AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 4907–4929.
DOI: 10.3934/math.2023245
Received: 22 August 2022
Revised: 30 October 2022
Accepted: 08 November 2022
Published: 09 December 2022

Research article

An innovative fuzzy parameterized MADM approach to site selection for dam construction based on sv-complex neutrosophic hypersoft set

Atiqe Ur Rahman[1,*], **Muhammad Saeed**[1], **Mazin Abed Mohammed**[2], **Alaa S Al-Waisy**[3], **Seifedine Kadry**[4,5,6] **and Jungeun Kim**[7,*]

[1] Department of Mathematics, University of Management and Technology, Lahore 54000, Pakistan
[2] College of Computer Science and Information Technology, University of Anbar, Anbar 31001, Iraq
[3] Computer Technologies Engineering Department, Information Technology College, Imam Ja'afar Al-Sadiq University, Baghdad, Iraq
[4] Department of Applied Data Science, Noroff University College, Kristiansand 4608, Norway
[5] Artificial Intelligence Research Center (AIRC), College of Engineering and Information Technology, Ajman University, Ajman, United Arab Emirates
[6] Department of Electrical and Computer Engineering, Lebanese American University, Byblos, Lebanon
[7] Department of Software, Kongju National University, Cheonan 31080, Korea

* **Correspondence:** Email: jekim@kongju.ac.kr.

Abstract: Dams are water reservoirs that provide adequate freshwater to residential, industrial, and mining sites. They are widely used to generate electricity, control flooding, and irrigate agricultural lands. Due to recent urbanization trends, industrialization, and climatic changes, the construction of dams is in dire need, which is planning intensive, quite expensive, and time-consuming. Moreover, finding an appropriate site to construct dams is also considered a challenging task for decision-makers. The dam site selection problem (DSSP) has already been considered a multi-criteria decision-making (MCDM) problem under uncertain (fuzzy set) environments by several researchers. However, they ignored some essential evaluating features (e.g., (a) fuzzy parameterized grades, which assess the vague nature of parameters and sub-parameters, (b) the hypersoft setting, which provides multi-argument-based domains for the approximation of alternatives, (c) the complex setting which tackles the periodicity of data, and (d) the single-valued neutrosophic setting which facilitates the decision makers to provide their opinions in three-dimensional aspects) that can be used in DSSP to make it more reliable and trustworthy. Thus this study aims to employ a robust fuzzy parameterized algebraic approach which starts with the characterization of a novel structure "fuzzy parameterized single valued complex neutrosophic hypersoft set ($\tilde{\lambda}$-set)" that is competent to deal with the above-mentioned features jointly. After that, it integrates the concept of fuzzy parameterization, decision-makers opinions in terms of single-valued complex neutrosophic numbers, and the classical matrix theory to compute the score values for evaluating alternatives. Based on the stages of the proposed approach, an algorithm is proposed, which is further explained by an illustrative example in which DSSP is considered a multiple attributes decision-making (MADM) scenario. The computed score values are then used to evaluate some suitable sites (regions) for dam construction. The computational results of the proposed algorithm are found to be precise and consistent through their comparison with some already developed approaches.

Keywords: dam construction; dam site selection problem; complex setting; hypersoft setting; single-valued neutrosophic setting; fuzzy parameterization
Mathematics Subject Classification: 03B52, 03E72, 90B50

1. Introduction

The method used to find the most suitable entity from the set of objects under examination by considering the appropriate multi attributes is known as multiple attributes decision making (MADM). It may be taken as a form of multi-criteria decision-making (MCDM) due to the replacement of criteria with attributes [1–6]. It is observed in several real-world problems that the decision-making methods, as mentioned earlier or others, may have various types of information or data-based uncertainties. Therefore, in this regard, many researchers have already attempted to tackle expected information-based uncertainties in data by developing different algebraic models. The single-valued neutrosophic set (svNS) [7] is one of them, which is an extension of fuzzy set-like models such as fuzzy set (FS) [8], intuitionistic fuzzy set (IFS) [9], picture fuzzy set (PFS) [10], and neutrosophic set (NS) [11]. The svNS provides a belonging mapping with a triad (truth, indeterminacy, falsity). Every triad component lies within [0,1], so their sum should be within [0,3]. The svNS has been a subject of great interest for several researchers. However, the research argumentations made by the researchers [12–18] are found to be more prominent regarding the formulation of distance, similarity, and entropy measures and the development of aggregation operators of svNS. Later, a need for developing a particular algebraic model is felt, which may collectively tackle the periodic nature of data and uncertainties; thus, single-valued complex neutrosophic set (svCNS) [19] is characterized as a general case of svNS. The svCNS transforms the belonging mapping of svNS to complex-valued belonging mapping, i.e., the triad components are complex-valued functions characterized by relevant amplitude and phase values. All amplitudes values must lie within [0,1], and their sum must lie within [0,3]. Similarly, all phase values must lie within $[0, 2\pi]$. Every triad component lies within [0,1], so their sum should be within [0,3]. The svCNS is the general case of a complex fuzzy set (CFS) [20], and a complex intuitionistic fuzzy set (CIFS) [21]. Sometimes parameters are needed to evaluate objects under examination, which demands a specific parameterization tool. Thus fulfilling this need, a soft set (SOS) [22] is conceptualized, which provides the approximations of objects under examination concerning suitable parameters by using soft approximate mapping. Zhan & Alcantud [23] integrated the concept of SOS and rough set to develop soft-rough covering and discussed its application in multi-criteria group decision-making. The single-valued neutrosophic soft set (svNSS) [24] is characterized by combining SOS with svNS. It generalizes the concepts of the fuzzy soft set (FSS) [25], intuitionistic fuzzy soft set (IFSS) [26], and neutrosophic soft set (NSS) [27]. As an extension of a complex fuzzy soft set (CFSS) [28], and complex intuitionistic fuzzy soft set (CIFSS) [29], complex neutrosophic soft set (CNSS) [30] is initiated, which combines SOS with NS.

To tackle the uncertain nature of criteria (attributes) and sub-criteria (sub-attributes), an innovative context of the study, fuzzy parameterization, is initiated. The fuzzy parameterized fuzzy soft set (FpFSS) [31], fuzzy parameterized intuitionistic fuzzy soft set (FpIFSS) [32], and fuzzy parameterized neutrosophic soft set (FpNSS) have been put forward by considering fuzzy parameters in the domain of approximate mappings of FSS, IFSS, and NSS respectively. An emerging field of study, hypersoft set (HSOS) [33], is presented as an extension of SOS to tackle decision-making situations where parameters are likely to be categorized into disjoint sub-classes consisting of their corresponding sub-parametric values. The HSOS considers a multi-argument domain for its approximate mapping, which is the Cartesian product of such subclasses. The concepts of CFSS, CIFSS, and CNSS have been modified for the HSOS environment [34]. Recently, researchers have applied the concept of neutrosophic parameterization in various decision-making problems [35, 36].

1.1. Motivation

After analyzing the works of literature discussed earlier, we can observe that the existing fuzzy set or neutrosophic set-like researches are not sufficient to cope with the following problems collectively that may be faced by decision-makers (DMs):

(1) Fuzzy belonging grade base parameterization: it assists the DMs in monitoring the indecisive nature of parameters and their related sub-parameters.
(2) Entitlement of amplitude and phase terms: it aims to deal with episodic trends of the raw data collected through different sources.
(3) Deliberation of neutrosophic settings: it aims to make a flexible environment available to DMs to provide expert opinions while approximating the objects under inspection.
(4) Consideration of hypersoft settings: it is aimed to provide a suitable approximate mapping that may tackle multi-argument-based approximation of alternatives concerning parameters and the respective sub-parameters.

Keeping in mind the above limitations of the research analyzed earlier; this study aims to characterize the notions of a new mathematical model, i.e., fuzzy parameterized single valued complex neutrosophic hypersoft set ($\tilde{\lambda}$-set), which is capable of addressing these limitations efficiently.

1.2. Significant contributions

Some significant contributions of the paper are highlighted as:

(1) With the aggregation operations of $\tilde{\lambda}$-set, a MADM-based algorithm is put forward, which assists the decision-makers in finding an optimized region (site) for constructing a mini dam.
(2) An example is presented which explains the steps (input, construction, computation, and output) of the proposed algorithm with self-explanatory calculations.
(3) Appropriate algebraic criteria are employed to transform the single-valued complex neutrosophic numbers (svCNNs) and complex fuzzy numbers (CFNs) into reduced fuzzy values.

(4) The fuzzy parameterized grades are determined to assess the imprecise depiction of input variables (parameters or sub-parametric tuples). A straightforward mathematical criterion is used in this regard.

The sectional structure of the remaining part of the paper is that the second section presents a detailed review of relevant literature and describes common challenges that motivate the proposed study. The third section recalls some necessary definitions to make the concept understandable to the readers. The fourth section elaborates on the stages of the adopted methodology and compares the proposed study with some relevant already-developed models. The last section summarizes the study with a description of the limitations.

2. Relevant literature

With the rapid increase in human population and urbanization, different but rough water use has been reported, creating many social and economic problems. The water resource management problem is the most significant one in this regard. Much water is used in farming for cultivation, construction projects, household consumption, and energy production. Farming for cultivation meets the vegetational requirements of society through agricultural yields, construction projects fulfill residential needs, household consumption is for domestic use like washing, bathing, etc., and energy production is for fulfilling energy needs like electricity. One problem humans may encounter in the forthcoming years is the water shortage problem (WSP). The infertile and semi-infertile areas covering the central portion of any country will be affected by WSP. Several attempts have been made to find appropriate solutions to tackle this problem. One of these solutions is to store water for future use by constructing water reservoirs, i.e., dams. The selection of a location for the construction of dams is another challenging task for policymakers [37]. Several factors (attributes) and sub-factors (sub-attributes) are to be considered to locate appropriate areas for the construction of dams. Thus, the dam site selection problem (DSSP) is the MADM problem that several researchers have already discussed. In this regard, the argumentations made by researchers [38–40] are worth noting. Moreover, due to the possibility of various expected uncertainties in selecting criteria (attributes) and sub-criteria (sub-attributes), many scholars have discussed the DSSP as a fuzzy MADM problem. Still, the contributions of Chien et al. [41], Esavi et al. [42], Narayanamoorthy et al. [43], Janjua & Hassan [44], and Adeyanju & Adedeji [45] are worth noting in this regard. Recently Deveci et al. [46] discussed the DSSP using modified LAAW and RAFSI techniques based on fuzzy rough numbers. Ahmadi et al. [47], and Heidarimozaffar & Shahavand [48] presented fuzzy logic decision-making models for DSSP. Haghshenas et al. [49], Jozaghi et al. [50], and Tufail et al. [51] conferred the DSSP by using modified TOPSIS based on fuzzy settings. Akram et al. [52] characterized the notions of m-polar fuzzy soft expert set (mFSES) and designed a multi-criteria group decision-making model for DSSP. The state of the art review of relevant DSSP literature is presented in Table 1. The notations and abbreviations used in Table 1 are explained below:
TOPSIS= technique for order preference by similarity to ideal solution; AHP= analytic hierarchy process; FANP= fuzzy analytical network process; HFSDV-MOORA= hesitant fuzzy standard deviation with multi-objective optimization method by ratio analysis; HFSDV-TOPSIS= hesitant fuzzy standard deviation with technique, for order preference by similarity to an ideal solution; HFSDV-VIKOR= hesitant fuzzy standard deviation with VIsekriterijumsko Kompromisno Rangiranje;

MCDA= multi-criteria decision analysis; RAFSI= ranking of alternatives through functional mapping of criterion subintervals into a single interval; LAAW = logarithmic additive assessment of the weight coefficients; GIS= geospatial information system; MCGDM= multi-criteria group decision making; FRNs= fuzzy rough numbers; TFNs = triangular fuzzy numbers and FNs= fuzzy numbers.

Table 1. Review of existing literature related DSSP.

