$ we get

$$\begin{aligned}|\mathcal{H}Z_1(t)-\mathcal{H}Z_2(t)|\le\sup_{t\in[0,a]}\Big\{&\frac{1}{\Gamma(\theta+\rho)}\int_0^t\Delta^{\theta+\rho-1}(t,u,\pi)\frac{|\phi(u,Z_1(u))-\phi(u,Z_2(u))|}{\sigma(u,\pi)}du\\&-\frac{\nu}{\Gamma(\rho)}\int_0^t\Delta^{\rho-1}(t,u,\pi)\frac{|Z_1(u)-Z_2|}{\sigma(u,\pi)}du+\frac{\Delta^\rho(t,0,\pi)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta\Delta^{\delta-1}(\eta,u,\pi)\frac{|Z_1(u)-Z_2|}{\sigma(u,\pi)}du\\&+\frac{\Delta^{\rho+1}(t,0,\pi)}{(k-\nu)\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2)}\Big(\frac{(\nu-k)}{\Gamma(\theta+\rho)}\int_0^a\Delta^{\theta+\rho-1}(a,u,\pi)\frac{|\phi(u,Z_1(u))-\phi(u,Z_2(u))|}{\sigma(u,\pi)}du\\&-\frac{1}{\Gamma(\theta)}\int_0^a\Delta^{\theta-1}(a,u,\pi)\frac{|\phi(u,Z_1(u))-\phi(u,Z_2(u))|}{\sigma(u,\pi)}du-\frac{\nu(\nu-k)}{\Gamma(\rho)}\int_0^a\Delta^{\rho-1}(a,u,\pi)\frac{|Z_1(u)-Z_2|}{\sigma(u,\pi)}du\\&+\frac{(\nu-k)\Delta^\rho(a,0,\pi)-\Gamma(\rho+1)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta\Delta^{\delta-1}(\eta,u,\pi)\frac{|Z_1(u)-Z_2|}{\sigma(u,\pi)}du\Big).\end{aligned}$$

$$\begin{aligned}|\mathcal{H}Z_1(t)-\mathcal{H}Z_2(t)|\le(\lambda\|Z_1-Z_2\|)\sup_{t\in[0,a]}\Big\{&\frac{1}{\Gamma(\theta+\rho)}\int_0^t\Delta^{\theta+\rho-1}(t,u,\pi)\frac{du}{\sigma(u,\pi)}\\&+\frac{|\nu-k|\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+\rho)}\int_0^a\Delta^{\theta+\rho-1}(a,u,\pi)\frac{du}{\sigma(u,\pi)}\\&+\frac{\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta)}\int_0^a\Delta^{\theta-1}(a,u,\pi)\frac{du}{\sigma(u,\pi)}\Big\}\\&+\|Z_1-Z_2\|\sup_{t\in[0,a]}\Big\{\Big(\frac{|\nu|}{\Gamma(\rho)}+\frac{\nu|\nu-k|\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho)}\Big)\int_0^a\Delta^{\rho-1}(a,u,\pi)\frac{du}{\sigma(u,\pi)}\\&+\Big(\frac{(|(\nu-k)\Delta^\rho(a,0,\pi)-\Gamma(\rho+1)|)\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta)}\\&+\frac{\Delta^{2\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta)}\Big)\int_0^\eta\Delta^{\delta-1}(\eta,u,\pi)\frac{du}{\sigma(u,\pi)}\Big\}\\\le\Big[\lambda\Big\{&\frac{\Lambda^{\theta+\rho}}{\Gamma(\theta+\rho+1)}+\frac{|\nu-k|\Lambda^{\theta+2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+\rho+1)}\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\Lambda^{\theta+\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+1)}\Big\} \\
&+ \Big\{\frac{|\nu|\Lambda^{\rho}}{\Gamma(\rho+1)} + \frac{\nu|\nu-k|\Lambda^{2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)} \\
&+ \frac{(|\nu-k|\Lambda^{\rho}+\Gamma(\rho+1))\Lambda^{\delta+\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)} \\
&+ \frac{\Lambda^{\delta+2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)}\Big\}\Big]\|Z_1-Z_2\| \\
&\leq (\lambda\mathfrak{L}_1+\mathfrak{L}_2)\|Z_1-Z_2\|,
\end{aligned}
$$

since $\lambda\mathfrak{L}_1+\mathfrak{L}_2<1$. Then $\mathcal{H}$ is contractive operator.

