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Research article

On the solvability of an initial-boundary value problem for a non-linear fractional diffusion equation

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Abstract: In this paper, we consider an initial-boundary value problem for a non-linear fractional diffusion equation on a bounded domain. The fractional derivative is defined in Caputo's sense with respect to the time variable and represents the case of sub-diffusion. Also, the equation involves a second order symmetric uniformly elliptic operator with time-independent coefficients. These initial-boundary value problems arise in applied sciences such as mathematical physics, fluid mechanics, mathematical biology and engineering. By using eigenfunction expansions and Banach fixed point theorem, we establish the existence, uniqueness and regularity properties of the solution of the problem.

Keywords: solvability; non-linear fractional diffusion equation; initial-boundary value problem; eigenfunction expansions; Banach fixed point theorem
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1. Introduction

The classical diffusion equation describes the random motion of particles suspended in a medium. But when the particles do not obey a certain law, it may not give a correct result. Anomalous diffusion which occurs in a very heterogeneous aquifer, is an example of this case. Here, fractional diffusion equations can be used as an accurate model [1–3]. In recent years, these equations have been getting more attention, since they are able to create new models for a wide range of natural processes in physics, medicine, and so on [4].

Some of the significant works for initial-boundary value problems (IBVP) involving fractional-order diffusion equations are as follows: Luchko [2] solved this kind of problem for the homogeneous equation and by the same method, the inhomogeneous case was studied in [3]. As for the non-linear equations with fractional order $0 < \alpha < 1$ and Laplace operator, Luchko et al. [5] and Jin et al. [6, 7] considered existence, uniqueness and regularity of the solutions. Kian and Yamamoto [8] studied a more general operator for $1 < \alpha < 2$. In [4, 9], some systems involving non-linear fractional diffusion equations were examined. In this work, we generalize these results by taking a more general elliptic operator with $0 < \alpha < 1$.

As for the recent algorithms for the non-linear partial differential equations which include fractional derivatives and integrals, we refer to [10–12].

2. Preliminaries

We use eigenfunction expansions in order to establish the weak solution of the problem, which is a fundamental technique for finding solutions of IBVP for partial differential equations. In the literature, this method was used by [1, 2] for a homogeneous linear fractional diffusion equation. In [2], since the operator in the equation is a symmetric uniformly elliptic operator with time-independent coefficients, the problem for a fractional partial differential equation was solved by transforming into two different problems for fractional ordinary differential equations. Their solutions were obtained by using Laplace transform and the theory of boundary value problems for elliptic equations [13]. By using the result of [2] in the case of inhomogeneous classical partial differential equations, Sakamoto and Yamamoto [3] obtained the solution of the IBVP for an inhomogeneous linear fractional diffusion equation. Then, Luchko et al. [5] and Jin et al. [7] used this idea for a non-linear fractional diffusion equation.

In order to investigate the existence, uniqueness and regularity properties of the solution of the problem, we use a priori estimates in $L^2(\Omega)$. Sakamoto and Yamamoto [3] proved regularity of solution of an IBVP for an inhomogeneous linear diffusion equation. For non-linear fractional diffusion equations, Jin [7] applied a generalized method of Bielecki [14] which is widely used in the theory of functional equations.

Let $d \in \{1, 2, 3\}$ and

$$\Omega = \begin{cases} (0, r_1), & \text{if } d = 1, \\ (0, r_1) \times (0, r_2), & \text{if } d = 2, \\ (0, r_1) \times (0, r_2) \times (0, r_3), & \text{if } d = 3 \end{cases} \tag{2.1}$$

be a bounded domain in $\mathbb{R}^d$ such that r_1, r_2, r_3 are positive real numbers. See Table 1 for the symbols and notations used in the paper.

We consider a partial differential equation with the Caputo fractional derivative in time t for $0 < \alpha < 1$, and the fractional derivative is defined by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_\tau u(x,\tau)}{(t-\tau)^\alpha} d\tau, \tag{2.2}$$

see [15, 16]. Let us assume that

$$F(x, t, u(x, t)) = g(u(x, t)) + s(t)p(x), \tag{2.3}$$

and g, s, p are given functions in $\Omega \times (0, T)$ and $T > 0$ is a fixed value.

Table 1. Nomenclature.

