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Research article

Differential sandwich theorems involving Riemann-Liouville fractional integral of q-hypergeometric function

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Abstract: The development of certain aspects of geometric function theory after incorporating fractional calculus and q-calculus aspects is obvious and indisputable. The study presented in this paper follows this line of research. New results are obtained by applying means of differential subordination and superordination theories involving an operator previously defined as the Riemann-Liouville fractional integral of the q-hypergeometric function. Numerous theorems are stated and proved involving the fractional q-operator and differential subordinations for which the best dominants are found. Associated corollaries are given as applications of those results using particular functions as best dominants. Dual results regarding the fractional q-operator and differential superordinations are also considered and theorems are proved where the best subordinants are given. Using certain functions known for their remarkable geometric properties applied in the results as best subordinant, interesting corollaries emerge. As a conclusion of the investigations done by applying the means of the two dual theories considering the fractional q-operator, several sandwich-type theorems combine the subordination and superordiantion established results.

Keywords: Riemann-Liouville fractional integral of q-confluent hypergeometric function; differential subordination; differential superordination; best dominant; subordinant; best subordinant
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1. Introduction

The benefits of the fractional calculus and q-calculus added to geometric function theory underlined in the recent review paper published by Srivastava [1] have encouraged and motivated new studies connecting the two prolific tools with univalent functions theory. The same paper highlights the added value given to the studies by the use of convolution and fractional operators and also the applicability of q- hypergeometric functions and q- hypergeometric polynomials in various mathematical fields and in geometric function theory.

Fractional calculus applied in studies regarding geometric function theory has provided a tremendous amount of new and interesting results in recent years. Mittag-Leffler–confluent hypergeometric function is associated with fractional calculus in [2, 3], convexity aspects are investigated using fractional calculus in [4, 5] and certain inequality results can also be listed [6–8]. Riemann-Liouville fractional integral is a particularly interesting function that was combined with many powerful functions for defining new operators involved in studies. The confluent hypergeometric function was combined with Riemann-Liouville fractional integral in [9, 10], Ruscheweyh and Sălăgean operators were considered for a new operator involving Riemann-Liouville fractional integral in [11] and Riemann-Liouville fractional integral of Gaussian hypergeometric function was applied in the study seen in [12].

Research including quantum calculus aspects in studies related to geometric function theory started to develop when the general context for such studies was described by Srivastava in a book chapter in 1989 [13]. The same publication highlights q-hypergeometric function as potentially important in studies related to univalent functions.

Indeed, the q-hypergeometric function is associated with many studies in geometric function theory. New operators were defined using q-hypergeometric function [14, 15], subclasses of meromorphic functions were introduced and studied using q-hypergeometric function [16,17] and q-hypergeometric polynomials are defined in [18].

Some basic notations and definitions in geometric function theory are the folowing:

$U = \{z \in \mathbb{C} : |z| < 1\}$ denotes the unit disc of the complex plane and $\mathcal{H}(U)$ represents the class of holomorphic functions in U. Particularly interesting subclasses of $\mathcal{H}(U)$ are written as:

$$\mathcal{A}_n = \left\{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U \right\}, \quad \text{with } \mathcal{A}_1 = \mathcal{A},$$

and

$$\mathcal{H}(a,n) = \left\{ f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots, z \in U \right\},$$

when $a \in \mathbb{C}$, $n \in \mathbb{N}^*$.

Riemann-Liouville fractional integral is considered as defined in [19, 20].

Definition 1. *([19, 20]) For a function f the fractional integral of order α ($\alpha > 0$) is defined by*

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(t)}{(z-t)^{1-\alpha}} dt.$$

Riemann-Liouville fractional integral of q-hypergeometric function is considered in this study in the context of differential subordination [21, 22] and superordination [23] theories.

Definition 2. *([21, 22]) The analytic function f is subordinate to the analytic function g, written $f \prec g$, if there exists an analytic Schwarz function u, with $u(0) = 0$ and $|u(z)| < 1$, for all $z \in U$, such that $f(z) = g(u(z))$, for all $z \in U$. When the function g is univalent in U, the subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.*

Definition 3. *([24]) Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h a univalent function in U. When p is an analytic function in U which verifies the second order differential subordination*

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad z \in U, \tag{1.1}$$

then p is a solution of the differential subordination. The univalent function g is a dominant of the solutions of the differential subordination, when $p \prec g$ for all p verifies the relation (1.1). A dominant $\widetilde{g}$ with the property $\widetilde{g} \prec g$ for all dominants g of (1.1) is the best dominant of (1.1).

Definition 4. *([23]) Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ and h an analytic function in U.*

When p and $\varphi\left(p(z), zp'(z), z^2p''(z); z\right)$ are univalent functions in U that verify the differential superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \qquad z \in U, \tag{1.2}$$

then p is a solution of the differential superordination. An analytic function g is a subordinant of the solutions of the differential superordination when $g \prec p$ for all p verifying relation (1.2). A subordinant $\widetilde{g}$ with the property $g \prec \widetilde{g}$ for all subordinants g of (1.2) is the best subordinant of (1.2).

