Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 4875–4891.
DOI: 10.3934/math.2023243
Received: 22 August 2022
Revised: 06 November 2022
Accepted: 16 November 2022
Published: 09 December 2022

Research article

Analytical approximation of European option prices under a new two-factor non-affine stochastic volatility model

Shou-de Huang[1] **and Xin-Jiang He**[2,*]

[1] School of Mathematics and Computer Science, Anshun University, Guizhou, China
[2] School of Economics, Zhejiang University of Technology, Hangzhou, China

* **Correspondence:** Email: xinjiang@zjut.edu.cn.

Abstract: In this paper, the pricing of European options under a new two-factor non-affine stochastic volatility model is studied. In order to reduce the computational complexity, we use the Taylor expansion and Fourier-cosine method to derive an analytical approximation formula for European option prices. Numerical experiments prove that the proposed formula is fast and efficient for pricing European options compared with Monte Carlo simulations. The sensitivity of the parameters is analyzed to explain the rationality of the model. Finally, we present some preliminary empirical analysis revealing that the pricing performance of our proposed model is superior to that of the single-factor model.

Keywords: European option; Fourier-cosine method; two-factor stochastic volatility; non-affine
Mathematics Subject Classification: 91G20

1. Introduction

Although the Black-Scholes-Merton (BS) model [1, 2] proposed in 1973 has attracted a lot of attention from theoretical researchers and financial practitioners because of its simplicity and tractability, the assumptions of this model are far away from reality. For example, the model assumes that the volatility of the underlying asset price is constant, which is inconsistent with the fact that the implied volatility extracted from real trading data often presents a "smile" or "smirk". Therefore, a number of researchers have improved the BS model by relaxing the model assumption that the volatility is constant. Among them, the stochastic volatility model has received a lot of attention.

Hull and White [3] first proposed the concept of stochastic volatility and used the Taylor expansion to obtain the pricing formula of European options. However, this model assumes a zero correlation between the asset price and volatility process, which contradicts the "leverage effect" demonstrated by Bakshi et al. [4]. Stein and Stein [5] derived an analytical pricing formula for European options by assuming that the underlying asset price follows the Ornstein-Uhlenbeck process. However, this model produces negative volatility values, which is clearly contrary to reality. Heston [6] made a breakthrough by proposing to use the Cox-Ingersoll-Ross (CIR) process to describe the dynamic process of the volatility since it satisfies a series of properties, including the non-negativity and mean reversion. Under this model, the analytical solution of European options can also be derived, so that model calibration can be carried out at a reasonable speed.

Furthermore, Christoffersen et al. [7] proposed the double Heston model and their empirical results demonstrated that the double Heston model was more flexible than the Heston model in establishing the volatility term structure and could provide a better fit to market option data than the Heston model. Moreover, the double Heston model, which consists of two unrelated processes, keeps the characteristics of the Heston model that it is easy to calculate, and it is possible to get analytical or semi-analytical solutions when pricing path-dependent options under the double Heston model. However, it should be remarked that the square root process of the Heston model ignores the nonlinear properties of option prices observed in the real market, which has prompted many researchers to improve the Heston stochastic volatility model in recent years [8–10]. In particular, Christoffersen et al. [11] and Chourdakis [12] found that the non-affine stochastic volatility model was better than other stochastic volatility models (including the Heston model) in describing the nonlinear characteristics observed from the trading data of the options market.

Motivated by the advantages of the non-affine stochastic volatility model as well as the double Heston model, we propose a two-factor non-affine stochastic volatility model for the price process of the underlying asset, and consider the pricing problem of European options under the newly proposed model. Due to the complicated model dynamics, it is not possible to derive an analytical solution to the characteristic function of the underlying log price, and we adopt the perturbation method to approximate the characteristic function, so that European options can be analytically evaluated with a Fourier cosine series through the COS method [13–17]. The economic implication of our proposed two-factor non-affine stochastic volatility model can be illustrated from two aspects; a) volatility smile or smirk can be better captured by multi-factor stochastic volatility models [7], and (b) non-affine stochastic volatility is able to not only describe the mean reversion characteristics of the time series of option price volatility, but also describe the nonlinear characteristics observed from the trading data of the options market.

The rest of this paper is organized as follows. A two-factor non-affine stochastic volatility pricing model is proposed in Section 2. In Section 3, an analytical pricing formula is derived using the Taylor expansion and COS method which can reduce the computational complexity. In Section 4, Some numerical examples and calibration analysis are provided to test our results, after which we conclude the paper.

2. Model specification

Let $\{\Omega, \mathcal{F}_t, \mathbb{Q}\}$ be a complete probability space with a filtration continuous on the right, where $\mathbb{Q}$ is a risk-neutral probability measure. The price process of the underlying asset, S_t, and the processes of the volatility, v_{1t} and v_{2t}, are specified under $\mathbb{Q}$ as follows

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sqrt{v_{1t}}\mathrm{d}W_{1s}(t) + \sqrt{v_{2t}}\mathrm{d}W_{2s}(t), \tag{2.1}$$

$$\mathrm{d}v_{1t} = \kappa_1(\theta_1 - v_{1t})\,\mathrm{d}t + \sigma_1 v_{1t}\mathrm{d}W_{1v}(t), \quad (2.2)$$

$$\mathrm{d}v_{2t} = \kappa_2(\theta_2 - v_{2t})\,\mathrm{d}t + \sigma_2 v_{2t}\mathrm{d}W_{2v}(t), \quad (2.3)$$

where $\mathrm{Cov}\,(\mathrm{d}W_{1s}(t), \mathrm{d}W_{1v}(t)) = \rho_1\mathrm{d}t$ and $\mathrm{Cov}\,(\mathrm{d}W_{2s}(t), \mathrm{d}W_{2v}(t)) = \rho_2\mathrm{d}t$. The pairs of $W_{1s}(t)$ and $W_{2s}(t)$, $W_{1s}(t)$ and $W_{2v}(t)$, as well as $W_{2s}(t)$ and $W_{1v}(t)$, are uncorrelated. k_1, θ_1, σ_1 and k_2, θ_2, σ_2 represent the mean-reversion speed, long-term mean and instantaneous volatility of the volatility processes v_{1t} and v_{2t}, respectively.