Reference	Fuzzy set-like structure	Decision-making approach
Wang et al. [1]	FS	Fuzzy hierarchical TOPSIS with FNs based simplified parameterized metric distance
Noori et al. [3]	FS	Fuzzy TOPSIS with TFNs based features
Chien et al. [41]	FS	FANP and TOPSIS with TFNs based features
Esavi et al. [42]	FS	AHP and fuzzy-AHP
Narayanamoorthy et al. [43]	Hesitant FS	HFSDV-MOORA, HFSDV-TOPSIS and HFSDV-VIKOR
Janjua & Hassan [44]	FS	AHP-TOPSIS with TFNs based features
Adeyanju & Adedeji [45]	FS	MCDA-TOPSIS with TFNs based features
Deveci et al. [46]	Fuzzy rough set	LAAW and RAFSI with FRNs based features
Ahmadi et al. [47]	FS	Fuzzy ANP and Fuzzy VIKOR
Heidarimozaffar & Shahavand [48]	FS	Fuzzy logic and GIS
Haghshenas et al. [49]	FS	Fuzzy TOPSIS with risk analysis
Jozaghi et al. [50]	FS	AHP and TOPSIS using GIS
Tufail et al. [51]	Bipolar soft rough set	PROMETHEE and TOPSIS using covering based rough sets
Akram et al. [52]	mFSES	Set operations based MCGDM

After integrating the analyses of literature provided in Section 1, the Subsection 1.1, and this section, it is observed that the existing literature has ignored some challenges that may be encountered by DMs while dealing with DSSP. These expected challenges are outlined below:

(1) How can the vague and uncertain aspects of criteria and sub-criteria be dealt with for approximating the geographical regions scrutinized for dam construction?
(2) How can the information-based periodicity of opinions provided by decision-makers be tackled to get a reliable and consistent evaluation of sites for dam construction?
(3) Suppose decision-makers demand a particular arrangement that facilitates them to provide their opinions independently in terms of a triad (truth, indeterminacy, falsity) while approximating the dam construction sites. How can such an arrangement be managed in DSSP?
(4) How can categorizing parameters into related sub-parametric non-overlapping classes be managed in DSSP to have multi-argument-based domains?

The first challenge points out that a specific fuzzy grade is required to attach to each attribute or sub-attribute to cope with its uncertain nature. It leads to the entitlement of fuzzy parameterization. The second one indicates the use of complex plane setting for the approximation process. The third one is meant for the employment of single-valued neutrosophic settings, and the last one demands a hypersoft setting.

This study aims to develop a MADM-based DSSP model using the notions of the fuzzy parameterized single-valued complex neutrosophic hypersoft set ($\tilde{\lambda}$-set). The $\tilde{\lambda}$-set is a novel algebraic structure capable of collectively coping with all the above challenges.

3. Basic definitions

The key purpose of this section is to review some definitions of elementary nature for a clear understanding of the proposed approach.

An svNS is the generalization of NS, which makes the NS compatible with real-life applications. It transforms $]^{-}0, 3^{+}[$ to $[0, 3]$ that can easily be interpreted.

Definition 3.1. [7] Let a mapping $\tilde{\zeta} : \tilde{\mho} \rightarrow [0, 1]^3$ defined by a triad $\tilde{\zeta}(\tilde{n}) = \langle \hat{T}_{\tilde{\aleph}}(\tilde{n}), \hat{I}_{\tilde{\aleph}}(\tilde{n}), \hat{F}_{\tilde{\aleph}}(\tilde{n})\rangle$ for all $\tilde{n} \in \tilde{\mho}$ such that all three components of triad $\hat{T}_{\tilde{\aleph}}(\tilde{n}), \hat{I}_{\tilde{\aleph}}(\tilde{n}), \hat{F}_{\tilde{\aleph}}(\tilde{n}) \in [0, 1]$ with $\hat{T}_{\tilde{\aleph}}(\tilde{n}) + \hat{I}_{\tilde{\aleph}}(\tilde{n}) + \hat{F}_{\tilde{\aleph}}(\tilde{n}) \in [0, 3]$. The components $\hat{T}_{\tilde{\aleph}}(\tilde{n}), \hat{I}_{\tilde{\aleph}}(\tilde{n})$ and $\hat{F}_{\tilde{\aleph}}(\tilde{n})$ are named as true, indeterminate and false belonging grades of $\tilde{n} \in \tilde{\mho}$. An svNS $\tilde{\aleph}$ on $\tilde{\mho}$ is stated as $\tilde{\aleph} = \left\{ \left(\tilde{n}, \langle \hat{T}_{\tilde{\aleph}}(\tilde{n}), \hat{I}_{\tilde{\aleph}}(\tilde{n}), \hat{F}_{\tilde{\aleph}}(\tilde{n})\rangle\right) : \tilde{n} \in \tilde{\mho} \right\}$. The symbol $\Theta(\tilde{\aleph})$ is regarded as family of svNSs on $\tilde{\mho}$.

As svNS cannot tackle information-based periodicity, this limitation is addressed by developing the svCNS, which may manage this periodicity with the help of a complex-valued true-belonging function, indeterminate-belonging function, and false belonging function having amplitude and phase values.

Definition 3.2. [19] A svCNS $\tilde{\aleph}_c$ is characterized by three complex valued belonging functions: true-belonging function $\hat{T}_{\tilde{\aleph}_c}$, indeterminate-belonging function $\hat{I}_{\tilde{\aleph}_c}$ and false belonging function $\hat{F}_{\tilde{\aleph}_c}$ which are defined by $\alpha_{\hat{T}}(\tilde{n})Exp(j\beta_{\hat{T}}(\tilde{n}))$, $\alpha_{\hat{I}}(\tilde{n})Exp(j\beta_{\hat{I}}(\tilde{n}))$ and $\alpha_{\hat{F}}(\tilde{n})\ Exp(j\beta_{\hat{F}}(\tilde{n}))$ respectively such that $\alpha_{\hat{T}}(\tilde{n})+\alpha_{\hat{I}}(\tilde{n})+\alpha_{\hat{F}}(\tilde{n}) \in [0, 3]$ and $\beta_{\hat{T}}(\tilde{n}), \beta_{\hat{I}}(\tilde{n}), \beta_{\hat{F}}(\tilde{n}) \in [0, 2\pi]$ for all $\tilde{n} \in \tilde{\mho}$ and $\sqrt{j} = -1$. The values $\alpha_{\hat{T}}(\tilde{n}), \alpha_{\hat{I}}(\tilde{n})$ and $\alpha_{\hat{F}}(\tilde{n})$ are named as amplitude values and the values $\beta_{\hat{T}}(\tilde{n}), \beta_{\hat{I}}(\tilde{n})$ and $\beta_{\hat{F}}(\tilde{n})$ are named as phase values of $\tilde{\aleph}_c$. A svCNS $\tilde{\aleph}_c$ can be stated jointly as $\tilde{\aleph}_c = \left\{ \left(\tilde{n}, \langle \hat{T}_{\tilde{\aleph}_c}(\tilde{n}), \hat{I}_{\tilde{\aleph}_c}(\tilde{n}), \hat{F}_{\tilde{\aleph}_c}(\tilde{n})\rangle\right) : \tilde{n} \in \tilde{\mho} \right\}$. The symbol $\Theta(\tilde{\aleph}_c)$ is regarded as family of svCNSs on $\tilde{\mho}$.

The existing fuzzy set-like models are not projected to entitle parameterization tool, which is a necessary mode for many decision-making situations; therefore, SOS is developed, which provides a proper arrangement for the entitlement of such method in the form of approximate mapping.

Definition 3.3. [22] Let an approximate mapping $\tilde{\pi} : \tilde{\Lambda}_1 \rightarrow 2^{\tilde{\mho}}$ defined by $\tilde{\pi}(\tilde{a}) \subseteq \tilde{\mho}$ for all $\tilde{a} \in \tilde{\Lambda}_1$ and $\tilde{\Lambda}_1 \subseteq \tilde{\Lambda}$ where $\tilde{\Lambda}$ is a set of parameters and $2^{\tilde{\mho}}$ is the power set of $\tilde{\mho}$. A SOS $\tilde{\mathfrak{R}}$ on $\tilde{\mho}$ is stated as $\tilde{\mathfrak{R}} = \left\{(\tilde{a}, \tilde{\pi}(\tilde{a})) : \tilde{a} \in \tilde{\Lambda}_1\right\}$ where $\tilde{\pi}(\tilde{a})$ is regarded as $\tilde{a}$-approximate element of $\tilde{\mathfrak{R}}$.

It is a matter of common observation that only parameters are not sufficient for evaluating a particular alternative; they need to be classified into respective non-overlapping classes consisting of their sub-parametric values. This limitation of literature is addressed by the development of HSOS, which employs multi-argument approximate mapping to tackle this sub-parametric valued classification.

Definition 3.4. [11] Let an approximate mapping $\tilde{\varpi} : \tilde{\Xi} \rightarrow 2^{\tilde{\mho}}$ defined by $\tilde{\varpi}(\tilde{e}) \subseteq \tilde{\mho}$ for all $\tilde{e} \in \tilde{\Xi}$ and $\tilde{\Xi} = \prod_{i=1}^{n} \tilde{\Xi}_i$ where $\tilde{\Xi}_i$ are non overlapping sets consisting of sub parametric values for all entries in

$\tilde{\Lambda}$. A HSOS $\tilde{\hbar}$ on $\tilde{\mho}$ is stated as $\tilde{\hbar} = \left\{(\tilde{e}, \tilde{\varpi}(\tilde{e})) : \tilde{e} \in \tilde{\Xi}\right\}$ where $\tilde{\varpi}(\tilde{e})$ is regarded as $\tilde{e}$-multi approximate element of $\tilde{\hbar}$.

According to arguments provided in [11], a HSOS $\tilde{\hbar}$ (as in Definition 3.4) is claimed to be called svNHSOS (single-valued neutrosophic hypersoft set) and svCNHSOS (single-valued complex neutrosophic hypersoft set) on $\tilde{\mho}$ if the mapping $\tilde{\varpi} : \tilde{\Xi} \to 2^{\tilde{\mho}}$ is replaced with $\tilde{\varpi} : \tilde{\Xi} \to \Theta(\tilde{\aleph})$ and $\tilde{\varpi} : \tilde{\Xi} \to \Theta(\tilde{\aleph}_c)$ respectively.

4. The proposed approach and its stages

This section briefly describes the various stages involved in the adopted methodology. Figure 1 presents a brief description of the steps involved in the adopted process.

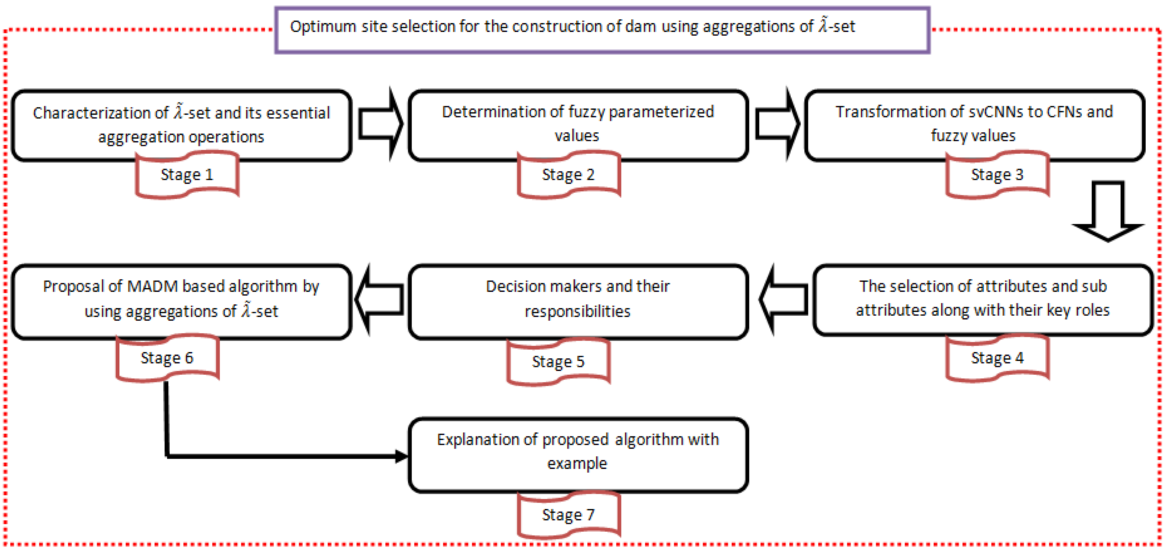

Figure 1. Brief description of stages involved in adopted methodology.

4.1. Characterization of $\tilde{\lambda}$-set and its essential aggregation operations

In this part of the paper, the elementary notions of $\tilde{\lambda}$-set are characterized and explained with the support of illustrative examples. Throughout the remaining paper, the symbol $\tilde{\Xi}$ will be used as it is described in Definition 3.4.

