Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 4442–4466.
DOI: 10.3934/math.2023221
Received: 27 July 2022
Revised: 13 November 2022
Accepted: 23 November 2022
Published: 05 December 2022

Research article

A new spectral method with inertial technique for solving system of nonlinear monotone equations and applications

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Abstract: Many problems arising from science and engineering are in the form of a system of nonlinear equations. In this work, a new derivative-free inertial-based spectral algorithm for solving the system is proposed. The search direction of the proposed algorithm is defined based on the convex combination of the modified long and short Barzilai and Borwein spectral parameters. Also, an inertial step is introduced into the search direction to enhance its efficiency. The global convergence of the proposed algorithm is described based on the assumption that the mapping under consideration is Lipschitz continuous and monotone. Numerical experiments are performed on some test problems to depict the efficiency of the proposed algorithm in comparison with some existing ones. Subsequently, the proposed algorithm is used on problems arising from robotic motion control.

Keywords: derivative-free method; spectral gradient method; inertial step; nonlinear monotone equations
Mathematics Subject Classification: 90C30, 90C06, 90C56

1. Introduction

Spectral gradient methods are among the widely known first-order methods for unconstrained optimization problems $\min_{m\in\mathbb{R}^n} f(m)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth nonlinear function that is bounded below. These methods generate a sequence of approximations using the iterative formula:

$$m_{k+1} = m_k + \Lambda_k t_k, \tag{1.1}$$

where Λ_k is called the step size and is obtained using some line search procedures, and t_k is called a search direction, defined as

$$t_k := \begin{cases} -g_k, & \text{if } k = 0, \\ -\overline{\gamma}_k g_k, & \text{if } k \geq 1, \end{cases} \tag{1.2}$$

where $g_k := \nabla f(m_k)$, and the coefficient $\overline{\gamma}_k$ is a scalar known as the spectral parameter, which differentiates any two spectral gradient methods (see Barzilai and Borwein (BB) [1]). The classical forms of the parameter $\overline{\gamma}_k$ given in [1] are

$$\overline{\gamma}_k^{long} = \frac{\|m_{k+1} - m_k\|^2}{\langle g(m_{k+1}) - g(m_k), m_{k+1} - m_k\rangle}, \tag{1.3}$$

$$\overline{\gamma}_k^{short} = \frac{\langle g(m_{k+1}) - g(m_k), m_{k+1} - m_k\rangle}{\|g(m_{k+1}) - g(m_k)\|^2}. \tag{1.4}$$

Among the advantages of the spectral gradient method is its simplicity in implementation and low storage requirement; thus, it is suitable for large-scale problems. As a result, researchers have extended the spectral gradient method to solve systems of nonlinear equations (see, [2–4]). A system of nonlinear equations involves finding a vector m in a nonempty, closed and convex set $C \subset \mathbb{R}^n$ such that

$$\Gamma(m) = 0, \tag{1.5}$$

where $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. When the mapping Γ is monotone, i.e.,

$$\langle \Gamma(m) - \Gamma(\overline{m}), m - \overline{m}\rangle \geq 0, \quad \forall m, \overline{m} \in \mathbb{R}^n,$$

Problem (1.5) becomes a system of nonlinear monotone equations. Many problems arising from science and engineering can be translated into the form of problem (1.5). For example, applications of system of nonlinear equations (1.5) have appeared in different fields [5–10]. In recent years, algorithms for solving systems of nonlinear monotone equations have been used in signal and image recovery, (see [11–17]).

Motivated by the hyperplane projection strategy of Solodov and Svaiter [18], spectral gradient-like methods for solving (1.5) have gained more attention. Zhang and Zhou [19] developed a spectral gradient projection method for unconstrained nonlinear equations based on the modified version of the $\overline{\gamma}_k^{long}$ (1.3). Later on, Yu et al. [20] extended the work in [19] to solve convex constrained nonlinear equations. In addition, Yu et al. [21] proposed another spectral method for solving a system of nonlinear equations. The search direction in their work uses a convex combination of the modified $\overline{\gamma}_k^{long}$ (1.3) and $\overline{\gamma}_k^{short}$ (1.4). Their work revealed that combining the modified BB parameters gives better numerical performance than deploying them separately. This means that the efficiency of the convex combination of the $\overline{\gamma}_k^{long}$ (1.3) and $\overline{\gamma}_k^{short}$ (1.4) can be further explored.