If we make the transformation of $x_t = \ln(S_t/K)$, Eqs (2.1)–(2.3) can be reformulated as

$$\mathrm{d}x_t = (r - \frac{v_{1t} + v_{2t}}{2})\mathrm{d}t + \sqrt{v_{1t}}\mathrm{d}W_{1s}(t) + \sqrt{v_{2t}}\mathrm{d}W_{2s}(t), \quad (2.4)$$

$$\mathrm{d}v_{1t} = \kappa_1(\theta_1 - v_{1t})\,\mathrm{d}t + \sigma_1 v_{1t}\left(\rho_1 dW_{1s}(t) + \sqrt{\left(1-\rho_1^2\right)}dW_{1v}^{\perp}(t)\right), \quad (2.5)$$

$$\mathrm{d}v_{2t} = \kappa_2(\theta_2 - v_{2t})\,\mathrm{d}t + \sigma_2 v_{2t}\left(\rho_2 dW_{2s}(t) + \sqrt{\left(1-\rho_2^2\right)}dW_{2v}^{\perp}(t)\right). \quad (2.6)$$

3. The pricing of European options

In this section, we derive an approximation to the characteristic function of the underlying log-price, based on which an analytical formula for European option prices with the COS method is obtained.

3.1. An approximate characteristic function

It is well-known that if we are able to derive the characteristic function of the underlying log price, then it would be fairly straightforward to derive the European option pricing formula. With the definition of the characteristic function as

$$\Phi(x, v_1, v_2, \tau; u) = E^{\mathbb{Q}}\left[e^{iux_T} | x_t = x, v_{1t} = v_1, v_{2t} = v_2\right],$$

where $T \geq t, \tau = T - t, i = \sqrt{-1}$, its analytical approximation is presented in the following theorem.

Theorem 1. If we assume that the price process of the underlying asset and the volatility processes satisfy Eqs (2.1)–(2.3), the characteristic function of x_T can be approximated by

$$\Phi(x, v_1, v_2, \tau; u) = exp\left\{iux + A(u,\tau) + B_1(u,\tau)v_1 + B_2(u,\tau)v_2\right\}, \quad (3.1)$$

where

$$A(u,\tau) = riu\tau - \frac{1}{4}(iu + u^2)(\theta_1 + \theta_2)\tau - \frac{\alpha_3}{\alpha_2}\left[\beta_1\tau + \ln\left(\frac{-\beta_2 + \beta_1 e^{-\alpha\tau}}{\alpha}\right)\right]$$
$$- \frac{\xi_3}{\xi_2}\left[\eta_1\tau + \ln\left(\frac{-\eta_2 + \eta_1 e^{-\xi\tau}}{\xi}\right)\right] - \frac{1}{2}\left(B_1(u,\tau)\theta_1 + B_2(u,\tau)\theta_2\right),$$
$$B_1(u,\tau) = \alpha_0\frac{1 - e^{-\alpha\tau}}{-\beta_2 + \beta_1 e^{-\alpha\tau}}, \quad B_2(u,\tau) = \alpha_0\frac{1 - e^{-\xi\tau}}{-\eta_2 + \eta_1 e^{-\xi\tau}},$$

with

$$
\begin{aligned}
&\alpha_0 = -\frac{1}{2}(iu+u^2),\ \alpha_1 = \frac{3}{2}\theta_1^{\frac{1}{2}}\sigma_1\rho_1 iu - \kappa_1,\\
&\alpha_2 = \sigma_1^2\theta_1, \beta_1 = \frac{\alpha_1+\alpha}{2},\ \beta_2 = \frac{\alpha_1-\alpha}{2},\ \eta_2 = \frac{\xi_1-\xi}{2},\\
&\alpha = \sqrt{\alpha_1^2 - 4\alpha_0\alpha_2},\ \xi_1 = \frac{3}{2}\theta_2^{\frac{1}{2}}\sigma_2\rho_2 iu - \kappa_2,\ \xi_2 = \sigma_2^2\theta_2, \eta_1 = \frac{\xi_1+\xi}{2},\\
&\xi = \sqrt{\xi_1^2 - 4\alpha_0\xi_2},\ \alpha_3 = \frac{1}{2}\theta_1\kappa_1 + \frac{1}{4}\sigma_1\rho_1 iu\theta_1^{\frac{3}{2}},\ \xi_3 = \frac{1}{2}\theta_2\kappa_2 + \frac{1}{4}\sigma_2\rho_2 iu\theta_2^{\frac{3}{2}}.
\end{aligned}
$$

Proof. Applying the Feynman-Kac theorem yields the partial differential equation (PDE) governing $\Phi(x, v_1, v_2, \tau; u)$ as

$$
\begin{aligned}
&-\frac{\partial\Phi}{\partial\tau} + \left(r - \frac{v_1+v_2}{2}\right)\frac{\partial\Phi}{\partial x} + \frac{v_1+v_2}{2}\frac{\partial^2\Phi}{\partial x^2} + \sum_{j=1}^{2}\left(\kappa_j(\theta_j - v_j)\frac{\partial\Phi}{\partial v_j}\right.\\
&\left.+\frac{1}{2}\sigma_j^2 v_j^2\frac{\partial\Phi^2}{\partial v_j^2} + v_j^{\frac{3}{2}}\sigma_j\rho_j\frac{\partial^2\Phi}{\partial x\partial v_j}\right) = 0,
\end{aligned}
\tag{3.2}
$$

with the boundary condition given by

$$\Phi\left(x, v_1, v_2, 0; u\right) = e^{iux}.$$

As the above PDE is clearly nonlinear, it does not admit a closed-form solution, and thus we try to first linearize it. The idea is to approximate $v^{\frac{3}{2}}$ and v^2 in the PDE using the Taylor expansion around the long-term mean of the volatility as follows:

$$v_j^2 \approx 2\theta_j v_j - \theta_j^2, \tag{3.3}$$

$$v_j^{\frac{3}{2}} \approx \frac{3}{2}\theta_j^{\frac{1}{2}} v_j - \frac{1}{2}\theta_j^{\frac{3}{2}} \tag{3.4}$$

where $j = 1, 2$. Substituting Eqs (3.3) and (3.4) into (3.2) leads to

$$
\begin{aligned}
&-\frac{\partial\Phi}{\partial\tau} + \left(r - \frac{v_1+v_2}{2}\right)\frac{\partial\Phi}{\partial x} + \frac{v_1+v_2}{2}\frac{\partial^2\Phi}{\partial x^2} + \sum_{j=1}^{2}\left(\kappa_j(\theta_j - v_j)\frac{\partial\Phi}{\partial v_j}\right.\\
&\left.+\frac{1}{2}\sigma_j^2(2\theta_j v_j - \theta_j^2)\frac{\partial\Phi^2}{\partial v_j^2} + \frac{\partial^2\Phi}{\partial x\partial v_j}(\frac{3}{2}\theta_j^{\frac{1}{2}} v_j - \frac{1}{2}\theta_j^{\frac{3}{2}})\sigma_j\rho_j\right) = 0.
\end{aligned}
\tag{3.5}
$$