Definition 5. *([24]) Q is the set of all analytic and injective functions f on $\overline{U}\backslash E(f)$, with $f'(z) \neq 0$ for $z \in \partial U \backslash E(f)$ and $E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$.*

Definition 6. *([25]) The q-hypergeometric function $\phi(m, n; q, z)$ is defined by*

$$\phi(m, n; q, z) = \sum_{j=0}^{\infty} \frac{(m, q)_j}{(q, q)_j (n, q)_j} z^j,$$

where

$$(m, q)_j = \begin{cases} 1, & j = 0, \\ (1-m)(1-mq)\left(1-mq^2\right)...\left(1-mq^{j-1}\right), & j \in \mathbb{N}, \end{cases}$$

and $0 < q < 1$.

Considering Definitions 1 and 6, Riemann-Liouville fractional integral of q-hypergeometric function in introduced as:

Definition 7. *([26]) The Riemann-Liouville fractional integral of q-confluent hypergeometric function is*

$$D_z^{-\alpha}\phi(m, n; q, z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{\phi(m, n; q, t)}{(z-t)^{1-\alpha}} dt \tag{1.3}$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{\infty} \frac{(m, q)_j}{(q, q)_j (n, q)_j} \int_0^z \frac{t^j}{(z-t)^{1-\alpha}} dt,$$

where $m, n \in \mathbb{C}$ with $n \neq 0, -1, -2, ...$ and $\alpha > 0$, $0 < q < 1$.

After a simple calculation, it can be written using the following form

$$D_z^{-\alpha}\phi(m, n; q, z) = \sum_{j=0}^{\infty} \frac{(m, q)_j}{(q, q)_j (n, q)_j (j+1)_\alpha} z^{\alpha+j}, \tag{1.4}$$

and $D_z^{-\alpha}\phi(m, n; q, z) \in \mathcal{H}[0, \alpha]$.

The next two lemmas are used in the proofs of the new results presented in the next section.

Lemma 1. *([24]) Consider the univalent function g in U and the analytic functions φ, γ in a domain $D \supset g(U)$ with $\gamma(u) \neq 0$ when $u \in g(U)$. Set $G(z) = z\gamma(g(z))g'(z)$ and $h(z) = G(z) + \varphi g(z)$. Assume that G is starlike univalent in U and $Re\left(\frac{zh'(z)}{G(z)}\right) > 0$ for $z \in U$. When p is analytic with the properties $p(0) = g(0)$, $p(U) \subseteq D$ and $\varphi(p(z)) + z\gamma(p(z))p'(z) \prec \varphi(g(z)) + z\gamma(g(z))g'(z)$, then $p \prec g$ and g is the best dominant.*

Lemma 2. *([27]) Consider the convex univalent function g in U and the analytic functions φ, γ in a domain $D \supset g(U)$. Assume that $Re\left(\frac{\varphi'(g(z))}{\gamma(g(z))}\right) > 0$ for $z \in U$ and $G(z) = z\gamma(g(z))g'(z)$ is starlike univalent in U. When $p(z) \in \mathcal{H}[g(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\varphi(p(z)) + z\gamma(p(z))p'(z)$ is univalent in U and $\varphi(g(z)) + z\gamma(g(z))g'(z) \prec \varphi(p(z)) + z\gamma(p(z))p'(z)$, then $g \prec p$ and g is the best subordinant.*

The first results obtained using the operator given by (1.3) and (1.4) are related to differential subordination (see also [28]). The first theorem proved is followed by two corollaries obtained by using certain functions as best dominant of the differential subordination considered in Theorem 1. Theorem 4 contains the dual results regarding differential superordinations associated with the operator given in Definition 7. Associated corollaries are derived by using particular functions with nice geometrical properties as the best subordinant provided in Theorem 4. Theorem 7 and the corollaries which accompany it combine the dual results previously obtained into the first sandwich-type results stated in this paper, familiary to geometric function theory (see also [29]). The research is continued by choosing different functions for obtaining another series of theorems and corollaries related to differential subordinations and superordinations. The study is completed by the statement of the sandwich-type theorem and associated corollaries combining the dual results obtained in Theorems 10 and 13 and in the corollaries following them.

2. Main results

The first theorem stated is related to a differential subordination obtained by using the Riemann-Liouville fractional integral of q-hypergeometric function presented in Definition 7. For this differential subordination, the best dominant is provided.

Theorem 1. *Consider the analytic and univalent function g in U with $g(z) \neq 0$, for all $z \in U$ and $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \in \mathcal{H}(U)$, where $m, n \in \mathbb{C}$, $n \neq 0, -1, -2, ...$ and $\alpha > 0$, $0 < q < 1$. Assume that $\frac{zg'(z)}{g(z)}$ is a starlike univalent function in U and*

$$Re\left(\frac{\beta}{\psi}g(z) + \frac{2\delta}{\psi}g^2(z) + 1 - z\frac{g'(z)}{g(z)} + z\frac{g''(z)}{g'(z)}\right) > 0, \tag{2.1}$$

for $\varepsilon, \beta, \psi, \delta \in \mathbb{C}$, $\psi \neq 0$, $z \in U$ and

$$\Psi_\alpha^{m,n,q}(\varepsilon, \beta, \delta, \psi; z) := \varepsilon + \psi + (\beta - \psi)\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} + \tag{2.2}$$

$$\delta\left(\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}\right)^2 + \delta\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)''}{\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}.$$

When g verifies the differential subordination

$$\Psi_{\alpha}^{m,n,q}(\varepsilon,\beta,\delta,\psi;z) \prec \varepsilon + \beta g(z) + \delta (g(z))^2 + \psi \frac{zg'(z)}{g(z)}, \tag{2.3}$$

for $\varepsilon,\beta,\psi,\delta \in \mathbb{C}$, $\psi \neq 0$, *then*

$$\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \prec g(z), \tag{2.4}$$

and the best dominant is the function g.

