Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 4551–4574.
DOI: 10.3934/math.2023225
Received: 28 July 2022
Revised: 19 October 2022
Accepted: 19 October 2022
Published: 06 December 2022

*Research article*

# Bifurcation and chaos in a discrete activator-inhibitor system

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**Abstract:** In this paper, we explore local dynamic characteristics, bifurcations and control in the discrete activator-inhibitor system. More specifically, it is proved that discrete-time activator-inhibitor system has an interior equilibrium solution. Then, by using linear stability theory, local dynamics with different topological classifications for the interior equilibrium solution are investigated. It is investigated that for the interior equilibrium solution, discrete activator-inhibitor system undergoes Neimark-Sacker and flip bifurcations. Further chaos control is studied by the feedback control method. Finally, numerical simulations are presented to validate the obtained theoretical results.

**Keywords:** activator-inhibitor model; Neimark-Sacker bifurcations; flip bifurcation; numerical simulation; chaos
**Mathematics Subject Classification:** 70K50, 40A05

## 1. Introduction

Moderators or modifiers are the compounds that affect the rate of enzyme catalyzed reactions. Usually, the influence which reduces the rate of reaction is called inhibition, while activation is the process in which the rate of enzyme reaction is increased. The compounds which are responsible for the inhibition and activation are termed as inhibitors and activators, respectively. Chemical inhibitors are the particles which diminish the productivity of a compound by restricting it to its dynamic site.

We can say that there is an inverse relationship between the amount of product produced and concentration of enzyme inhibitors. This is because, whenever the concentration of enzyme inhibitors increases, the rate of enzyme activity decreases. The inhibitors block catalyst action, which is liable for killing a pathogen or correcting a metabolic imbalance, and that is why many medications work as enzyme inhibitors.

The inhibitor can prevent a substrate from entering the dynamic site of proteins, or it can hinder the enzyme from undergoing a catalyzing reaction. Catalyst inhibitors are utilized as drug specialists in both human and veterinary medications, and furthermore as herbicides as well as pesticides. Inhibitors are mainly classified into two classes. The first contains reversible inhibitors that do not form covalent bonds with various pieces of the catalyst surface; that is the reason that they can be converted by dialysis easily. The other type comprises irreversible inhibitors which form strong noncovalent bonds with different areas of enzyme surfaces, and these bonds are so strong that they do not break and can even survive the complex breakdown of protein. A reversible inhibitor restricts a compound completely or to some extent by minimizing the action of the catalyst. Many medicines are compound inhibitors, so their revelation and improvement are a functioning space of exploration in organic chemistry and pharmacology. A therapeutic compound inhibitor is generally represented by its resistance to different proteins and its constituents, which demonstrates the concentration needed to inhibit the enzyme. A high particularity and intensity guarantee that a medication will not have many incidental effects and accordingly, low toxicity.

In systems biology, the activator-inhibitor system of an enzyme is an important phenomenon which describes the catalytic activity of the enzyme within a living organism. An enzyme is a group of biologically active proteins that catalyze all of the chemical reactions occurring in a living system. Activators and inhibitors are special substances which control the activity of an enzyme via a feedback mechanism. An inhibitor decreases or stops the production of an enzyme, while an activator accelerates or enhances the production depending upon the need of a living cell. Enzyme inhibitors occur naturally and are responsible for the regulation of metabolism. For example, chemicals in a metabolic process can be repressed by flowing materials. Other cellular protein inhibitors are proteins that particularly interact with and hinder a protein target. This will offer assistance control proteins which will be harmful to a cell, like proteases or nucleases. A well-known illustration of this particular one is the ribonuclease inhibitor, which interacts with ribonuclease in one of the strongest known forms of protein-protein bonding. Naturally occurring protein inhibitors can also be harmful, and they are utilized as protection against predators, or as a way to kill prey.

Enzyme inhibition is widely analyzed due to its extraordinary significance in both chemical and pharmacological fields. On the other hand, the chemical activation of enzymes is a rarely discussed subject. Real usage of enzyme inhibition mechanisms is qualitative, although quantitative understanding of enzyme inhibition is also important. Experimental results obtained from the molecular events should be understood while studying the phenomena. Numbers of techniques are available to obtain the fundamental information about these mechanisms. Although the processes of enzyme inhibition and activation are similar to each other, they are mostly studied separately [1].

Due to the great importance of activator inhibitor systems, many scientists have been working on it. For activator-inhibitor models, Pasemann et al. [2] proposed a theory for diffusivity estimation. Chen et al. [3] investigated spatial pattern formation in activator-inhibitor models with nonlocal dispersal. Chen [4] studied certain characteristics for the solution of activator inhibitor system. The molecular evidence for an activator-inhibitor mechanism in the development of embryonic feather branching was disclosed by Harris et al. [5]. Ni et al. [6] studied stability analysis for an activator-inhibitor model. Edelstein-Keshet [7] proposed the following continuous-time activator inhibitor system

$$\frac{dx}{dt}=\rho+\frac{x^2}{y}-x,\ \frac{dy}{dt}=x^2-\gamma y, \tag{1.1}$$

where $x$ and $y$ represent concentrations of the activator and inhibitor, respectively, $\rho$ is the strength of self-activation of the activator with the gross activation of the inhibitor and $\gamma$ measures the strength of the production of the activator and that of itself [8]. Now, it is important here to mention that discrete-time models described by difference equations are more appropriate than continuous-time models described by differential equations, and also discrete-time models provide more efficient computational results for numerical simulation [9]. For instance, in recent years, many mathematicians have investigated the dynamical characteristics of discrete-time biological models instead of continuous-time models [10–27]. So, motivated by the aforementioned studies, the purpose of this paper is to investigate the dynamical characteristics of an activator-inhibitor system that is a discrete analogue of the continuous-time model (1.1), by using a non-standard finite difference scheme [28]. In order to get the discrete version of the continuous-time model (1.1), we replace $\frac{dx}{dt}$ by $\frac{x_{t+1}-x_t}{h}$, $\frac{dy}{dt}$ by $\frac{y_{t+1}-y_t}{h}$ and $x^2$ by $x_tx_{t+1}$; thus, the continuous-time activator inhibitor system (1.1) takes the form

$$\frac{x_{t+1}-x_t}{h}=\rho+\frac{x_tx_{t+1}}{y_t}-x_t,\ \frac{y_{t+1}-y_t}{h}=x_tx_{t+1}-\gamma y_t, \tag{1.2}$$

where $h$ is the step size. After straightforward manipulation, the desired discrete-time activator-inhibitor system becomes

$$x_{t+1}=\frac{((1-h)x_t+h\rho)y_t}{y_t-hx_t},\ y_{t+1}=\frac{y_t((y_t-hx_t)(1-h\gamma)+hx_t((1-h)x_t+h\rho))}{y_t-hx_t}. \tag{1.3}$$

More precisely, our main contributions of this paper are as follows:

- Local dynamical behaviors of the equilibrium solution of the discrete activator-inhibitor model (1.3) are identified.
- Bifurcation analysis of the equilibrium solution by bifurcation theory.
- Study of chaos via a feedback control method.
- Validation of the obtained results numerically.

The next section is about the study of local dynamical classifications at the fixed point of the discrete activator-inhibitor system (1.3). The bifurcation analysis of the equilibrium solution is given in Section 3. Section 4 is about the investigation of chaos via a feedback control method for the activator-inhibitor system (1.3). Theoretical results are numerically verified in Section 5. The conclusions, along with future work, are given in Section 6.

## 2. Local dynamical classifications of the equilibrium solution of the discrete activator-inhibitor system (1.3)

In this section, local dynamical classifications of the equilibrium solution of the discrete activator-inhibitor system (1.3) are studied. For this, first we explore the existence of the equilibrium solution and variational matrix evaluated at equilibrium solution of the discrete activator-inhibitor system (1.3). It is easy to verify that $\forall\ \gamma, \rho, h > 0$, the discrete activator-inhibitor system (1.3) has the interior equilibrium solution $E_{xy}^{+}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$. The variational matrix $\Omega|_{E_{xy}(x,y)}$ evaluated at $E_{xy}(x, y)$ under the map

$$(f_1, f_2) \mapsto (x_{t+1}, y_{t+1}), \tag{2.1}$$

where

$$f_1 = \frac{((1-h)x + h\rho)y}{y - hx}, f_2 = \frac{y((y-hx)(1-h\gamma) + hx((1-h)x + h\rho))}{y - hx}, \tag{2.2}$$

is

$$\Omega|_{F_{xy}(x,y)} = \begin{pmatrix} \frac{y^2 - hy^2 + h^2\rho y}{(y-hx)^2} & \frac{hx(hx - x - h\rho)}{(y-hx)^2} \\ \frac{hy(2xy - 2hxy + h\rho y - hx^2 + h^2x^2)}{(y-hx)^2} & \frac{\begin{array}{c} y^2 - h\gamma y^2 - 2hxy + 2h^2\gamma xy + \\ h^2x^2 - h^3\gamma x^2 - h^2x^3 \\ - h^2x^3 + h^3x^3 - h^3\rho x^2 \end{array}}{(y-hx)^2} \end{pmatrix}. \tag{2.3}$$

Moreover, for the interior equilibrium solution $E_{xy}^{+}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$, (2.3) takes the form

$$\Omega|_{F_{xy}^{+}=\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)} = \begin{pmatrix} \frac{\gamma+\rho-h\rho}{\gamma+\rho-h\gamma} & -\frac{h\gamma^2}{(\rho+\gamma)(\rho+\gamma-h\gamma)} \\ \frac{h(2-h)(\rho+\gamma)^2}{\gamma+\rho-h\gamma} & \frac{\rho-h\rho\gamma-\gamma(h-1+h\gamma)}{\gamma+\rho-h\gamma} \end{pmatrix}. \tag{2.4}$$

From (2.4), the characteristic equation of $\Omega|_{E_{xy}^{+}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)}$ evaluated at $E_{xy}^{+}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ is

$$\lambda^2 - p\lambda + q = 0, \tag{2.5}$$

where

$$\begin{aligned} p &= \frac{(\rho+\gamma)(2-h-h\gamma)}{\rho+\gamma-h\gamma}, \\ q &= \frac{\rho(h-1)(h\gamma-1) + \gamma(1-h\gamma+h^2\gamma)}{\rho+\gamma-h\gamma}. \end{aligned} \tag{2.6}$$

Finally, the roots of (2.5) are

$$\lambda_{1,2} = \frac{p \pm \sqrt{\Delta}}{2}, \tag{2.7}$$

where

$$\begin{aligned} \Delta &= p^2 - 4q, \\ &= \left(\frac{(\rho+\gamma)(2-h-h\gamma)}{\rho+\gamma-h\gamma}\right)^2 - 4\left(\frac{\rho(h-1)(h\gamma-1) + \gamma(1-h\gamma+h^2\gamma)}{\rho+\gamma-h\gamma}\right). \end{aligned} \tag{2.8}$$

Based on the above computation, local dynamical classifications at $E_{xy}^{+}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of the discrete activator-inhibitor system (1.3) are presented according to the sign of $\Delta$, i.e., $\Delta < 0$ and $\Delta \geq 0$, respectively.