Definition 4.1. Let $\tilde{\Xi} = \{\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_r\}$ be a set and $\tilde{\mathscr{F}} = \{\tilde{e}_1/\mu(\tilde{e}_1), \tilde{e}_2/\mu(\tilde{e}_2), ..., \tilde{e}_r/\mu(\tilde{e}_r)\}$ be a FS over $\tilde{\Xi}$ then a $\tilde{\lambda}$-set $\tilde{\Omega}$ is stated as $\tilde{\Omega} = \left\{\left(\tilde{e}_i/\mu(\tilde{e}_i), \tilde{\psi}(\tilde{e}_i/\mu(\tilde{e}_i))\right) : \tilde{e}_i/\mu(\tilde{e}_i) \in \tilde{\mathscr{F}}\right\}$ where $\tilde{\psi} : \tilde{\mathscr{F}} \to \Theta(\tilde{\aleph}_c)$ defined by $\tilde{\psi}(\tilde{e}_i/\mu(\tilde{e}_i)) = \langle \hat{T}_{\tilde{\aleph}_c}(\tilde{e}_i/\mu(\tilde{e}_i)), \hat{I}_{\tilde{\aleph}_c}(\tilde{e}_i/\mu(\tilde{e}_i)), \hat{F}_{\tilde{\aleph}_c}(\tilde{e}_i/\mu(\tilde{e}_i))\rangle$ with $\hat{T}_{\tilde{\aleph}_c}(\tilde{e}_i/\mu(\tilde{e}_i)) = \alpha_{\hat{T}}(\tilde{e}_i/\mu(\tilde{e}_i))Exp(j\beta_{\hat{T}}(\tilde{e}_i/\mu(\tilde{e}_i)))$, $\hat{I}_{\tilde{\aleph}_c}(\tilde{e}_i/\mu(\tilde{e}_i)) = \alpha_{\hat{I}}(\tilde{e}_i/\mu(\tilde{e}_i))Exp(j\beta_{\hat{I}}(\tilde{e}_i/\mu(\tilde{e}_i)))$ and $\hat{F}_{\tilde{\aleph}_c}(\tilde{e}_i/\mu(\tilde{e}_i)) = \alpha_{\hat{F}}(\tilde{e}_i/\mu(\tilde{e}_i))Exp(j\beta_{\hat{F}}(\tilde{e}_i/\mu(\tilde{e}_i)))$ are complex valued true-belonging, indeterminate-belonging and false-belonging grades for all $\tilde{e}_i/\mu(\tilde{e}_i) \in \tilde{\mathscr{F}}$ in $\tilde{\Omega}$ provided that $\alpha_{\hat{T}}(\tilde{e}_i/\mu(\tilde{e}_i)) + \alpha_{\hat{I}}(\tilde{e}_i/\mu(\tilde{e}_i)) + \alpha_{\hat{F}}(\tilde{e}_i/\mu(\tilde{e}_i)) \in [0, 3]$ and $\beta_{\hat{T}}(\tilde{e}_i/\mu(\tilde{e}_i)), \beta_{\hat{I}}(\tilde{e}_i/\mu(\tilde{e}_i)), \beta_{\hat{F}}(\tilde{e}_i/\mu(\tilde{e}_i)) \in [0, 2\pi]$. The $\tilde{\lambda}$-set $\tilde{\Omega}$ can jointly be expressed as

$$\tilde{\Omega}=\left\{\left(\tilde{e}_i/\mu(\tilde{e}_i),\tilde{\psi}(\tilde{e}_i/\mu(\tilde{e}_i))=\left\langle\begin{array}{c}\alpha_{\hat{T}}(\tilde{e}_i/\mu(\tilde{e}_i))Exp(j\beta_{\hat{T}}(\tilde{e}_i/\mu(\tilde{e}_i))),\\ \alpha_{\hat{I}}(\tilde{e}_i/\mu(\tilde{e}_i))Exp(j\beta_{\hat{I}}(\tilde{e}_i/\mu(\tilde{e}_i))),\\ \alpha_{\hat{F}}(\tilde{e}_i/\mu(\tilde{e}_i))Exp(j\beta_{\hat{F}}(\tilde{e}_i/\mu(\tilde{e}_i)))\end{array}\right\rangle\right):\tilde{e}_i/\mu(\tilde{e}_i)\in\tilde{\mathscr{F}}\right\}. \tag{4.1}$$

The family of $\tilde{\lambda}$-sets is abbreviated as $\Theta(\tilde{\Omega})$.

Example 4.1. Let $\tilde{\mho}=\{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3,\tilde{n}_4\}$ be the collection of objects under observation as initial universe and $\tilde{\Xi}_1=\{\tilde{e}_{11},\tilde{e}_{12}\}$, $\tilde{\Xi}_2=\{\tilde{e}_{21},\tilde{e}_{22}\}$ and $\tilde{\Xi}_3=\{\tilde{e}_{31},\tilde{e}_{32}\}$ are sub parametric valued non overlapping classes such that $\tilde{\Xi}=\tilde{\Xi}_1\times\tilde{\Xi}_2\times\tilde{\Xi}_3=\{(\tilde{e}_{11},\tilde{e}_{21},\tilde{e}_{31}),(\tilde{e}_{11},\tilde{e}_{21},\tilde{e}_{32}),(\tilde{e}_{11},\tilde{e}_{22},\tilde{e}_{31}),(\tilde{e}_{11},\tilde{e}_{22},\tilde{e}_{32}),(\tilde{e}_{12},\tilde{e}_{21},\tilde{e}_{31}),(\tilde{e}_{12},\tilde{e}_{21},\tilde{e}_{32}),(\tilde{e}_{12},\tilde{e}_{22},\tilde{e}_{31}),(\tilde{e}_{12},\tilde{e}_{22},\tilde{e}_{32})\}=\{\tilde{e}_1,\tilde{e}_2,\tilde{e}_3,\tilde{e}_4,\tilde{e}_5,\tilde{e}_6,\tilde{e}_7,\tilde{e}_8\}$. Take $\tilde{\Xi}_\circ=\{\tilde{e}_2,\tilde{e}_4,\tilde{e}_6,\tilde{e}_8\}\subseteq\tilde{\Xi}$ with its related fuzzy set $\tilde{\mathscr{F}}=\{\tilde{e}_2/0.2,\tilde{e}_4/0.4,\tilde{e}_6/0.6,\tilde{e}_8/0.8\}$. The decision makers provide their opinions in terms of single valued complex neutrosophic numbers while approximating the alternatives based on fuzzy parameterized sub parametric tuples $\tilde{e}_2/0.2$, $\tilde{e}_4/0.4$, $\tilde{e}_6/0.6$ and $\tilde{e}_8/0.8$. Therefore a $\tilde{\lambda}$-set $\tilde{\Omega}$ can be constructed as

$$\tilde{\Omega}=\left\{\begin{array}{l}
\left(\frac{\tilde{e}_2}{0.2},\left\{\begin{array}{c}\frac{\tilde{n}_1}{\langle 0.5Exp(j2\pi0.2),0.6Exp(j2\pi0.3),0.8Exp(j2\pi0.5)\rangle},\frac{\tilde{n}_2}{\langle 0.9Exp(j2\pi0.6),0.7Exp(j2\pi0.4),0.6Exp(j2\pi0.4)\rangle},\\ \frac{\tilde{n}_3}{\langle 0.9Exp(j2\pi0.4),0.8Exp(j2\pi0.4),0.7Exp(j2\pi0.3)\rangle},\frac{\tilde{n}_4}{\langle 0.6Exp(j2\pi0.2),0.5Exp(j2\pi0.1),0.4Exp(j2\pi0.1)\rangle}\end{array}\right\}\right)\\
\left(\frac{\tilde{e}_4}{0.4},\left\{\begin{array}{c}\frac{\tilde{n}_1}{\langle 0.8Exp(j2\pi0.1),0.6Exp(j2\pi0.2),0.7Exp(j2\pi0.3)\rangle},\frac{\tilde{n}_2}{\langle 0.7Exp(j2\pi0.2),0.5Exp(j2\pi0.3),0.6Exp(j2\pi0.4)\rangle},\\ \frac{\tilde{n}_3}{\langle 0.6Exp(j2\pi0.4),0.6Exp(j2\pi0.5),0.8Exp(j2\pi0.4)\rangle},\frac{\tilde{n}_4}{\langle 0.5Exp(j2\pi0.1),0.9Exp(j2\pi0.4),0.8Exp(j2\pi0.2)\rangle}\end{array}\right\}\right)\\
\left(\frac{\tilde{e}_6}{0.6},\left\{\begin{array}{c}\frac{\tilde{n}_1}{\langle 0.8Exp(j2\pi0.4),0.8Exp(j2\pi0.5),0.7Exp(j2\pi0.6)\rangle},\frac{\tilde{n}_2}{\langle 0.6Exp(j2\pi0.4),0.6Exp(j2\pi0.4),0.7Exp(j2\pi0.5)\rangle},\\ \frac{\tilde{n}_3}{\langle 0.5Exp(j2\pi0.2),0.5Exp(j2\pi0.2),0.9Exp(j2\pi0.5)\rangle},\frac{\tilde{n}_4}{\langle 0.7Exp(j2\pi0.1),0.7Exp(j2\pi0.1),0.9Exp(j2\pi0.6)\rangle}\end{array}\right\}\right)\\
\left(\frac{\tilde{e}_8}{0.8},\left\{\begin{array}{c}\frac{\tilde{n}_1}{\langle 0.9Exp(j2\pi0.3),0.9Exp(j2\pi0.3),0.8Exp(j2\pi0.3)\rangle},\frac{\tilde{n}_2}{\langle 0.7Exp(j2\pi0.7),0.8Exp(j2\pi0.6),0.8Exp(j2\pi0.5)\rangle},\\ \frac{\tilde{n}_3}{\langle 0.6Exp(j2\pi0.6),0.7Exp(j2\pi0.5),0.7Exp(j2\pi0.4)\rangle},\frac{\tilde{n}_4}{\langle 0.5Exp(j2\pi0.5),0.6Exp(j2\pi0.2),0.9Exp(j2\pi0.2)\rangle}\end{array}\right\}\right)
\end{array}\right\},$$

and its matrix representation is

$$\tilde{\Omega}=\left(\begin{array}{ccccc}
\tilde{\Xi}\setminus\tilde{\mho} & \tilde{n}_1 & \tilde{n}_2 & \tilde{n}_3 & \tilde{n}_4\\
\frac{\tilde{e}_2}{0.2} & (0.5,0.2)\,(0.6,0.3)\,(0.8,0.5) & (0.9,0.6)\,(0.7,0.4)\,(0.6,0.4) & (0.9,0.4)\,(0.8,0.4)\,(0.7,0.3) & (0.6,0.2)\,(0.5,0.1)\,(0.4,0.1)\\
\frac{\tilde{e}_4}{0.4} & (0.8,0.1)\,(0.6,0.2)\,(0.7,0.3) & (0.7,0.2)\,(0.5,0.3)\,(0.6,0.4) & (0.6,0.4)\,(0.6,0.5)\,(0.9,0.5) & (0.7,0.1)\,(0.7,0.1)\,(0.9,0.6)\\
\frac{\tilde{e}_6}{0.6} & (0.8,0.4)\,(0.8,0.5)\,(0.7,0.6) & (0.6,0.4)\,(0.6,0.4)\,(0.7,0.5) & (0.5,0.2)\,(0.5,0.2)\,(0.7,0.4) & (0.5,0.5)\,(0.6,0.2)\,(0.9,0.2)\\
\frac{\tilde{e}_8}{0.8} & (0.9,0.3)\,(0.9,0.3)\,(0.8,0.3) & (0.7,0.7)\,(0.8,0.6)\,(0.8,0.5) & (0.6,0.6)\,(0.7,0.5)\,(0.7,0.4) & (0.5,0.5)\,(0.6,0.2)\,(0.9,0.2)
\end{array}\right).$$

The value $0.5Exp(j2\pi0.2)$ implies that the amplitude value 0.5 has period $2\pi0.2$ i.e. the value 0.5 will be repeated after $2\pi0.2=2\pi/5\,radians=72^\circ$ in complex plane with respect to fuzzy parameterized sub parametric tuple $\frac{\tilde{e}_2}{0.2}$.

4.1.1. Set theoretic operations of $\tilde{\lambda}$-set

In this part, some necessary set theoretic operations of $\tilde{\lambda}$-set are investigated. It is worth noting that every operation generates another of $\tilde{\lambda}$-set.