Type	Notation	Description
	d	dimension of the space domain,
Symbol	k	a fixed positive number,
	r_1, r_2, r_3	positive real numbers,
	t	the time variable,
	x	the space variable,
	A, L	the second order symmetric uniformly elliptic operators,
	C_j	positive constants for $1 \leq j \leq 29$,
	$E_{\alpha,\beta}$	the Mittag-Leffer function,
	T	the upper limit of time domain,
	∂_t^α	α-th Caputo fractional derivative.
Greek Symbol	α	the order of the fractional derivative,
	β	a positive real number,
	δ, τ	parameters,
	λ_j	the j-th eigenvalue,
	μ	a parameter for Mittag-Leffler function,
	ξ	the vector $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$,
	φ_j	the j-th eigenfunction,
	Γ	the Gamma function,
	Σ	the summation symbol,
	Ω	domain of the solution according to space coordinates.
Abbreviation	IBVP	Initial-Boundary Value Problem

We aim to solve the equation

$$\partial_t^\alpha u(x,t) = Lu(x,t) + F(x,t,u(x,t)), \quad (x,t) \in \Omega \times (0,T) \tag{2.4}$$

satisfying the following initial and boundary conditions:

$$u(x,0) = b(x), \; x \in \overline{\Omega}, \tag{2.5}$$

$$u(x,t) = 0, \; (x,t) \in \partial\Omega \times [0,T]. \tag{2.6}$$

We also define the second order symmetric uniformly elliptic operator by

$$Lu = \sum_{i,j=1}^{d} \partial_{x_i} \left(a_{ij}(x) \partial_{x_j} u \right), x \in \overline{\Omega}. \tag{2.7}$$

Here, we assume that the coefficients have the following properties:

$$a_{ij} \in C^\infty\left(\overline{\Omega}\right), \; a_{ij} = a_{ji}, \tag{2.8}$$

for every integer $0 \leq i, j \leq d$, and there exists a constant $v > 0$ satisfying

$$v \sum_{i=1}^{d} \xi_i^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j. \tag{2.9}$$

In this paper, $H_0^1(\Omega)$ and $H^2(\Omega)$ denote the usual Sobolev spaces [17, 18]. Additionally, the space $\widetilde{H}^s(\Omega)$ is associated with the elliptic operator

$$A : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega), \tag{2.10}$$

where it is assumed that $s \geq 0$ is a real number, A is defined as $A = -L$. The spectrum of A consists entirely of eigenvalues $\left\{\lambda_j\right\}_{j=1}^{\infty}$. By Section 6.5 of [19], there exists an orthonormal basis $\left\{\varphi_j\right\}_{j=1}^{\infty}$ of $L^2(\Omega)$ such that

$$A\varphi_j = \lambda_j \varphi_j, \ \varphi_j\big|_{\partial\Omega} = 0, \tag{2.11}$$

thus, $\varphi_j \in \widetilde{H}^2(\Omega)$ is an eigenfunction corresponding to j-th eigenvalue λ_j.

For any $s \geq 0$, the space $\widetilde{H}^s(\Omega)$ is defined by

$$\widetilde{H}^s(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^s \left|\left(v, \varphi_j\right)\right|^2 < \infty \right\}, \tag{2.12}$$

and it is a Hilbert space with the norm

$$\|v\|_{\widetilde{H}^s(\Omega)}^2 = \sum_{j=1}^{\infty} \lambda_j^s \left|\left(v, \varphi_j\right)\right|^2 = \left\|A^{s/2} v\right\|_{L^2(\Omega)}^2. \tag{2.13}$$

Moreover, we have $\widetilde{H}^s(\Omega) \subset H^s(\Omega)$ for $s > 0$ and

$$\widetilde{H}^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega). \tag{2.14}$$

Since $\widetilde{H}^s(\Omega) \subset L^2(\Omega)$, by identifying the dual $\left(L^2(\Omega)\right)'$ with $L^2(\Omega)$, we have

$$\widetilde{H}^s(\Omega) \subset L^2(\Omega) \subset \left(\widetilde{H}^s(\Omega)\right)', \tag{2.15}$$

see [3, 7]. Here, we can write

$$\widetilde{H}^{-s}(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{j=1}^{\infty} \frac{1}{\lambda_j^s} \left|\left(v, \varphi_j\right)\right|^2 < \infty \right\} = \left(\widetilde{H}^s(\Omega)\right)'. \tag{2.16}$$

Since our solution will be written by using the orthonormal basis, we will work in the space $L^2(\Omega)$.

Furthermore, for $0 < \alpha < 1$, the space $C^{0,\alpha}\left([0, T]; L^2(\Omega)\right)$ is defined as

$$\left\{ u \in C\left([0, T]; L^2(\Omega)\right) : \sup_{0 \leq t < s \leq T} \frac{\|u(., t) - u(., s)\|_{L^2(\Omega)}}{|t - s|^{\alpha}} < \infty \right\} \tag{2.17}$$

with the norm

$$\|u\|_{C^{0,\alpha}\left([0,T];L^2(\Omega)\right)} = \|u\|_{C\left([0,T];L^2(\Omega)\right)} + \sup_{0 \leq t < s \leq T} \frac{\|u(., t) - u(., s)\|_{L^2(\Omega)}}{|t - s|^{\alpha}}, \tag{2.18}$$

see [3].