**Lemma 2.1.** If $\Delta < 0$, then for the interior equilibrium solution $E^{+}_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of the activator-inhibitor system (1.3), the following dynamical classifications hold

(i) $E^{+}_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ is a stable focus if

$$0 < h < \frac{\rho+\rho\gamma-\gamma+\gamma^2}{\rho\gamma+\gamma^2}, \tag{2.9}$$

with

$$\rho > \frac{\gamma-\gamma^2}{1+\gamma}; \tag{2.10}$$

(ii) $E^{+}_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ is an unstable focus if (2.10) holds and

$$h > \frac{\rho+\rho\gamma-\gamma+\gamma^2}{\rho\gamma+\gamma^2}; \tag{2.11}$$

(iii) $E^{+}_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ is non-hyperbolic if

$$h = \frac{\rho+\rho\gamma-\gamma+\gamma^2}{\rho\gamma+\gamma^2}. \tag{2.12}$$

*Proof.* If $\Delta < 0$, then from (2.7) one gets $|\lambda_{1,2}| = \sqrt{\frac{\rho(h-1)(h\gamma-1)+\gamma(1-h\gamma+h^2\gamma)}{\rho+\gamma-h\gamma}} < 1$ which gives the fact that if $0 < h < \frac{\rho+\rho\gamma-\gamma+\gamma^2}{\rho\gamma+\gamma^2}$ with $\rho > \frac{\gamma-\gamma^2}{1+\gamma}$ then $E^{+}_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ is a stable focus. In a similar way, it is easy to prove that $E^{+}_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of the activator-inhibitor system (1.3) is an unstable focus (non-hyperbolic) if $h > \frac{\rho+\rho\gamma-\gamma+\gamma^2}{\rho\gamma+\gamma^2}$ ($h = \frac{\rho+\rho\gamma-\gamma+\gamma^2}{\rho\gamma+\gamma^2}$). □

**Lemma 2.2.** If $\Delta \geq 0$, then for the interior equilibrium solution $E^{+}_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of the activator-inhibitor system (1.3), the following dynamical classifications hold

(i) $E^{+}_{xy}\left(\gamma+\rho, \frac{(\rho+\gamma)^2}{\gamma}\right)$ is a stable node if

$$0 < h < \min\left\{2, \frac{2}{\gamma}\right\}; \tag{2.13}$$

(ii) $E^{+}_{xy}\left(\gamma+\rho, \frac{(\rho+\gamma)^2}{\gamma}\right)$ is an unstable node if

$$h > \max\left\{2, \frac{2}{\gamma}\right\}; \tag{2.14}$$

(iii) $E^{+}_{xy}\left(\gamma+\rho, \frac{(\rho+\gamma)^2}{\gamma}\right)$ is non-hyperbolic if

$$h = 2, \tag{2.15}$$

or

$$h = \frac{2}{\gamma}. \tag{2.16}$$

*Proof.* If $\Delta \geq 0$, then $E^{+}_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of system (1.3) is a stable node if $|\lambda_{1,2}|$ <1 which implies that if $0 < h < \min\left\{2, \frac{2}{\gamma}\right\}$ then $E^{+}_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of the discrete activator-inhibitor system (1.3) is a stable node. In a similar way one can prove that $E^{+}_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of the discrete activator-inhibitor system (1.3) is an unstable node if $h > \max\left\{2, \frac{2}{\gamma}\right\}$, and non-hyperbolic if $h = 2$ or $h = \frac{2}{\gamma}$. □

## 3. Bifurcation analysis

We will study bifurcation analysis of the equilibrium solution $E^+_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of the activator-inhibitor system (1.3) in the present section by using bifurcation theory [29, 30].

### 3.1. Neimark-Sacker bifurcation analysis of $E^+_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$

If $\Delta = \left(\frac{(\rho+\gamma)(2-h-h\gamma)}{\rho+\gamma-h\gamma}\right)^2 - 4\left(\frac{\rho(h-1)(h\gamma-1)+\gamma(1-h\gamma+h^2\gamma)}{\rho+\gamma-h\gamma}\right) < 0$, then in view of (2.7) and (2.12) the simple computation yields $\left|\lambda_{1,2}\right|_{(2.12)} = 1$. This implies that the activator-inhibitor system (1.3) may undergo Neimark-Sacker bifurcation if $(\gamma, h, \rho)$ are located in the set

$$\mathcal{N}|_{E^+_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)} = \left\{(\gamma, h, \rho),\ h = \frac{\rho+\rho\gamma-\gamma+\gamma^2}{\rho\gamma+\gamma^2}\right\}. \tag{3.1}$$

But in the following theorem it is proved that for $E^+_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$, the discrete activator-inhibitor system (1.3) must undergo the Neimark-Sacker bifurcation.

**Theorem 3.1.** For $E^+_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$, the discrete activator-inhibitor system (1.3) undergoes the Neimark-Sacker bifurcation if $(\gamma, h, \rho) \in \mathcal{N}|_{E^+_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)}$, as achieved by choosing $h$ as a bifurcation parameter.

*Proof.* If $h$ varies in a small neighborhood of $h^*$, i.e., $h = h^* + \epsilon$ where $\epsilon << 1$, then the activator-inhibitor system (1.3) becomes

$$\begin{aligned} x_{t+1} &= \frac{((1-(h^*+\epsilon))x_t + (h^*+\epsilon)\rho)y_t}{y_t-(h^*+\epsilon)x_t}, \\ y_{t+1} &= \frac{y_t((y_t-(h^*+\epsilon)x_t)(1-(h^*+\epsilon)\gamma) + (h^*+\epsilon)x_t((1-(h^*+\epsilon))x_t + (h^*+\epsilon)\rho))}{y_t-(h^*+\epsilon)x_t}. \end{aligned} \tag{3.2}$$

Further, for the $\epsilon$-dependence model (3.2), from (2.7) one gets

$$\lambda_{1,2} = \frac{p(\epsilon) \pm \iota\sqrt{4q(\epsilon)-p^2(\epsilon)}}{2}, \tag{3.3}$$

where

$$\begin{aligned} p(\epsilon) &= \frac{(\rho+\gamma)(2-(h^*+\epsilon)-(h^*+\epsilon)\gamma)}{\rho+\gamma-(h^*+\epsilon)\gamma}, \\ q(\epsilon) &= \frac{\rho((h^*+\epsilon)-1)((h^*+\epsilon)\gamma-1) + \gamma(1-(h^*+\epsilon)\gamma+(h^*+\epsilon)^2\gamma)}{\rho+\gamma-(h^*+\epsilon)\gamma}. \end{aligned} \tag{3.4}$$

From (3.3), the following computation shows that the non-degenerate condition holds, i.e.,

$$\frac{d|\lambda_{1,2}|}{d\epsilon}|_{\epsilon=0} = \frac{-\gamma^2+\rho^3(1+\gamma)+\rho\gamma(2-2\gamma+\gamma^2)+\rho^2(2\gamma^2-\gamma-1)}{(\rho+\gamma)(2\gamma-\gamma^2-\rho\gamma)^2} \neq 0. \tag{3.5}$$

Further, for the existence of Neimark-Sacker bifurcation, it is also required that $\lambda^m_{1,2} \neq 1,\ m = 1, \cdots, 4$ if $\epsilon = 0$, which corresponds to $p(0) \neq -2, 0, 1, 2$. But, if (2.12) holds then from (3.4), one gets $q(0) = 1$. Therefore, $p(0) \neq -2, 2$; thus, it is only required that $p(0) \neq 0, 1$. For this, the computation yields

$$\rho \neq \frac{2\gamma^2-\gamma^3\pm\gamma\sqrt{1+2\gamma+4\gamma^2-2\gamma^3}}{1+\gamma^2}, \frac{\gamma^2-2\gamma^3-\gamma\pm\sqrt{(2\gamma^3-\gamma^2-\gamma)^2-4(1+\gamma^2+\gamma)(\gamma^4+\rho^2)}}{2(\gamma^2+\gamma+1)}. \tag{3.6}$$

Now it is noted that (3.2) takes the form

$$\begin{aligned} u_{t+1} &= \frac{(v_t+y^*)((1-(h^*+\epsilon))(u_t+x^*)+(h^*+\epsilon)\rho)}{(v_t+y^*)-(h^*+\epsilon)(u_t+x^*)}-x^*, \\ v_{t+1} &= \frac{(v_t+y^*)\begin{pmatrix}((v_t+y^*)-(h^*+\epsilon)(u_t+x^*))(1-(h^*+\epsilon)\gamma)+\\ (h^*+\epsilon)(u_t+x^*)(1-(h^*+\epsilon))(u_t+x^*)+(h^*+\epsilon)\rho\end{pmatrix}}{(v_t+y^*)-(h^*+\epsilon)(u_t+x^*)}-y^*, \end{aligned} \tag{3.7}$$

using

$$u_t = x_t - x^*,\ v_t = y_t - y^*, \tag{3.8}$$

where $x^* = \rho+\gamma,\ y^* = \frac{(\rho+\gamma)^2}{\gamma}$. Hereafter, if $\epsilon = 0$, the normal form is investigated. So, by Taylor series expansion about $E_{00}\,(0,0)$, (3.7) takes the form

$$\begin{aligned} u_{t+1} &= \alpha_{11}u_t+\alpha_{12}v_t+\alpha_{13}u_t^2+\alpha_{14}u_tv_t+\alpha_{15}v_t^2+\alpha_{16}u_t^3+\alpha_{17}u_tv_t^2+\alpha_{18}u_t^2v_t+\alpha_{19}v_t^3, \\ u_{t+1} &= \alpha_{21}u_t+\alpha_{22}v_t+\alpha_{23}u_t^2+\alpha_{24}u_tv_t+\alpha_{25}v_t^2+\alpha_{26}u_t^3+\alpha_{27}u_tv_t^2+\alpha_{28}u_t^2v_t+\alpha_{29}v_t^3, \end{aligned} \tag{3.9}$$