Following Duffie et al. [18], we now assume that the solution to PDE (3.5) takes the form of

$$\Phi(x, v_1, v_2, \tau; u) = exp\left\{iux + A(u,\tau) + B_1(u,\tau)v_1 + B_2(u,\tau)v_2\right\}, \tag{3.6}$$

with the boundary conditions

$$A(u, 0) = B_1(u, 0) = B_2(u, 0) = 0.$$

The substitution of Eq (3.6) into (3.5) yields

$$
-\left(\frac{\partial A}{\partial \tau}+\frac{\partial B_1}{\partial \tau}v_1+\frac{\partial B_2}{\partial \tau}v_2\right)+\left(r-\frac{v_1+v_2}{2}\right)iu+\frac{v_1+v_2}{2}(iu)^2
$$

$$
+\sum_{j=1}^{2}\left(\kappa_j(\theta_j-v_j)B_j+\frac{1}{2}\sigma_j^2(2\theta_j v_j-\theta_j^2)B_j^2+(\frac{3}{2}\theta_j^{\frac{1}{2}}v_j-\frac{1}{2}\theta_j^{\frac{3}{2}})\sigma_j\rho_j iuB_j\right)=0. \tag{3.7}
$$

By matching the coefficients, we can derive the following three ordinary differential equations (ODEs)

$$
\frac{\partial B_1}{\partial \tau} = \sigma_1^2\theta_1 B_1^2+\left(\frac{3}{2}\sigma_1\rho_1\theta_1^{\frac{1}{2}}iu-\kappa_1\right)B_1-\frac{1}{2}(iu+u^2), \tag{3.8}
$$

$$
\frac{\partial B_2}{\partial \tau} = \sigma_2^2\theta_2 B_1^2+\left(\frac{3}{2}\sigma_2\rho_2\theta_2^{\frac{1}{2}}iu-\kappa_2\right)B_2-\frac{1}{2}(iu+u^2), \tag{3.9}
$$

$$
\frac{\partial A}{\partial \tau} = riu+\sum_{j=1}^{2}\left[\kappa_j\theta_j B_j-\frac{1}{2}\sigma_j^2\theta_j^2 B_j^2-\frac{1}{2}\sigma_j\rho_j\theta_j^{\frac{3}{2}}iuB_j\right]. \tag{3.10}
$$

ODEs (3.8) and (3.9) are clearly Riccati equations, whose solutions can be respectively written as

$$
B_1(u,\tau)=\alpha_0\frac{1-e^{-\alpha\tau}}{-\beta_2+\beta_1 e^{-\alpha\tau}},
$$

$$
B_2(u,\tau)=\alpha_0\frac{1-e^{-\xi\tau}}{-\eta_2+\eta_1 e^{-\xi\tau}}.
$$

Multiplying $\theta_1/2$ and $\theta_2/2$ respectively on both sides of Eqs (3.8) and (3.9), and substituting them into (3.10), we can obtain

$$
\frac{\partial A}{\partial \tau}=riu+\sum_{j=1}^{2}\left((\frac{1}{2}\theta_j\kappa_j+\frac{1}{4}\sigma_j\rho_j iu\theta_j^{\frac{3}{2}})B_j-\frac{1}{4}(iu+u^2)\theta_j\right)-\frac{1}{2}\left(\frac{\partial B_1}{\partial \tau}\theta_1+\frac{\partial B_2}{\partial \tau}\theta_2\right),
$$

integrating on both sides of which leads to the final expression of $A(\tau,u)$. This completes the proof.

It should be pointed that once the analytic expression of the characteristic function has been obtained, the cumulants of lnS_T can be computed, which will be used in the truncation of the computational domain of option pricing [13]. In particular, the n-th cumulant of lnS_T is given by

$$
c_n=\frac{1}{i^n}\frac{\partial^n(\ln\Phi(u))}{\partial u^n}\bigg|_{u=0}.
$$

3.2. *An analytical pricing formula*

The risk-neutral pricing rule implies that European options can be evaluated through [19]

$$
P(x,v_1,v_2,t_0)=e^{-r\Delta t}\int_{-\infty}^{\infty}p(y,T)f(y|x,v_1,v_2)dy, \tag{3.11}
$$

where $x=\ln(S_0/K)$, $y=\ln(S_T/K)$, $f(y|x,v_1,v_2)$ is the probability function of y given x,v_1,v_2, and $p(y,T)$ is the payoff function of a European option at maturity given by

$$
p(y,T)=g(y)=[\alpha K(e^y-1)]^+,\quad \alpha=\begin{cases}1, & \text{for a call}\\ -1, & \text{for a put.}\end{cases}
$$

Although the expression of $f(y|x, v_1, v_2)$ is unknown, we can formulate it with a Fourier cosine series [10] as

$$f(y|x, v_1, v_2) \approx \frac{2}{b-a} \sum_{k=0}^{N-1}{}' \operatorname{Re}\left\{\Phi\left(x, v_1, v_2, \tau; \frac{k\pi}{b-a}\right) e^{ik\pi\frac{x-a}{b-a}}\right\} \cos\left(k\pi\frac{y-a}{b-a}\right) \tag{3.12}$$

where $\sum'$ means the first term of the summation is multiplied by $1/2$, $\operatorname{Re}\{\cdot\}$ denotes taking the real part of a complex number, and $\Phi(x, v_1, v_2, \tau; u)$ is the characteristic function of $f(y|x, v_1, v_2)$. a, b are respectively the lower and upper bounds used for the integration interval in the Fourier cosine method.