Nowadays, there is a growing interest in incorporating the inertial technique into algorithms for solving systems of nonlinear equations (see, for example, [22, 23]). The inertial step defined as $i_{k+1} := m_{k+1} + \alpha_k(m_{k+1} - m_k),\ \alpha_k \in (0, 1)$, was proposed by Polyak [24] in order to speed up the performance of iterative algorithms. It can be observed that the inertial technique is using two previous iterates to compute the current iterate. This has also been shown to accelerate the iteration process of algorithms for solving nonlinear problems such as the proximal point method [25, 26] and auxiliary problem principle [27].

Motivated by the contributions of the above-mentioned literature, this work seeks to explore the effect of the inertial technique on the convex combination of the $\overline{\gamma}_k^{long}$ and $\overline{\gamma}_k^{short}$ based on some modifications. This idea can be viewed as the modification of the work of Yu et al. [21]. Numerical experiments conducted in Section 3 reveal some level of improvement in numerical performance. Some of the notable contributions of this work include the following:

- A new spectral method for solving a system of nonlinear monotone equations based on the inertial technique is proposed.
- This work generalizes some existing algorithms in the literature.
- The global convergence of the proposed method is discussed under standard conditions.
- To depict the efficiency of the new method, a numerical experiment on a collection of test problems in comparison with some existing methods is presented.
- Subsequently, the proposed algorithm is applied to problems arising from robotic motion control.

In the remaining part of this work, the next section gives some definitions, details of the algorithms and global convergence. Section three gives some numerical experiments, while the application part is in Section four. In the final section, some concluding remarks are given.

2. Proposed algorithm and convergence analysis

We begin this section by recalling the definition of the projection operator as follows:

Definition 2.1. *Suppose $C \subset \mathbb{R}^n$ is a convex, nonempty and closed set. Then, any point $m \in \mathbb{R}^n$ can be projected onto the set C using*

$$P_C(m) = \arg\min\{\|m - \overline{m}\| \ : \overline{m} \in C\}. \tag{2.1}$$

The relation (2.1) satisfies the following useful property:

$$\|P_C(m) - \overline{m}\| \leq \|m - \overline{m}\|, \quad \forall \overline{m} \in C. \tag{2.2}$$

Definition 2.2. *Any vector-valued map that satisfies*

$$\|\Gamma(m) - \Gamma(\overline{m})\| \leq L\|m - \overline{m}\|,\ \forall m, \overline{m} \in \mathbb{R}^n,\ L > 0,$$

is said to be Lipschitz continuous.

In what follows, we present the proposed algorithm and subsequently give some remarks.

Algorithm 1: Derivative-Free Spectral Method with Inertia (iSDFM).

Input : Choose $m_{-1},\ m_0 \in C$, $0 < \eta < 2$, $\kappa > 0$, $\sigma, \mu, \varsigma, \alpha_k \in (0,1)$, $r > 0$ and $Tol > 0$.

Step 0: Set $k = 0$, compute $t_0 := -\Gamma(m_0)$ and $i_0 := m_0 + \alpha_0(m_0 - m_{-1})$.

Step 1: If $\|\Gamma(m_k)\| \leq Tol$, then terminate; else, continue with **Step 2**.

Step 2: Set

$$p_k := m_k + \Lambda_k t_k, \quad \Lambda_k = \kappa\varsigma^j, \tag{2.3}$$

where j is the least non–negative integer such that

$$-\langle \Gamma(m_k + \kappa\varsigma^j t_k), t_k\rangle \geq \sigma\kappa\varsigma^j \|t_k\|^2 \min\{1,\ \|\Gamma(m_k + \kappa\varsigma^j t_k)\|^{\frac{1}{c}}\},\ c \geq 1. \tag{2.4}$$

Step 3: If $\|\Gamma(p_k)\| = 0$, stop. Else, compute

$$m_{k+1} := P_C\left[m_k - \eta\frac{\langle \Gamma(p_k), m_k - p_k\rangle}{\|\Gamma(p_k)\|^2}\Gamma(p_k)\right]. \tag{2.5}$$