with

$$\begin{aligned} \alpha_{11} &= \frac{y^{*2}-hy^{*2}+h^2\rho y^*}{(y^*-hx^*)^2},\ \alpha_{12} = \frac{hx^*(hx^*-x^*-h\rho)}{(y^*-hx^*)^2}, \\ \alpha_{13} &= \frac{hy^{*2}-h^2y^{*2}+h^3\rho y^*}{(y^*-hx^*)^3},\ \alpha_{14} = \frac{-h^2\rho y^*-2hx^*y^*+2h^2x^*y^*-h^3\rho x^*}{(y^*-hx^*)^3}, \\ \alpha_{15} &= \frac{hx^{*2}-h^2x^{*2}+h^2\rho x^*}{(y^*-hx^*)^3},\ \alpha_{16} = \frac{h^2y^{*2}-h^3y^{*2}+h^4\rho y^*}{(y^*-hx^*)^4}, \\ \alpha_{17} &= \frac{hx^*y^*-h^3x^{*2}+2h^3x^*\rho+h^{*2}x^{*2}-2h^{*2}x^*y^*+h^2\rho y^*}{(y^*-hx^*)^4}, \\ \alpha_{18} &= \frac{-h^2x^*y^*+h^2y^{*2}-hy^{*2}-h^4x\rho+2h^3x^*y^*-2h^{*3}\rho y^*}{(y^*-hx^*)^4}, \\ \alpha_{19} &= \frac{h^2x^{*2}-hx^{*2}-h^2\rho x^*}{(y^*-hx^*)^4}, \\ \alpha_{21} &= \frac{hy^*(2x^*y^*-2hx^*y^*+h\rho y^*-hx^{*2}+h^2x^{*2})}{(y^*-hx^*)^2}, \\ \alpha_{22} &= \frac{y^{*2}-h\gamma y^{*2}-2hx^*y^*+2h^2\gamma x^*y^*+h^2x^{*2}-h^3\gamma x^{*2}-h^2x^{*3}+h^3x^{*3}-h^3\rho x^{*2}}{(y^*-hx^*)^2}, \\ \alpha_{23} &= \frac{hy^{*3}-h^2y^{*3}+h^3\rho y^{*2}}{(y^*-hx^*)^3}, \\ \alpha_{24} &= \frac{-3h^2x^{*2}y^*+3h^3x^{*2}y*-2h^3\rho x^*y^*+h^3x^3-h^4x^3}{(y^*-hx^*)^3}, \\ \alpha_{25} &= \frac{h^2x^{*3}-h^3x^{*3}+h^3\rho x^{*2}}{(y^*-hx^*)^3},\ \alpha_{26} = \frac{h^2y^{*3}-h^3y^{*3}+h^4\rho y^{*2}}{(y^*-hx^*)^4}, \\ \alpha_{27} &= \frac{h^3\rho x^*y^*+3h^2x^{*2}y^*-3h^3x^{*2}y^*+h^4\rho x^{*2}}{(y^*-hx^*)^4}, \end{aligned} \tag{3.10}$$

$$\alpha_{28} = \frac{-h^3\rho y^{*2} - 3h^2x^*y^{*2} + 3h^3x^*y^{*2} - 2h^4\rho x^*y^*}{(y^* - hx^*)^4},$$
$$\alpha_{29} = \frac{-h^2x^{*3} + h^3x^{*3} - h^3x^{*2}\rho}{(y^* - hx^*)^4}.$$

Now in order to obtain the linear part of (3.9) in canonical form, we use the following transformation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \alpha_{12} & 0 \\ \eta - \alpha_{11} & -\zeta \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \tag{3.11}$$

with

$$\begin{aligned} \eta =& \frac{(\rho+\gamma)(2-h-h\gamma)}{2(\rho+\gamma-h\gamma)}, \\ \zeta =& \frac{1}{2}\sqrt{\frac{4\rho(h-1)(h\gamma-1)+4\gamma(1-h\gamma+h^2\gamma)}{\rho+\gamma-h\gamma} - \left(\frac{(\rho+\gamma)(2-h-h\gamma)}{\rho+\gamma-h\gamma}\right)^2}. \end{aligned} \tag{3.12}$$

In view of (3.11), (3.9) takes the following form

$$\begin{aligned} x_{t+1} =& \quad \eta x_t - \zeta y_t + \bar{P}(x_t, y_t), \\ y_{t+1} =& \quad \zeta x_t + \eta y_t + \bar{Q}(x_t, y_t), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \bar{P}(x_t, y_t) &= r_{11}x_t^3 + r_{12}x_t^2 + r_{13}x_t^2y_t + r_{14}x_ty_t + r_{15}x_ty_t^2 + r_{16}y_t^3 + r_{17}y_t^2, \\ \bar{Q}(x_t, y_t) &= r_{21}x_t^3 + r_{22}x_t^2 + r_{23}x_t^2y_t + r_{24}x_ty_t + r_{25}x_ty_t^2 + r_{26}y_t^3 + r_{27}y_t^2, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} r_{11} &= \alpha_{16}\alpha_{12}^2 + \alpha_{17}(\eta-\alpha_{11})^2 + \alpha_{18}\alpha_{12}(\eta-\alpha_{11}) + \frac{\alpha_{19}(\eta-\alpha_{11})}{\alpha_{12}}, \\ r_{12} &= \alpha_{13}\alpha_{12} + \alpha_{14}(\eta-\alpha_{11}) + \frac{\alpha_{15}(\eta-\alpha_{11})^2}{\alpha_{12}}, \\ r_{13} &= -2\alpha_{17}\xi(\eta-\alpha_{11}) - \xi\alpha_{12}\alpha_{18} - \frac{3\alpha_{19}\xi(\eta-\alpha_{11})^2}{\alpha_{12}}, \\ r_{14} &= -\alpha_{14}\xi - \frac{2\xi\alpha_{15}(\eta-\alpha_{11})}{\alpha_{12}}, \\ r_{15} &= \alpha_{17}\xi^2 + \frac{3\alpha_{19}\xi^2(\eta-\alpha_{11})}{\alpha_{12}}, \\ r_{16} &= -\frac{\xi^3\alpha_{19}}{\alpha_{12}}, \\ r_{17} &= \frac{\xi^2\alpha_{15}}{\alpha_{12}}, \\ r_{21} &= \frac{\alpha_{16}\alpha_{12}^2(\eta-\alpha_{11})}{\xi} + \frac{\alpha_{17}(\eta-\alpha_{11})^3}{\xi} + \frac{\alpha_{18}\alpha_{12}(\eta-\alpha_{11})^2}{\xi} + \frac{\alpha_{19}(\eta-\alpha_{11})^2}{\xi\alpha_{12}} - \frac{2\alpha_{26}\alpha_{12}^3}{\xi} \end{aligned} \tag{3.15}$$

$$
-\frac{\alpha_{27}\alpha_{12}(\eta-\alpha_{11})^2}{\xi}-\frac{\alpha_{28}\alpha_{12}^2(\eta-\alpha_{11})}{\xi}-\frac{\alpha_{29}(\eta-\alpha_{11})}{\xi},
$$

$$
\begin{aligned}
r_{22} &= \frac{\alpha_{13}\alpha_{12}(\eta-\alpha_{11})}{\xi}+\frac{\alpha_{14}(\eta-\alpha_{11})^2}{\xi}+\frac{\alpha_{15}(\eta-\alpha_{11})^3}{\xi\alpha_{12}}-\frac{\alpha_{23}\alpha_{12}^2}{\xi}-\frac{\alpha_{24}\alpha_{12}(\eta-\alpha_{11})}{\xi}\\
&\quad -\frac{\alpha_{25}(\eta-\alpha_{11})^2}{\xi},\\
r_{23} &= -2\alpha_{17}(\eta-\alpha_{11})^2-\alpha_{12}\alpha_{18}(\eta-\alpha_{11})-\frac{3\alpha_{19}(\eta-\alpha_{11})^3}{\alpha_{12}}+2\alpha_{27}\alpha_{12}(\eta-\alpha_{11})+\alpha_{12}^2\alpha_{28}\\
&\quad +3\alpha_{29}(\eta-\alpha_{11})^2,\\
r_{24} &= -\alpha_{14}(\eta-\alpha_{11})-\frac{2\alpha_{15}(\eta-\alpha_{11})^2}{\alpha_{12}}+\alpha_{24}\alpha_{12}+2\alpha_{25}(\eta-\alpha_{11}),\\
r_{25} &= \xi\alpha_{17}(\eta-\alpha_{11})+\frac{3\alpha_{19}(\eta-\alpha_{11})^2\xi}{\alpha_{12}}-\xi\alpha_{27}\alpha_{12}-3\alpha_{29}\xi(\eta-\alpha_{11}),\\
r_{26} &= -\frac{\xi^2\alpha_{19}(\eta-\alpha_{11})}{\alpha_{12}}+\xi\alpha_{29},\\
r_{27} &= \frac{\xi\alpha_{15}(\eta-\alpha_{11})}{\alpha_{12}}-\alpha_{25}\xi.
\end{aligned}
$$

From (3.14) the computation yields

$$
\begin{aligned}
&\frac{\partial^2\bar{P}}{\partial x_t^2}\Big|_{E_{00}(0,0)}=2r_{12},\ \frac{\partial^2\bar{P}}{\partial x_t\partial y_t}\Big|_{E_{00}(0,0)}=r_{14},\ \frac{\partial^2\bar{P}}{\partial y_t^2}\Big|_{E_{00}(0,0)}=2r_{17},\ \frac{\partial^3\bar{P}}{\partial x_t^3}\Big|_{E_{00}(0,0)}=6r_{11},\\
&\frac{\partial^3\bar{P}}{\partial x_t^2\partial y_t}\Big|_{E_{00}(0,0)}=2r_{13},\ \frac{\partial^3\bar{P}}{\partial x_t\partial y_t^2}\Big|_{E_{00}(0,0)}=2r_{15},\ \frac{\partial^3\bar{P}}{\partial y_t^3}\Big|_{E_{00}(0,0)}=6r_{16},\ \frac{\partial^2\bar{Q}}{\partial x_t^2}\Big|_{E_{00}(0,0)}=2r_{22},\\
&\frac{\partial^2\bar{Q}}{\partial x_t\partial y_t}\Big|_{E_{00}(0,0)}=r_{24},\ \frac{\partial^2\bar{Q}}{\partial y_t^2}\Big|_{E_{00}(0,0)}=2r_{27},\ \frac{\partial^3\bar{Q}}{\partial x_t^3}\Big|_{E_{00}(0,0)}=6r_{21},\ \frac{\partial^3\bar{Q}}{\partial x_t^2\partial y_t}\Big|_{E_{00}(0,0)}=2r_{23},\\
&\frac{\partial^3\bar{Q}}{\partial x_t\partial y_t^2}\Big|_{E_{00}(0,0)}=2r_{25},\ \frac{\partial^3\bar{Q}}{\partial y_t^3}\Big|_{E_{00}(0,0)}=6r_{26}.
\end{aligned}
\tag{3.16}
$$