In this work, $E_{\alpha,\beta}(z)$ denotes the Mittag-Leffler function, for $\alpha,\beta > 0$ and $z \in \mathbb{C}$. The function $E_{\alpha,\beta}(z)$ satisfies the following properties (see Section 1.2 of [16]):

(i) Let $0 < \alpha < 2$, β be an arbitrary real number and μ satisfy $\pi\alpha < \mu < \min\{\pi, \pi\alpha\}$. Then there exists a real constant $C_1 = C_1(\alpha, \beta, \mu)$ such that

$$\left|E_{\alpha,\beta}(z)\right| \leq \frac{C_1}{1+|z|},\ \mu \leq \left|\arg(z)\right| \leq \pi. \tag{2.19}$$

(ii) We have

$$\frac{d^m}{dt^m} E_{\alpha,1}\left(-\lambda_n t^\alpha\right) = -\lambda_n t^{\alpha-m} E_{\alpha,\alpha-m+1}\left(-\lambda_n t^\alpha\right) \tag{2.20}$$

for $t, \alpha, \lambda_n > 0$ and positive $m \in \mathbb{Z}$.

From now on, we assume that C_j, $1 \leq j \leq 29$ are positive constants which are independent of the function F in (2.4) and the initial condition b in (2.5). But it may depend on the fractional order α, the coefficients of the operator L and the domain of the solution.

3. Main result

By the method of eigenfunction expansions, the solution is sought in the form of

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\varphi_n(x), \tag{3.1}$$

where the functions $\varphi_n(x)$ are the solution of the following problem:

$$L\varphi_n = -\lambda_n\varphi_n,\ \varphi_n\big|_{\partial\Omega} = 0. \tag{3.2}$$

From (3.1), we see that

$$u_n(t) = \left(u(.,t), \varphi_n\right)_{L^2(\Omega)}, \tag{3.3}$$

where $(.,.)_{L^2(\Omega)}$ denotes the usual inner product of the space $L^2(\Omega)$. We multiply both sides of Eq (2.4) by φ_n and integrate it with respect to the space variable and we get

$$\left(\partial_t^\alpha u(.,t), \varphi_n\right)_{L^2(\Omega)} = \left(Lu(.,t), \varphi_n\right)_{L^2(\Omega)} + \left(F(.,t,u(.,t)), \varphi_n\right)_{L^2(\Omega)},\ t \in (0,T). \tag{3.4}$$

By analyzing the terms in (3.4), we have

$$\left(\partial_t^\alpha u(.,t), \varphi_n\right)_{L^2(\Omega)} = \partial_t^\alpha u_n(t), \tag{3.5}$$

$$\left(Lu(.,t), \varphi_n\right)_{L^2(\Omega)} = -\lambda_n u_n(t). \tag{3.6}$$

For simplicity, we can denote

$$F_n(u(t)) = \left(F(.,t,u(.,t)), \varphi_n\right)_{L^2(\Omega)}. \tag{3.7}$$

Similarly, for the initial condition (2.5), we can write

$$u_n(0) = \left(b, \varphi_n\right)_{L^2(\Omega)}. \tag{3.8}$$

Now, $u_n(t)$ can be found by the means of Laplace transform, and then we have

$$u_n(t) = (b,\varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha) F_n(u(\tau))\,d\tau. \tag{3.9}$$

By substituting (3.9) into (3.1), solution of the problem can be written as follows:

$$u(x,t) = I_1(x,t) + I_2(x,t), \tag{3.10}$$

where

$$I_1(x,t) = \sum_{n=1}^{\infty} (b,\varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n(x), \tag{3.11}$$

$$I_2(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^\alpha) F_n(u(\tau))\,d\tau \right) \varphi_n(x). \tag{3.12}$$

Since solution (3.10) is in the form of an integral equation, we can use the Banach fixed point theorem, see [20, 21].

The main result of this paper is given in the following theorem:

Theorem 3.1. *Let $b \in \widetilde{H}^2(\Omega)$, $s \in C[0,T]$ and $p \in L^2(\Omega)$. We also assume that for any $u, v \in C\left([0,T]; L^2(\Omega)\right)$ there exists a positive real constant C_2 such that*

$$\|g(u(.,t)) - g(v(.,t))\|_{L^2(\Omega)} \le C_2 \|u(.,t) - v(.,t)\|_{L^2(\Omega)}. \tag{3.13}$$

Then, for problem (2.4)–(2.6), there exists a unique solution u and a constant $C_3 > 0$ such that

$$\begin{aligned} & \|u\|_{C\left((0,T];\widetilde{H}^2(\Omega)\right)} + \|\partial_t u\|_{C\left((0,T];L^2(\Omega)\right)} + \left\|\partial_t^\alpha u\right\|_{C\left((0,T];L^2(\Omega)\right)} + \|u\|_{C^{0,\alpha}\left([0,T];L^2(\Omega)\right)} \\ \le\ & C_3 \left(\|b\|_{H^2(\Omega)} + \|s\|_{C[0,T]} \|p\|_{L^2(\Omega)} \right). \end{aligned} \tag{3.14}$$