Substituting Eq (3.12) into (3.11) and interchanging integration and summation, the approximate price of European options $P(x, v_1, v_2, t_0)$ can be derived as

$$\hat{P}(x, v_1, v_2, t_0) = e^{-r\Delta t} \sum_{k=0}^{N-1}{}' \operatorname{Re}\left\{\Phi\left(x, v_1, v_2, \tau; \frac{k\pi}{b-a}\right) e^{ik\pi\frac{x-a}{b-a}}\right\} V_k$$

where

$$V_k = \frac{2}{b-a}\int_a^b p(y, T) \cos\left(k\pi\frac{y-a}{b-a}\right) dy.$$

This means that the remaining task is to calculate the coefficients V_k. For a European call option, we denote

$$\begin{aligned} \chi_k(x_1, x_2) &= \int_{x_1}^{x_2} e^x \cos\left(k\pi\frac{x-a}{b-a}\right) dx \\ &= \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[\cos\left(k\pi\frac{x_2-a}{b-a}\right)e^{x_2} - \cos\left(k\pi\frac{x_1-a}{b-a}\right)e^{x_1}\right. \\ &\quad \left. + \frac{k\pi}{b-a}\sin\left(k\pi\frac{x_2-a}{b-a}\right)e^{x_2} + \frac{k\pi}{b-a}\sin\left(k\pi\frac{x_1-a}{b-a}\right)e^{x_1}\right] \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \psi_k(x_1, x_2) &= \int_{x_1}^{x_2} \cos\left(k\pi\frac{x-a}{b-a}\right) dx \\ &= \begin{cases} \frac{b-a}{k\pi}\left[\sin\left(k\pi\frac{x_2-a}{b-a}\right) - \sin\left(k\pi\frac{x_1-a}{b-a}\right)\right], & k \neq 0 \\ x_2 - x_1, & k = 0 \end{cases}, \end{aligned} \tag{3.14}$$

so that V_k can be calculated as:

$$\begin{aligned} V_k^{call} &= \frac{2}{b-a}\int_0^b K(e^y - 1)\cos\left(k\pi\frac{y-a}{b-a}\right) dy \\ &= \frac{2}{b-a}K\left(\chi_k(0, b) - \psi_k(0, b)\right). \end{aligned}$$

Similarly, the European put option price can be derived as:

$$\begin{aligned} V_k^{put} &= \frac{2}{b-a}\int_0^b K\left(e^y - 1\right)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}K\left(-\chi_k(a,0) + \psi_k(a,0)\right). \end{aligned}$$

4. Numerical analysis

4.1. Accurate tests

In this subsection, some numerical examples are performed to show the accuracy of the newly derived formula by comparing the results produced from the formula and those obtained from Monte Carlo simulation. Without loss of generality, European call options will be used as an example to demonstrate this, and the integration interval [a, b] is chosen as [13]

$$[a,b] = \left[c_1 + a_0 - L\sqrt{c_2 + \sqrt{c_4}}, c_1 + a_0 + L\sqrt{c_2 + \sqrt{c_4}}\right]$$

with $a_0 = \ln S_0, L = 10$ and c_n being the n-th cumulant of lnS_T. The number of the sample paths and time steps for Monte Carlo simulation is 100,000 and 200, respectively, and $N = 2^{10}$. The default parameter values are listed in Table 1 for all our numerical examples. The computer used in the experiments equips an Intel Core *i*5 CPU with a 1.6+2.1 GHz processor. All of our numerical examples were performed with Matlab 2020a.

Tables 2 and 3 indicate that the approximation formula is fairly accurate, with the absolute relative error (Abs R.E.) between European option prices obtained from our formula and those from the Monte Carlo (MC) simulation across a wide range of strike prices and different maturities being less than 0.7%. On the other hand, one can clearly observe that our approach is significantly faster than the MC simulation. These demonstrate the accuracy and efficiency of our proposed approach.

Table 1. Parameter values for the numerical experiments.

Parameter	r	θ_1	θ_2	ρ_1	ρ_2	S_0	σ_1	σ_2	v_{20}	v_{10}
value	0.05	0.05	0.08	-0.5	-0.5	100	0.09	0.09	0.02	0.05

Table 2. Our price vs. Monte Carlo price when $\kappa_1 = 12, \kappa_2 = 16$.

T	K	FC method	MC simulation	Abs R.E.
1/6	90	12.1669	12.2143	0.39%
	95	8.6172	8.6609	0.50%
	100	5.7773	5.8104	0.57%
	105	3.6629	3.6830	0.55%
	110	2.1979	2.2092	0.51%
	Time(sec.)	0.935	3.311	
1/4	90	13.4270	13.4877	0.45%
	95	10.1043	10.1586	0.53%
	100	7.3614	7.4044	0.58%
	105	5.1948	5.2250	0.58%
	110	3.5547	3.5744	0.55%
	Time(sec.)	1.092	2.968	
1/2	90	16.6070	16.6933	0.52%
	95	13.6062	13.6841	0.57%
	100	11.0102	11.0760	0.59%
	105	8.8058	8.8598	0.61%
	110	6.9664	7.0081	0.59%
	Time(sec.)	1.076	2.788	
1	90	21.4996	21.6192	0.55%
	95	18.7751	18.8837	0.58%
	100	16.3295	16.4282	0.60%
	105	14.1503	14.2381	0.62%
	110	12.2214	12.2968	0.61%
	Time(sec.)	1.093	2.772	

Table 3. Our price vs. Monte Carlo price when $\kappa_1 = 2, \kappa_2 = 3$.