Finally, in order to determine the map (3.13) undergoes the Neimark-Sacker bifurcation following quantity should be non-zero

$$
\chi=-\Re\left(\frac{(1-2\bar{\lambda})\bar{\lambda}^2}{1-\lambda}\varrho_{11}\varrho_{20}\right)-\frac{1}{2}\|\varrho_{11}\|^2-\|\varrho_{02}\|^2+\Re\left(\bar{\lambda}\varrho_{21}\right),
\tag{3.17}
$$

where

$$
\begin{aligned}
\varrho_{02} &= \frac{1}{8}\left(\frac{\partial^2\bar{P}}{\partial x_t^2}-\frac{\partial^2\bar{Q}}{\partial y_t^2}+2\frac{\partial^2\bar{Q}}{\partial x_t\partial y_t}+\iota\left(\frac{\partial^2\bar{Q}}{\partial x_t^2}-\frac{\partial^2 Q}{\partial y_t^2}+2\frac{\partial^2\bar{P}}{\partial x_t\partial y_t}\right)\right)\Big|_{E_{00}(0,0)},\\
\varrho_{11} &= \frac{1}{4}\left(\frac{\partial^2\bar{P}}{\partial x_t^2}+\frac{\partial^2\bar{P}}{\partial y_t^2}+\iota\left(\frac{\partial^2\bar{Q}}{\partial x_t^2}+\frac{\partial^2\bar{Q}}{\partial y_t^2}\right)\right)\Big|_{E_{00}(0,0)},\\
\varrho_{20} &= \frac{1}{8}\left(\frac{\partial^2\bar{P}}{\partial x_t^2}-\frac{\partial^2\bar{P}}{\partial y_t^2}+2\frac{\partial^2\bar{Q}}{\partial x_t\partial y_t}+\iota\left(\frac{\partial^2\bar{Q}}{\partial x_t^2}-\frac{\partial^2\bar{Q}}{\partial y_t^2}-2\frac{\partial^2\bar{P}}{\partial x_t\partial y_t}\right)\right)\Big|_{E_{00}(0,0)},
\end{aligned}
\tag{3.18}
$$

$$\varrho_{21} = \frac{1}{16}\left(\frac{\partial^3 \bar{P}}{\partial x_t^3} + \frac{\partial^3 \bar{P}}{\partial y_t^3} + \frac{\partial^3 \bar{Q}}{\partial x_t^2 \partial y_t} + \frac{\partial^3 \bar{Q}}{\partial y_t^3} + \iota\left(\frac{\partial^3 \bar{Q}}{\partial x_t^3} + \frac{\partial^3 \bar{Q}}{\partial x_t \partial y_t^2} - \frac{\partial^3 \bar{P}}{\partial x_t^2 \partial y_t} - \frac{\partial^3 \bar{P}}{\partial y_t^3}\right)\right)\Bigg|_{E_{00}(0,0)}.$$

In view of (3.16), from (3.18) one gets

$$\begin{aligned}
\varrho_{02} &= \frac{1}{4}\left(r_{12} - r_{17} + r_{24} + \iota(r_{22} - r_{27} + r_{14})\right), \\
\varrho_{11} &= \frac{1}{2}\left(r_{12} + r_{17} + \iota(r_{22} + r_{27})\right), \\
\varrho_{20} &= \frac{1}{4}\left(r_{12} - r_{17} + r_{24} + \iota(r_{22} - r_{27} - r_{14})\right), \\
\varrho_{21} &= \frac{1}{8}\left(3r_{11} + 3r_{16} + r_{23} + 3r_{26} + \iota(3r_{21} + r_{25} - r_{15} - 3r_{16})\right).
\end{aligned} \tag{3.19}$$

Finally, incorporating (3.19) into (3.17) if one gets $\chi \neq 0$ as $(\rho, h, \gamma) \in \mathcal{N}|_{E_{xy}^+\left(\gamma+\rho, \frac{(\rho+\gamma)^2}{\gamma}\right)}$ then for interior equilibrium solution the activator-inhibitor system (1.3) undergoes Neimark-Sacker bifurcation. Further supercritical (subcritical) Neimark-Sacker bifurcation occurs if $\chi < 0$ ($\chi > 0$). □

### *3.2. Flip bifurcation of $E_{xy}^+\left(\rho + \gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$*

If $\Delta = \left(\frac{(\rho+\gamma)(2-h-h\gamma)}{\rho+\gamma-h\gamma}\right)^2 - 4\left(\frac{\rho(h-1)(h\gamma-1)+\gamma(1-h\gamma+h^2\gamma)}{\rho+\gamma-h\gamma}\right) > 0$, then, from (2.15) and (2.7) one gets $\lambda_1|_{(2.15)} = -1$ but $\lambda_2|_{(2.15)} = \frac{-2\rho\gamma-2\gamma^2+\rho-\gamma}{\rho-\gamma} \neq 1$ or $-1$. This implies that the activator-inhibitor system (1.3) may undergo flip bifurcation if $(\gamma, h, \rho)$ are located in the set

$$\mathcal{F}|_{E_{xy}^+\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)} = \{(\gamma, h, \rho), h = 2\}. \tag{3.20}$$

But, the following theorem guarantees that flip bifurcation will occur for $E_{xy}^+\left(\rho + \gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of the activator-inhibitor system (1.3).

**Theorem 3.2.** If $(\gamma, h, \rho) \in \mathcal{F}|_{E_{xy}^+\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)}$, then for $E_{xy}^+\left(\rho + \gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$, the activator-inhibitor system (1.3) undergoes the flip bifurcation.

*Proof.* If $h$ varies in a small neighborhood of $h^*$, i.e., $h = h^* + \epsilon$, then the activator-inhibitor system (1.3) takes the form (3.2). Further, the activator-inhibitor system (1.3) takes the form

$$\begin{aligned}
u_{t+1} =& \widehat{\alpha_{11}} u_t + \widehat{\alpha_{12}} v_t + \widehat{\alpha_{13}} u_t^2 + \widehat{\alpha_{14}} u_t v_t + \widehat{\alpha_{15}} v_t^2 + \gamma_{01} u_t \epsilon + \gamma_{02} v_t \epsilon + \gamma_{03} u_t^2 \epsilon + \\
& \gamma_{04} u_t v_t \epsilon + \gamma_{05} v_t^2 \epsilon, \\
v_{t+1} =& \widehat{\alpha_{21}} u_t + \widehat{\alpha_{22}} v_t + \widehat{\alpha_{23}} u_t^2 + \widehat{\alpha_{24}} u_t v_t + \widehat{\alpha_{25}} v_t^2 + \gamma_{06} u_t \epsilon + \gamma_{07} v_t \epsilon + \gamma_{08} u_t^2 + \\
& \gamma_{09} u_t v_t \epsilon + \gamma_{10} v_t^2 \epsilon,
\end{aligned} \tag{3.21}$$

where

$$\widehat{\alpha_{11}} = \frac{y^{*2} - h^*y^{*2} + h^{*2}\rho y^*}{(y^* - h^*x^*)^2},\ \widehat{\alpha_{12}} = \frac{h^*x^*(h^*x^* - x^* - h^*\rho)}{(y^* - h^*x^*)^2},$$
$$\widehat{\alpha_{13}} = \frac{h^*y^{*2} - h^{*2}y^{*2} + h^{*3}\rho y^*}{(y^* - h^*x^*)^3},\ \widehat{\alpha_{14}} = \frac{-h^{*2}\rho y^* - 2h^*x^*y^* + 2h^{*2}x^*y^* - h^{*3}\rho x^*}{(y^* - h^*x^*)^3},$$
$$\widehat{\alpha_{15}} = \frac{h^*x^{*2} - h^{*2}x^{*2} + h^{*2}\rho x^*}{(y^* - h^*x^*)^3},$$
$$\gamma_{01} = \frac{2x^*y^{*2}}{h^{*3}} - \frac{4x^*y^{*2}}{h^{*2}} - \frac{y^{*3}}{h^{*3}} + \frac{6\rho x^*y^*}{h^*} + \frac{2\rho y^{*2}}{h^{*2}},$$
$$\gamma_{02} = -\frac{6\rho x^{*2}}{h^*} - \frac{4x^{*3}}{h^{*2}} + \frac{6x^{*3}}{h^*} - \frac{2\rho x^*y^*}{h^{*2}} - \frac{x^{*2}y^*}{h^{*3}} - \frac{2x^{*2}y^*}{h^{*2}},$$
$$\gamma_{03} = \frac{12\rho x^*y^*}{h^*} + \frac{3\rho y^{*2}}{h^{*2}} + \frac{6x^*y^{*2}}{h^{*3}} - \frac{9x^*y^{*2}}{h^{*2}} + \frac{y^{*3}}{h^{*4}} - \frac{2y^{*3}}{h^{*3}},$$
$$\gamma_{04} = -\frac{12\rho x^{*2}}{h^*} - \frac{12x^*y^*}{h^{*2}} - \frac{12x^{*2}y^*}{h^{*3}} + \frac{18x^{*2}y^*}{h^{*2}} - \frac{2\rho y^{*2}}{h^{*3}} - \frac{2x^*y^{*2}}{h^{*4}} + \frac{4x^*y^{*2}}{h^{*3}},$$
$$\gamma_{05} = \frac{9\rho x^{*2}}{h^{*2}} + \frac{6x^{*3}}{h^{*3}} - \frac{9x^{*3}}{h^{*2}} + \frac{2\rho x^*y^*}{h^{*3}} + \frac{x^{*2}y^*}{h^{*4}} - \frac{2x^{*2}y^*}{h^{*3}},$$
$$\widehat{\alpha_{21}} = \frac{h^*y^*(2x^*y^* - 2h^*x^*y^* + h^*\rho y^* - h^*x^{*2} + h^{*2}x^{*2})}{(y^* - h^*x^*)^2}, \tag{3.22}$$
$$\widehat{\alpha_{22}} = \frac{y^{*2} - h^*\gamma y^{*2} - 2hx^*y^* + 2h^{*2}\gamma x^*y^* + h^{*2}x^{*2} - h^{*3}\gamma x^{*2}}{(y^* - h^*x^*)^2} + \frac{h^{*3}x^{*3} - h^{*2}x^{*3} - h^{*2}x^{*3} - h^{*3}\rho x^{*2}}{(y^* - h^*x^*)^2},$$
$$\widehat{\alpha_{23}} = \frac{h^*y^{*3} - h^{*2}y^{*3} + h^{*3}\rho y^{*2}}{(y^* - h^*x^*)^3},$$
$$\widehat{\alpha_{24}} = \frac{-3h^{*2}x^{*2}y^* + 3h^{*3}x^{*2}y * -2h^{*3}\rho x^*y^* + h^{*3}x^3 - h^{*4}x^3}{(y^* - h^*x^*)^3},$$
$$\widehat{\alpha_{25}} = \frac{h^{*2}x^{*3} - h^{*3}x^{*3} + h^{*3}\rho x^{*2}}{(y^* - h^*x^*)^3},$$
$$\gamma_{06} = 8x^{*3}y^* - \frac{6x^{*3}y^*}{h^*} + \frac{6\rho x^*y^{*2}}{h^*} + \frac{6x^{*2}y^{*2}}{h^{*2}} - \frac{9x^{*2}y^{*2}}{h^*} + \frac{2\rho y^{*3}}{h^{*2}} + \frac{2x^*y^{*3}}{h^{*3}} - \frac{4x^*y^{*3}}{h^{*2}},$$
$$\gamma_{07} = \frac{6x^{*3}}{h^*} - 8\rho x^{*3} - 8\gamma x^{*3} + 8x^{*4} - \frac{6x^{*4}}{h^*2} - \frac{6x^{*2}y^*}{h^{*2}} - \frac{3\rho x^{*2}y^*3}{h^*} + \frac{9\gamma x^{*2}y^*}{h^*} - \frac{2x^{*3}y^*}{h^{*2}} + \frac{3x^{*3}y^*}{h^*} - \frac{\gamma y^{*2}}{h^{*3}},$$

$$\gamma_{08} = \frac{12\rho x^*y^{*2}}{h^*} + \frac{3\rho y^2}{h^{*2}} + \frac{6x^*y^{*3}}{h^{*3}} - \frac{9x^*y^{*3}}{h^{*2}} + \frac{y^{*4}}{h^{*4}} - \frac{2y^{*4}}{h^{*3}},$$
$$\gamma_{09} = -15x^{*4} + \frac{12x^{*4}}{h^*} + \frac{24\rho x^2y^*}{h^*} - \frac{24x^{*3}y^*}{h^{*2}} + \frac{32x^{*3}y^*}{h^*} - \frac{6\rho x^*y^{*2}}{h^{*2}} - \frac{6x^{*2}y^{*2}}{h^{*3}} + \frac{9x^{*2}y^{*2}}{h^{*2}}, \tag{3.23}$$
$$\gamma_{10} = \frac{12\rho x^{*3}}{h^*} + \frac{9x^4}{h^{*2}} - \frac{12x^{*4}}{h^*} + \frac{3\rho x^{*2}y^*}{h^{*2}} + \frac{2x^{*3}y^*}{h^{*3}} - \frac{3x^{*3}y^*}{h^{*2}},$$

by using the transformation shown in (3.8). Again, it is noted that (3.21) becomes