Step 4: Set $k := k + 1$ and redo the task from step 1, where the inertial step is updated as $i_{k+1} := m_{k+1} + \alpha_k(m_{k+1} - m_k)$ with the search direction as

$$t_k := -\gamma_k \Gamma(m_k), \tag{2.6}$$

where

$$\gamma_k = (1 - \theta_k)\overline{\beta}_k + \theta_k\widehat{\beta}_k, \tag{2.7}$$

$$\overline{\beta}_k = \frac{\|i_{k+1} - i_k\|^2}{\langle i_{k+1} - i_k,\ \Gamma(i_{k+1}) - \Gamma(i_k) + r(i_{k+1} - i_k)\rangle}, \tag{2.8}$$

$$\widehat{\beta}_k = \frac{\langle i_{k+1} - i_k,\ \Gamma(i_{k+1}) - \Gamma(i_k) + r(i_{k+1} - i_k)\rangle}{\|\Gamma(i_{k+1}) - \Gamma(i_k) + r(i_{k+1} - i_k)\|^2}, \tag{2.9}$$

$$\theta_k = 1 - \mu \cdot \frac{\langle \Gamma(m_k), i_{k+1} - i_k\rangle^2}{\overline{\Gamma(m_k)}^2 \|i_{k+1} - i_k\|^2}, \tag{2.10}$$

$$\overline{\Gamma(m_k)} = \max\{\|\Gamma(m_{k-1})\|, \|\Gamma(m_k)\|\}. \tag{2.11}$$

Remark 2.3. *The choice of the* θ_k *defined by* (2.10) *is prompted by the work of Awwal et al. [28]. Now, since* $\max\{\|\Gamma(m_{k-1})\|, \|\Gamma(m_k)\|\} \geq \|\Gamma(m_k)\|$, *then by the Cauchy-Schwarz inequality we have*

$$0 \leq \frac{\langle \Gamma(m_k),\ i_{k+1} - i_k\rangle^2}{\|\overline{\Gamma(m_k)}\|^2 \|i_{k+1} - i_k\|^2} \leq \frac{\|\Gamma(m_k)\|^2 \|i_{k+1} - i_k\|^2}{\|\Gamma(m_k)\|^2 \|i_{k+1} - i_k\|^2} = 1, \tag{2.12}$$

and by the fact that $\mu \in (0,1)$, *we get*

$$0 \leq \theta_k \leq 1. \tag{2.13}$$

Remark 2.4. *We note that if for all k,* $i_k = m_k$ (2.8) *and* (2.9) *and* $\theta_k \in [0,1]$, *then the search direction* (2.6) *reduces to that of Yu et al. [21]. Moreover, if for all k,* $i_k = m_k$ *in* (2.8) *and* $\theta_k = 0$, *then* *the search direction* (2.6) *reduces to that of Yu et al. [20]. Furthermore, the new search direction* (2.6) *also reduces to DAIS1 and DAIS2 of Awwal et al. in [22] if* $\theta_k = 1$ *and* $\theta_k = 0$*, respectively. Hence, our proposed work is regarded as an extension of the algorithms proposed in [21, 22].*

To show the global convergence of the proposed algorithms, we assumed the following:

(A_1) The mapping Γ is monotone.
(A_2) The mapping Γ is Lipschitz continuous.
(A_3) The solution set of problem (1.5) is nonempty.

Lemma 2.5. *Suppose that the assumptions* (A_1)*–*(A_3) *hold, and that* $\{m_k\}$ *and* $\{i_k\}$ *are produced by Algorithm 1. Then,*

(i) $\lim_{k\to\infty} \|m_k - \widehat{m}\|$ *exists.*
(ii) $\{m_k\}$, $\{i_k\}$ *and* $\|\Gamma(m_k)\|$ *are bounded.*
(iii) The search direction is bounded, i.e.,

$$\|t_k\| \leq c_1,\ c_1 > 0. \tag{2.14}$$

(iv) The search direction satisfies

$$\langle \Gamma(m_k), t_k \rangle \leq -c_2\|\Gamma(m_k)\|^2,\ c_2 > 0. \tag{2.15}$$