Definition 4.2. Let $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are $\tilde{\lambda}$-set with their representations as

$$\tilde{\Omega}_1=\left\{\begin{array}{l}\left(\tilde{e}_i/\mu_1(\tilde{e}_i),\tilde{\psi}_1(\tilde{e}_i/\mu_1(\tilde{e}_i))=\left\langle\begin{array}{c}\alpha^1_{\hat{T}}(\tilde{e}_i/\mu_1(\tilde{e}_i))Exp(j\beta^1_{\hat{T}}(\tilde{e}_i/\mu_1(\tilde{e}_i))),\\ \alpha^1_{\hat{I}}(\tilde{e}_i/\mu_1(\tilde{e}_i))Exp(j\beta^1_{\hat{I}}(\tilde{e}_i/\mu_1(\tilde{e}_i))),\\ \alpha^1_{\hat{F}}(\tilde{e}_i/\mu_1(\tilde{e}_i))Exp(j\beta^1_{\hat{F}}(\tilde{e}_i/\mu_1(\tilde{e}_i)))\end{array}\right\rangle\right):\\ \tilde{e}_i/\mu_1(\tilde{e}_i)\in\tilde{\mathscr{F}}_1;\tilde{e}_i\in\tilde{\Xi}'\subseteq\tilde{\Xi}\end{array}\right\},$$

and

$$\tilde{\Omega}_2 = \left\{ \begin{array}{l} \left(\tilde{e}_i/\mu_2(\tilde{e}_i), \tilde{\psi}_2(\tilde{e}_i/\mu_2(\tilde{e}_i)) = \left\langle \begin{array}{l} \alpha^2_{\hat{T}}(\tilde{e}_i/\mu_2(\tilde{e}_i))Exp(j\beta^2_{\hat{T}}(\tilde{e}_i/\mu_2(\tilde{e}_i))), \\ \alpha^2_{\hat{I}}(\tilde{e}_i/\mu_2(\tilde{e}_i))Exp(j\beta^2_{\hat{I}}(\tilde{e}_i/\mu_2(\tilde{e}_i))), \\ \alpha^2_{\hat{F}}(\tilde{e}_i/\mu_2(\tilde{e}_i))Exp(j\beta^2_{\hat{F}}(\tilde{e}_i/\mu_2(\tilde{e}_i))) \end{array} \right\rangle \right) : \\ \tilde{e}_i/\mu_2(\tilde{e}_i) \in \mathscr{F}_2; \tilde{e}_i \in \tilde{\Xi}'' \subseteq \tilde{\Xi} \end{array} \right\},$$

then,

(1) for any $\tilde{e}_i \in \tilde{\Xi}$, their union $\tilde{\Omega}_1 \cup \tilde{\Omega}_2$ has multi argument approximate element $\tilde{\psi}_3(\tilde{e}_i/\mu_3(\tilde{e}_i))$ which can be determined as

$$\tilde{\psi}_3(\tilde{e}_i/\mu_3(\tilde{e}_i)) = \begin{cases} \tilde{\psi}_1(\tilde{e}_i/\mu_1(\tilde{e}_i)) & ; \tilde{e}_i \in \tilde{\Xi}' \setminus \tilde{\Xi}'' \\ \tilde{\psi}_2(\tilde{e}_i/\mu_2(\tilde{e}_i)) & ; \tilde{e}_i \in \tilde{\Xi}'' \setminus \tilde{\Xi}' \\ \tilde{\psi}_3(\tilde{e}_i/\mu'(\tilde{e}_i)) = \tilde{\psi}_1(\tilde{e}_i/\mu_1(\tilde{e}_i)) \cup \tilde{\psi}_2(\tilde{e}_i/\mu_2(\tilde{e}_i)) & ; \tilde{e}_i \in \tilde{\Xi}' \cap \tilde{\Xi}'' \end{cases}$$

where,

$$\tilde{\psi}_3(\tilde{e}_i/\mu'(\tilde{e}_i)) = \left\langle \begin{array}{l} max\{\alpha^1_{\hat{T}}(\tilde{e}_i/\mu_1(\tilde{e}_i)), \alpha^2_{\hat{T}}(\tilde{e}_i/\mu_2(\tilde{e}_i))\}Exp[j\max\{\beta^1_{\hat{T}}(\tilde{e}_i/\mu_1(\tilde{e}_i)), \beta^2_{\hat{T}}(\tilde{e}_i/\mu_2(\tilde{e}_i))\}], \\ min\{\alpha^1_{\hat{I}}(\tilde{e}_i/\mu_1(\tilde{e}_i)), \alpha^2_{\hat{I}}(\tilde{e}_i/\mu_2(\tilde{e}_i))\}Exp[j\max\{\beta^1_{\hat{I}}(\tilde{e}_i/\mu_1(\tilde{e}_i)), \beta^2_{\hat{I}}(\tilde{e}_i/\mu_2(\tilde{e}_i))\}], \\ min\{\alpha^1_{\hat{F}}(\tilde{e}_i/\mu_1(\tilde{e}_i)), \alpha^2_{\hat{F}}(\tilde{e}_i/\mu_2(\tilde{e}_i))\}Exp[j\max\{\beta^1_{\hat{F}}(\tilde{e}_i/\mu_1(\tilde{e}_i)), \beta^2_{\hat{F}}(\tilde{e}_i/\mu_2(\tilde{e}_i))\}], \end{array} \right\rangle,$$

and $\mu'(\tilde{e}_i) = max\{\mu_1(\tilde{e}_i), \mu_2(\tilde{e}_i)\}$.

(2) Similarly, their intersection $\tilde{\Omega}_1 \cap \tilde{\Omega}_2$ has multi argument approximate element $\tilde{\psi}_4(\tilde{e}_i/\mu''(\tilde{e}_i))$, which can be determined as $\tilde{\psi}_4(\tilde{e}_i/\mu''(\tilde{e}_i)) = \tilde{\psi}_1(\tilde{e}_i/\mu_1(\tilde{e}_i)) \cap \tilde{\psi}_2(\tilde{e}_i/\mu_2(\tilde{e}_i))$ for all $\tilde{e}_i \in \tilde{\Xi}' \cap \tilde{\Xi}''$ where,

$$\begin{aligned} &\tilde{\psi}_1(\tilde{e}_i/\mu_1(\tilde{e}_i)) \cap \tilde{\psi}_2(\tilde{e}_i/\mu_2(\tilde{e}_i)) \\ &= \left\langle \begin{array}{l} min\{\alpha^1_{\hat{T}}(\tilde{e}_i/\mu_1(\tilde{e}_i)), \alpha^2_{\hat{T}}(\tilde{e}_i/\mu_2(\tilde{e}_i))\}Exp[j\min\{\beta^1_{\hat{T}}(\tilde{e}_i/\mu_1(\tilde{e}_i)), \beta^2_{\hat{T}}(\tilde{e}_i/\mu_2(\tilde{e}_i))\}], \\ max\{\alpha^1_{\hat{I}}(\tilde{e}_i/\mu_1(\tilde{e}_i)), \alpha^2_{\hat{I}}(\tilde{e}_i/\mu_2(\tilde{e}_i))\}Exp[j\min\{\beta^1_{\hat{I}}(\tilde{e}_i/\mu_1(\tilde{e}_i)), \beta^2_{\hat{I}}(\tilde{e}_i/\mu_2(\tilde{e}_i))\}], \\ max\{\alpha^1_{\hat{F}}(\tilde{e}_i/\mu_1(\tilde{e}_i)), \alpha^2_{\hat{F}}(\tilde{e}_i/\mu_2(\tilde{e}_i))\}Exp[j\min\{\beta^1_{\hat{F}}(\tilde{e}_i/\mu_1(\tilde{e}_i)), \beta^2_{\hat{F}}(\tilde{e}_i/\mu_2(\tilde{e}_i))\}], \end{array} \right\rangle \end{aligned}$$

and $\mu''(\tilde{e}_i) = min\{\mu_1(\tilde{e}_i), \mu_2(\tilde{e}_i)\}$.

Example 4.2. After reassuming the data from Example 4.1, the matrix representations of $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are given as

$$\tilde{\Omega}_1 = \begin{pmatrix} \tilde{\Xi} \setminus \tilde{\mho} & \tilde{n}_1 & \tilde{n}_2 & \tilde{n}_3 & \tilde{n}_4 \\ \frac{\tilde{e}_2}{0.2} & (0.7,0.1)(0.8,0.2)(0.9,0.3) & (0.5,0.4)(0.6,0.5)(0.7,0.6) & (0.8,0.7)(0.9,0.8)(0.5,0.9) & (0.6,0.2)(0.7,0.3)(0.8,0.4) \\ \frac{\tilde{e}_4}{0.4} & (0.6,0.2)(0.7,0.3)(0.8,0.4) & (0.9,0.5)(0.9,0.6)(0.8,0.7) & (0.7,0.8)(0.6,0.7)(0.5,0.6) & (0.6,0.5)(0.7,0.4)(0.8,0.3) \\ \frac{\tilde{e}_6}{0.6} & (0.6,0.3)(0.7,0.4)(0.8,0.5) & (0.9,0.6)(0.8,0.7)(0.7,0.8) & (0.6,0.7)(0.5,0.6)(0.4,0.5) & (0.3,0.4)(0.9,0.3)(0.8,0.2) \\ \frac{\tilde{e}_8}{0.8} & (0.7,0.4)(0.8,0.5)(0.9,0.6) & (0.8,0.7)(0.7,0.8)(0.6,0.7) & (0.5,0.6)(0.6,0.5)(0.7,0.4) & (0.8,0.3)(0.9,0.2)(0.6,0.1) \end{pmatrix},$$

and

$$\tilde{\Omega}_2 = \begin{pmatrix} \tilde{\Xi} \setminus \tilde{\mho} & \tilde{n}_1 & \tilde{n}_2 & \tilde{n}_3 & \tilde{n}_4 \\ \frac{\tilde{e}_2}{0.2} & (0.6,0.5)(0.7,0.6)(0.7,0.7) & (0.8,0.8)(0.9,0.7)(0.8,0.6) & (0.7,0.5)(0.6,0.4)(0.5,0.3) & (0.9,0.2)(0.7,0.1)(0.6,0.9) \\ \frac{\tilde{e}_4}{0.4} & (0.5,0.6)(0.9,0.7)(0.8,0.8) & (0.7,0.9)(0.6,0.1)(0.7,0.2) & (0.8,0.3)(0.9,0.4)(0.7,0.5) & (0.6,0.6)(0.5,0.7)(0.7,0.8) \\ \frac{\tilde{e}_6}{0.6} & (0.6,0.7)(0.7,0.8)(0.7,0.9) & (0.6,0.8)(0.5,0.7)(0.6,0.6) & (0.7,0.5)(0.8,0.4)(0.9,0.3) & (0.8,0.2)(0.7,0.1)(0.6,0.9) \\ \frac{\tilde{e}_8}{0.8} & (0.4,0.8)(0.9,0.9)(0.9,0.1) & (0.8,0.2)(0.8,0.3)(0.8,0.4) & (0.7,0.5)(0.7,0.6)(0.7,0.7) & (0.6,0.8)(0.6,0.9)(0.8,0.1) \end{pmatrix}.$$

Therefore,

$$\tilde{\Omega}_1 \cup \tilde{\Omega}_2 = \begin{pmatrix} \tilde{\Xi} \setminus \tilde{\mho} & \tilde{n}_1 & \tilde{n}_2 & \tilde{n}_3 & \tilde{n}_4 \\ \frac{\tilde{e}_2}{0.2} & (0.7,0.5)(0.7,0.6)(0.7,0.7) & (0.8,0.8)(0.6,0.7)(0.7,0.6) & (0.8,0.7)(0.6,0.8)(0.5,0.9) & (0.9,0.2)(0.7,0.3)(0.6,0.9) \\ \frac{\tilde{e}_4}{0.4} & (0.6,0.6)(0.7,0.7)(0.8,0.8) & (0.9,0.9)(0.6,0.6)(0.7,0.7) & (0.8,0.8)(0.6,0.7)(0.5,0.6) & (0.6,0.6)(0.5,0.7)(0.7,0.8) \\ \frac{\tilde{e}_6}{0.6} & (0.6,0.7)(0.7,0.8)(0.7,0.9) & (0.9,0.8)(0.5,0.7)(0.6,0.8) & (0.7,0.7)(0.5,0.6)(0.4,0.5) & (0.8,0.4)(0.7,0.3)(0.6,0.9) \\ \frac{\tilde{e}_8}{0.8} & (0.7,0.8)(0.8,0.9)(0.9,0.6) & (0.8,0.7)(0.7,0.8)(0.6,0.7) & (0.7,0.6)(0.6,0.6)(0.7,0.7) & (0.8,0.8)(0.6,0.9)(0.6,0.1) \end{pmatrix},$$

and

$$\tilde{\Omega}_1 \cap \tilde{\Omega}_2 = \begin{pmatrix} \tilde{\Xi} \setminus \tilde{\mho} & \tilde{n}_1 & \tilde{n}_2 & \tilde{n}_3 & \tilde{n}_4 \\ \frac{\tilde{e}_2}{0.2} & (0.6, 0.1)\,(0.8, 0.2)\,(0.9, 0.3) & (0.5, 0.4)\,(0.9, 0.5)\,(0.8, 0.6) & (0.7, 0.5)\,(0.9, 0.4)\,(0.5, 0.3) & (0.6, 0.2)\,(0.7, 0.1)\,(0.8, 0.4) \\ \frac{\tilde{e}_4}{0.4} & (0.5, 0.2)\,(0.9, 0.3)\,(0.8, 0.4) & (0.7, 0.5)\,(0.9, 0.1)\,(0.8, 0.2) & (0.7, 0.3)\,(0.9, 0.4)\,(0.7, 0.5) & (0.6, 0.5)\,(0.7, 0.4)\,(0.8, 0.3) \\ \frac{\tilde{e}_6}{0.6} & (0.6, 0.3)\,(0.7, 0.4)\,(0.8, 0.5) & (0.6, 0.6)\,(0.8, 0.7)\,(0.7, 0.6) & (0.6, 0.5)\,(0.8, 0.4)\,(0.9, 0.3) & (0.3, 0.2)\,(0.9, 0.1)\,(0.8, 0.2) \\ \frac{\tilde{e}_8}{0.8} & (0.4, 0.4)\,(0.9, 0.5)\,(0.9, 0.1) & (0.8, 0.2)\,(0.8, 0.3)\,(0.8, 0.4) & (0.5, 0.5)\,(0.7, 0.5)\,(0.7, 0.4) & (0.6, 0.3)\,(0.9, 0.2)\,(0.8, 0.1) \end{pmatrix}.$$