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} \widehat{X}(u_t, v_t, \epsilon) \\ \widehat{Y}(u_t, v_t, \epsilon) \end{pmatrix}, \tag{3.24}$$

where

$$\begin{aligned}
\widehat{X} =& \frac{\widehat{\alpha_{13}}\left(\lambda_2 - \widehat{\alpha_{11}}\right) - \widehat{\alpha_{12}}\widehat{\alpha_{23}}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} u_t^2 + \frac{\widehat{\alpha_{14}}\left(\lambda_2 - \widehat{\alpha_{11}}\right) - \widehat{\alpha_{12}}\widehat{\alpha_{24}}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} u_t v_t \\
&+ \frac{\widehat{\alpha_{15}}\left(\lambda_2 - \widehat{\alpha_{11}}\right) - \widehat{\alpha_{12}}\widehat{\alpha_{25}}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} v_t^2 + \frac{\gamma_{01}\left(\lambda_2 - \widehat{\alpha_{11}}\right) - \widehat{\alpha_{12}}\gamma_{06}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} u_t \epsilon \\
&+ \frac{\gamma_{02}\left(\lambda_2 - \widehat{\alpha_{11}}\right) - \widehat{\alpha_{12}}\gamma_{07}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} v_t \epsilon + \frac{\gamma_{03}\left(\lambda_2 - \widehat{\alpha_{11}}\right) - \widehat{\alpha_{12}}\gamma_{08}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} u_t^2 \epsilon \\
&+ \frac{\gamma_{04}\left(\lambda_2 - \widehat{\alpha_{11}}\right) - \widehat{\alpha_{12}}\gamma_{09}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} u_t v_t \epsilon + \frac{\gamma_{05}\left(\lambda_2 - \widehat{\alpha_{11}}\right) - \widehat{\alpha_{12}}\gamma_{10}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} v_t^2 \epsilon, \\
\widehat{Y} =& \frac{\widehat{\alpha_{13}}\left(1 + \widehat{\alpha_{11}}\right) + \widehat{\alpha_{12}}\widehat{\alpha_{23}}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} u_t^2 + \frac{\widehat{\alpha_{14}}\left(1 + \widehat{\alpha_{11}}\right) + \widehat{\alpha_{12}}\widehat{\alpha_{24}}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} u_t v_t \\
&+ \frac{\widehat{\alpha_{15}}\left(1 + \widehat{\alpha_{11}}\right) + \widehat{\alpha_{12}}\widehat{\alpha_{25}}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} v_t^2 + \frac{\gamma_{01}\left(1 + \widehat{\alpha_{11}}\right) + \widehat{\alpha_{12}}\gamma_{06}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} u_t \epsilon \\
&+ \frac{\gamma_{02}\left(1 + \widehat{\alpha_{11}}\right) + \widehat{\alpha_{12}}\gamma_{07}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} v_t \epsilon + \frac{\gamma_{03}\left(1 + \widehat{\alpha_{11}}\right) + \widehat{\alpha_{12}}\gamma_{08}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} u_t^2 \epsilon \\
&+ \frac{\gamma_{05}\left(1 + \widehat{\alpha_{11}}\right) + \widehat{\alpha_{12}}\gamma_{10}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} v_t^2 \epsilon + \frac{\gamma_{04}\left(1 + \widehat{\alpha_{11}}\right) + \widehat{\alpha_{12}}\gamma_{9}}{\widehat{\alpha_{12}}\left(1+\lambda_2\right)} u_t v_t \epsilon, \qquad (3.25) \\
u_t =& \widehat{\alpha_{12}} x_t + \widehat{\alpha_{12}} y_t, \\
v_t =& -(1 + \widehat{\alpha_{11}}) x_t + \left(\lambda_2 - \widehat{\alpha_{11}}\right) y_t, \\
u_t^2 =& \widehat{\alpha_{12}}^2 \left(x_t^2 + 2 x_t y_t + y_t^2\right), \\
v_t^2 =& (1 + \widehat{\alpha_{11}})^2 x_n^2 + \left(\lambda_2 - \widehat{\alpha_{11}}\right)^2 x_t^2 - 2\left(1 + \widehat{\alpha_{11}}\right)\left(\lambda_2 - \widehat{\alpha_{11}}\right) x_t y_t, \\
u_t v_t =& -\widehat{\alpha_{12}}\left(1 + \widehat{\alpha_{11}}\right) x_t^2 + \left(\widehat{\alpha_{12}}\left(\lambda_2 - \widehat{\alpha_{11}}\right) - \widehat{\alpha_{12}}\left(1 + \widehat{\alpha_{11}}\right)\right) x_t y_t \\
&+ \widehat{\alpha_{12}}\left(\lambda_2 - \widehat{\alpha_{11}}\right) y_t^2, \\
u_t \epsilon =& \widehat{\alpha_{12}} x_t \epsilon + \widehat{\alpha_{12}} y_t \epsilon, \\
v_t \epsilon =& -(1 + \widehat{\alpha_{11}}) x_t \epsilon + \left(\lambda_1 - \widehat{\alpha_{11}}\right) y_t \epsilon, \\
v_t^2 \epsilon =& (1 + \widehat{\alpha_{11}})^2 x_t^2 \epsilon + \left(\lambda_1 - \widehat{\alpha_{11}}\right)^2 y_t^2 \epsilon - 2\left(1 + \widehat{\alpha_{11}}\right)\left(\lambda_2 - \widehat{\alpha_{11}}\right) x_t y_t \epsilon, \\
u_t^2 \epsilon =& \epsilon \widehat{\alpha_{12}}^2 \left(x_t^2 + 2 x_t y_t + y_t^2\right), \\
u_t v_t \epsilon =& -\widehat{\alpha_{12}}\left(1 + \widehat{\alpha_{11}}\right) x_t^2 \epsilon + \left(\widehat{\alpha_{12}}\left(\lambda_1 - \widehat{\alpha_{11}}\right) - \widehat{\alpha_{12}}\left(1 + \widehat{\alpha_{11}}\right)\right) x_t y_t \epsilon \\
&+ \widehat{\alpha_{12}}\left(\lambda_1 - \widehat{\alpha_{11}}\right) y_t^2 \epsilon,
\end{aligned}$$

as obtained by using

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} := \begin{pmatrix} \widehat{\alpha_{12}} & \widehat{\alpha_{12}} \\ -1 - \widehat{\alpha_{11}} & \lambda_2 - \widehat{\alpha_{11}} \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}. \tag{3.26}$$

Hereafter, for (3.24), the center manifold $M^c E_{00}(0,0)$ around $E_{00}(0,0)$ is explored in a neighborhood of $\epsilon$; hence, one can write $M^c E_{00}(0,0)$ as the following mathematical expression

$$M^c E_{00}(0,0) = \left\{(x_t, y_t) : y_t = A_0 \epsilon + A_1 x_t^2 + A_2 x_t \epsilon + A_3 \epsilon^3 + O\left((|x_t| + |\epsilon|)^3\right)\right\}. \tag{3.27}$$

The computation yields

$$
\begin{aligned}
A_0 &= 0,\\
A_1 &= \frac{1}{1-\lambda_2^2}\Big(\widehat{\alpha_{12}}\widehat{\alpha_{13}}(1+\widehat{\alpha 11})+\widehat{\alpha_{23}}\widehat{\alpha_{12}^2}-\widehat{\alpha 14}(1+\widehat{\alpha 11})^2\Big)\\
&\quad+\frac{\widehat{\alpha_{15}}(1+\widehat{\alpha_{11}})^3+\widehat{\alpha_{12}}\widehat{\alpha_{25}}(1+\widehat{\alpha 11})^2}{\widehat{\alpha_{12}}(1-\lambda_2^2)},\\
A_2 &= \frac{1}{1-\lambda_2^2}\Big(\frac{2\widehat{\alpha_{12}}^2 c_0(\widehat{\alpha_{13}}(1+\widehat{\alpha_{11}})+\widehat{\alpha_{23}}\widehat{\alpha_{12}})}{\widehat{\alpha_{12}}}+c_0(\lambda_2-\widehat{\alpha_{11}})\\
&\qquad-(1+\widehat{\alpha_{11}})(\widehat{\alpha_{14}}(1+\widehat{\alpha_{11}})+\widehat{\alpha_{24}}\widehat{\alpha_{12}}))\\
&\qquad-\frac{2c_0(1+\widehat{\alpha_{11}})(\lambda_2-\widehat{\alpha_{11}})(\widehat{\alpha_{15}}(1+\widehat{\alpha_{11}})+\widehat{\alpha_{25}}\widehat{\alpha_{12}})}{\widehat{\alpha_{12}}}\\
&\qquad+\frac{\gamma_{01}\widehat{\alpha_{12}}(1+\widehat{\alpha_{11}})+\gamma_{06}\widehat{\alpha_{12}}^2}{\widehat{\alpha_{12}}}-\frac{\gamma_{02}(1+\widehat{\alpha_{11}})^2+\gamma_{07}\widehat{\alpha_{12}}(1+\widehat{\alpha_{11}})}{\widehat{\alpha_{12}}}\Big),\\
A_3 &= 0.
\end{aligned}
\tag{3.28}
$$