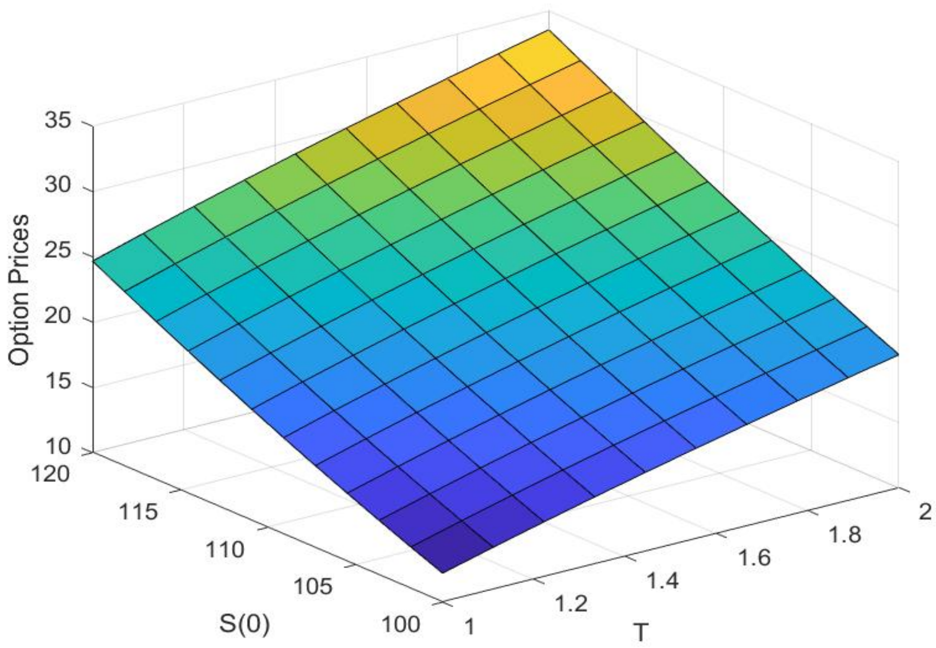
T	K	FC method	MC simulation	Abs R.E.
1/6	90	11.7335	11.7755	0.36%
	95	8.0186	8.0582	0.49%
	100	5.0906	5.1190	0.55%
	105	2.9922	3.0061	0.46%
	110	1.6279	1.6341	0.38%
	Time(sec.)	1.638	4.443	
1/4	90	12.8147	12.8671	0.41%
	95	9.3383	9.3862	0.51%
	100	6.5101	6.5461	0.55%
	105	4.3404	4.3605	0.46%
	110	2.7700	2.7818	0.42%
	Time(sec.)	1.581	4.648	
1/2	90	15.8152	15.8931	0.49%
	95	12.7023	12.7719	0.54%
	100	10.0342	10.0906	0.56%
	105	7.8015	7.8421	0.52%
	110	5.9750	6.0040	0.48%
	Time(sec.)	1.613	4.613	
1	90	20.7803	20.8963	0.56%
	95	17.9864	18.0928	0.59%
	100	15.4882	15.5807	0.59%
	105	13.2740	13.3527	0.59%
	110	11.3273	11.3926	0.57%
	Time(sec.)	1.970	4.637	

4.2. Sensitivity analysis

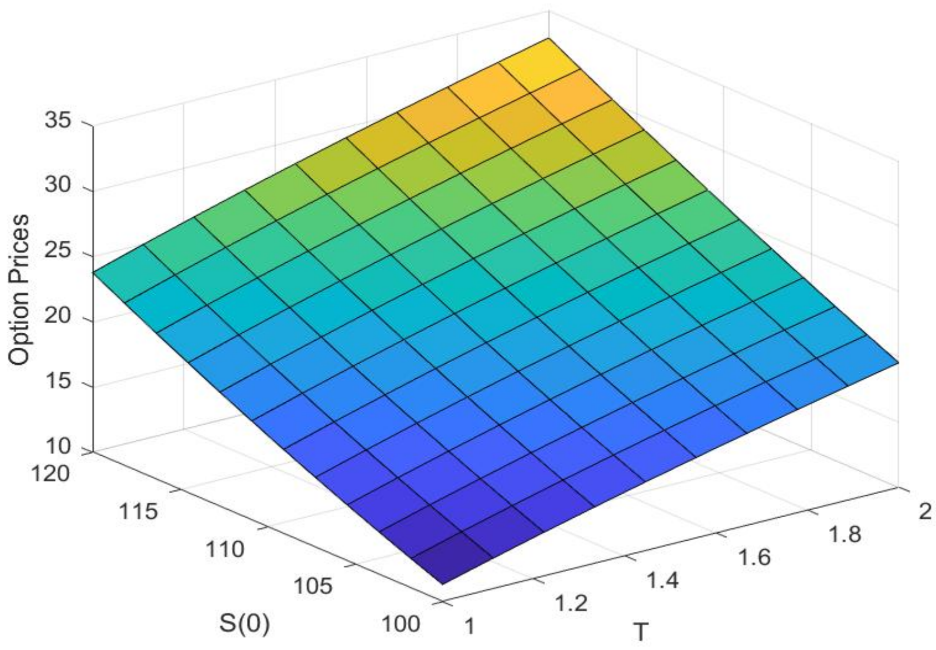
In this subsection, we will assess the effect of the following parameters on the prices of European call options: (*i*) the underlying asset price S_0 and time to maturity T with $t = 0$), (*ii*) correlation coefficients ρ_1 and ρ_2, (*iii*) the long-term mean level θ_1 and θ_2.

Figure 1 shows the variation of European call option prices with respect to the underlying asset prices S_0 and time to maturity $T - t$. Clearly, both a higher S_0 and a higher option remaining time result in a higher European call price, which is expected since the final return of the European call option increases with the increase of S_0, and the time value of the option increases with time.

Figure 2 displays the effects of ρ_1 and ρ_2 on call option prices, and one can clearly observe that option prices increase with either ρ_1 or ρ_2, which is as expected since a higher correlation between the underlying and volatility implies that a positive increase in the volatility will result in a greater climb in the underlying price, leading to a higher option premium. On the other hand, depicted in Figure 3 is the sensitivity of call option prices with respect to the long-term mean of the volatility θ_1 and θ_2. It is not difficult to find that an increase in the long-term mean typically contributes to higher option, and this can be understood from the fact that increasing the level of the long-term mean is equivalent to raising the level of volatility in the long run, which implies higher risk and in turn leads call options to be more expensive.

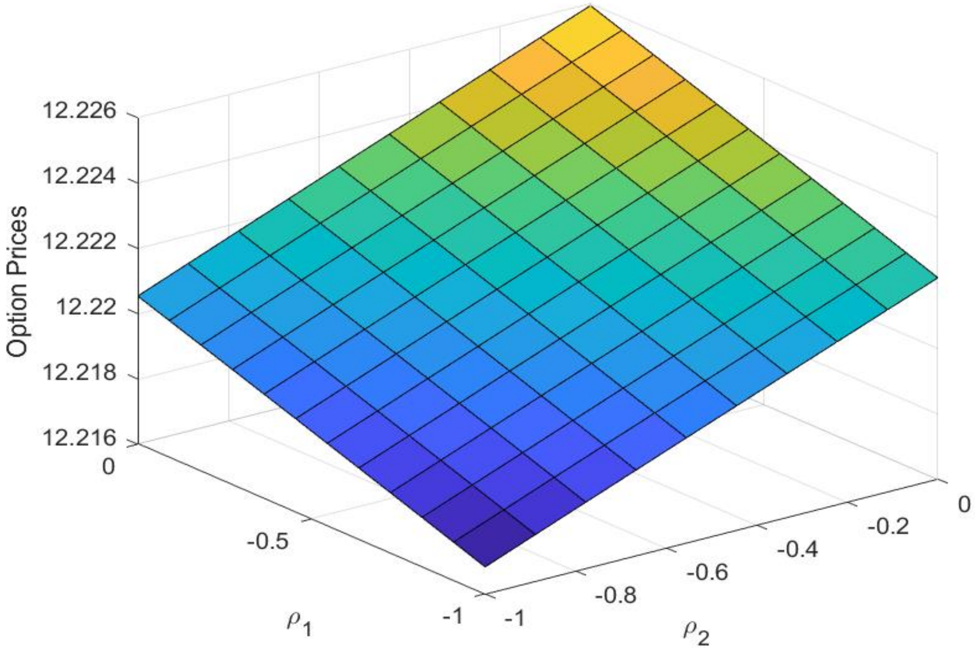


(a) $\kappa_1 = 12, \kappa_2 = 16$.