Moreover, we have

$$u \in C^{0,\alpha}\left([0,T]; L^2(\Omega)\right) \cap C\left((0,T]; \widetilde{H}^2(\Omega)\right), \tag{3.15}$$

$$\partial_t u \in C\left((0,T]; L^2(\Omega)\right), \partial_t^\alpha u \in C\left((0,T]; L^2(\Omega)\right). \tag{3.16}$$

In the proof, we will use the same method as [7]. It can be seen as a generalization of Bielecki's method [14] which is used to investigate solvability of initial value problems for ordinary differential equations.

Proof. We will show the existence of the solution by defining a map in the following form:

$$M : C\left([0,T]; L^2(\Omega)\right) \to C\left([0,T]; L^2(\Omega)\right), \tag{3.17}$$

$$M(u(x,t)) = u(x,t). \tag{3.18}$$

Here, instead of the usual norm of $C\left([0,T]; L^2(\Omega)\right)$, we consider

$$\|u\|_k = \max_{t\in[0,T]} \left\{ \left\| e^{-kt} u(t) \right\|_{L^2(\Omega)} \right\} \tag{3.19}$$

for any fixed $k > 0$. In order to use the Banach fixed point theorem, we will determine the value of k later. The function u is a solution of problem (2.4)–(2.6) if and only if u is a fixed point of the map M. We have

$$\begin{aligned} & \|M(u(.,t)) - M(v(.,t))\|^2_{L^2(\Omega)} \\ = & \sum_{n=1}^{\infty} \left| \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^{\alpha}) \left(g(u(.,\tau)) - g(v(.,\tau)), \varphi_n \right)_{L^2(\Omega)} d\tau \right|^2, \end{aligned} \quad (3.20)$$

and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} e^{-2kt} \|M(u(.,t)) - M(v(.,t))\|^2_{L^2(\Omega)} & \leq \sum_{n=1}^{\infty} \left| \int_0^t \left[(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^{\alpha}) \right]^2 d\tau \right| \left\| \varphi_n \right\|^2_{L^2(\Omega)} \\ & \quad \times \left| e^{-2kt} \int_0^t \|g(u(.,\tau)) - g(v(.,\tau))\|^2_{L^2(\Omega)} d\tau \right| \\ & = I_3(t) \times I_4(t), \end{aligned} \quad (3.21)$$

for any $u, v \in C\left([0,T]; L^2(\Omega)\right)$. Using (2.19), the properties for function φ_n and the fact that $\lambda_n \geq C_4 n^{2/d}$, $n \in \mathbb{N}$ by [3], we evaluate

$$\begin{aligned} I_3(t) & = \sum_{n=1}^{\infty} \left| \int_0^t \left[w^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n w^{\alpha}) \right]^2 dw \right| \left\| \varphi_n \right\|^2_{L^2(\Omega)} \\ & \leq C_1^2 \sum_{n=1}^{\infty} \frac{1}{(\lambda_n)^{2(\alpha-1)/\alpha}} \left| \int_0^t \left[\frac{(\lambda_n w^{\alpha})^{(\alpha-1)/\alpha}}{1+\lambda_n w^{\alpha}} \right]^2 dw \right| \left\| \varphi_n \right\|^2_{L^2(\Omega)} \\ & \leq t C_5^2. \end{aligned} \quad (3.22)$$

Now we consider I_4 with (3.13), (3.19), we get

$$\begin{aligned} I_4(t) & \leq e^{-2kt} \int_0^t C_2^2 \|u(.,\tau) - v(.,\tau)\|^2_{L^2(\Omega)} d\tau \\ & \leq C_2^2 e^{-2kt} \int_0^t \max_{\tau \in [0,T]} \left\{ e^{-2k\tau} \|u(.,\tau) - v(.,\tau)\|^2_{L^2(\Omega)} \right\} e^{2k\tau} d\tau \\ & = C_2^2 \frac{1}{2k} \left(1 - \frac{1}{e^{2kt}} \right) \|u - v\|^2_k. \end{aligned} \quad (3.23)$$

Multiplying the terms I_3 and I_4, we get

$$\begin{aligned}\|M(u)-M(v)\|_k^2 &= \max_{t\in[0,T]}\left\{e^{-2kt}\|M(u(.,t))-M(v(.,t))\|_{L^2(\Omega)}^2\right\}\\ &\le \max_{t\in[0,T]}\left\{tC_5^2C_2^2\frac{1}{2k}\left(1-\frac{1}{e^{2kt}}\right)\|u-v\|_k^2\right\}\\ &\le C_5^2TC_2^2\frac{1}{2k}\left(1-\frac{1}{e^{2kT}}\right)\|u-v\|_k^2\\ &\le C_6^2\frac{T}{k}\|u-v\|_k^2. \end{aligned} \tag{3.24}$$

With the choice of

$$k > C_6^2 T, \tag{3.25}$$

inequality (3.24) becomes a contraction on the space $C\left([0,T]\,;L^2(\Omega)\right)$ with the norm $\|.\|_k$. From the Banach fixed point theorem, we conclude that the transform M has a fixed point, which is the solution u of the integral equation (3.10).