Finally, we will express (3.24), restricted to $M^cE_{00}(0,0)$ as follows

$$
f(x_t) = -x_t + p_1x_t^2 + p_2x_t\epsilon + p_3x_t^2\epsilon + p_4x_t\epsilon^2 + p_5x_t^3 + O\left((|x_t|+|\epsilon|)^4\right), \tag{3.29}
$$

where

$$
\begin{aligned}
p_1 &= \frac{1}{1+\lambda_2}\Big[(\lambda_2-\widehat{\alpha_{11}})\widehat{\alpha_{13}}\widehat{\alpha_{12}}-\widehat{\alpha_{23}}\widehat{\alpha_{12}}^2-\widehat{\alpha_{14}}(\lambda_2-\widehat{\alpha_{11}})(1+\widehat{\alpha_{11}})\\
&\qquad+\widehat{\alpha_{12}}\widehat{\alpha_{24}}(1+\widehat{\alpha_{11}})+\frac{\widehat{\alpha_{15}}(\lambda_2-\widehat{\alpha_{11}})(1+\widehat{\alpha_{11}})^2-\widehat{\alpha_{12}}\widehat{\alpha_{25}}(1+\widehat{\alpha_{11}})^2}{\widehat{\alpha_{12}}}\Big],\\
p_2 &= \frac{1}{(1+\lambda_2)}\left[(\gamma_{01}(\lambda_2-\widehat{\alpha_{11}})-\gamma_{06}\widehat{\alpha_{12}})-\frac{(1+\widehat{\alpha_{11}})(\gamma_{02}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\gamma_{07})}{\widehat{\alpha_{12}}}\right],\\
p_3 &= \frac{1}{\widehat{\alpha_{12}}(1+\lambda_2)}\Big[2C_2\widehat{\alpha_{12}}^2(\widehat{\alpha_{13}}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\widehat{\alpha_{23}})\\
&\qquad+C_2\widehat{\alpha_{12}}((\lambda_2-\widehat{\alpha_{11}})-(1+\widehat{\alpha_{11}}))(\widehat{\alpha_{15}}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\widehat{\alpha_{25}}))\\
&\qquad-2C_2\widehat{\alpha_{15}}(\lambda_2-\widehat{\alpha_{11}})^2(1+\widehat{\alpha_{11}})+2C_2\widehat{\alpha_{25}}\widehat{\alpha_{12}}(\lambda_2-\widehat{\alpha_{11}})(1+\widehat{\alpha_{11}})\\
&\qquad+(\gamma_{01}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\gamma_{06})C_1\widehat{\alpha_{12}}\\
&\qquad+C_1(\lambda_2-\widehat{\alpha_{11}})(\gamma_{02}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\gamma_{07})\\
&\qquad+(\gamma_{03}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\gamma_{08})\widehat{\alpha_{12}}^2\\
&\qquad-((1+\widehat{\alpha_{11}})\widehat{\alpha_{12}})(\lambda_2-\widehat{\alpha_{11}})\gamma_{04}-\gamma_{09}\widehat{\alpha_{12}})\\
&\qquad+\gamma_{05}(\lambda_2-\widehat{\alpha_{11}})(1+\widehat{\alpha_{11}})^2-\widehat{\alpha_{12}}(1+\widehat{\alpha_{11}})^2\gamma_{10}\Big],\\
p_4 &= \frac{C_2}{(1+\lambda_2)}\left[(\gamma_{01}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\gamma_{06})+\frac{(\lambda_2-\widehat{\alpha_{11}})(\gamma_{02}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\gamma_{07})}{\widehat{\alpha_{12}}}\right],\\
p_5 &= \frac{C_1}{(1+\lambda_1)}\left[2\widehat{\alpha_{12}}(\widehat{\alpha_{13}}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\widehat{\alpha_{23}})+(\widehat{\alpha_{14}}(\lambda_2-\widehat{\alpha_{11}})-\widehat{\alpha_{12}}\widehat{\alpha_{24}})\right.
\end{aligned}
\tag{3.30}
$$

$$\times\ ((\lambda_2 - \widehat{\alpha_{11}}) - (1 + \widehat{\alpha_{11}})) - \frac{2\widehat{\alpha_{15}}(1 + \widehat{\alpha_{11}})(\lambda_2 - \widehat{\alpha_{11}})^2\widehat{\alpha_{15}} - \widehat{\alpha_{12}}\widehat{\alpha_{25}}}{\widehat{\alpha_{12}}}\Bigg].$$

Now, in order for the map (3.29) to undergo flip bifurcation it should be required that $\Gamma_1$ and $\Gamma_2$ are non-zero

$$\begin{aligned}\Gamma_1 &= \left(\frac{\partial^2 f}{\partial x_t \partial \epsilon} + \frac{1}{2}\frac{\partial f}{\partial \epsilon}\frac{\partial^2 f}{\partial x_t^2}\right)|_{E_{00}(0,0)},\\ \Gamma_2 &= \left(\frac{1}{6}\frac{\partial^3 f}{\partial x_t^3} + \left(\frac{1}{2}\frac{\partial^2 f}{\partial x_t^2}\right)^2\right)|_{E_{00}(0,0)}.\end{aligned} \tag{3.31}$$

Therefore the calculation yields

$$\Gamma_1 \ = \tfrac{-1}{8}(\rho-\gamma)^2\gamma^3(\rho+\gamma)^3(\rho+7\gamma) + 4\rho\gamma - 4\rho\gamma^2 - 4\gamma^3 + \tfrac{1}{4}\gamma^2(\rho+\gamma)^4(\rho^2+4\rho\gamma-5\gamma^2), \tag{3.32}$$

and

$$\Gamma_2 = \frac{16\gamma^7(\gamma^2+\gamma-1)(\rho+\gamma)}{(\rho-\gamma)^4(\rho+\gamma)^4(\gamma^2+\gamma-\rho+\rho\gamma)}. \tag{3.33}$$

Finally, from (3.33), if $\Gamma_2 \neq 0$ as $(\gamma, h, \rho) \in \mathcal{F}|_{E^+_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)}$, then for $E^+_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$, the activator-inhibitor system (1.3) undergoes flip bifurcation. Further if $\Gamma_2 > 0$ ($\Gamma_2 < 0$) then the period-2 points bifurcating from $E^+_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ are stable (unstable). □

## 4. Chaos control

Now, the state feedback control method is utilized to stabilize chaos at the state of unstable trajectories by adding $u_t$ as a control force to the discrete-time activator-inhibitor system [31,32]

$$\begin{aligned}x_{t+1} &= \frac{((1-h)x_t + h\rho)y_t}{y_t - hx_t} + u_t,\\ y_{t+1} &= \frac{y_t((y_t - hx_t)(1-h\gamma) + hx_t((1-h)x_t + h\rho))}{y_t - hx_t},\end{aligned} \tag{4.1}$$

where $u_t = -k_1(x_t - x) - k_2(y_t - y)$, $k_1$ and $k_2$ denote feedback gains, $x = \rho + \gamma$ and $y = \frac{(\rho+\gamma)^2}{\gamma}$. Now, for the control system (4.1), the variational matrix $\Omega^C|_{E_{xy}(x,y)}$ takes the following form

$$\Omega^C|_{E_{xy}(x,y)} = \begin{pmatrix} \ell_{11} - k_1 & \ell_{12} - k_2 \\ \ell_{21} & \ell_{22} \end{pmatrix}, \tag{4.2}$$

where

$$\begin{aligned}\ell_{11} &= \frac{\gamma+\rho-h\rho}{\gamma+\rho-h\gamma},\ \ell_{12} = -\frac{h\gamma^2}{(\rho+\gamma)(\gamma+\rho-h\gamma)},\\ \ell_{21} &= \frac{h(2-h)(\rho+\gamma)^2}{\gamma+\rho-h\gamma},\ \ell_{22} = \frac{\rho-h\rho\gamma-\gamma(h-1+h\gamma)}{\gamma+\rho-h\gamma}.\end{aligned} \tag{4.3}$$

Now, if the roots of the characteristic equation of $\Omega^C|_{E_{xy}(x,y)}$ are $\lambda_{1,2}$ then

$$\lambda_1 + \lambda_2 = \ell_{11} + \ell_{22} - k_1, \tag{4.4}$$

$$\lambda_1\lambda_2 = \ell_{22}(\ell_{11} - k_1) - \ell_{21}(\ell_{12} - k_2). \tag{4.5}$$

Now, it is noted here that marginal stability is determined from the restrictions $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$, which give the fact that $|\lambda_{1,2}| < 1$. If $\lambda_1\lambda_2 = 1$ then from (4.5), we get

$$\begin{aligned} L_1 : (\rho - h\rho\gamma - \gamma(h - 1 + h\gamma))(\rho + \gamma - h\rho - k_1(\rho + \gamma - h\gamma)) - ((2h - h^2)(\rho + \gamma)) \\ \times(-h\gamma^2 - k_2(\rho + \gamma)(\rho + \gamma - h\gamma)) - (\rho + \gamma - h\gamma)^2 = 0. \end{aligned} \tag{4.6}$$

If $\lambda_1 = 1$ then from (4.4) and (4.5) we get:

$$\begin{aligned} L_2 : (h\rho\gamma + h\gamma^2)k_1 + ((2h - h^2)(\rho + \gamma)^2)k_2 + (\rho + \gamma)(h - 2 + h\gamma) + (h\rho - \rho)(h\gamma - 1) \\ +\gamma(1 - h\gamma + h^2\gamma) - (\rho + \gamma - h\gamma) = 0. \end{aligned} \tag{4.7}$$

Finally if $\lambda_1 = -1$ then from (4.4) and (4.5) one has

$$\begin{aligned} L_3 : k_1(2\rho + 2\gamma - 2h\gamma - h\rho\gamma - h\gamma^2) - k_2(2h - h^2)(\rho + \gamma)^2) - ((h\rho - \rho)(h\gamma - 1) \\ +\gamma(1 - h\gamma + h^2\gamma)) - (3\rho + 3\gamma - 2h\gamma - h\rho - \rho h\gamma - h\gamma^2) = 0. \end{aligned} \tag{4.8}$$

Therefore from (4.6)–(4.8), the lines $L_1$–$L_3$ in the $(k_1, k_2)$-plane gives the triangular region, which further gives the fact that $|\lambda_{1,2}| < 1$.

## 5. Numerical simulations

The main results are numerically verified in this section for fixing suitable values of the involved parameters. Here, the following two cases are to be considered for the completeness of this section.

**Case 1:** Let $\gamma = 0.37$ and $\rho = 0.45 > \frac{\gamma-\gamma^2}{1+\gamma} = 0.17014598540145986$ then from (2.12) one gets $h = 1.2636783124588005$. So, Lemma 2.1 implies that $E^+_{xy}\left(\rho + \gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of the activator-inhibitor system (1.3) is a stable focus if $h < 1.2636783124588005$ and has exchange stability if $h = 1.2636783124588005$; and meanwhile it is an unstable focus if $h > 1.2636783124588005$. In order to show this fact deeply, if $h = 0.5 < 1.2636783124588005$ then Figure 1a shows that $E^+_{xy}(0.82, 1.8172972972972976)$ of the discrete activator-inhibitor system (1.3) is a stable focus. Moreover, Figure 1b–1f also show that the interior equilibrium solution $E^+_{xy}(0.83, 1.8128947368421056)$ of the discrete activator-inhibitor system (1.3) is a stable focus if $h = 0.8, 1.23, 1.254, 1.261, 1.263 < 1.2636783124588005$. On the other hand, if $h = 1.27 > 1.2636783124588005$ then Figure 2a shows that $E^+_{xy}(0.82, 1.8172972972972976)$ of the discrete activator-inhibitor system (1.3) changes behavior, i.e., it is an unstable focus and as a consequence stable curves appear. Now, numerically we have to show that, if $\gamma = 1.27 > 1.2636783124588005$ then the discrete system (1.3) undergoes supercritical Neimark-Sacker bifurcation, that is, from (3.17) the discriminator quantity $\chi < 0$. So, if $\gamma = 1.27$ then from (3.5) one gets $\frac{d|\lambda_{1,2}|}{d\epsilon}|_{\epsilon=0} = -0.009311101383197325 \neq 0$, i.e., the non-degenerate condition holds. Moreover, from (3.3) and (3.19) one gets

$$\lambda_{1,2} = 0.30460154241645243 \pm 0.9561252598212974\iota, \tag{5.1}$$

and

$$\begin{aligned}
\varrho_{02} &= 1.5733628161382742 + 0.7852920800715116\iota,\\
\varrho_{11} &= -0.5900461579192877 + 0.7144295009123655\iota,\\
\varrho_{20} &= 1.5733628161382742 + 0.20701539785842238\iota,\\
\varrho_{21} &= -1.1693108796562233 + 0.5003042167226045\iota.
\end{aligned} \tag{5.2}$$

Using (5.1) and (5.2) in (3.17) one gets $\chi = -2.0038147757104667 < 0$, which confirms that our theoretical results are mathematically correct and, hence, that the activator-inhibitor system (1.3) undergoes a supercritical Neimark-Sacker bifurcation. Similarly, for others chosen bifurcation values $h = 1.28,\ 1.29,\ 1.3,\ 1.34,\ 1.343 > 1.2636783124588005$ Figure 2b–2f indicate that stable curves appear and therefore, the discrete activator-inhibitor system (1.3) undergoes a supercritical Neimark-Sacker bifurcation, particularly for said bifurcation values: $\chi < 0$ (see Table 1). Finally, bifurcation diagrams are presented in Figure 3, and the Maximum Lyapunov exponent corresponding to Figure 3 are drawn in Figure 4.