4.2. Determination of fuzzy parameterized values

Let $\tilde{\mho} = \{\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, ..., \tilde{n}_p\}$ be an initial universe and $\tilde{\mathscr{F}} = \{\tilde{e}_1/\mu(\tilde{e}_1), \tilde{e}_2/\mu(\tilde{e}_2), ..., \tilde{e}_r/\mu(\tilde{e}_r)\}$ be a FS over $\tilde{\Xi} = \{\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_r\}$. Let $\alpha^i_{\hat{T}}(\tilde{e}_1/\mu(\tilde{e}_1)), \alpha^i_{\hat{I}}(\tilde{e}_1/\mu(\tilde{e}_1))$ and $\alpha^i_{\hat{F}}(\tilde{e}_1/\mu(\tilde{e}_1))$ are amplitude values of $\tilde{n}_i, i = 1, 2, ..., p$ in true-belonging, indeterminate-belonging and false-belonging components of svCNNs with respect to $\tilde{e}_1/\mu(\tilde{e}_1)$. Similarly let $\beta^i_{\hat{T}}(\tilde{e}_1/\mu(\tilde{e}_1)), \beta^i_{\hat{I}}(\tilde{e}_1/\mu(\tilde{e}_1))$ and $\beta^i_{\hat{F}}(\tilde{e}_1/\mu(\tilde{e}_1))$ are phase values of $\tilde{n}_i$ in true-belonging, indeterminate-belonging and false-belonging components of svCNNs with respect to $\tilde{e}_1/\mu(\tilde{e}_1)$. Then fuzzy parameterized value $\mu(\tilde{e}_1)$ of $\tilde{e}_1$ can be computed as

$$\mu(\tilde{e}_1) = \frac{1}{2}\left\{ \begin{array}{c} \frac{max\{\alpha^i_{\hat{T}}(\tilde{e}_1)\}+min\{\alpha^i_{\hat{I}}(\tilde{e}_1)\}+min\{\alpha^i_{\hat{F}}(\tilde{e}_1)\}}{3}+ \\ \frac{max\{\beta^i_{\hat{T}}(\tilde{e}_1)\}+min\{\beta^i_{\hat{I}}(\tilde{e}_1)\}+min\{\beta^i_{\hat{F}}(\tilde{e}_1)\}}{3} \end{array} \right\}. \tag{4.2}$$

4.3. Transformation of svCNNs to CFNs and fuzzy values

If $\left\langle \alpha_{\hat{T}}(\tilde{e}^{'}_i))Exp(j\beta_{\hat{T}}(\tilde{e}^{'}_i)),\ \alpha_{\hat{I}}(\tilde{e}^{'}_i)Exp(j\beta_{\hat{I}}(\tilde{e}^{'}_i)),\ \alpha_{\hat{F}}(\tilde{e}^{'}_i)Exp(j\beta_{\hat{F}}(\tilde{e}^{'}_i))\right\rangle$ be a svCNN for alternative $\tilde{n} \in \tilde{\mho}$ corresponding to fuzzy parameterized tuples $\tilde{e}^{'}_i \in \tilde{\mathscr{F}}$ then it can be transformed to CFN by the following criterion:

$$\mathfrak{A}_{CFN} = \left\langle \frac{|\alpha_{\hat{T}}(\tilde{e}^{'}_i)) - \alpha_{\hat{I}}(\tilde{e}^{'}_i) - \alpha_{\hat{F}}(\tilde{e}^{'}_i)|}{3}, \frac{\beta_{\hat{T}}(\tilde{e}^{'}_i) + \beta_{\hat{I}}(\tilde{e}^{'}_i) + \beta_{\hat{F}}(\tilde{e}^{'}_i)}{6\pi} \right\rangle. \tag{4.3}$$

If $\mathfrak{A}_{CFN} = \langle \eta, \tau \rangle$ be a CFN then it can be transformed to fuzzy number (FN) by employing the following criterion:

$$\mathfrak{A}_{FN} = \frac{|\eta - \tau|}{2}. \tag{4.4}$$

4.4. The selection of criteria and sub-criteria along with their key roles

The attributes (criteria) and sub-attributes (sub-criteria) are the key factors that directly relate to the MADM problem and may affect the decisions drastically. Therefore, it is advisable to adopt an intelligent approach for selecting parameters and sub-parameters. Deeply analyzing existing literature, conducting questionnaire-based surveys, and interviewing the individuals are considered appropriate ways to collect information leading to selecting parameters and sub-parameters. However, the parameters and sub-parameters compatible with the adopted algebraic model are likely to be considered. Perusing the relevant literature and pursuing the proposed model, only those parameters (criteria) are adopted, which are likely to be partitioned into disjoint sub-classes having sub-parametric values (sub-criteria). Many approaches and criteria have already been discussed for the dam site selection problem (already reviewed in Introduction 1); however, the criteria and sub-criteria discussed in [51] are the most relevant to this study. Therefore these criteria, along with sub-criteria, are adopted and modified to make them compatible with the proposed model $\tilde{\lambda}$-set. The modified parameters and sub-parameters are presented in Figure 2. In order to have a brief description of the operational roles of these parameters and sub-parameters in the dam site selection problem, the research work of Tufail et al. [51] can be consulted.

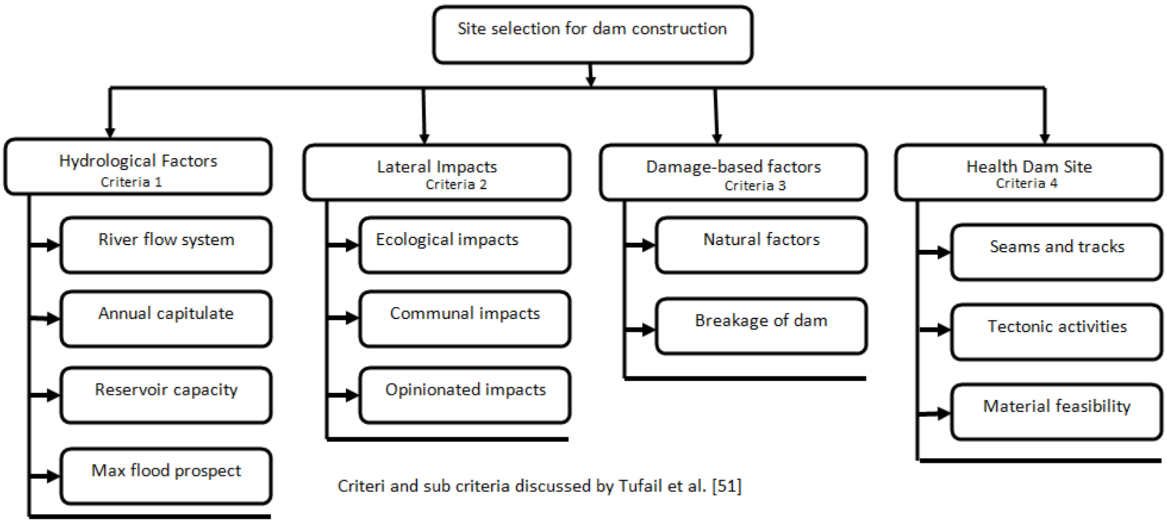

Figure 2. Criteria and sub criteria for dam site selection.

4.5. Decision makers and their responsibilities

In MADM practice, the most significant figures are the decision-makers responsible for accomplishing the evaluation process. Their mutual conflicts of interest or related differences may lead to biased decisions. Therefore, it is usual practice to hire experts from different sources (departments) with multi-disciplinary areas of expertise. The following are some key responsibilities of decision-makers in MADM:

(1) Analysis of raw information (data) collected through various sources.
(2) Statistical processing of analyzed data.
(3) Exploration of alternatives and parameters.
(4) Scrutiny of processed data based on parameters.
(5) Short-listing of suitable parameters for the evaluation of alternatives.
(6) Employment of a suitable algebraic model to provide opinions for the approximation of alternatives on the basis of parameters.
(7) Ranking of alternatives.

As far as the present research is concerned, the site selection for dam construction is usually endorsed and conferred by the Ministry of Water Resources (MOWR) and the Water and Power Development Authority (WAPDA) jointly in most Asian countries like Pakistan. Therefore, it is the responsibility of these departments to provide decision-makers for this task. Thus, officers like the Project Manager (PM), Executive Engineer (XEN) from WAPDA, Deputy Chief Hydro Power (Dy.CHP), and Deputy Chief Development (Dy.CD) from MOWR have been nominated as decision makers as they have good profiles with rich, relevant expertise.

4.6. Proposal of MADM based algorithm by using aggregations of $\tilde{\lambda}$-set

This segment of the paper proposes an algorithm for choosing an appropriate site for the construction of dam.

Algorithm 1. An optimised site selection for dam construction based on aggregations of $\tilde{\lambda}$-set.

(1) Assume the sets $\tilde{\mho} = \{\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, ..., \tilde{n}_p\}$, $\tilde{\Lambda} = \{\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n\}$ and $\tilde{\mathbb{D}} = \{\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, ..., \tilde{d}_q\}$ as a set of initial universe consisting of sites under observation, a set of parameters (criteria) and a set of decision makers deputed for the evaluation process respectively. Explore the sub parametric values of each parameter $\tilde{a}_\epsilon, \epsilon = 1, 2, ..., n$ and enclosed them in non overlapping sets $\tilde{\Xi}_\epsilon$. Determine $\tilde{\Xi} = \tilde{\Xi}_1 \times \tilde{\Xi}_2 \times ... \times \tilde{\Xi}_n = \{\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_r\}$ where r is the product of cardinalities of $\tilde{\Xi}_\epsilon$.

(2) Collect the joint opinions of decision makers $\tilde{d}_\varsigma, \varsigma = 1, 2, ..., q$ for alternatives based on sub parametric tuples $\tilde{e}_k, k = 1, 2, ..., r$ in terms of svCNNs and express them in matrix $\Gamma_1 = [\alpha_{ij}]_{r \times p}$ where α_{ij} denotes svCNN corresponding to ith row and jth column.

(3) Determine fuzzy parameterized values of each $\tilde{e}_k$ by using Eq (4.2) and then express $\tilde{\lambda}$-set in matrix $\Gamma_2 = [\beta_{ij}]_{r \times p}$.

(4) Transform the entries i.e svCNNs of Γ_2 to CFNs by using Eq (4.3) and obtain new matrix $\Gamma_3 = [\gamma_{ij}]_{r \times p}$. Again transform CFNs of Γ_3 to fuzzy values by using Eq (4.4) and obtain another matrix $\Gamma_4 = [\theta_{ij}]_{r \times p}$.

(5) Multiply each fuzzy parameterized value to each fuzzy value lying in its respective row of matrix $\Gamma_4 = [\theta_{ij}]_{r \times p}$ and obtain decision matrix $\Gamma_D = [\vartheta_{ij}]_{r \times p}$. Compute the score values $\mathbb{S}(\tilde{n}_\epsilon), \epsilon = 1, 2, ..., n$ of each alternative by adding respective fuzzy values in its column. Select the alternative with maximum score as optimum selection.

Figure 3 represents the depiction of Algorithm 1.