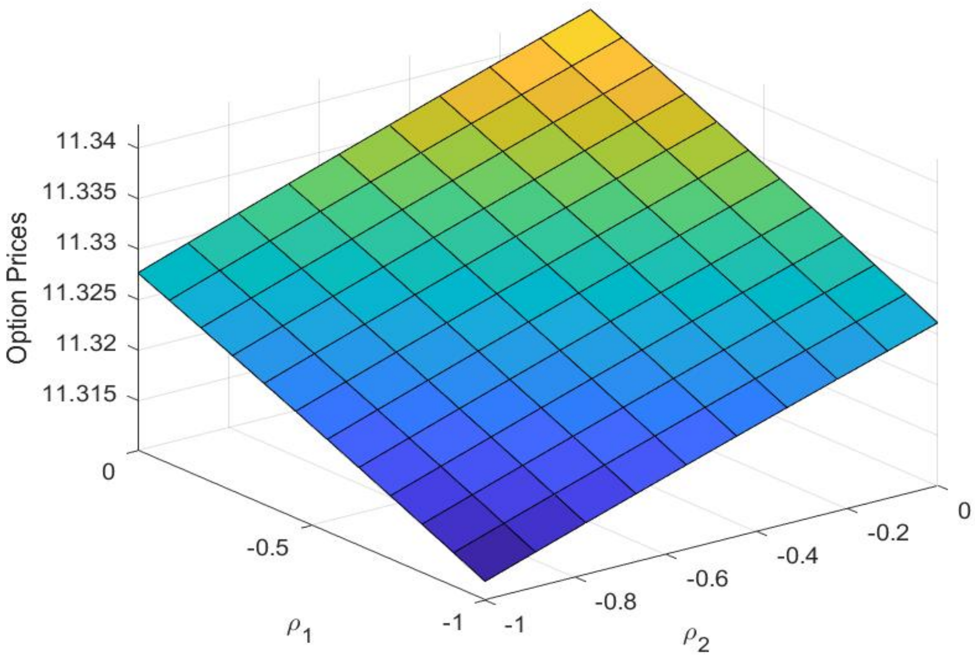


(b) $\kappa_1 = 2, \kappa_2 = 3$.

Figure 1. European option prices for different S_0 and T.

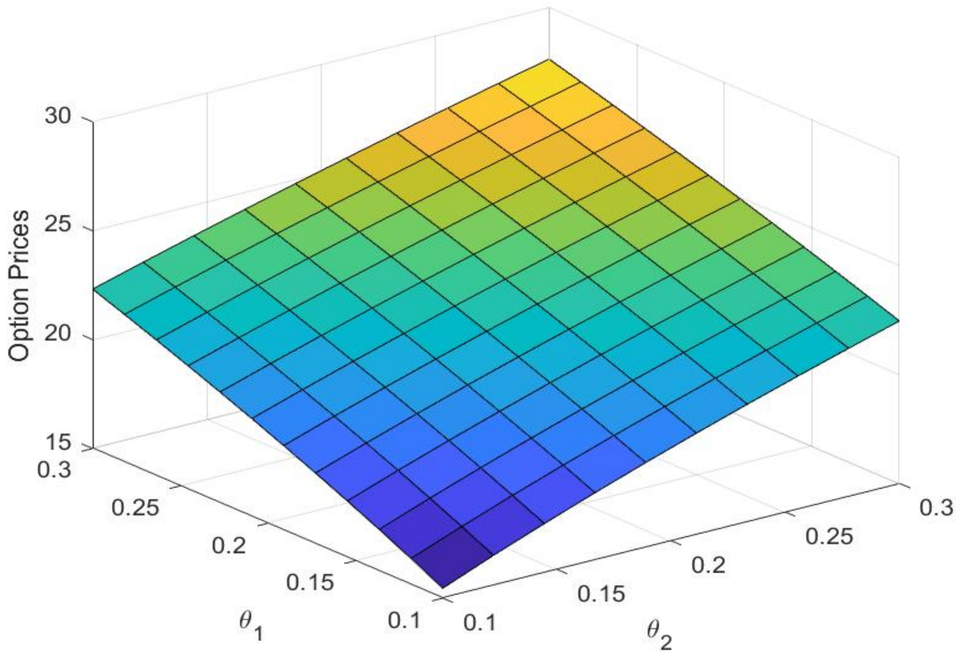


(a) $\kappa_1 = 12, \kappa_2 = 16$.