Next, we will show the uniqueness of the solution. Let us assume that u and $\tilde{u}$ are two solutions of initial-value problem (2.4)–(2.6). We set

$$C_7^2 = C_6^2\frac{T}{k}. \tag{3.26}$$

By (3.18), we can write

$$\|M(u)-M(\tilde{u})\|_k^2 = \|u-\tilde{u}\|_k^2 \le C_7^2\|u-\tilde{u}\|_k^2, \tag{3.27}$$

and

$$(1-C_7)\|u-\tilde{u}\|_k \le 0. \tag{3.28}$$

Therefore, we have

$$\|u-\tilde{u}\|_k = 0, \tag{3.29}$$

which implies $u=\tilde{u}$. Thus, the solution u is unique.

Finally, we will examine the regularity property of the solution. We can divide this part of the proof into five steps.

Step 1. From (3.10), we write

$$\|u(.,t)\|_{L^2(\Omega)}^2 \le 2\|I_1(.,t)\|_{L^2(\Omega)}^2 + 2\|I_2(.,t)\|_{L^2(\Omega)}^2. \tag{3.30}$$

Here, using (2.19), (2.20) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|I_1(.,t)\|_{L^2(\Omega)}^2 &= \sum_{n=1}^{\infty}\left|(b,\varphi_n)E_{\alpha,1}(-\lambda_n t^\alpha)\right|^2\\ &\le C_1^2\|-Lb\|_{L^2(\Omega)}^2\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\\ &\le C_8^2\|b\|_{H^2(\Omega)}^2 \end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
\|I_2(.,t)\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty}\left|\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n (t-\tau)^{\alpha}\right) F_n\left(u(\tau)\right) d\tau\right|^2 \\
&\le \|F(u)\|^2_{C\left([0,T];L^2(\Omega)\right)} \sum_{n=1}^{\infty}\left|\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n (t-\tau)^{\alpha}\right) d\tau\right|^2 \\
&= \|F(u)\|^2_{C\left([0,T];L^2(\Omega)\right)} \sum_{n=1}^{\infty}\left|\int_0^t \frac{d}{dw}\left[-\frac{1}{\lambda_n} E_{\alpha,1}\left(-\lambda_n w^{\alpha}\right)\right] dw\right|^2 \\
&\le 2\|F(u)\|^2_{C\left([0,T];L^2(\Omega)\right)} \left[\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2}\left[1+\left(E_{\alpha,1}\left(-\lambda_n t^{\alpha}\right)\right)^2\right]\right] \\
&\le C_9^2 \|F(u)\|^2_{C\left([0,T];L^2(\Omega)\right)}. \qquad (3.32)
\end{aligned}$$

Now, we examine the right hand side of (3.32). By the hypotheses of Theorem 3.1, we can write

$$\begin{aligned}
\|F(u(.,t))\|^2_{L^2(\Omega)} &= \|F(u(.,t)) - F(u(0,0)) + F(u(0,0))\|^2_{L^2(\Omega)} \\
&\le 2\|s(t)\,p(x) + g(0)\|^2_{L^2(\Omega)} + 2\|g(u(.,t)) - g(u(0,0))\|^2_{L^2(\Omega)} \\
&\le 4[s(t)]^2\|p\|^2_{L^2(\Omega)} + 4[g(0)]^2\|1\|^2_{L^2(\Omega)} + 2C_2^2\|u(.,t) - u(0,0)\|^2_{L^2(\Omega)} \\
&\le C_{10}^2 + 4[s(t)]^2\|p\|^2_{L^2(\Omega)} + 2C_2^2\|u(.,t) - u(0,0)\|^2_{L^2(\Omega)}, \qquad (3.33)
\end{aligned}$$

and by (3.19), (3.24), we have

$$e^{-2kt}\|u(.,t) - \tilde{u}(.,t)\|^2_{L^2(\Omega)} \le \max_{t\in[0,T]}\left\{e^{-2kt}\|u(.,t) - \tilde{u}(.,t)\|^2_{L^2(\Omega)}\right\} \le C_6^2\frac{T}{k}\|u-\tilde{u}\|_k^2. \qquad (3.34)$$