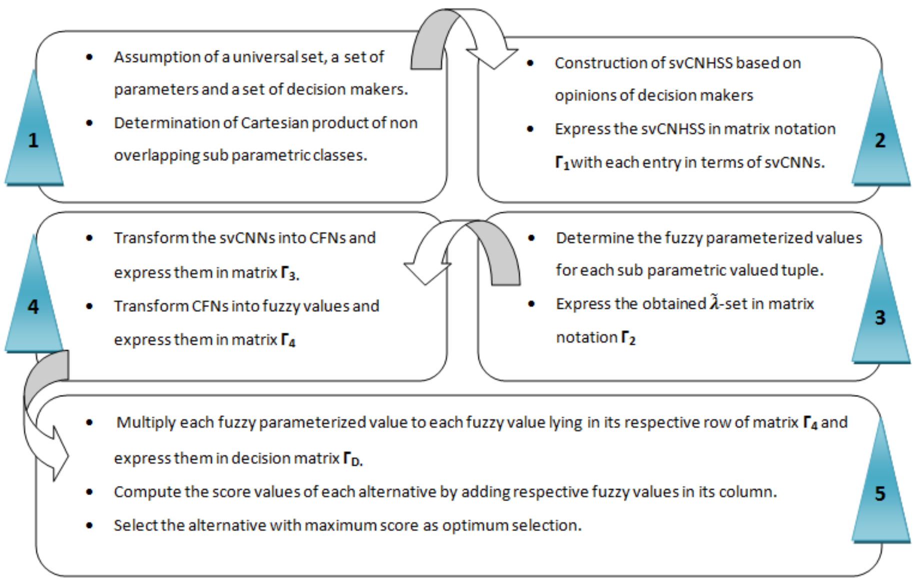

Figure 3. Pictorial description of proposed algorithm.

5. Illustrative example: a case study

This portion of the paper provides an explanation of various steps of the algorithm with a self-explanatory example.

A dam is a constructed barrier that holds back water and raises its level. The important uses of dams have made it a matter of great concern for human civilization. Some of its uses are water supply, irrigation, electrical generation, flood control, water storage, mine tailings, debris control, navigation, and recreation. With the increasing problems of urbanization and environmental variations, the need for water storage for generating electricity, availability of water for irrigation through the net of canals, and controlling of floods have increased to a great extent. These issues lead to the demand for dam construction at the appropriate locations. In the following example, a case study is presented in which an appropriate location is evaluated for the construction of the mini dam. This case study is based on hypothetical data e.g. the opinions of experts are hypothetical. The data collection is performed through the surveying process and literature review.

Example 5.1. In order to meet the water needs of a certain area, the administrations of MOWR and WAPDA plan to initiate a joint project for the construction of a mini dam in the nearby area of river "XAX" (a hypothetical name) that is flowing along the bottom areas of mountains. After various surveys done by many survey teams of both concerned departments, the four most suitable sites (regions) $\tilde{n}_1 = Reg\,1$, $\tilde{n}_2 = Reg\,2$, $\tilde{n}_3 = Reg\,3$ and $\tilde{n}_4 = Reg\,4$ have been scrutinized for further evaluation to construct a mini dam there. These regions constitute an initial space of alternatives $\tilde{\mho} = \{Reg\,1, Reg\,2, Reg\,3, Reg\,4\}$. After mutual consultation, four experts (decision makers) have been deputed from both departments for the evaluation of these four regions. The brief profiles of these experts are presented in Table 2.

Table 2. Profiles of decision makers.

Experts	Professional qualification	Minimum relevant experience	Department
Project Manager (PM)	M.Sc Business Administration or Project Management	10 years	WAPDA
Executive Engineer (XEN)	M.Sc engineering (preferable civil)	15 years	WAPDA
Deputy Chief Hydro Power (Dy.CHP)	M.Sc hydro power engineering	10 years	MOWR
Deputy Chief Development (Dy.CD)	M.Sc development studies	10 years	MOWR

With mutual consensus, the parameters "$\tilde{a}_1$ =hydrological factors", "$\tilde{a}_2$ =lateral impacts", "$\tilde{a}_3$ =damage-based factors" and "$\tilde{a}_4$ =health dam site" are finalized for the assessment of dam sites. With keen analysis, the respective sub parametric values of these parameters are enclosed in non overlapping classes,

$$\tilde{\Xi}_1 = \{\tilde{a}_{11} = reservoir\,capacity,\ \tilde{a}_{12} = maximum\,flood\,prospect\},$$

$$\tilde{\Xi}_2 = \{\tilde{a}_{21} = ecological\, impacts,\ \tilde{a}_{22} = opinionated\, impacts\},$$
$$\tilde{\Xi}_3 = \{\tilde{a}_{31} = natural\, factors\},$$
$$\tilde{\Xi}_4 = \{\tilde{a}_{41} = material\, feasibility\}.$$

In order to cope with multi-argument parametric values, the set $\tilde{\Xi} = \tilde{\Xi}_1 \times \tilde{\Xi}_2 \times \tilde{\Xi}_3 \times \tilde{\Xi}_4 = \{\tilde{e}_1 = (\tilde{a}_{11}, \tilde{a}_{21}, \tilde{a}_{31}, \tilde{a}_{41}),\ \tilde{e}_2 = (\tilde{a}_{11}, \tilde{a}_{22}, \tilde{a}_{31}, \tilde{a}_{41}),\ \tilde{e}_3 = (\tilde{a}_{12}, \tilde{a}_{21}, \tilde{a}_{31}, \tilde{a}_{41}),\ \tilde{e}_4 = (\tilde{a}_{12}, \tilde{a}_{22}, \tilde{a}_{31}, \tilde{a}_{41})\}$ is determined which consists of 4-argument sub parametric valued tuples. Figure 4 presents the method of determination of these multi argument tuples.

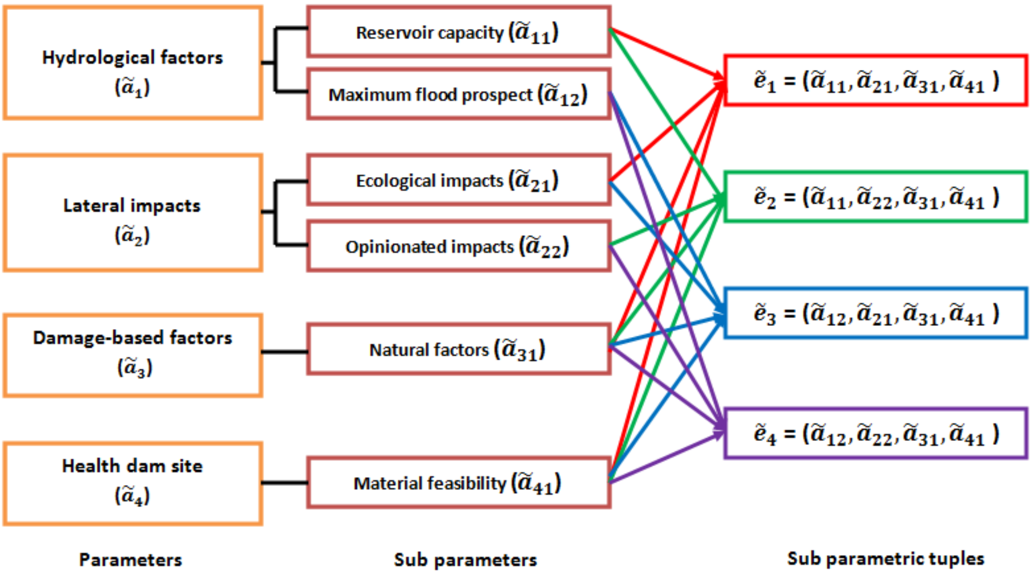

Figure 4. Determination of multi argument tuples.

Now decision-makers provide their opinions for approximations of alternatives (sites) by considering these 4-argument sub-parametric valued tuples. The opinions are in terms of svCNNs $\langle truth, indeterminacy, falsity \rangle$ with a description of their related amplitude and phase values. The mean of their opinions are computed and the set "single-valued complex neutrosophic hypersoft set (svCNHSOS)" is constructed and expressed in matrix notation $\Gamma_1 = [\alpha_{ij}]_{4\times4}$ which is tabulated in Table 3.

Table 3. Tabulation of matrix $\Gamma_1 = [\alpha_{ij}]_{4\times4}$.

Γ_1	Reg 1	Reg 2	Reg 3	Reg 4
$\tilde{e}_1$	(0.92, 0.5) (0.94, 0.4) (0.82, 0.3)	(0.84, 0.2) (0.72, 0.1) (0.74, 0.8)	(0.62, 0.7) (0.64, 0.6) (0.52, 0.5)	(0.54, 0.4) (0.42, 0.3) (0.83, 0.2)
$\tilde{e}_2$	(0.83, 0.6) (0.95, 0.8) (0.86, 0.9)	(0.75, 0.6) (0.96, 0.4) (0.87, 0.3)	(0.78, 0.5) (0.96, 0.4) (0.73, 0.4)	(0.65, 0.6) (0.73, 0.3) (0.84, 0.1)
$\tilde{e}_3$	(0.66, 0.7) (0.76, 0.5) (0.85, 0.3)	(0.63, 0.1) (0.83, 0.8) (0.75, 0.5)	(0.64, 0.2) (0.86, 0.6) (0.92, 0.3)	(0.39, 0.1) (0.91, 0.4) (0.84, 0.6)
$\tilde{e}_4$	(0.49, 0.5) (0.91, 0.6) (0.91, 0.7)	(0.89, 0.8) (0.87, 0.7) (0.85, 0.6)	(0.59, 0.5) (0.71, 0.4) (0.75, 0.3)	(0.68, 0.2) (0.92, 0.1) (0.81, 0.1)

Now fuzzy parameterized values of each $\tilde{e}_k$ are determined by using Eq (4.2) which are given in Table 4, and then $\tilde{\lambda}$-set is constructed whose matrix notation $\Gamma_2 = [\beta_{ij}]_{4\times4}$ is tabulated in Table 5.

Table 4. Fuzzy parameterized values of $\tilde{e}_k$.

$\tilde{e}_k$	$\mu(\tilde{e}_k)$	$\tilde{e}_k$	$\mu(\tilde{e}_k)$
$\tilde{e}_1$	$\frac{1}{2}\left\{\frac{0.92+0.42+0.52}{3}+\frac{0.7+0.1+0.2}{3}\right\}=0.475$	$\tilde{e}_3$	$\frac{1}{2}\left\{\frac{0.66+0.76+0.75}{3}+\frac{0.7+0.4+0.3}{3}\right\}=0.595$
$\tilde{e}_2$	$\frac{1}{2}\left\{\frac{0.83+0.73+0.73}{3}+\frac{0.6+0.3+0.1}{3}\right\}=0.548$	$\tilde{e}_4$	$\frac{1}{2}\left\{\frac{0.89+0.71+0.75}{3}+\frac{0.8+0.1+0.1}{3}\right\}=0.558$

Table 5. Tabulation of $\tilde{\lambda}$-set with matrix $\Gamma_2 = [\beta_{ij}]_{4\times4}$.

Γ_2	Reg 1	Reg 2	Reg 3	Reg 4
$\frac{\tilde{e}_1}{0.475}$	(0.92, 0.5) (0.94, 0.4) (0.82, 0.3)	(0.84, 0.2) (0.72, 0.1) (0.74, 0.8)	(0.62, 0.7) (0.64, 0.6) (0.52, 0.5)	(0.54, 0.4) (0.42, 0.3) (0.83, 0.2)
$\frac{\tilde{e}_2}{0.548}$	(0.83, 0.6) (0.95, 0.8) (0.86, 0.9)	(0.75, 0.6) (0.96, 0.4) (0.87, 0.3)	(0.78, 0.5) (0.96, 0.4) (0.73, 0.4)	(0.65, 0.6) (0.73, 0.3) (0.84, 0.1)
$\frac{\tilde{e}_3}{0.595}$	(0.66, 0.7) (0.76, 0.5) (0.85, 0.3)	(0.63, 0.1) (0.83, 0.8) (0.75, 0.5)	(0.64, 0.2) (0.86, 0.6) (0.92, 0.3)	(0.39, 0.1) (0.91, 0.4) (0.84, 0.6)
$\frac{\tilde{e}_4}{0.558}$	(0.49, 0.5) (0.91, 0.6) (0.91, 0.7)	(0.89, 0.8) (0.87, 0.7) (0.85, 0.6)	(0.59, 0.5) (0.71, 0.4) (0.75, 0.3)	(0.68, 0.2) (0.92, 0.1) (0.81, 0.1)

Now the entries i.e svCNNs of Γ_2 are transformed to CFNs by using Eq (4.3) and tabulated in Table 6 with matrix notation matrix $\Gamma_3 = [\gamma_{ij}]_{4\times4}$ such that fuzzy parameterized values remain unchanged. The entries i.e. CFNs of Γ_3 are now reduced to fuzzy values by using Eq (4.4) and obtain another matrix $\Gamma_4 = [\theta_{ij}]_{4\times4}$ which is tabulated in Table 7.