(v)

$$\lim_{k\to\infty} \Lambda_k \|t_k\| = 0. \tag{2.16}$$

Proof. Let $\widehat{m}$ be a solution of problem (1.5), and by the assumption (A_1), we have

$$\begin{aligned}\langle \Gamma(p_k),\ m_k - \widehat{m}\rangle &= \langle \Gamma(p_k),\ m_k - p_k + p_k - \widehat{m}\rangle \\ &= \langle \Gamma(p_k),\ m_k - p_k\rangle + \langle \Gamma(p_k) - \Gamma(\widehat{m}),\ p_k - \widehat{m}\rangle \\ &\geq \langle \Gamma(p_k),\ m_k - p_k\rangle. \end{aligned} \tag{2.17}$$

Now, since $0 < \eta < 2$, from (2.2), (2.5) and (2.17) we have

$$\begin{aligned}\|m_{k+1} - \widehat{m}\|^2 &= \left\| P_C\left[m_k - \eta\frac{\langle \Gamma(p_k),\ m_k - p_k\rangle}{\|\Gamma(p_k)\|^2}\Gamma(p_k)\right] - \widehat{m}\right\|^2 \\ &\leq \left\| m_k - \widehat{m} - \eta\frac{\langle \Gamma(p_k),\ m_k - p_k\rangle}{\|\Gamma(p_k)\|^2}\Gamma(p_k)\right\|^2 \\ &= \|m_k - \widehat{m}\|^2 - 2\eta\frac{\langle \Gamma(p_k),\ m_k - p_k\rangle}{\|\Gamma(p_k)\|^2}\langle \Gamma(p_k),\ m_k - \widehat{m}\rangle + \eta^2\frac{\langle \Gamma(p_k),\ m_k - p_k\rangle^2}{\|\Gamma(p_k)\|^2} \\ &\leq \|m_k - \widehat{m}\|^2 - 2\eta\frac{\langle \Gamma(p_k),\ m_k - p_k\rangle}{\|\Gamma(p_k)\|^2}\langle \Gamma(p_k),\ m_k - p_k\rangle + \eta^2\frac{\langle \Gamma(p_k),\ m_k - p_k\rangle^2}{\|\Gamma(p_k)\|^2} \\ &= \|m_k - \widehat{m}\|^2 - \eta(2-\eta)\frac{\langle \Gamma(p_k),\ m_k - p_k\rangle^2}{\|\Gamma(p_k)\|^2} \\ &\leq \|m_k - \widehat{m}\|^2. \end{aligned} \tag{2.18}$$

This means that $\|m_k - \widehat{m}\| \le \|m_{k-1} - \widehat{m}\| \le \cdots \le \|m_0 - \widehat{m}\|$, and thus $\lim_{k\to\infty} \|m_k - \widehat{m}\|$ exists.

(ii) Since $\lim_{k\to\infty} \|m_k - \widehat{m}\|$ exists, $\{m_k\}$ is bounded. Combined with the fact that $0 < \alpha_k < 1,\ \forall k$, this gives the boundedness of $\{i_k\}$.

In addition, since the mapping Γ is Lipschitz continuous and $\{m_k\}$ is bounded, we can find a positive constant $c_3 > 0$ such that

$$\|\Gamma(m_k)\| \le c_3. \tag{2.19}$$

(iii) To prove that the search direction defined by (2.6) is bounded, we need to show that the parameter γ_k (2.7) is bounded.

From assumption (A_1), the mapping Γ is monotone. This gives us $\langle \Gamma(i_{k+1}) - \Gamma(i_k),\ i_{k+1} - i_k\rangle \ge 0$, and thus

$$\begin{aligned} \langle \Gamma(i_{k+1}) - \Gamma(i_k) + r(i_{k+1} - i_k),\ i_{k+1} - i_k\rangle &= \langle \Gamma(i_{k+1}) - \Gamma(i_k),\ i_{k+1} - i_k\rangle + r\|i_{k+1} - i_k\|^2 \\ &\ge r\|i_{k+1} - i_k\|^2. \end{aligned} \tag{2.20}$$