Taking a sufficiently large k implies

$$e^{-2kt}\|u(.,t) - \tilde{u}(.,t)\|^2_{L^2(\Omega)} \le C_{11}^2, \qquad (3.35)$$

and

$$\|u(.,t) - \tilde{u}(.,t)\|^2_{L^2(\Omega)} \le C_{11}^2 e^{2kt} \le C_{12}^2. \qquad (3.36)$$

Therefore, it yields

$$\|F(u(.,t))\|^2_{L^2(\Omega)} \le C_{10}^2 + 4[s(t)]^2\|p\|^2_{L^2(\Omega)} + 2C_2^2C_{12}^2, \qquad (3.37)$$

and taking the maximum with respect to the time variable t on $[0,T]$, we obtain

$$\|F(u)\|_{C\left([0,T];L^2(\Omega)\right)} \le C_{13}\|s\|_{C[0,T]}\|p\|_{L^2(\Omega)}. \qquad (3.38)$$

From (3.31) and (3.32), we get

$$\|u(.,t)\|_{L^2(\Omega)} \le C_{14}\left\{\|b\|_{H^2(\Omega)} + \|s\|_{C[0,T]}\|p\|_{L^2(\Omega)}\right\}, \qquad (3.39)$$

which results in

$$\|u\|_{C\left([0,T];L^2(\Omega)\right)} \le C_{15}\left(\|b\|_{H^2(\Omega)} + \|s\|_{C[0,T]}\|p\|_{L^2(\Omega)}\right) \qquad (3.40)$$

and $u \in C\left([0,T];L^2(\Omega)\right)$.

Step 2. In this step, we will show the inequality

$$\|u\|_{C\left((0,T];\widetilde{H}^2(\Omega)\right)} \le C_{16}\left(\|b\|_{H^2(\Omega)} + \|s\|_{C[0,T]}\|p\|_{L^2(\Omega)}\right). \tag{3.41}$$

We apply the second order operator L to both sides of (3.10) by using (3.2), then we have

$$\|Lu(.,t)\|^2_{L^2(\Omega)} \le 2\|I_5(.,t)\|^2_{L^2(\Omega)} + 2\|I_6(.,t)\|^2_{L^2(\Omega)}, \tag{3.42}$$

where

$$\|I_5(.,t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty}\left|-\lambda_n(b,\varphi_n)E_{\alpha,1}(-\lambda_n t^{\alpha})\right|^2 \tag{3.43}$$

and

$$\|I_6(.,t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty}\left|-\lambda_n\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})F_n(u(\tau))\,d\tau\right|^2. \tag{3.44}$$

On the other hand, with a similar technique to the one used in the previous steps, we can write

$$\|I_5(.,t)\|^2_{L^2(\Omega)} \le C_{17}^2 t^{-2\alpha}\|b\|^2_{H^2(\Omega)},\ t>0, \tag{3.45}$$

$$\|I_6(.,t)\|^2_{L^2(\Omega)} \le C_{18}^2\|F(u)\|^2_{C\left([0,T];L^2(\Omega)\right)} \le C_{19}^2\|s\|^2_{C[0,T]}\|p\|^2_{L^2(\Omega)}. \tag{3.46}$$

By adding (3.45) and (3.46) and taking the maximum of both sides with respect to t, we reach (3.41) and therefore we obtain

$$u \in C\left((0,T];H^2(\Omega)\cap H_0^1(\Omega)\right). \tag{3.47}$$

Step 3. Here, we will prove the inequality

$$\|\partial_t u\|_{C\left((0,T];L^2(\Omega)\right)} \le C_{20}\left(\|b\|_{H^2(\Omega)} + \|s\|_{C[0,T]}\|p\|_{L^2(\Omega)}\right). \tag{3.48}$$

By calculating the classical derivative of (3.10) with respect to t and making use of the Leibnitz integral rule, we get

$$\|\partial_t u(.,t)\|^2_{L^2(\Omega)} \le 2\|I_7(.,t)\|^2_{L^2(\Omega)} + 2\|I_8(.,t)\|^2_{L^2(\Omega)}, \tag{3.49}$$

where

$$\|I_7(.,t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty}\left|(b,\varphi_n)(-\lambda_n)t^{\alpha-1}E_{\alpha,1}(-\lambda_n t^{\alpha})\right|^2 \tag{3.50}$$

and

$$\|I_8(.,t)\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty}\left|\int_0^t \partial_t\left[(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})\right]F_n(u(\tau))\,d\tau\right|^2. \tag{3.51}$$

Then, we see that

$$\|I_7(.,t)\|^2_{L^2(\Omega)} \le C_{21}^2 t^{-2}\|b\|^2_{H^2(\Omega)},\ t>0, \tag{3.52}$$

and

$$\|I_8(.,t)\|^2_{L^2(\Omega)} \le C_{22}^2 t^{-2}\|F(u)\|^2_{C\left([0,T];L^2(\Omega)\right)},\ t>0. \tag{3.53}$$

By (3.52), (3.53), considering (3.38) and taking the maximum of both sides with respect to t, we get (3.48).