Finally we show that $\mathcal{H}\overline{\mathcal{B}}_{\epsilon_1}$ is bounded for each $t\in[0,a]$ and $Z\in\overline{\mathcal{B}}_{\epsilon_1}$

$$
\begin{aligned}
|\phi(t,Z(t))| &= |\phi(t,Z(t))-\phi(t,0)+\phi(t,0)| \leq |\phi(t,Z(t))-\phi(t,0)|+|\phi(t,0)| \\
&\leq \lambda|Z(t)|+\chi \leq \lambda\epsilon_1+\chi.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\mathcal{H}Z(t)| \leq& \sup_{t\in[1,e]}\Big\{\frac{1}{\Gamma(\theta+\rho)}\int_0^t \Delta^{\theta+\rho-1}(t,u,\pi)\frac{|\phi(u,Z(u))|}{\sigma(u,\pi)}du - \frac{\nu}{\Gamma(\rho)}\int_0^t \Delta^{\rho-1}(t,u,\pi)\frac{|Z(u)|}{\sigma(u,\pi)}du \\
&+ \frac{\Delta^{\rho+1}(t,0,\pi)}{(k-\nu)\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2)}\Big(\frac{(\nu-k)}{\Gamma(\theta+\rho)}\int_0^a \Delta^{\theta+\rho-1}(a,u,\pi)\frac{|\phi(u,Z(u))|}{\sigma(u,\pi)}du \\
&- \frac{1}{\Gamma(\theta)}\int_0^a \Delta^{\theta-1}(a,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du - \frac{\nu(\nu-k)}{\Gamma(\rho)}\int_0^a \Delta^{\rho-1}(a,u,\pi)\frac{|Z(u)|}{\sigma(u,\pi)}du \\
&+ \frac{(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{|Z(u)|}{\sigma(u,\pi)}du\Big) \\
&+ \frac{\Delta^{\rho}(t,0,\pi)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{|Z(u)|}{\sigma(u,\pi)}du\Big\} \\
\leq& (\lambda\epsilon_1+\chi)\Big\{\frac{\Lambda^{\theta+\rho}}{\Gamma(\theta+\rho+1)} + \frac{|\nu-k|\Lambda^{\theta+2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+\rho+1)} \\
&+ \frac{\Lambda^{\theta+\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+1)}\Big\} \\
&+ \epsilon_1\Big\{\frac{|\nu|\Lambda^{\rho}}{\Gamma(\rho+1)} + \frac{\nu|\nu-k|\Lambda^{2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)} \\
&+ \frac{(|\nu-k|\Lambda^{\rho}+\Gamma(\rho+1))\Lambda^{\delta+\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)} \\
&+ \frac{\Lambda^{\delta+2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)}\Big\} \\
\leq& (\lambda\epsilon_1+\chi)\mathfrak{L}_g+\epsilon_1\mathfrak{L}_y<\epsilon_1.
\end{aligned}
$$

This shows that $\|\mathcal{H}w\|$ bounded i.e., $\mathcal{H}\overline{\mathcal{B}}_{\epsilon_1}\subseteq\overline{\mathcal{B}}_{\epsilon_1}$.

Hence, it implies that $\mathcal{H}$ has a single fixed point. Consequently, the Fractal Langevin Eq (3.1) has just one solution. □

4. Hyres-Ulam stability

This section talks about the Hyres-Ulam (H. U.) stability of the fractal Langevin Eq (3.1). The following is what "H. U. stability" means.

Definition 4.1. *The Langevin Eq (3.9) is stated to be H.U. stable. If for a specific constant* $\mathfrak{R} > 0$ $\exists \hbar \geq 0$ *such that, if*

$$\begin{aligned}
|Z(t) - \Big(&\frac{1}{\Gamma(\theta+\rho)}\int_0^t \Delta^{\theta+\rho-1}(t,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du - \frac{\nu}{\Gamma(\rho)}\int_0^t \Delta^{\rho-1}(t,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du \\
&+ \frac{\Delta^{\rho+1}(t,0,\pi)}{(k-\nu)\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2)}\Big(\frac{(\nu-k)}{\Gamma(\theta+\rho)}\int_0^a \Delta^{\theta+\rho-1}(a,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du \\
&- \frac{1}{\Gamma(\theta)}\int_0^a \Delta^{\theta-1}(a,u,\pi)\frac{\phi(u,Z(u))}{\sigma(u,\pi)}du - \frac{\nu(\nu-k)}{\Gamma(\rho)}\int_0^a \Delta^{\rho-1}(a,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du \\
&+ \frac{(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du\Big) \\
&+ \frac{\Delta^{\rho}(t,0,\pi)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{Z(u)}{\sigma(u,\pi)}du\Big)| \leq \mathfrak{R}.
\end{aligned}$$