**Case 2:** If $\gamma = 2.31$ then from (2.13) one gets $0 < h < \min\{2, 0.8658008658008658\}$. From theoretical discussion, if $\rho = 1.9$ then the interior fixed point $E^+_{xy}(3.5, 7.65625)$ of the discrete activator-inhibitor system (1.3) is a stable node if $0 < h < \min\{2, 0.8658008658008658\}$, and an unstable node if $h > \max\{2, 0.8658008658008658\}$ according to (2.14). Moreover, from (2.15) if $h = 2$, then the discrete activator-inhibitor system (1.3) changes its stability and in fact, the model undergoes flip bifurcation. So in this case, the flip bifurcation diagram along with the maximum Lyapunov exponents are plotted and presented in Figures 5 and 6. Finally, the complex dynamics with orbits of period-$11, -13, -14$ and $-15$ are shown if Figure 7.

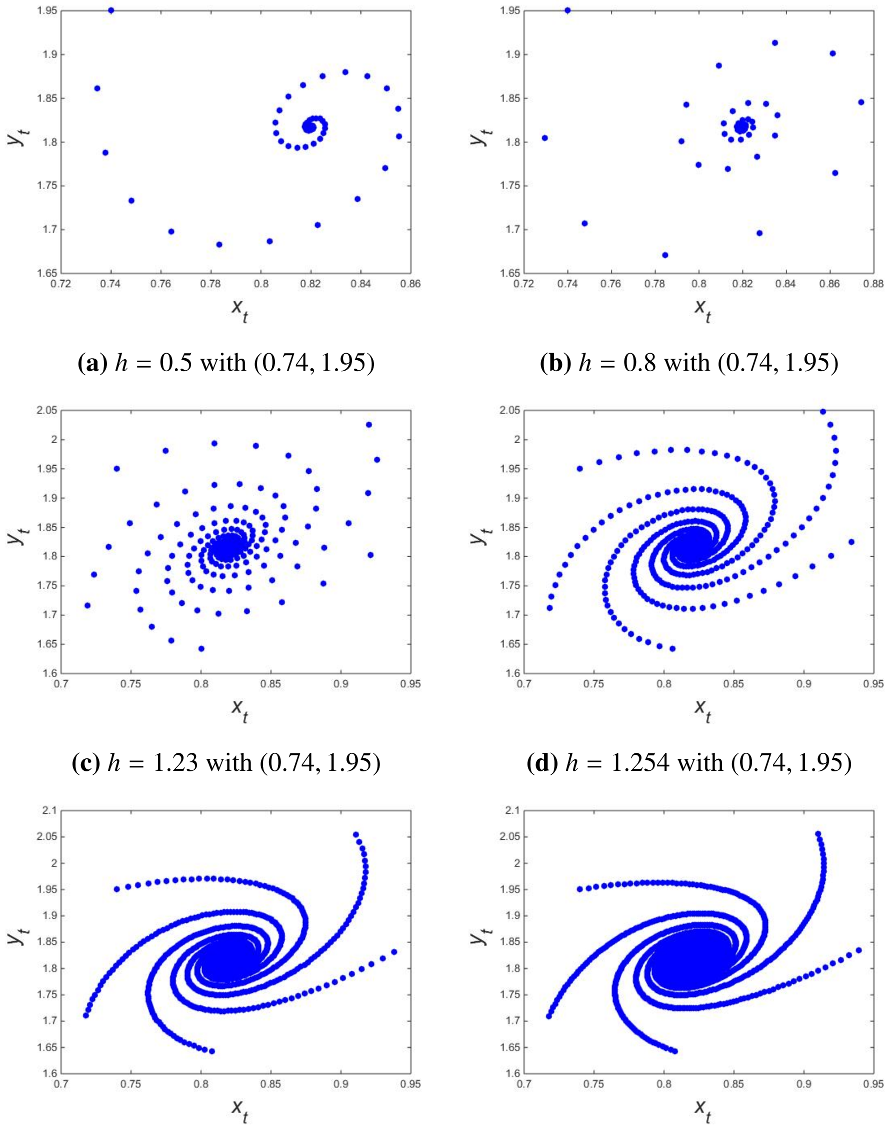
Hereafter, we will prove the validity of the obtained results in Section 4. For instance, if $h = 2.23$, $\gamma = 2.31$ and $\rho = 1.9$, then from (4.6)–(4.8) one gets

$$L_1 : 15.06875009k_1 + 29.28570767099999k_2 + 2.5381501089999965 = 0, \tag{5.3}$$

$$L_2 : 10.697610000000001k_1 + 4.1678999999999995k_2 + 8.429371000000003 = 0, \tag{5.4}$$

$$L_3 : -7.359610000000002k_1 - 17.546859k_2 + 2.049849000000002. \tag{5.5}$$

Hence lines that are presented in (5.3)–(5.5) determine the triangular region that gives $|\lambda_{1,2}| < 1$ (see Figure 8).

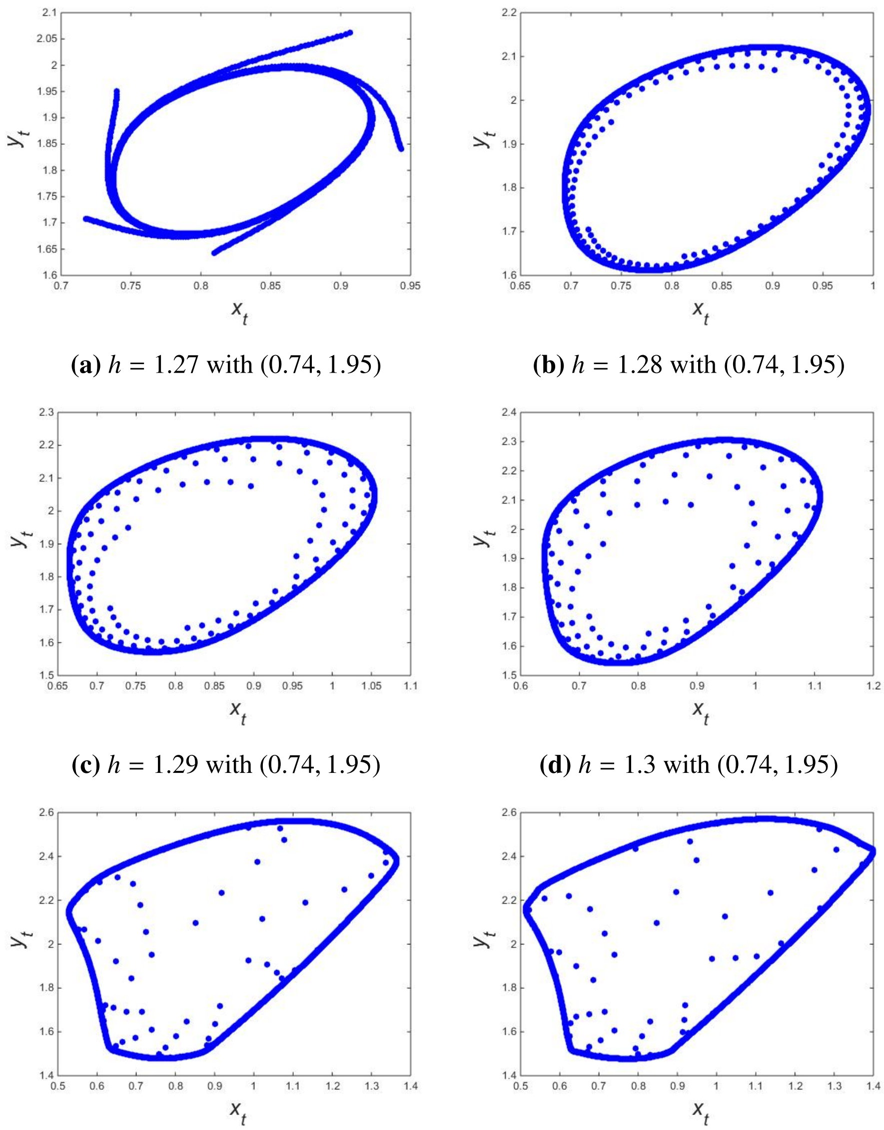


**(a)** $h$ = 0.5 with (0.74, 1.95) **(b)** $h$ = 0.8 with (0.74, 1.95)

**(c)** $h$ = 1.23 with (0.74, 1.95) **(d)** $h$ = 1.254 with (0.74, 1.95)

**(e)** $\gamma$ = 1.261 with (0.74, 1.95) **(f)** $\gamma$ = 1.263 with (0.74, 1.95)

**Figure 1.** Stable focus of the discrete activator-inhibitor system (1.3).

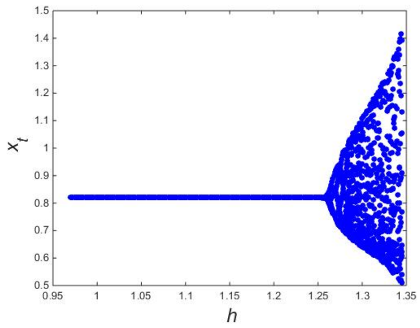



**(a)** $h$ = 1.27 with (0.74, 1.95) **(b)** $h$ = 1.28 with (0.74, 1.95)

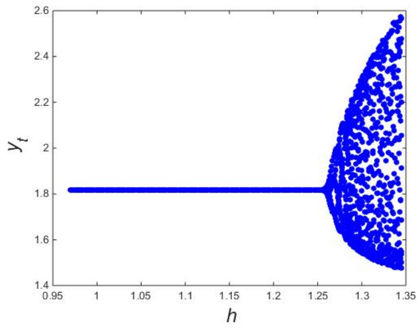
**(c)** $h$ = 1.29 with (0.74, 1.95) **(d)** $h$ = 1.3 with (0.74, 1.95)

**(e)** $h$ = 1.34 with (0.74, 1.95) **(f)** $h$ = 1.343 with (0.74, 1.95)

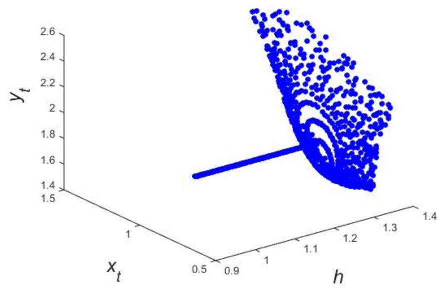
**Figure 2.** Stable closed curves of the discrete activator-inhibitor system (1.3).