Proof. Define $p(z) := \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}$, $z \in U$, $z \neq 0$ and differentiating it we get $p'(z) = \frac{\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} - z\left(\frac{\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}\right)^2 + z\frac{\left(D_z^{-\alpha}\phi(m,n;q,z)\right)''}{D_z^{-\alpha}\phi(m,n;q,z)}$ and

$$\frac{zp'(z)}{p(z)} = 1 - z\frac{\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} + z\frac{\left(D_z^{-\alpha}\phi(m,n;q,z)\right)''}{\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}. \tag{2.5}$$

Considering $\varphi(u) = \delta u^2 + \beta u + \varepsilon$ analytic in $\mathbb{C}$, and $\gamma(u) = \frac{\psi}{u}$, analytic in $\mathbb{C}\backslash\{0\}$ with $\gamma(u) \neq 0$, $u \in \mathbb{C}\backslash\{0\}$, we define the starlike univalent function in U, $G(z) = z\gamma(g(z))g'(z) = \psi\frac{zg'(z)}{g(z)}$ and $h(z) = G(z) + \varphi(g(z)) = \varepsilon + \beta g(z) + \delta(g(z))^2 + \psi\frac{zg'(z)}{g(z)}$.

Differentiating it, we get $h'(z) = \beta g'(z) + 2\delta g(z) g'(z) + \psi\frac{g'(z)}{g(z)} - \psi z\left(\frac{g'(z)}{g(z)}\right)^2 + \psi z\frac{g''(z)}{g(z)}$ and $\frac{zh'(z)}{G(z)} = \frac{\beta}{\psi}g(z) + \frac{2\delta}{\psi}g^2(z) + 1 - z\frac{g'(z)}{g(z)} + z\frac{g''(z)}{g(z)}$ and we have
$Re\left(\frac{zh'(z)}{G(z)}\right) = Re\left(\frac{\beta}{\psi}g(z) + \frac{2\delta}{\psi}g^2(z) + 1 - z\frac{g'(z)}{g(z)} + z\frac{g''(z)}{g(z)}\right) > 0$ by relation (2.1).

Using relation (2.5), we can write $\varepsilon + \beta p(z) + \delta(p(z))^2 + \psi\frac{zp'(z)}{p(z)} =$
$\varepsilon + \psi + (\beta - \psi)\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} + \delta\left(\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}\right)^2 + \delta\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)''}{\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}$.

Taking into account the differential subordination (2.3), we get $\varepsilon + \beta p(z) + \delta(p(z))^2 + \psi\frac{zp'(z)}{p(z)} \prec \varepsilon + \beta g(z) + \delta(g(z))^2 + \psi\frac{zg'(z)}{g(z)}$ and applying Lemma 1, we obtain $p \prec g$, i.e. $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \prec g(z)$, $z \in U$ and g is the best dominant. □

Corollary 2. *Considering* $g(z) = \frac{Mz+1}{Nz+1}$, $-1 \leq N < M \leq 1$ *and relation (2.1) is true, when*

$$\Psi_{\alpha}^{m,n,q}(\varepsilon,\beta,\delta,\psi;z) \prec \varepsilon + \beta\frac{Mz+1}{Nz+1} + \delta\left(\frac{Mz+1}{Nz+1}\right)^2 + \frac{\psi(M-N)z}{(Mz+1)(Nz+1)},$$

where $m, n \in \mathbb{C}$, $n \neq 0, -1, -2, \ldots$; $\alpha > 0$, $0 < q < 1$, $\varepsilon,\beta,\psi,\delta \in \mathbb{C}$, $\psi \neq 0$ *and* $\psi_{\alpha}^{m,n,q}$ *defined by relation (2.2), then*

$$\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \prec \frac{Mz+1}{Nz+1},$$

with the best dominant $\frac{Mz+1}{Nz+1}$.

Corollary 3. *Considering* $g(z) = \left(\frac{z+1}{1-z}\right)^k$, $0 < k \leq 1$, *and relation (2.1) is true, when*

$$\Psi_{\alpha}^{m,n,q}(\varepsilon,\beta,\delta,\psi;z) \prec \varepsilon + \beta\left(\frac{z+1}{1-z}\right)^k + \delta\left(\frac{z+1}{1-z}\right)^{2k} + \frac{2k\psi z}{1-z^2},$$

where $m,\ n \in \mathbb{C}$, $n \neq 0,-1,-2,...;$ $\alpha > 0$, $0 < q < 1$, $\varepsilon,\beta,\psi,\delta \in \mathbb{C}$, $\psi \neq 0$ *and* $\psi_{\alpha}^{m,n,q}$ *defined by relation (2.2), then*

$$\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \prec \left(\frac{z+1}{1-z}\right)^k,$$

with the best dominant $\left(\frac{z+1}{1-z}\right)^k$.