Table 6. Tabulation of matrix $\Gamma_3 = [\gamma_{ij}]_{4\times4}$.

Γ_3	Reg 1	Reg 2	Reg 3	Reg 4
$\frac{\tilde{e}_1}{0.475}$	(0.280, 0.400)	(0.207, 0.367)	(0.180, 0.600)	(0.237, 0.333)
$\frac{\tilde{e}_2}{0.548}$	(0.327, 0.767)	(0.360, 0.433)	(0.303, 0.433)	(0.307, 0.333)
$\frac{\tilde{e}_3}{0.595}$	(0.317, 0.500)	(0.317, 0.467)	(0.380, 0.367)	(0.453, 0.367)
$\frac{\tilde{e}_4}{0.558}$	(0.443, 0.600)	(0.277, 0.700)	(0.290, 0.400)	(0.350, 0.133)

Table 7. Tabulation of matrix $\Gamma_4 = [\theta_{ij}]_{4\times4}$.

Γ_4	Reg 1	Reg 2	Reg 3	Reg 4
$\frac{\tilde{e}_1}{0.475}$	0.06	0.08	0.21	0.048
$\frac{\tilde{e}_2}{0.548}$	0.22	0.0365	0.065	0.013
$\frac{\tilde{e}_3}{0.595}$	0.0915	0.075	0.0065	0.043
$\frac{\tilde{e}_4}{0.558}$	0.0785	0.2115	0.055	0.1085

Each fuzzy parameterized value is multiplied to each fuzzy value lying in its respective row of matrix Γ_4 to construct decision matrix $\Gamma_D = [\vartheta_{ij}]_{4\times4}$ whose entries are tabulated in Table 8. The score values $\mathbb{S}(\tilde{n}_\epsilon), \epsilon = 1, 2, ..., n$ of each alternative is computed by adding respective fuzzy values in its column. The computed score values are presented in Table 9. According to Table 9 and Figure 5, the maximum score 0.2473 is achieved by Reg 1 ($\tilde{n}_1$) therefore it is recommended by the decision makers for the construction of mini dam. The ranking of the sites is $Reg\,1(\tilde{n}_1) > Reg\,2(\tilde{n}_2) > Reg\,3(\tilde{n}_3) > Reg\,4(\tilde{n}_4)$.

Table 8. Tabulation of matrix $\Gamma_D = [\vartheta_{ij}]_{4\times4}$.

Γ_D	Reg 1	Reg 2	Reg 3	Reg 4
$\tilde{e}_1$	0.0285	0.0380	0.0998	0.0228
$\tilde{e}_2$	0.1206	0.0200	0.0356	0.0071
$\tilde{e}_3$	0.0544	0.0446	0.0039	0.0256
$\tilde{e}_4$	0.0438	0.1180	0.0307	0.0605

Table 9. Score values $\mathbb{S}(\tilde{n}_\epsilon), \epsilon = 1, 2, ..., n$ of sites.

Sites	Score values $\mathbb{S}(\tilde{n}_\epsilon)$
Reg 1 ($\tilde{n}_1$)	$\sum_{i=1}^{4} \vartheta_{i1} = 0.0285 + 0.1206 + 0.0544 + 0.0438 = \mathbf{0.2473}$
Reg 2 ($\tilde{n}_2$)	$\sum_{i=1}^{4} \vartheta_{i2} = 0.0380 + 0.0200 + 0.0446 + 0.1180 = 0.2206$
Reg 3 ($\tilde{n}_3$)	$\sum_{i=1}^{4} \vartheta_{i3} = 0.0998 + 0.0356 + 0.0039 + 0.0307 = 0.1700$
Reg 4 ($\tilde{n}_4$)	$\sum_{i=1}^{4} \vartheta_{i4} = 0.0228 + 0.0071 + 0.0256 + 0.0605 = 0.1160$

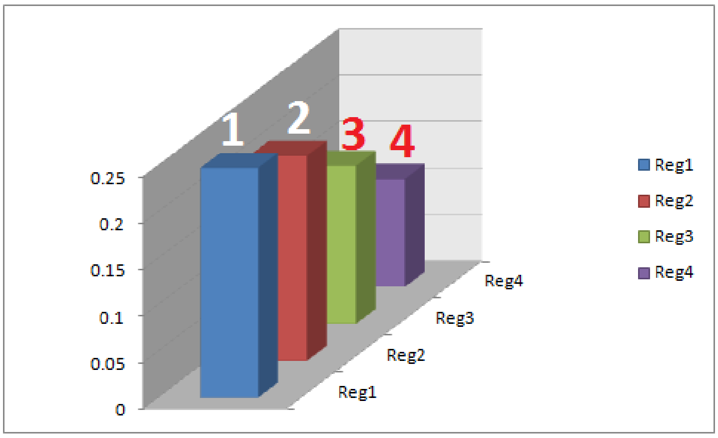

Figure 5. Ranking of sites based on score values.

6. Discussion and comparison

As the water demand has increased enormously due to the increased human population, urbanization, industrialization, and the use of modern agricultural modes have led to the construction of mini dams. However, finding an appropriate region/site for dam construction is a challenging task because it involves various factors that may be political, environmental, social, and economic. Also, it involves various kinds of ambiguities and uncertainties well. Many researchers have already contributed their share in this regard by using different algebraic and non-algebraic approaches like fuzzy hierarchical TOPSIS (Wang et al. [1], Noori et al. [3]), AHP and fuzzy-AHP (Esavi et al. [42]), Fuzzy AHP-TOPSIS (Janjua & Hassan [44]) and Hybrid Fuzzy-TOPSIS (Adeyanju & Adedeji [45]). Although these approaches are important in literature, there are some limitations concerning the dam site selection problem. The consideration of parameterization mode, the independency of decision-makers regarding the provision of their opinions in three-dimensional graded aspects like truth, indeterminate, and falsity, the consideration of multi-argument domain and multi-argument function for the approximation of sites, the consideration of amplitude and phase values to tackle the periodicity of data involved in approximation and the entitlement of fuzzy parameterization to tackle vague attitude of parameters and sub-parameters are those important features that play a vital role in every decision making process in general and dam site selection problem in specific for having reliable and consistent decisions. Nevertheless, the models mentioned above and other related models have ignored such features. This study's advantage is that all the above limitations of existing research have been managed by introducing a novel algebraic model, i.e., $\tilde{\lambda}$-set which uses complex hypersoft, fuzzy parameterized, and single-valued neutrosophic setting jointly for the accomplishing this challenge. As the proposed "$\tilde{\lambda}$-set approach" is novel in literature, it cannot be compared comprehensively with other similar algebraic models based on computational results. However, Table 10 presents its ranking-based comparison with the most relevant approaches employed by Adeyanju & Adedeji [45], Tufail et al. [51] and Akram et al. [52] As per computational rules, the smaller numerical values lead to more reliability, preciseness, and accuracy of the concept because they reduce the chances of expected numerical errors. After the analysis of computational results provided in Table 10, it is obvious and intelligible that the computational score values of the proposed approach are more precise and consistent as compared to other mentioned models due to smaller score values with fewer chances of computational errors.

In order to prove the claim of the flexibility and reliability of the proposed model, its structural comparison is presented in Table 11, in which some most essential features like consideration of fuzzy parameterization (COFP), management of information-based periodicity (MIBP), the entitlement of three-dimensional membership function (ETDMF), the entitlement of multi argument-based opinions (EMAO) and the consideration of parameterization mode (COPM) are considered. It can easily be observed that the proposed model $\tilde{\lambda}$-set is tackling all such important features, whereas the mentioned relevant models have some limitations for these features. The acronyms ✓ and × are meant for "valid" and "not valid" respectively.

Table 10. Analysis of preferential aspects based on sites ranking.

References	Score values achieved by sites	Ranking of sites
Adeyanju & Adedeji [45]	$\begin{cases} Reg\,1 = 0.547, \\ Reg\,2 = 0.532, \\ Reg\,3 = 0.469, \\ Reg\,4 = 0.534 \end{cases}$	$Reg\,1 > Reg\,4 > Reg\,2 > Reg\,3$
Tufail et al. [51]	$\begin{cases} Reg\,1 = 0.347, \\ Reg\,2 = 0.319, \\ Reg\,3 = 0.840, \\ Reg\,4 = 0.577 \end{cases}$	$Reg\,3 > Reg\,4 > Reg\,1 > Reg\,2$
Akram et al. [52]	$\begin{cases} Reg\,1 = 0.777, \\ Reg\,2 = 0.975, \\ Reg\,3 = 0.925, \\ Reg\,4 = 0.725 \end{cases}$	$Reg\,2 > Reg\,3 > Reg\,1 > Reg\,4$
Proposed approach	$\begin{cases} Reg\,1 = 0.2473, \\ Reg\,2 = 0.2206, \\ Reg\,3 = 0.1700, \\ Reg\,4 = 0.1160 \end{cases}$	$Reg\,1 > Reg\,2 > Reg\,3 > Reg\,4$

Table 11. The flexibility of proposed model.

Authors	Structure	COFP	MIBP	ETDMF	EMAO	COPM
Thirunavukarasu et al. [28]	CFSS	×	✓	×	×	✓
Kumar et al. [29]	CIFSS	×	✓	×	×	✓
Smarandache et al. [30]	CNSS	×	✓	✓	×	✓
Zhu & Zhan [31]	FpFSS	✓	✓	×	×	✓
Sulukan et al. [32]	FpIFSS	✓	✓	×	×	✓
Rahman et al. [34]	CNHSS	×	✓	✓	✓	✓
Proposed model	$\tilde{\lambda}$-set	✓	✓	✓	✓	✓

6.1. The merits of the proposed model

The proposed model is more preferable and trustworthy due to the following aspects:

(1) The choice of parameters and their sub-parameters-based classification is not always definite for decision-makers in numerous situations. They face various kinds of uncertainties in this regard. The proposed model copes with such situations by applying the idea of fuzzy parameterization, which assesses the uncertain behavior of parameters and their sub-parameters.

(2) The proposed model facilitates the decision makers to furnish their approximations in terms of three-dimensional membership function, allowing them to be agreed with or indeterminate or disagree with the approximations independently.

(3) It is commonly observed that the approximations of parameters and their sub-parameters may have repeated values (periodicity) with the depiction of uncertainties; the proposed model tackles such periodicity by considering complex valued three-dimensional membership function, which

employs phase values to cope with such information-based repetitions.

(4) The provision of approximations based on a single parameter is a time-taking process. The proposed model assists the decision makers in furnishing their approximations by observing multi parameters collectively by considering hypersoft settings.

7. Conclusions

In the present study, the novel approach called the "$\tilde{\lambda}$-set approach" is employed to find the most suitable region (site) for the construction of a mini dam which is initiated with the characterization of an algebraic model "$\tilde{\lambda}$-set" and its essential aggregation operations. This model is flexible as it can collectively cope with the periodicity of information, the entitlement of multi-argument mapping, fuzzy parameterization mode, and single-valued neutrosophic setting. The input variables are those parameters (criteria) that have appropriate sub-parameters (sub-criteria). A proper algebraic criterion is utilized to find the fuzzy parameterized degrees of multi-argument tuples obtained through the Cartesian product of sub-parametric non-overlapping classes. Based on the aggregation operations of $\tilde{\lambda}$-set, a novel MADM algorithm is proposed which is further explained by discussing the problem of dam site selection. In the proposed decision support framework, the decision makers are experienced officers of relevant departments. They are free to assess the alternatives by considering fuzzy parameterized multi-argument-based criteria, i.e., sub-parametric tuples capable of tackling their expected uncertainties about selecting parameters and their sub-parameters. A flexible environment is provided to independently record their expert opinions in the form of agree-based, disagree-based, and indeterminate-based grades.

Moreover, it provides a specific setting known as a complex setting to manage the periodicity of data after regular intervals of time by introducing amplitude and phase-graded values. The proposed approach is a general framework that utilizes hypothetical data-based opinions of decision-makers. It can easily be applied to all real-life situations by taking real data sets with the help of the proposed algorithm. As in the proposed approach, the concept of fuzzy parameterization is utilized to assess the parameters- based uncertainties, but it has limitations for those decision-making situations where such parameters-based uncertainties are found in terms of intuitionistic fuzzy numbers, picture fuzzy numbers, neutrosophic fuzzy numbers, and plithogenic numbers therefore as future task the concepts of such parameterizations may be introduced which may further be integrated with complex, neutrosophic and hypersoft settings to evaluate the sites for dam construction. Moreover, the proposed model may also be utilized efficiently to resolve other decision-making problems like risk analysis projects or investment selection, medical diagnosis, assessment of renewable energy resources, and many others. Many algebraic structures, like topological spaces, metric spaces, vector spaces, etc., may also be developed using the basic notions of $\tilde{\lambda}$-set.