(a)




(b)




(c)

**Figure 3.** Neimark-Sacker bifurcation diagrams of the discrete activator-inhibitor system (1.3) with $h \in [0.97, 1.5]$ and the initial value (0.74, 1.95).

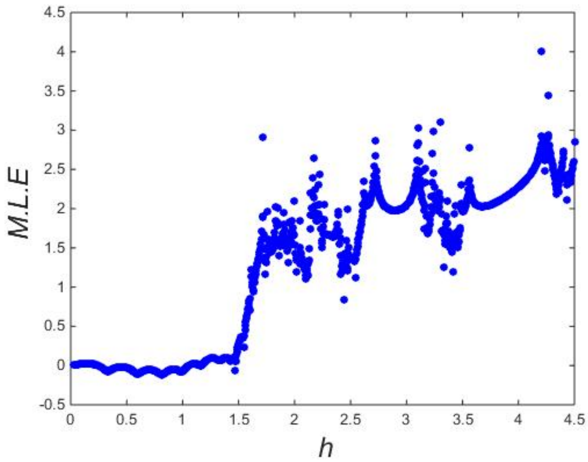



**Figure 4.** Maximum Lyapunov exponents corresponding to Figure 3.

**Table 1.** Corresponding values of $\chi$ for $h > 1.2636783124588005$.

| Different values of $h$ if $h > 1.2636783124588005$ | Corresponding $\chi$ values |
|---|---|
| 1.27 | $-2.0038147757104667 < 0$ |
| 1.28 | $-2.450393091374341 < 0$ |
| 1.29 | $-2.995700277824937 < 0$ |
| 1.3 | $-3.6613654157639792 < 0$ |
| 1.34 | $-8.151946700055621 < 0$ |
| 1.343 | $-8.655495416630451 < 0$ |

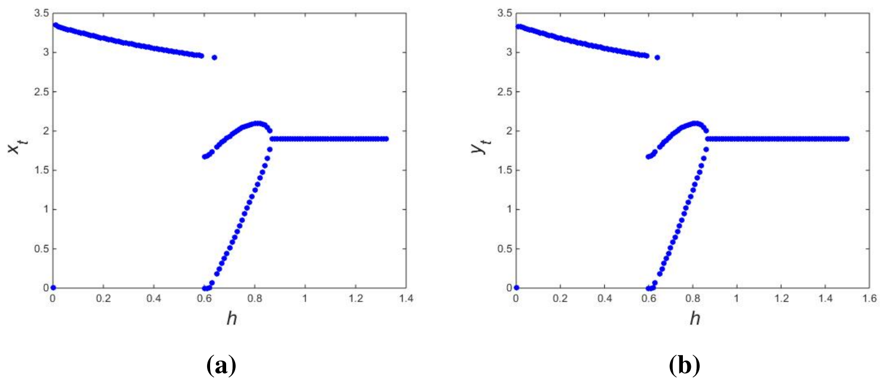



**Figure 5.** Flip bifurcation diagrams of discrete activator-inhibitor system (1.3) with $h \in [0.000001, 2.8]$ and the initial value $(0.0009, 17.9)$.

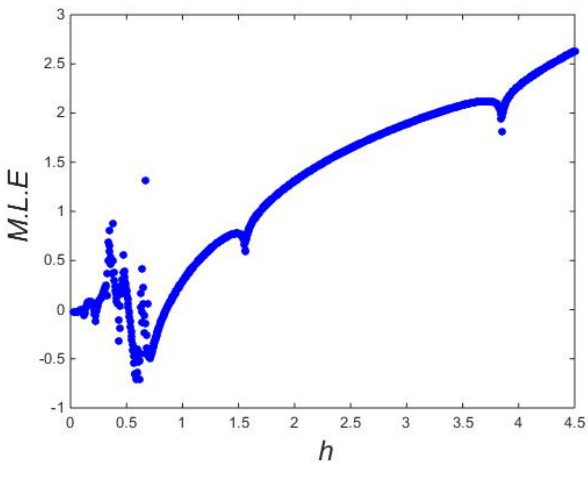
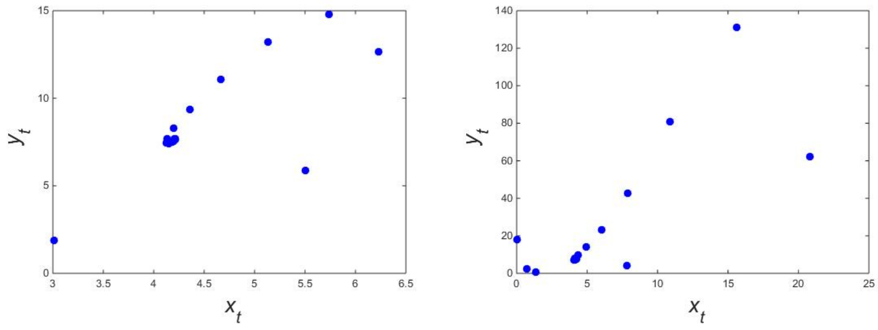
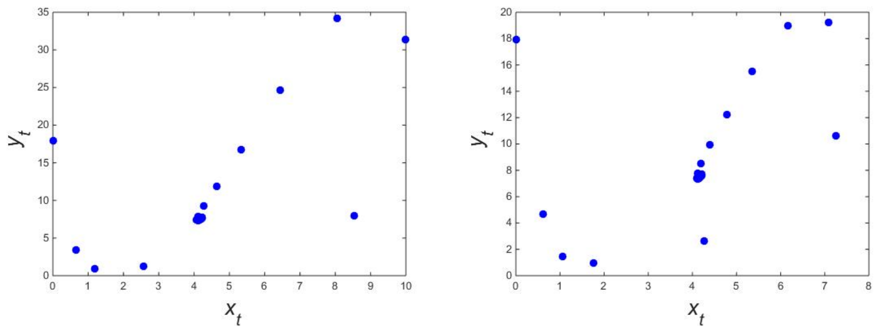


**Figure 6.** Maximum Lyapunov exponents corresponding to Figure 5.



**(a)** $h$ = 2.2 with (0.0009, 17.9) **(b)** $h$ = 2.45 with (0.009, 17.9)



**(c)** $h$ = 2.5 with (0.09, 17.9) **(d)** $h$ = 2.654 with (3.009, 1.9)

**Figure 7.** Complex dynamics of the discrete activator-inhibitor system (1.3).

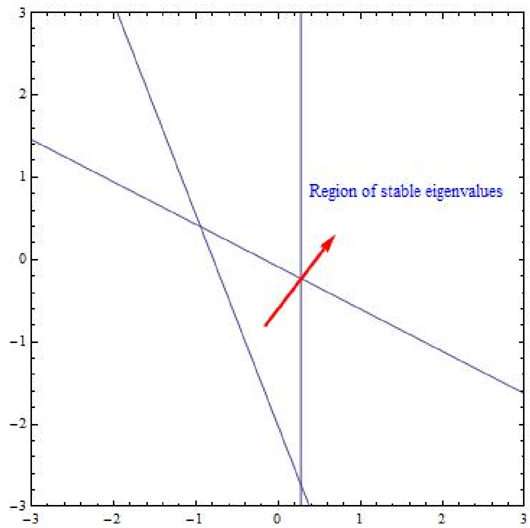



**Figure 8.** Region of stability where $|\lambda_{1,2}| < 1$.

## 6. Conclusions and future work

The work was about the local dynamic characteristics at the interior equilibrium solution, bifurcations and chaos control in the discrete activator-inhibitor system (1.3). We proved that $\forall\ h, \gamma$ and $\rho$, the activator-inhibitor system (1.3) has the interior equilibrium solution $E^+_{xy}\left(\gamma+\rho, \frac{(\rho+\gamma)^2}{\gamma}\right)$. Further, we studied the local stability with different topological classifications for $E^+_{xy}\left(\gamma+\rho, \frac{(\rho+\gamma)^2}{\gamma}\right)$. It was investigated that the interior equilibrium solution $E^+_{xy}\left(\gamma+\rho, \frac{(\rho+\gamma)^2}{\gamma}\right)$ is a stable focus if $0 < h < \frac{\rho+\rho\gamma-\gamma+\gamma^2}{\rho\gamma+\gamma^2}$ with $\rho > \frac{\gamma-\gamma^2}{1+\gamma}$, an unstable focus if (2.10) and $h > \frac{\rho+\rho\gamma-\gamma+\gamma^2}{\rho\gamma+\gamma^2}$ hold, and non-hyperbolic if $h = \frac{\rho+\rho\gamma-\gamma+\gamma^2}{\rho\gamma+\gamma^2}$. Further, it was also proved that the interior equilibrium solution $E^+_{xy}\left(\gamma+\rho, \frac{(\rho+\gamma)^2}{\gamma}\right)$ of system (1.3) is a stable node if $0 < h < \min\left\{2, \frac{2}{\gamma}\right\}$, an unstable node if $h > \max\left\{2, \frac{2}{\gamma}\right\}$ and non hyperbolic if $h = 2$ or $h = \frac{2}{\gamma}$. We have also studied the existence of possible bifurcations of $E^+_{xy}\left(\rho+\gamma, \frac{(\rho+\gamma)^2}{\gamma}\right)$, and proved that for the interior equilibrium solution $E^+_{xy}\left(\gamma+\rho, \frac{(\rho+\gamma)^2}{\gamma}\right)$, the activator-inhibitor system (1.3) undergoes Neimark-Sacker and flip bifurcations if $\mathcal{N}|_{E^+_{xy}\left(\gamma+\rho, \frac{(\rho+\gamma)^2}{\gamma}\right)} = \left\{(\gamma, h, \rho),\ h = \frac{\rho+\rho\gamma-\gamma+\gamma^2}{\rho\gamma+\gamma^2}\right\}$ and $\mathcal{F}|_{E^+_{xy}\left(\gamma+\rho, \frac{(\rho+\gamma)^2}{\gamma}\right)} = \{(\gamma, h, \rho), h = 2\}$, respectively. Biologically, the occurrence of the Neimark-Sacker bifurcation means that there exist periodic or quasiperiodic oscillations between the activator and inhibitor concentrations. Further, the state feedback control method was utilized in order to stabilize the chaos existing in the discrete-time activator-inhibitor system (1.3). Finally, simulations were performed to not only validate the obtained results, but also to show the complex dynamics with orbits of period-11, −13, −14 and −15. Investigation of the global dynamics, calculation of the forbidden set and global bifurcation analysis for the under consideration discrete activator-inhibitor system are our next goals of study.

## Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Larg groups (Project under a grant number RGP.2/47/43/1443).

## Conflict of interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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