The next theorem gives the best subordinant of a differential superordination studied in connection to Riemann-Liouville fractional integral of q-hypergeometric function given in Definition 7.

Theorem 4. *Consider the analytic and univalent function g in U such that $g(z) \neq 0$ and $\frac{zg'(z)}{g(z)}$ starlike univalent in U. Assume that*

$$Re\left(\frac{\beta}{\psi}g(z)g'(z) + \frac{2\delta}{\psi}g^2(z)g'(z)\right) > 0,\ for\,\beta,\psi,\delta \in \mathbb{C},\ \psi \neq 0. \tag{2.6}$$

When $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \in \mathcal{H}[g(0),1] \cap Q$ *and* $\Psi_{\alpha}^{m,n,q}(\varepsilon,\beta,\delta,\psi;z)$ *defined by relation (2.2) is univalent in* U, $\alpha > 0$, $0 < q < 1$, $m,\ n \in \mathbb{C}$, $n \neq 0,-1,-2,...$, *then*

$$\varepsilon + \beta g(z) + \delta(g(z))^2 + \frac{\psi z g'(z)}{g(z)} \prec \Psi_{\alpha}^{m,n,q}(\varepsilon,\beta,\delta,\psi;z) \tag{2.7}$$

implies

$$g(z) \prec \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}, \quad z \in U, \tag{2.8}$$

and g is the best subordinant.

Proof. Define $p(z) := \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}$, $z \in U$, $z \neq 0$. and consider the analytic functions $\varphi(u) = \delta u^2 + \beta u + \varepsilon$ in $\mathbb{C}$, and $\gamma(u) = \frac{\psi}{u}$, respectively in $\mathbb{C}\backslash\{0\}$ with $\gamma(u) \neq 0$, $u \in \mathbb{C}\backslash\{0\}$. Differentiating it, we can write $\frac{\varphi'(g(z))}{\gamma(g(z))} = \frac{[\beta+2\delta g(z)]g(z)g'(z)}{\psi}$, and
$Re\left(\frac{\varphi'(g(z))}{\gamma(g(z))}\right) = Re\left(\frac{\beta}{\psi}g(z)g'(z) + \frac{2\delta}{\psi}g^2(z)g'(z)\right) > 0$, for $\beta,\psi,\delta \in \mathbb{C}$, $\psi \neq 0$, taking account of the relation (2.6).

Differential subordination (2.7) can be written using relation (2.5) as follows

$$\varepsilon + \beta g(z) + \delta(g(z))^2 + \frac{\psi z g'(z)}{g(z)} \prec \varepsilon + \beta p(z) + \delta(p(z))^2 + \frac{\tau\psi z p'(z)}{p(z)},$$

and applying Lemma 2, we obtain

$$g(z) \prec p(z) = \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}, \quad z \in U,$$

and g is the best subordinant. □

Corollary 5. *Considering* $g(z) = \frac{Mz+1}{Nz+1}$, $-1 \leq N < M \leq 1$ *and relation (2.6) is true, when* $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \in \mathcal{H}[g(0),1] \cap Q$ *and*

$$\varepsilon + \beta\frac{Mz+1}{Nz+1} + \delta\left(\frac{Mz+1}{Nz+1}\right)^2 + \frac{\psi(M-N)z}{(Mz+1)(Nz+1)} \prec \Psi_\alpha^{m,n,q}(\varepsilon,\beta,\delta,\psi;z),$$

where $m,\ n \in \mathbb{C}$, $n \neq 0,-1,-2,...$; $\alpha > 0$, $0 < q < 1$, $\varepsilon,\beta,\psi,\delta \in \mathbb{C}$, $\psi \neq 0$ *and* $\Psi_\alpha^{m,n,q}$ *defined by the relation (2.2), then*

$$\frac{Mz+1}{Nz+1} \prec \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)},$$

with the best subordinant $\frac{Mz+1}{Nz+1}$.

Corollary 6. *Considering* $g(z) = \left(\frac{z+1}{1-z}\right)^k$, $0 < k \leq 1$, *and relation (2.6) is true, when* $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \in \mathcal{H}[g(0),1] \cap Q$ *and*

$$\varepsilon + \beta\left(\frac{z+1}{1-z}\right)^k + \delta\left(\frac{z+1}{1-z}\right)^{2k} + \frac{2k\psi z}{1-z^2} \prec \Psi_\alpha^{m,n,q}(\varepsilon,\beta,\delta,\psi;z),$$

where $m,\ n \in \mathbb{C}$, $n \neq 0,-1,-2,...$; $\alpha > 0$, $0 < q < 1$, $\varepsilon,\beta,\psi,\delta \in \mathbb{C}$, $\psi \neq 0$ *and* $\Psi_\alpha^{m,n,q}$ *defined by the relation (2.2), then*

$$\left(\frac{z+1}{1-z}\right)^k \prec \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)},$$

with the best subordinant $\left(\frac{z+1}{1-z}\right)^k$.