Step 4. For this step, we will obtain (3.14), (3.16) and the inequality

$$\left\|\partial_t^\alpha u\right\|_{C\left((0,T];L^2(\Omega)\right)} \leq C_{23}\left(\|b\|_{H^2(\Omega)} + \|s\|_{C[0,T]}\|p\|_{L^2(\Omega)}\right). \tag{3.54}$$

Since, the terms on the right-hand side of Eq (2.4) are examined in the first and second step, we can easily write inequality (3.54). As for (3.16), the first part is obvious from (2.14) and (3.47). We can write the second part by (3.54).

Step 5. Finally, at this step, we will prove

$$\|u\|_{C^{0,\alpha}\left([0,T];L^2(\Omega)\right)} \leq C_{24}\left(\|b\|_{H^2(\Omega)} + \|s\|_{C[0,T]}\|p\|_{L^2(\Omega)}\right) \tag{3.55}$$

and (3.15). By taking $s = t + h$, we can rewrite the norm of the space as

$$\|u\|_{C^{0,\alpha}\left([0,T];L^2(\Omega)\right)} = \|u\|_{C\left([0,T];L^2(\Omega)\right)} + \sup_{0\leq t<s\leq T}\frac{\|u(.,t+h) - u(.,t)\|_{L^2(\Omega)}}{h^\alpha}. \tag{3.56}$$

We know from the previous steps that the first term on the right hand side is finite. Now, we consider the second one and we set

$$u(.,t+h) - u(.,t) = I_9(.,t)\,h^\alpha + I_{10}(.,t)\,h^\alpha + I_{11}(.,t)\,h^\alpha, \tag{3.57}$$

where

$$I_9(.,t)\,h^\alpha = \sum_{n=1}^{\infty}(b,\varphi_n)\left[E_{\alpha,1}\left(-\lambda_n(t+h)^\alpha\right) - E_{\alpha,1}\left(-\lambda_n t^\alpha\right)\right]\varphi_n(x), \tag{3.58}$$

$$I_{10}(.,t)\,h^\alpha = \sum_{n=1}^{\infty}\left(\int_t^{t+h}(t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t+h-\tau)^\alpha\right)F_n(u(\tau))\,d\tau\right)\varphi_n(x), \tag{3.59}$$

$$I_{11}(.,t)\,h^\alpha = \sum_{n=1}^{\infty}\left(\int_0^t W(t,\tau)\cdot F_n(u(\tau))\,d\tau\right)\varphi_n(x), \tag{3.60}$$

and

$$W(t,\tau) = \left[(t+h-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t+h-\tau)^\alpha\right) - (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^\alpha\right)\right]. \tag{3.61}$$

Since

$$\left|E_{\alpha,1}\left(-\lambda_n(t+h)^\alpha\right) - E_{\alpha,1}\left(-\lambda_n t^\alpha\right)\right|^2 = \left|\int_t^{t+h}\frac{-\lambda_n\tau^{\alpha-1}}{1+\lambda_n\tau^\alpha}d\tau\right|^2 \leq C_{25}\left(\frac{h}{t}\right)^{2\alpha} \leq C_{25}\left(\frac{h}{\delta}\right)^{2\alpha}, \tag{3.62}$$

where δ is a number such that $0 < \delta \leq t \leq T$, and

$$|W(t,\tau)| = \left|\int_{t-\tau}^{t+h-\tau}\tau^{\alpha-2}E_{\alpha,\alpha-1}\left(-\lambda_n\tau^\alpha\right)d\tau\right| \leq \frac{C_{26}h}{\lambda_n(t-\tau)(t+h-\tau)}, \tag{3.63}$$

we can evaluate (3.58)–(3.60) and obtain the following results.

There are two cases for the term I_9, which are

$$\|I_9(.,t)\|^2_{L^2(\Omega)} h^{2\alpha} \le \frac{C^2_{27} \|b\|^2_{H^2(\Omega)} h^{2\alpha}}{\delta^{2\alpha}} \tag{3.64}$$

for $0 < \delta \le t \le T$ and

$$\|I_9(.,t)\|^2_{L^2(\Omega)} h^{2\alpha} \le \frac{4C^2_1}{h^{2\alpha}} \|b\|^2_{H^2(\Omega)} h^{2\alpha} \tag{3.65}$$

for $t = 0$. We also have

$$\|I_{10}(.,t)\|^2_{L^2(\Omega)} h^{2\alpha} \le C^2_{28} \|F(u)\|^2_{C\left([0,T];L^2(\Omega)\right)} h^{2\alpha} \tag{3.66}$$

and

$$\|I_{11}(.,t)\|^2_{L^2(\Omega)} h^{2\alpha} \le C^2_{29} \|F(u)\|^2_{C\left([0,T];L^2(\Omega)\right)} h^{2\alpha}. \tag{3.67}$$

By considering (3.64)–(3.67), we get (3.55). Therefore, we can write

$$u \in C^{0,\alpha}\left([0,T]\,;L^2(\Omega)\right). \tag{3.68}$$

Using (3.41), (3.48), (3.54) and (3.55), we obtain (3.14). Additionally, taking into account of (3.47) and (3.68), we have (3.15).