Acknowledgements

This research was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1A4A1031509).

Conflict of interest

The authors declare no conflicts of interest.

References

1. J. W. Wang, C. H. Cheng, K. C. Huang, Fuzzy hierarchical TOPSIS for supplier selection, *Appl. Soft Comput.*, **9** (2009), 377–386. https://doi.org/10.1016/j.asoc.2008.04.014
2. S. Q. Weng, G. H. Huang, Y. P. Li, An integrated scenario-based multi-criteria decision support system for water resources management and planning: a case study in the Haihe River Basin, *Expert Syst. Appl.*, **37** (2010), 8242–8254. https://doi.org/10.1016/j.eswa.2010.05.061
3. A. Noori, H. Bonakdari, K. Morovati, B. Gharabaghi, The optimal dam site selection using a group decision-making method through fuzzy TOPSIS model, *Environ. Syst. Decisions*, **38** (2018), 471–488. https://doi.org/10.1007/s10669-018-9673-x
4. Z. L. Yang, M. W. Lin, Y. C. Li, W. Zhou, B. Xu, Assessment and selection of smart agriculture solutions using an information error-based Pythagorean fuzzy cloud algorithm, *Int. J. Intell. Syst.*, **36** (2021), 6387–6418. https://doi.org/10.1002/int.22554
5. M. W. Lin, Z. Y. Chen, R. Q. Chen, H. Fujita, Evaluation of startup companies using multicriteria decision making based on hesitant fuzzy linguistic information envelopment analysis models, *Int. J. Intell. Syst.*, **36** (2021), 2292–2322. https://doi.org/10.1002/int.22379
6. J. Deng, J. M. Zhan, Z. Sh. Xu, E. Herrera-Viedma, Regret-theoretic multiattribute decision-making model using three-way framework in multiscale information systems, *IEEE Trans. Cybernetics*, 2022, 1–14. https://doi.org/10.1109/TCYB.2022.3173374
7. H. B. Wang, F. Smarandache, Y. Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, *Tech. Sci. Appl. Math.*, 2010.
8. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
9. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, **20** (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3
10. B. C. Cuong, Picture fuzzy sets, *J. Comput. Sci. Cybernetics,* **30** (2014), 409–420. http://doi.org/10.15625/1813-9663/30/4/5032
11. F. Smarandache, *Neutrosophy neutrosophic probability, set and logic: analytic synthesis synthetic analysis*, American Research Press, 1998.
12. R. Şahin, A. Küçük, Subsethood measure for single valued neutrosophic sets, *J. Intell. Fuzzy Syst.*, **29** (2015), 525–530. http://doi.org/10.3233/IFS-141304
13. H. L. Huang, New distance measure of single-valued neutrosophic sets and its application, *Int. J. Intell. Syst.*, **31** (2016), 1021–1032. https://doi.org/10.1002/int.21815
14. S. Pramanik, S. Dalapati, S. Alam, F. Smarandache, T. K. Roy, NS-cross entropy-based MAGDM under single-valued neutrosophic set environment. *Information*, **9** (2018), 37. https://doi.org/10.3390/info9020037

15. A. Aydoğdu, On similarity and entropy of single valued neutrosophic sets, *Gen. math. notes*, **29** (2015), 67–74.

16. P. Biswas, S. Pramanik, B. C. Giri, TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment, *Neural comput. Appl.*, **27** (2016), 727–737. https://doi.org/10.1007/s00521-015-1891-2

17. H. M. A. Farid, M. Riaz, Single-valued neutrosophic Einstein interactive aggregation operators with applications for material selection in engineering design: case study of cryogenic storage tank, *Complex Intell. Syst.*, **8** (2022), 2131–2149.

18. J. Ling, M. W. Lin, L. L. Zhang, Medical waste treatment scheme selection based on single-valued neutrosophic numbers, *AIMS Mathematics*, **6** (2021), 10540–10564. https://doi.org/10.3934/math.2021612

19. M. Ali, F. Smarandache, Complex neutrosophic set, *Neural comput. Appl.*, **28** (2017), 1817–1834.

20. D. Ramot, R. Milo, M. Friedman, A. Kandel, Complex fuzzy sets, *IEEE Trans. Fuzzy Syst.*, **10** (2022), 171–186. https://doi.org/10.1109/91.995119

21. A. Alkouri, A. R. Salleh, Complex intuitionistic fuzzy sets, In *International conference on fundamental and applied sciences*, **1482** (2012), 464–470. https://doi.org/10.1063/1.4757515

22. D. Molodtsov, Soft set theory-first results, *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5

23. J. M. Zhan, J. C. R. Alcantud, A novel type of soft rough covering and its application to multicriteria group decision making, *Artif. Intell. Rev.*, **52** (2019), 2381–2410. https://doi.org/10.1007/s10462-018-9617-3

24. C. Jana, M. Pal, A robust single-valued neutrosophic soft aggregation operators in multi-criteria decision making, *Symmetry*, **11** (2019), 110. https://doi.org/10.3390/sym11010110

25. P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, *J. Fuzzy Math.*, **9** (2001), 589–602.

26. N. Çağman, S. Karataş, Intuitionistic fuzzy soft set theory and its decision making, *J. Intell. Fuzzy Syst.*, **24** (2013), 829–836. https://doi.org/10.3233/IFS-2012-0601

27. P. K. Maji, Neutrosophic soft set, *Ann. Fuzzy Math. Inform.*, **5** (2013), 157–168.

28. P. Thirunavukarasu, R. Suresh, V. Ashokkumar, Theory of complex fuzzy soft set and its applications, *Int. J. Innov. Res. Sci. Technol.*, **3** (2017), 13–18.

29. T. Kumar, R. K. Bajaj, On complex intuitionistic fuzzy soft sets with distance measures and entropies, *J. Math.*, **2014** (2014), 1–12. http://doi.org/10.1155/2014/972198

30. S. Broumi, A. Bakali, F. Smarandache, M. Talea, M. Ali, G. Selvachandran, Complex neutrosophic soft set, In: *2017 FUZZ-IEEE Conference on Fuzzy Systems*, 2017.

31. K. Y. Zhu, J. M. Zhan, Fuzzy parameterized fuzzy soft sets and decision making, *Int. J. Machine Learn. Cyb.*, **7** (2016), 1207–1212.

32. E. Sulukan, N. Çağman, T. Aydin, Fuzzy parameterized intuitionistic fuzzy soft sets and their application to a performance-based value assignment problem, *J. New Theory*, **29** (2019), 79–88.

33. F. Smarandache, Extension of soft set of hypersoft set, and then to plithogenic hypersoft set, *Neutrosophic Sets Syst.*, **22** (2018), 168–170.

34. A. U. Rahman, M. Saeed, F. Smarandache, M. R. Ahmad, Development of hybrids of hypersoft set with complex fuzzy set, complex intuitionistic fuzzy set and complex neutrosophic set, *Neutrosophic Sets Syst.*, **38** (2020), 335–355.

35. A. U. Rahman, M. Saeed, S. S. Alodhaibi, H. A. E. W. Khalifa, Decision making algorithmic approaches based on parameterization of neutrosophic set under hypersoft set environment with fuzzy, intuitionistic fuzzy and neutrosophic settings, *CMES Comput. Model. Eng. Sci.*, **128** (2021), 743–777. https://doi.org/10.32604/cmes.2021.016736

36. A. U. Rahman, M. Saeed, A. Alburaikan, H. A. E. W. Khalifa, An intelligent multiattribute decision-support framework based on parameterization of neutrosophic hypersoft set, *Comput. Intel. Neurosc.*, 2022, 6229947. https://doi.org/10.1155/2022/6229947

37. P. Rezaei, K. Rezaie, S. Nazari-Shirkouhi, M. R. J. Tajabadi, Application of fuzzy multi-criteria decision making analysis for evaluating and selecting the best location for construction of underground dam, *Acta Polytech. Hung.*, **10** (2013), 187–205.

38. J. Chezgi, H. R. Pourghasemi, S. A. Naghibi, H. R. Moradi, M. K. Zarkesh, Assessment of a spatial multi-criteria evaluation to site selection underground dams in the Alborz Province, Iran, *Geocarto Int.*, **31** (2016), 628–646. https://doi.org/10.1080/10106049.2015.1073366

39. A. Dortaj, S. Maghsoudy, F. D. Ardejani, Z. Eskandari, A hybrid multi-criteria decision making method for site selection of subsurface dams in semi-arid region of Iran, *Groundwater Sustain. Dev.*, **10** (2020), 100284. https://doi.org/10.1016/j.gsd.2019.100284

40. C. B. Karakuş, S. Yıldız, Gis-multi criteria decision analysis-based land suitability assessment for dam site selection, *Int. J. Environ. Sci. Technol.*, **19** (2022), 12561–12580.

41. F. S. Chien, C. N. Wang, V. T Nguyen, V. T. Nguyen, K. Y. Chau, An evaluation model of quantitative and qualitative fuzzy multi-criteria decision-making approach for hydroelectric plant location selection, *Energies*, **13** (2020), 2783. https://doi.org/10.3390/en13112783

42. V. Esavi, J. Karami, A. Alimohammadi, S. A. Niknezhad, Comparison the AHP and fuzzy-AHP decision making methods in underground dam site selection in Taleghan basin, *Sci. Quart. J. Geosci.*, **22** (2012), 27–34.

43. S. Narayanamoorthy, V. Annapoorani, S. Kalaiselvan, D. Kang, Hybrid hesitant fuzzy multi-criteria decision making method: A symmetric analysis of the selection of the best water distribution system, *Symmetry*, **12** (2020), 2096. https://doi.org/10.3390/sym12122096

44. S. Janjua, I. Hassan, Fuzzy AHP-TOPSIS multi-criteria decision analysis applied to the Indus Reservoir system in Pakistan, *Water Supply*, **20** (2020), 1933–1949. https://doi.org/10.2166/ws.2020.103

45. O. I. Adeyanju, A. A. Adedeji, Application of hybrid fuzzy-topsis for decision making in dam site selection, *Int. J. Civ. Eng.*, **4** (2017), 40–52. https://doi.org/10.14445/23488352/IJCE-V4I7P106

46. M. Deveci, D. Pamucar, E. Oguz, Floating photovoltaic site selection using fuzzy rough numbers based LAAW and RAFSI model, *Appl. Energy*, **324** (2022), 119597. https://doi.org/10.1016/j.apenergy.2022.119597

47. S. H. R. Ahmadi, Y. Noorollahi, S. Ghanbari, M. Ebrahimi, H. Hosseini, A. Foroozani, et al., Hybrid fuzzy decision making approach for wind-powered pumped storage power plant site selection: a case study, *Sustain. Energy Techn.*, **42** (2020), 100838. https://doi.org/10.1016/j.seta.2020.100838

48. M. Heidarimozaffar, M. Shahavand, Spatial zoning of Kabodarahang plain using fuzzy logic in geospatial information system for the construction of an underground dam, *Geogr. Data.*, **30** (2021), 95–110.

49. S. S. Haghshenas, M. A. L. Neshaei, P. Pourkazem, S. S. Haghshenas, The risk assessment of dam construction projects using fuzzy TOPSIS (case study: Alavian Earth Dam), *Civ. Eng. J.*, **2** (2016), 158–167. http://doi.org/10.28991/cej-2016-00000022

50. A. Jozaghi, B. Alizadeh, M. Hatami, I. Flood, M. Khorrami, N. Khodaei, et al., A comparative study of the AHP and TOPSIS techniques for dam site selection using GIS: a case study of Sistan and Baluchestan Province, Iran, *Geosciences*, **8** (2018), 494. http://doi.org/10.20944/preprints201810.0773.v1

51. F. Tufail, M. Shabir, E. S. A. Abo-Tabl, A comparison of Promethee and TOPSIS techniques based on bipolar soft covering-based rough sets, *IEEE Access*, **10** (2022), 37586–37602. https://doi.org/10.1109/ACCESS.2022.3161470

52. M. Akram, G. Ali, M. A. Butt, J. C. R. Alcantud, Novel MCGDM analysis under m-polar fuzzy soft expert sets, *Neural Comput. Appl.*, **33** (2021), 12051–12071.