Combining Theorems 1 and 4, we state the following sandwich theorem.

Theorem 7. *Consider the analytic and univalent functions* g_1, g_2 *in* U *with the properties* $g_1(z) \neq 0$, $g_2(z) \neq 0$, *for all* $z \in U$, *and* $\frac{zg_1'(z)}{g_1(z)}$, $\frac{zg_2'(z)}{g_2(z)}$ *starlike univalent. Assuming that* g_1 *verifies (2.1) and* g_2 *verifies (2.6), when* $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \in \mathcal{H}[g(0),1] \cap Q$ *and* $\Psi_\alpha^{m,n,q}(\varepsilon,\beta,\delta,\psi;z)$ *defined by relation (2.2) is univalent in* U, $\alpha > 0$, $0 < q < 1$, $m,\ n \in \mathbb{C}$, $n \neq 0,-1,-2,...$; $\varepsilon,\beta,\psi,\delta \in \mathbb{C}$, $\psi \neq 0$, *then*

$$\varepsilon + \beta g_1(z) + \delta(g_1(z))^2 + \frac{\psi z g_1'(z)}{g_1(z)} \prec \Psi_\alpha^{m,n,q}(\varepsilon,\beta,\delta,\psi;z) \prec \varepsilon + \beta g_2(z) + \delta(g_2(z))^2 + \frac{\psi z g_2'(z)}{g_2(z)},$$

implies

$$g_1(z) \prec \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \prec g_2(z),$$

and g_1 *and* g_2 *are respectively the best subordinant and the best dominant.*

Corollary 8. *Considering* $g_1(z) = \frac{M_1z+1}{N_1z+1}$, $g_2(z) = \frac{M_2z+1}{N_2z+1}$, *where* $-1 \leq N_2 < N_1 < M_1 < M_2 \leq 1$ *and relations (2.1) and (2.6) are true, if* $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \in \mathcal{H}[g(0),1] \cap Q$ *and*

$$\varepsilon + \beta\frac{M_1z+1}{N_1z+1} + \delta\left(\frac{M_1z+1}{N_1z+1}\right)^2 + \frac{\psi(M_1-N_1)z}{(M_1z+1)(N_1z+1)} \prec \Psi_\alpha^{m,n,q}(\varepsilon,\beta,\delta,\psi;z)$$

$$\prec \frac{M_2 z+1}{N_2 z+1}+\delta\left(\frac{M_2 z+1}{N_2 z+1}\right)^2+\frac{\psi\left(M_2-N_2\right) z}{\left(M_2 z+1\right)\left(N_2 z+1\right)},$$

where m, $n \in \mathbb{C}$, $n \neq 0,-1,-2,\ldots$; $\alpha>0$, $0<q<1$, $\varepsilon, \beta, \psi, \delta \in \mathbb{C}$, $\psi \neq 0$ and $\Psi_\alpha^{m,n,q}$ defined by the relation (2.2), then

$$\frac{M_1 z+1}{N_1 z+1} \prec \frac{z\left(D_z^{-\alpha} \phi(m, n ; q, z)\right)^{\prime}}{D_z^{-\alpha} \phi(m, n ; q, z)} \prec \frac{M_2 z+1}{N_2 z+1},$$

with the best dominant $\frac{M_2 z+1}{N_2 z+1}$ and the best subordinant $\frac{M_1 z+1}{N_1 z+1}$.

Corollary 9. *Considering $g_1(z)=\left(\frac{z+1}{1-z}\right)^{k_1}$, $g_2(z)=\left(\frac{z+1}{1-z}\right)^{k_2}$ $0<k_1<k_2 \leq 1$, and relations (2.1) and (2.6) are true, if $\frac{z\left(D_z^{-\alpha} \phi(m, n ; q, z)\right)^{\prime}}{D_z^{-\alpha} \phi(m, n ; q, z)} \in \mathcal{H}[g(0), 1] \cap Q$ and*

$$\varepsilon+\beta\left(\frac{z+1}{1-z}\right)^{k_1}+\delta\left(\frac{z+1}{1-z}\right)^{2 k_1}+\frac{2 k_1 \psi z}{1-z^2} \prec \Psi_\alpha^{m,n,q}(\varepsilon, \beta, \delta, \psi ; z)$$

$$\prec \varepsilon+\beta\left(\frac{z+1}{1-z}\right)^{k_2}+\delta\left(\frac{z+1}{1-z}\right)^{2 k_2}+\frac{2 k_2 \psi z}{1-z^2},$$

where m, $n \in \mathbb{C}$, $n \neq 0,-1,-2,\ldots$; $\alpha>0$, $0<q<1$, $\varepsilon, \beta, \psi, \delta \in \mathbb{C}$, $\psi \neq 0$ and $\Psi_\alpha^{m,n,q}$ defined by the relation (2.2), then

$$\left(\frac{z+1}{1-z}\right)^{k_1} \prec \frac{z\left(D_z^{-\alpha} \phi(m, n ; q, z)\right)^{\prime}}{D_z^{-\alpha} \phi(m, n ; q, z)} \prec\left(\frac{z+1}{1-z}\right)^{k_2},$$

with the best dominant $\left(\frac{z+1}{1-z}\right)^{k_2}$ and the best subordinant $\left(\frac{z+1}{1-z}\right)^{k_1}$.