This completes the proof of the theorem. □

4. Conclusions

In this study, we consider an initial-boundary value problem for a non-linear fractional diffusion equation. We prove the existence, uniqueness and regularity properties of the solution under some conditions on the non-linear function F and the initial condition.

Conflict of interest

The authors declare no conflicts of interest.

References

1. J. Cheng, J. Nakagawa, M. Yamamoto, T. Yamazaki, Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation, *Inverse Probl.,* **25** (2009), 1–16. https://doi.org/10.1088/0266-5611/25/11/115002
2. Y. Luchko, Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation, *Comput. Math. Appl.,* **59** (2010), 1766–1772. https://doi.org/10.1016/j.camwa.2009.08.015
3. K. Sakamoto, M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, *J. Math. Anal. Appl.,* **382** (2011), 426–447. https://doi.org/10.1016/j.jmaa.2011.04.058

4. L. Li, L. Y. Jin, S. M. Fang, Existence and uniqueness of the solution to a coupled fractional diffusion system, *Adv. Differ. Equ.,* **370** (2015), 1–14. https://doi.org/10.1186/s13662-015-0707-0
5. Y. Luchko, W. Rundell, M. Yamamoto, L. H. Zuo, Uniqueness and reconstruction of an unknown semilinear term in a time-fractional reaction-diffusion equation, *Inverse Probl.,* **29** (2013), 1–16. https://doi.org/10.1088/0266-5611/29/6/065019
6. B. T. Jin, B. Y. Li, Z. Zhou, Numerical analysis of nonlinear subdiffusion equations, *SIAM J. Numer. Anal.,* **56** (2018), 1–23. https://doi.org/10.1137/16M1089320
7. B. T. Jin, *Fractional differential equations*, Cham: Springer, 2021. https://doi.org/10.1007/978-3-030-76043-4
8. Y. Kian, M. Yamamoto, On existence and uniqueness of solutions for semilinear fractional wave equations, *Fract. Calc. App. Anal.* , **20** (2017), 117–138. https://doi.org/10.1515/fca-2017-0006
9. T. B. Ngoc, N. H. Tuan, D. O'Regan, Existence and uniqueness of mild solutions for a final value problem for nonlinear fractional diffusion systems, *Commun. Nonlinear Sci. Numer. Simul.,* **78** (2019), 104882. https://doi.org/10.1016/j.cnsns.2019.104882
10. M. H. Tiwana, K. Maqbool, A. B. Mann, Homotopy perturbation Laplace transform solution of fractional non-linear reaction diffusion system of Lotka-Volterra type differential equation, *Eng. Sci. Technol. Int. J.,* **20** (2017), 672–678. https://doi.org/10.1016/j.jestch.2016.10.014
11. W. L. Qiu, D. Xu, J. Guo, J. Zhou, A time two-grid algorithm based on finite difference method for the two-dimensional nonlinear time-fractional mobile/immobile transport model, *Numer. Algorithms,* **85** (2020), 39–58. https://doi.org/10.1007/s11075-019-00801-y
12. W. L. Qiu, X. Xiao, K. X. Li, Second-order accurate numerical scheme with graded meshes for the nonlinear partial integrodifferential equation arising from viscoelasticity, *Commun. Nonlinear Sci. Numer. Simul.,* **116** (2023), 106804. https://doi.org/10.1016/j.cnsns.2022.106804
13. V. S. Vladimirov, *Equations of mathematical physics*, New York: Marcel Dekker, 1971.
14. E. U. Tarafdar, M. S. R. Chowhury, *Topological methods for set-valued nonlinear analysis*, Singapore: World Scientific, 2008. https://doi.org/10.1142/6347
15. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam: Elsevier, 2006.
16. I. Podlubny, *Fractional differential equations*, Academic Press, 1998.
17. R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, 2 Eds., Elsevier, 2003.
18. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, New York: Springer, 2011. https://doi.org/10.1007/978-0-387-70914-7
19. L. C. Evans, *Partial differential equations*, Providence, RI: American Mathematical Society, 1998.
20. A. Henrot, *Extremum problems for eigenvalues of elliptic operators*, Basel, Switzerland: Birkhauser, 2006.
21. J. K. Hunter, B. Nachtergaele, *Applied analysis*, Singapore: World Scientific, 2001.