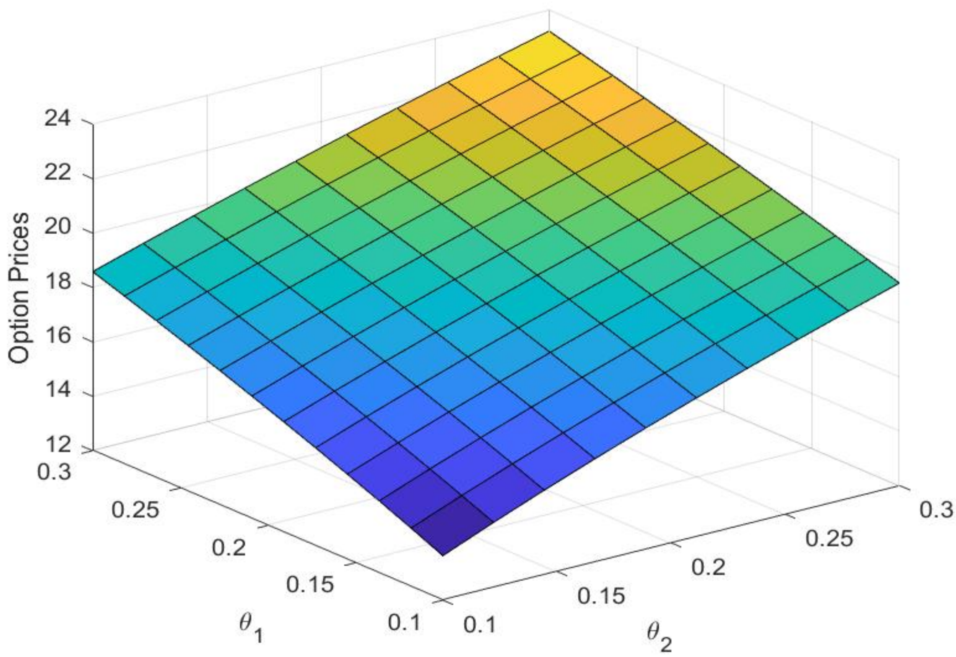


(b) $\kappa_1 = 2, \kappa_2 = 3$.

Figure 2. European option prices for different ρ_1 and ρ_2.



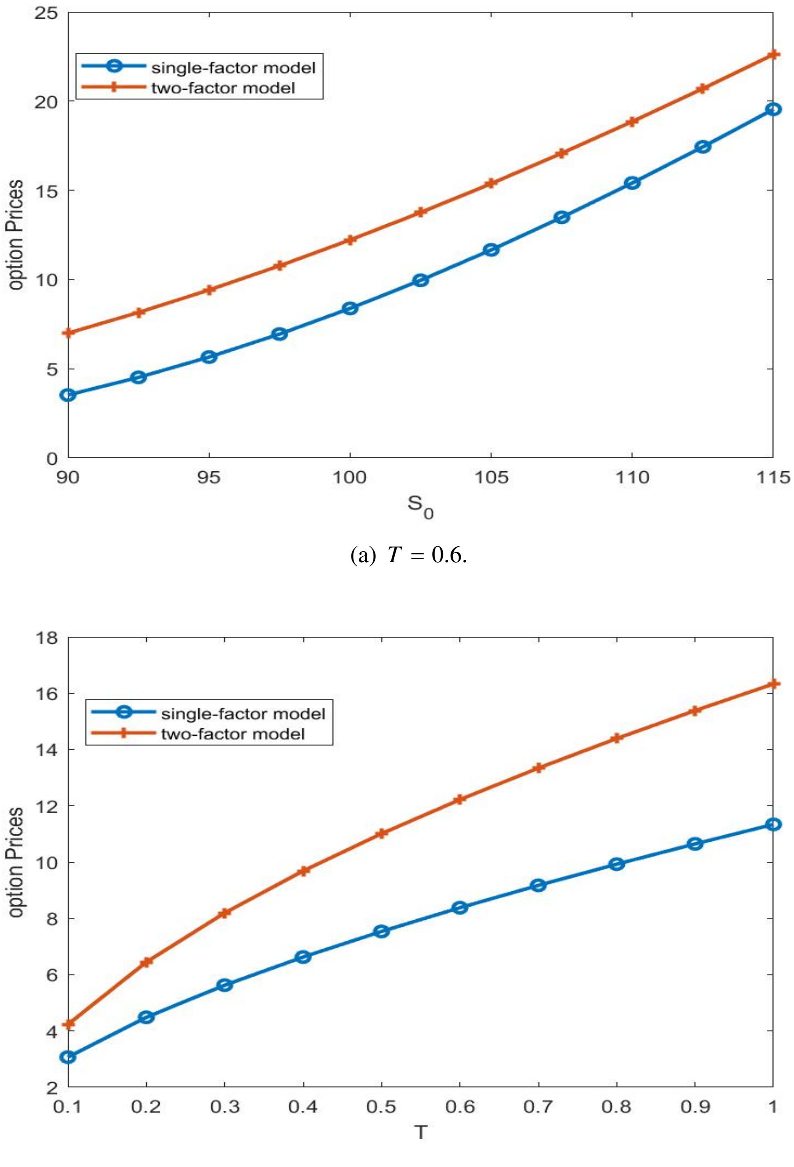
(a) $\kappa_1 = 12, \kappa_2 = 16.$



(b) $\kappa_1 = 2, \kappa_2 = 3.$

Figure 3. European option prices for different θ_1 and θ_2.

To further investigate the effect of the initial price of the underlying asset and the time to expiry on the option price, we make S_0 vary between 90 and 115 with other parameters kept unchanged to produce Figure 4(a,b) is plotted by assuming that the maturity time T changes between 0.1 and 1. One can observe that our price is larger than the single-factor model price under the current parameter settings. Of course, the comparison made between our model and the single-factor model in this section is based on the fact that the corresponding parameters in both models are kept the same, which is not the case in practice where models need to be calibrated so that model parameters can be determined from market data. Thus, we are still not sure about the performance of our model in real markets, and this will be discussed in the next subsection.



(a) T = 0.6.

(b) S_0 = 100.

Figure 4. Our price vs single-factor model price with respect to different S_0 and T.

4.3. Model calibration

In this subsection, we use the SSE 50ETF option trading data to calibrate the proposed model. We define the relative mean error sum of squares (RMSE) loss function as follows:

$$RMSE = \frac{1}{NT \times NK} \sum_{t=1}^{NT} \sum_{k=1}^{NK} \left(P_{tk}^{\Theta} - P_{tk}\right)^2 / P_{tk}$$

where P_{tk}^{Θ} and P_{tk} means the tth option prices obtained from the model and market with maturity time $T(t)$ and strike price $K(k)$, respectively. NT is the number of strike prices, and NK is the number of maturity times. Hence, the parameters can be estimated by solving the following nonlinear optimization problem:

$$\Theta^* = \arg\min RMSE,$$

where Θ^* is the optimal parameter vector. The risk-free interest rate is set to be 0.15. We choose the SSE 50ETF options as of 2 January 2020 for the calibration of the two-factor non-affine stochastic volatility model and the single-factor one, the estimated parameters for which are listed in Table 4.

Table 4. Parameter values of the two-factor non-affine stochastic volatility model and the single-factor model calibrated to SSE 50ETF options as of 2 January 2020.