Choosing the functions $\varphi(u)=\varepsilon u$ and $\gamma(u)=\psi$, $u \in U$, we obtain other subordination and superordination theorems and corollaries.

Theorem 10. *Let the convex and univalent function g in U with $g(0)=\alpha$ and $\frac{z\left(D_z^{-\alpha} \phi(m, n ; q, z)\right)^{\prime}}{D_z^{-\alpha} \phi(m, n ; q, z)} \in \mathcal{H}(U)$, $z \in U$, where $\alpha>0$, $0<q<1$, m, $n \in \mathbb{C}$, $n \neq 0,-1,-2,\ldots$. Suppose that*

$$Re\left(\frac{\varepsilon+\psi}{\psi}+z \frac{g^{\prime \prime}(z)}{g^{\prime}(z)}\right)>0, \tag{2.9}$$

for $\varepsilon, \psi \in \mathbb{C}$, $\psi \neq 0$, $z \in U$, and

$$\Psi_\alpha^{m,n,q}(\varepsilon, \psi ; z):=(\varepsilon+\psi) \frac{z\left(D_z^{-\alpha} \phi(m, n ; q, z)\right)^{\prime}}{D_z^{-\alpha} \phi(m, n ; q, z)}-\psi\left(\frac{z\left(D_z^{-\alpha} \phi(m, n ; q, z)\right)^{\prime}}{D_z^{-\alpha} \phi(m, n ; q, z)}\right)^2 \tag{2.10}$$

$$+\psi \frac{z^2\left(D_z^{-\alpha} \phi(m, n ; q, z)\right)^{\prime \prime}}{D_z^{-\alpha} \phi(m, n ; q, z)}.$$

When g verifies the differential subordination

$$\Psi_\alpha^{m,n,q}(\varepsilon, \psi ; z) \prec \varepsilon g(z)+\psi z g^{\prime}(z), \tag{2.11}$$

for $\varepsilon, \psi \in \mathbb{C}$, $\psi \neq 0$, $z \in U$, then

$$\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \prec g(z), \quad z \in U, \tag{2.12}$$

and g is the best dominant.

Proof. Define the analytic function $p(z) := \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}$, $z \in U$, $z \neq 0$ in U, with $p(0) = \alpha$. Differentiating it, we get $p'(z) = \frac{\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} - z\left(\frac{\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}\right)^2 + z\frac{\left(D_z^{-\alpha}\phi(m,n;q,z)\right)''}{D_z^{-\alpha}\phi(m,n;q,z)}$ and

$$zp'(z) = \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} - \left(\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}\right)^2 + \frac{z^2\left(D_z^{-\alpha}\phi(m,n;q,z)\right)''}{D_z^{-\alpha}\phi(m,n;q,z)}. \tag{2.13}$$

Considering the analytic functions $\varphi(u) = \varepsilon u$ in $\mathbb{C}$ and $\gamma(u) = \psi \neq 0$ in $\mathbb{C}\backslash\{0\}$, we define the starlike univalent function $G(z) = z\gamma(g(z))g'(z) = \psi z g'(z)$ in U and $h(z) = G(z) + \varphi(g(z)) = \varepsilon g(z) + \psi z g'(z)$. Relation (2.9) can be written $Re\left(\frac{zh'(z)}{G(z)}\right) = Re\left(\frac{\varepsilon+\psi}{\psi} + z\frac{g''(z)}{g'(z)}\right) > 0$ and using relation (2.13), we obtain $\varepsilon p(z) + \psi z p'(z) = (\varepsilon + \psi)\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} - \psi\left(\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}\right)^2 + \psi\frac{z^2\left(D_z^{-\alpha}\phi(m,n;q,z)\right)''}{D_z^{-\alpha}\phi(m,n;q,z)}$.

The differential subordination (2.11) can be written $\varepsilon p(z) + \psi z p'(z) \prec \varepsilon g(z) + \psi z g'(z)$ and applying Lemma 1, we get $p \prec g$, i.e. $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \prec g(z)$, $z \in U$, and g is the best dominant. □

Corollary 11. *Considering $g(z) = \frac{Mz+1}{Nz+1}$, $-1 \leq N < M \leq 1$, $z \in U$, and relation (2.9) is true, when*

$$\Psi_\alpha^{m,n,q}(\varepsilon, \psi; z) \prec \varepsilon\frac{Mz+1}{Nz+1} + \frac{\psi(M-N)z}{(Nz+1)^2},$$

where m, $n \in \mathbb{C}$, $n \neq 0, -1, -2, ...$; $\alpha > 0$, $0 < q < 1$, $\varepsilon, \psi \in \mathbb{C}$, $\psi \neq 0$ and $\psi_\alpha^{m,n,q}$ defined by the relation (2.10), then

$$\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \prec \frac{Mz+1}{Nz+1},$$

with the best dominant $\frac{Mz+1}{Nz+1}$.