Then, a continuous function $Z_1(t)$ *that is fulfilling*

$$\begin{aligned}
Z_1(t) = &\frac{1}{\Gamma(\theta+\rho)}\int_0^t \Delta^{\theta+\rho-1}(t,u,\pi)\frac{\phi(u,Z_1(u))}{\sigma(u,\pi)}du - \frac{\nu}{\Gamma(\rho)}\int_0^t \Delta^{\rho-1}(t,u,\pi)\frac{Z_1(u)}{\sigma(u,\pi)}du \\
&+ \frac{\Delta^{\rho+1}(t,0,\pi)}{(k-\nu)\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2)}\Big(\frac{(\nu-k)}{\Gamma(\theta+\rho)}\int_0^a \Delta^{\theta+\rho-1}(a,u,\pi)\frac{\phi(u,Z_1(u))}{\sigma(u,\pi)}du \\
&- \frac{1}{\Gamma(\theta)}\int_0^a \Delta^{\theta-1}(a,u,\pi)\frac{\phi(u,Z_1(u))}{\sigma(u,\pi)}du - \frac{\nu(\nu-k)}{\Gamma(\rho)}\int_0^a \Delta^{\rho-1}(a,u,\pi)\frac{Z_1(u)}{\sigma(u,\pi)}du \\
&+ \frac{(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{Z_1(u)}{\sigma(u,\pi)}du\Big) \\
&+ \frac{\Delta^{\rho}(t,0,\pi)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{Z_1(u)}{\sigma(u,\pi)}du,
\end{aligned}$$

meets the aforementioned prerequisit

$$|Z(t) - Z_1(t)| \leq \mathfrak{R}\hbar. \tag{4.1}$$

Theorem 4.1. *Consider the following continuous function* $\phi : [0,a]\times\mathbb{R} \to \mathbb{R}$ *fulfilling the assumptions (I) and (II). Then the Eq (3.1) is stable under H. U.*

Proof.

$$\begin{aligned}
|Z(t) - Z_1(t)| \leq \sup_{t\in[0,a]}\Big\{&\frac{1}{\Gamma(\theta+\rho)}\int_0^t \Delta^{\theta+\rho-1}(t,u,\pi)\frac{|\phi(u,Z(u)) - \phi(u,Z_1(u))|}{\sigma(u,\pi)}du \\
&- \frac{\nu}{\Gamma(\rho)}\int_0^t \Delta^{\rho-1}(t,u,\pi)\frac{|Z(u)-Z_1|}{\sigma(u,\pi)}du + \frac{\Delta^{\rho}(t,0,\pi)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^\eta \Delta^{\delta-1}(\eta,u,\pi)\frac{|Z(u)-Z_1|}{\sigma(u,\pi)}du
\end{aligned}$$

$$+\frac{\Delta^{\rho+1}(t,0,\pi)}{(k-\nu)\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2)}\Big(\frac{(\nu-k)}{\Gamma(\theta+\rho)}\int_0^a\Delta^{\theta+\rho-1}(a,u,\pi)\frac{|\phi(u,Z(u))-\phi(u,Z_1(u))|}{\sigma(u,\pi)}du$$
$$-\frac{1}{\Gamma(\theta)}\int_0^a\Delta^{\theta-1}(a,u,\pi)\frac{|\phi(u,Z(u))-\phi(u,Z_1(u))|}{\sigma(u,\pi)}du-\frac{\nu(\nu-k)}{\Gamma(\rho)}\int_0^a\Delta^{\rho-1}(a,u,\pi)\frac{|Z(u)-Z_1|}{\sigma(u,\pi)}du$$
$$+\frac{(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)}{\Gamma(\rho+1)\Gamma(\delta)}\int_0^{\eta}\Delta^{\delta-1}(\eta,u,\pi)\frac{|Z(u)-Z_1|}{\sigma(u,\pi)}du\Big),$$