Parameter value	Single-factor model	Two-factor model
κ_1	0.0846	0.0219
θ_1	0.261	0.9842
σ_1	0.1432	0.0127
ρ_1	−0.9974	−0.9840
v_{10}	0.0246	0.0012
κ_2		0.0106
θ_2		0.0286
σ_2		0.2455
ρ_2		−0.2183
v_{20}		0.0236

To test the performance of the proposed model, we give the following two measures, i.e., the relative mean absolute error (RAE) and the relative mean squared error (RSE), which are respectively defined as

$$RAE = \frac{1}{N} \sum_{k=1}^{N} \frac{\left|P_k^{\Theta} - P_k\right|}{P_k},$$

$$RSE = \frac{1}{N} \sum_{k=1}^{N} \frac{\left(P_k^{\Theta} - P_k\right)^2}{P_k},$$

where P_k and P_k^{Θ} denote the kth market price and model price, respectively, and N means the number of options used in calibration. We compute the above three errors of the two-factor non-affine stochastic volatility model and those of the single factor one using the obtained calibrated parameters listed in Table 4, the results of which are provided in Table 5. We can easily observe that our two-factor non-affine stochastic volatility model provides better performance when pricing European options, compared with the single-factor model.

Table 5. Comparison of the RAE and RSE between the two-factor non-affine stochastic volatility model and the single factor one for pricing SSE 50ETF options as of 2 January 2020.

	RAE		RSE	
	In-sample	Out-of-sample	In-sample	Out-of-sample
One-factor model	0.0328	0.0405	0.1067	0.1116
Two-factor model	0.0326	0.0400	0.1066	0.1113

5. Conclusions

Motivated by the nonlinear characteristics of the volatility as well as the advantages of multi-factor stochastic volatility models, this paper proposes a two-factor non-affine stochastic volatility model for option pricing. Based on the Taylor expansion and COS method, we derive an analytical approximation formula for European option prices, after the characteristic function of the underlying log price is successfully derived. Through numerical experiments, we verify our formula by comparing it against Monte Carlo simulation, and the influence of main model parameters on option prices under the newly proposed model is also shown. We also show through some empirical analysis that our two-factor non-affine stochastic volatility model performs better than the single-factor model does for the pricing of European options.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No. 12101554), the Fundamental Research Funds for Zhejiang Provincial Universities (No. GB202103001), the Advanced Research Funds of Zhejiang University of Technology (No. SKY-ZX-20220212) and Doctoral Scientific Fund Project of Anshun University (No. asxybsjj202202).

Conflict of interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

References

1. F. Black, M. Scholes, The pricing of options and corporate liabilities, *J. Polit. Econ.*, **81** (1973), 637–654. https://doi.org/10.1142/9789814759588_0001

2. R. C. Merton, Theory of rational option pricing, *Bell J. Econ. Manage. Sci.*, **4** (1973), 141–183. https://doi.org/10.2307/3003143

3. J. Hull, A. White, The pricing of options on assets with stochastic volatilities, *J. Finance*, **42** (1987), 281–300. https://doi.org/10.1111/j.1540-6261.1987.tb02568.x

4. G. Bakshi, C. Cao, Z. Chen, Empirical performance of alternative option pricing models, *J. Finance*, **52** (1997), 2003–2049. https://doi.org/10.1111/j.1540-6261.1997.tb02749.x

5. E. M. Stein, J. C. Stein, Stock price distributions with stochastic volatility: an analytic approach, *Rev. Financ. Stud.*, **4** (1991), 727–752. https://doi.org/10.1093/rfs/4.4.727

6. S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Rev. Financ. Stud.*, **6** (1993), 327–343. https://doi.org/10.1093/rfs/6.2.327

7. P. Christoffersen, S. Heston, K. Jacobs, The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well, *Manage. Sci.*, **55** (2009), 1914–1932. https://doi.org/10.1287/mnsc.1090.1065

8. X. J. He, W. Chen, Pricing foreign exchange options under a hybrid Heston-Cox-Ingersoll-Ross model with regime switching, *IMA J. Manag. Math.*, **33** (2022), 255–272. https://doi.org/10.1093/imaman/dpab013

9. X. J. He, S. Lin, A new nonlinear stochastic volatility model with regime switching stochastic mean reversion and its applications to option pricing, *Exp. Syst. Appl.*, **212** (2023), 118742. https://doi.org/10.1016/j.eswa.2022.118742

10. X. J. He, S. Lin, A closed-form pricing formula for European options under a new three-factor stochastic volatility model with regime switching, *Japan J. Indust. Appl. Math.*, 2022, 1–12. https://doi.org/10.1007/s13160-022-00538-7

11. P. Christoffersen, K. Jacobs, J. Mimouni, Volatility dynamics for the S&P500: evidence from realized volatility, daily returns, and option prices, *Rev. Financ. Stud.*, **23** (2010), 3141–3189. https://doi.org/10.1093/rfs/hhq032

12. K. Chourdakis, G. Dotsis, Maximum likelihood estimation of non-affine volatility processes, *J. Empir. Financ.*, **18** (2011), 533–545. https://doi.org/10.1016/j.jempfin.2010.10.006

13. F. Fang, C. W. Oosterlee, A novel pricing method for European options based on Fourier-cosine series expansions, *SIAM J. Sci. Comput.*, **31** (2008), 826–848. https://doi.org/10.1137/080718061

14. F. Fang, C. W. Oosterlee, Pricing early-exercise and discrete barrier options by Fourier-cosine series expansions, *Numer. Math.*, **114** (2009), 27–62. https://doi.org/10.1007/s00211-009-0252-4

15. X. J. He, S. Lin, An analytical approximation formula for barrier option prices under the Heston model, *Comput. Econ.*, **60** (2022), 1413–1425. https://doi.org/10.1007/s10614-021-10186-7

16. S. Huang, X. Guo, A Fourier-cosine method for pricing discretely monitored barrier options under stochastic volatility and double exponential Jump, *Math. Probl. Eng.*, **2020** (2020), 1–9. https://doi.org/10.1155/2020/4613536

17. S. Zhang, J. Geng, Fourier-cosine method for pricing forward starting options with stochastic volatility and jumps, *Commun. Stat.-Theory Methods*, **46** (2017), 9995–10004. https://doi.org/10.1080/03610926.2016.1228960

18. D. Duffie, J. Pan, K. Singleton, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica*, **68** (2000), 1343–1376. https://doi.org/10.1111/1468-0262.00164

19. P. Pasricha, X. J. He, A simple European option pricing formula with a skew Brownian motion, *Probab. Eng. Inform. Sci.*, 2022, 1–6. https://doi.org/10.1017/S0269964822000407