Corollary 12. *Considering $g(z) = \left(\frac{z+1}{1-z}\right)^k$, $0 < k \leq 1$, and relation (2.9) is true, when*

$$\Psi_\alpha^{m,n,q}(\varepsilon, \psi; z) \prec \varepsilon\left(\frac{z+1}{1-z}\right)^k + \frac{2k\psi z}{1-z^2}\left(\frac{z+1}{1-z}\right)^k,$$

where m, $n \in \mathbb{C}$, $n \neq 0, -1, -2, ...$; $\alpha > 0$, $0 < q < 1$, $\varepsilon, \psi \in \mathbb{C}$, $\psi \neq 0$ and $\Psi_\alpha^{m,n,q}$ defined by the relation (2.10), then

$$\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \prec \left(\frac{z+1}{1-z}\right)^k,$$

with the best dominant $\left(\frac{z+1}{1-z}\right)^k$.

Theorem 13. *Consider the convex and univalent function g in U with* $g(0)=\alpha$*, where* $\alpha>0$, $0<q<1$, m, $n\in\mathbb{C}$, $n\neq 0,-1,-2,\ldots$*. Suppose that*

$$Re\left(\frac{\varepsilon}{\psi}g'(z)\right)>0,\ for\ \varepsilon,\psi\in\mathbb{C},\psi\neq 0. \tag{2.14}$$

When $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}\in\mathcal{H}[g(0),1]\cap Q$ *and* $\Psi_\alpha^{m,n,q}(\varepsilon,\psi;z)$ *defined by relation (2.10) is univalent in U, then*

$$\varepsilon g(z)+\psi z g'(z)\prec\Psi_\alpha^{m,n,q}(\varepsilon,\psi;z) \tag{2.15}$$

implies

$$g(z)\prec\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)},\ \ z\in U, \tag{2.16}$$

and g is the best subordinant.

Proof. Define the analytic function $p(z)=\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}$, $z\in U$, $z\neq 0$, with $p(0)=\alpha$ and consider the analytic functions $\varphi(u)=\varepsilon u$ in $\mathbb{C}$ and $\gamma(u)=\psi\neq 0$ in $\mathbb{C}\backslash\{0\}$.

Differentiating it we obtain $\frac{\varphi'(g(z))}{\gamma(g(z))}=\frac{\varepsilon}{\psi}g'(z)$, and $Re\left(\frac{\varphi'(g(z))}{\gamma(g(z))}\right)=Re\left(\frac{\varepsilon}{\psi}g'(z)\right)>0$, for $\varepsilon,\psi\in\mathbb{C},\psi\neq 0$, taking account of relation (2.14).

The differential superordination (2.15) takes the following form

$$\varepsilon g(z)+\psi z g'(z)\prec\varepsilon p(z)+\psi z p'(z),\quad z\in U,$$

and applying Lemma 2, we obtain

$$g(z)\prec p(z)=\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)},\quad z\in U,$$

and g is the best subordinant. □

Corollary 14. *Considering* $g(z)=\frac{Mz+1}{Nz+1}$, $-1\leq N<M\leq 1$, $z\in U$*, and the relation (2.14) is true, when* $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)}\in\mathcal{H}[g(0),1]\cap Q$ *and*

$$\varepsilon\frac{Mz+1}{Nz+1}+\frac{\psi(M-N)z}{(Nz+1)^2}\prec\Psi_\alpha^{m,n,q}(\varepsilon,\psi;z),$$

where m, $n\in\mathbb{C}$, $n\neq 0,-1,-2,\ldots;$ $\alpha>0$, $0<q<1$, $\varepsilon,\psi\in\mathbb{C}$, $\psi\neq 0$ *and* $\Psi_\alpha^{m,n,q}$ *defined by the relation (2.10), then*

$$\frac{Mz+1}{Nz+1}\prec\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)},$$

with the best subordinant $\frac{Mz+1}{Nz+1}$.

Corollary 15. *Considering* $g(z) = \left(\frac{z+1}{1-z}\right)^k$, $0 < k \le 1$, *and the relation (2.14) is true, when* $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \in \mathcal{H}[g(0),1] \cap Q$ *and*

$$\varepsilon\left(\frac{z+1}{1-z}\right)^k + \frac{2k\psi z}{1-z^2}\left(\frac{z+1}{1-z}\right)^k \prec \Psi_\alpha^{m,n,q}(\varepsilon,\psi;z),$$

where $m, n \in \mathbb{C}$, $n \ne 0,-1,-2,...$; $\alpha > 0$, $0 < q < 1$, $\varepsilon, \psi \in \mathbb{C}$, $\psi \ne 0$ *and* $\Psi_\alpha^{m,n,q}$ *defined by the relation (2.10), then*

$$\left(\frac{z+1}{1-z}\right)^k \prec \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)},$$

with the best subordinant $\left(\frac{z+1}{1-z}\right)^k$.

Combining Theorem 10 and 13, we state the following sandwich theorem.