$$|Z(t)-Z_1(t)|\leq(\lambda\|Z-Z_1\|)\sup_{t\in[0,a]}\Big\{\frac{1}{\Gamma(\theta+\rho)}\int_0^t\Delta^{\theta+\rho-1}(t,u,\pi)\frac{du}{\sigma(u,\pi)}$$
$$+\frac{|\nu-k|\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+\rho)}\int_0^a\Delta^{\theta+\rho-1}(a,u,\pi)\frac{du}{\sigma(u,\pi)}$$
$$+\frac{\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta)}\int_0^a\Delta^{\theta-1}(a,u,\pi)\frac{du}{\sigma(u,\pi)}\Big\}$$
$$+\|Z-Z_1\|\sup_{t\in[0,a]}\Big\{\Big(\frac{|\nu|}{\Gamma(\rho)}+\frac{\nu|\nu-k|\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho)}\Big)\int_0^a\Delta^{\rho-1}(a,u,\pi)\frac{du}{\sigma(u,\pi)}$$
$$+\Big(\frac{(|(\nu-k)\Delta^{\rho}(a,0,\pi)-\Gamma(\rho+1)|)\Delta^{\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta)}$$
$$+\frac{\Delta^{2\rho+1}(t,0,\pi)}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta)}\Big)\int_0^{\eta}\Delta^{\delta-1}(\eta,u,\pi)\frac{du}{\sigma(u,\pi)}\Big\}$$
$$\leq\Big[\lambda\Big\{\frac{\Lambda^{\theta+\rho}}{\Gamma(\theta+\rho+1)}+\frac{|\nu-k|\Lambda^{\theta+2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+\rho+1)}$$
$$+\frac{\Lambda^{\theta+\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\theta+1)}\Big\}$$
$$+\Big\{\frac{|\nu|\Lambda^{\rho}}{\Gamma(\rho+1)}+\frac{\nu|\nu-k|\Lambda^{2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)}$$
$$+\frac{(|\nu-k|\Lambda^{\rho}+\Gamma(\rho+1))\Lambda^{\delta+\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)}$$
$$+\frac{\Lambda^{\delta+2\rho+1}}{(|k-\nu|\Delta^{\rho+1}(a,0,\pi)+\Delta(a,0,\pi)\Gamma(\rho+2))\Gamma(\rho+1)\Gamma(\delta+1)}\Big\}\Big]\|Z-Z_1\|$$
$$\leq(\lambda\mathfrak{L}_1+\mathfrak{L}_2)\|Z-Z_1\|\leq\hbar\|Z-Z_1\|. \tag{4.2}$$

Hence, Eq (3.9) can be made H. U. stable by utilizing Eq (4.2). The fractional Langevin Eq (3.1) is hence H. U. stable. □

5. Conclusions and future remarks

In modeling a variety of physical occurrences, fractional Langevin equations play a significant role. They have been applied, for example, to explain single-file spread [23] and the behavior of unchained nanoparticles propelled by interior sounds [24]. In this study, a new version of the fractional Langevin equation is investigated under nonlocal integral and nonperiodic boundary conditions. The fractional environment has been formed by modern extensive fractional operators which generalize a variety of fractional operators known frequently in literature. Considering these fractional operators, the Hyres-Ulam stability, existence, and uniqueness of the solution are defined and examined for the suggested equations. Further, the Banach contraction principle and the Krasnoselskii fixed point theorem have been used to get the study's results. It is worth mentioning that the outcomes of this paper incorporate several previously published results as particular cases. Indeed, if $\sigma(\tau,\pi) = \tau^{1-\pi}$ and $\pi = 1$, then $\Delta(\tau,u,1) = \tau - u$, the fractional operator ${}^{C}D_{\sigma}^{\theta,\pi}$ reduces to Caputo fractional derivative, and the results in Theorems 3.1, 3.2 and 4.1 coincide with the results of Rizwan and Zada in [25]. Also, if $\sigma(\tau,\pi) = \tau^{1-\pi}$ and $\pi \to 0$, then $\Delta(\tau,u,1) \to \ln\left(\frac{\tau}{u}\right)$, the fractional operator ${}^{C}D_{\sigma}^{\theta,\pi}$ approaches to Caputo–Hadamard fractional derivative, and the results in Theorems 3.1 and 3.2 reduce to the fractional results in [26] for the Langevin equation. Eventually, there are some future research ideas that are relevant to our outcomes and listed as follows:

- The solvability of nonlinear Langevin equation including the extensive fractional operators 2.3 can be examined with suitable boundary conditions and specific intervals.
- The distributed and boundary control issues for the fractional Langevin Eq (3.1) can be investigated in an appropriate space of admissible controls.
- The variational calculus technique can be applied to examine the existence of solution for the fractional Langevin Eq (3.1).

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Conflict of interest

The authors declare that they have no conflicts of interest.

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