Moreover, using the assumption that the mapping Γ is Lipschitz continuous, together with the Cauchy Schwarz inequality, we have

$$\begin{aligned} \langle \Gamma(i_{k+1}) - \Gamma(i_k) + r(i_{k+1} - i_k),\ i_{k+1} - i_k\rangle &= \langle \Gamma(i_{k+1}) - \Gamma(i_k),\ i_{k+1} - i_k\rangle + r\|i_{k+1} - i_k\|^2 \\ &\le (L + r)\|i_{k+1} - i_k\|^2. \end{aligned} \tag{2.21}$$

Thus, (2.20) and (2.21) imply

$$r\|i_{k+1} - i_k\|^2 \le \langle \Gamma(i_{k+1}) - \Gamma(i_k) + r(i_{k+1} - i_k),\ i_{k+1} - i_k\rangle \le (L + r)\|i_{k+1} - i_k\|^2, \tag{2.22}$$

and therefore

$$\frac{1}{L + r} \le \bar{\beta}_k \le \frac{1}{r}. \tag{2.23}$$

On the other hand,

$$\begin{aligned} &\|\Gamma(i_{k+1}) - \Gamma(i_k) + r(i_{k+1} - i_k)\|^2 \\ &= \langle \Gamma(i_{k+1}) - \Gamma(i_k) + r(i_{k+1} - i_k), \Gamma(i_{k+1}) - \Gamma(i_k) + r(i_{k+1} - i_k)\rangle \\ &= \|\Gamma(i_{k+1}) - \Gamma(i_k)\|^2 + 2r\langle \Gamma(i_{k+1}) - \Gamma(i_k), i_{k+1} - i_k\rangle + r^2\|i_{k+1} - i_k\|^2 \\ &\ge \|\Gamma(i_{k+1}) - \Gamma(i_k)\|^2 + r^2\|i_{k+1} - i_k\|^2 \\ &\ge r^2\|i_{k+1} - i_k\|^2. \end{aligned} \tag{2.24}$$

Using Lipschitz continuity, we have

$$\|\Gamma(i_{k+1}) - \Gamma(i_k) + r(i_{k+1} - i_k)\| \le (L + r)\|i_{k+1} - i_k\|. \tag{2.25}$$

Combining (2.24) and (2.25) gives

$$r^2\|i_{k+1} - i_k\|^2 \le \|\Gamma(i_{k+1}) - \Gamma(i_k) + r(i_{k+1} - i_k)\|^2 \le (L + r)^2\|i_{k+1} - i_k\|^2. \tag{2.26}$$

Using (2.22) and (2.26), we have

$$\frac{r}{(L + r)^2} \le \widehat{\beta}_k \le \frac{(L + r)}{r^2}. \tag{2.27}$$

Therefore, setting $M = \frac{1}{r} + \frac{L+r}{r^2}$ yields

$$\overline{\beta}_k + \widehat{\beta}_k \leq M. \tag{2.28}$$

Since $\forall k$, $\theta_k \leq 1$ (see Remark 1),

$$\gamma_k = (1-\theta_k)\overline{\beta}_k + \theta_k\widehat{\beta}_k \leq \overline{\beta}_k + \widehat{\beta}_k \leq M. \tag{2.29}$$

Combining this with (2.19) gives (2.14) with $c_1 = Mc_3$.

(iv) By the definition of θ_k, we have three possibilities, and thus the parameter γ_k may take any of the three different following forms: $\gamma_k = \overline{\beta}_k$, $\gamma_k = \widehat{\beta}_k$ and $\gamma_k = (1-\theta_k)\overline{\beta}_k + \theta_k\widehat{\beta}_k$ for $\theta_k = 0$, $\theta_k = 1$ and $\theta_k \in (0, 1)$, respectively. Therefore, we divide this proof into three cases:

Case I: If $\theta = 0$, $\forall k$, then the search direction (2.6) reduces to $t_k = -\overline{\beta}_k\Gamma_k(m_k)$, and therefore, using (2.23) gives

$$\langle \Gamma(m_k), t_k\rangle = -\overline{\beta}_k\|\Gamma(m_k)\|^2 \leq -\frac{1}{L+r}\|\Gamma(m_k)\|^2. \tag{2.30}$$