Theorem 16. *Consider the convex and univalent functions* g_1, g_2 *in* U *such that* $g_1(z) \ne 0$ *and* $g_2(z) \ne 0$, *for all* $z \in U$. *Assuming that* g_1 *satisfies (2.9) and* g_2 *satisfies (2.14), if* $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \in \mathcal{H}[g(0),1] \cap Q$, *and* $\Psi_\alpha^{m,n,q}(\varepsilon,\psi;z)$ *defined by relation (2.10) is univalent in* U, $\alpha > 0$, $0 < q < 1$, $m, n \in \mathbb{C}$, $n \ne 0,-1,-2,...$, *then*

$$\varepsilon g_1(z) + \psi z g_1'(z) \prec \Psi_\alpha^{m,n,q}(\varepsilon,\psi;z) \prec \varepsilon g_2(z) + \psi z g_2'(z),$$

for $\varepsilon, \psi \in \mathbb{C}$, $\psi \ne 0$, *implies*

$$g_1(z) \prec \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \prec g_2(z), \quad z \in U,$$

and g_1 *and* g_2 *are respectively the best subordinant and the best dominant.*

Corollary 17. *Considering* $g_1(z) = \frac{M_1 z+1}{N_1 z+1}$, $g_2(z) = \frac{M_2 z+1}{N_2 z+1}$, *where* $-1 \le N_2 < N_1 < M_1 < M_2 \le 1$ *and the relations (2.9) and (2.14) are true, if* $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \in \mathcal{H}[g(0),1] \cap Q$ *and*

$$\varepsilon\frac{M_1 z+1}{N_1 z+1} + \frac{\psi(M_1-N_1)z}{(N_1 z+1)^2} \prec \Psi_\alpha^{m,n,q}(\varepsilon,\psi;z) \prec \varepsilon\frac{M_2 z+1}{N_2 z+1} + \frac{\psi(M_2-N_2)z}{(N_2 z+1)^2},$$

where $m, n \in \mathbb{C}$, $n \ne 0,-1,-2,...$; $\alpha > 0$, $0 < q < 1$, $\varepsilon, \psi \in \mathbb{C}$, $\psi \ne 0$ *and* $\Psi_\alpha^{m,n,q}$ *defined by the relation (2.10), then*

$$\frac{M_1 z+1}{N_1 z+1} \prec \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \prec \frac{M_2 z+1}{N_2 z+1},$$

with the best dominant $\frac{M_2 z+1}{N_2 z+1}$ *and the best subordinant* $\frac{M_1 z+1}{N_1 z+1}$.

Corollary 18. *Considering* $g_1(z) = \left(\frac{z+1}{1-z}\right)^{k_1}$, $g_2(z) = \left(\frac{z+1}{1-z}\right)^{k_2}$ $0 < k_1 < k_2 \le 1$, *and the relations (2.9) and (2.14) are true, if* $\frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \in \mathcal{H}[g(0),1] \cap Q$ *and*

$$\varepsilon\left(\frac{z+1}{1-z}\right)^{k_1} + \frac{2k_1\psi z}{1-z^2}\left(\frac{z+1}{1-z}\right)^{k_1} \prec \Psi_\alpha^{m,n,q}(\varepsilon,\psi;z)$$

$$\prec \varepsilon \left(\frac{z+1}{1-z}\right)^{k_2} + \frac{2k_2 \psi z}{1-z^2}\left(\frac{z+1}{1-z}\right)^{k_2},$$

where $m, n \in \mathbb{C}$, $n \neq 0, -1, -2, ...$; $\alpha > 0$, $0 < q < 1$, $\varepsilon, \psi \in \mathbb{C}$, $\psi \neq 0$ *and* $\Psi_{\alpha}^{m,n,q}$ *defined by the relation (2.10), then*

$$\left(\frac{z+1}{1-z}\right)^{k_1} \prec \frac{z\left(D_z^{-\alpha}\phi(m,n;q,z)\right)'}{D_z^{-\alpha}\phi(m,n;q,z)} \prec \left(\frac{z+1}{1-z}\right)^{k_2},$$

with the best dominant $\left(\frac{z+1}{1-z}\right)^{k_2}$ *and the best subordinant* $\left(\frac{z+1}{1-z}\right)^{k_1}$.

3. Discussions

Motivated by the interesting and varied results obtained by incorporating fractional calculus and q-hypergeometric function in the studies regarding geometric function theory, the dual theories of differential subordination and superordination are applied for obtaining new results involving Riemann-Liouville fractional integral of q-hypergeometric function presented in Definition 7 and given in relations (1.3) and (1.4). Best dominants and best subordinants are obtained for each differential subordination and superordination considered, respectively, in the theorems and interesting corollaries emerge when functions well-known for their geometric properties as applied in the theorems as best dominant or best subordinant. The results obtained considering the two dual theories are connected by sandwich-type results which are familiar to geometric function theory.

Given the geometrical properties which can be interpreted from corollaries, future studies could consider introducing new subclasses of functions using the Riemann-Liouville fractional integral of q-hypergeometric function as seen in [27,30,31], for example and other differential subrodinations and superordinations could be obtained related to those functions. Also, strong differential subordination and superordination could be connected to the operator given in Definition 7 like seen in [32].

Conflict of interest

The authors declare that they have no competing interests. The authors drafted the manuscript, read and approved the final manuscript.

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