Case II: If $\theta = 1$, $\forall k$, then the search direction (2.6) becomes $t_k = -\widehat{\beta}_k\Gamma_k(m_k)$, and thus, using (2.27) yields

$$\langle \Gamma(m_k), t_k\rangle = -\widehat{\beta}_k\|\Gamma(m_k)\|^2 \leq -\frac{r}{(L+r)^2}\|\Gamma(m_k)\|^2. \tag{2.31}$$

Case III: If $0 < \theta < 1$, $\forall k$, then we can find some constant $c_4 > 0$ such that $\theta_k > c_4$ and $(1-\theta_k) > 0$. Therefore, from (2.7) and (2.27), we have

$$\gamma_k \geq \theta_k\widehat{\beta}_k \geq c_4\frac{r}{(L+r)^2} := c_5. \tag{2.32}$$

Thus, from the search direction (2.6) and (2.32), it holds that

$$\langle \Gamma(m_k), t_k\rangle = -\gamma_k\|\Gamma(m_k)\|^2 \leq -c_5\|\Gamma(m_k)\|^2. \tag{2.33}$$

Hence, from the three cases above, we see that (2.15) holds.

(v) Using the boundedness of $\{m_k\}$, (2.14) and the definition of p_k in (2.3), $\{p_k\}$ is bounded. Also, using the assumption that Γ is Lipschitz continuous, we get

$$\|\Gamma(p_k)\| \leq n_1,\ n_1 > 0. \tag{2.34}$$

Since $\min\left\{1, \|\Gamma(m_k + \Lambda_k t_k)\|^{\frac{1}{c}}\right\} \leq 1$, squaring from both sides of (2.4) yields

$$\sigma^2\Lambda_k^4\|t_k\|^4 \leq \langle \Gamma(p_k),\ \Lambda_k t_k\rangle^2. \tag{2.35}$$

Furthermore, since $0 < \eta < 2$, from (2.18) we obtain

$$\langle \Gamma(p_k),\ m_k - p_k\rangle^2 \leq \frac{\|\Gamma(p_k)\|^2(\|m_k - \widehat{m}\|^2 - \|m_{k+1} - \widehat{m}\|^2)}{\eta(2-\eta)}. \tag{2.36}$$

This together with (2.35) gives

$$\sigma^2\Lambda_k^4\|t_k\|^4 \leq \frac{\|\Gamma(p_k)\|^2(\|m_k - \widehat{m}\|^2 - \|m_{k+1} - \widehat{m}\|^2)}{\eta(2-\eta)}. \tag{2.37}$$

Recall that $\|\Gamma(p_k)\|$ is bounded by n_1 (see (2.34)), and we obtain

$$\sigma^2 \Lambda_k^4 \|t_k\|^4 \leq \frac{\|\Gamma(p_k)\|^2(\|m_k - \widehat{m}\|^2 - \|m_{k+1} - \widehat{m}\|^2)}{\eta(2-\eta)} \leq \frac{n_1^2(\|m_k - \widehat{m}\|^2 - \|m_{k+1} - \widehat{m}\|^2)}{\eta(2-\eta)}. \tag{2.38}$$

Since $\lim_{k\to\infty} \|m_k - \widehat{m}\|$ exists, taking the limit as $k \to \infty$ on both sides of (2.38) gives

$$\sigma^2 \lim_{k\to\infty} \Lambda_k^4 \|t_k\|^4 = 0,$$

which implies (2.16). □

Lemma 2.6. *Suppose that the Assumption (A_2) holds. Let the sequences $\{m_k\}$ and $\{p_k\}$ be generated by Algorithm 1. Then,*

$$\Lambda_k \geq \max\left\{\kappa, \frac{c_2 \varsigma \|\Gamma(m_k)\|^2}{(L+\sigma)\|t_k\|^2}\right\}. \tag{2.39}$$

Proof. From (2.4), if $\Lambda_k \neq \kappa$, then $\tilde{\Lambda}_k = \Lambda_k \varsigma^{-1}$ violates (2.4), that is,

$$-\langle \Gamma(m_k + \tilde{\Lambda}_k t_k), t_k \rangle < \sigma \|t_k\|^2 \tilde{\Lambda}_k \min\{1, \|\Gamma(m_k + \varsigma^j t_k)\|^{\frac{1}{c}}\}.$$

We know that $\min\{1, \|\Gamma(m_k + \varsigma^j t_k)\|^{\frac{1}{c}}\} \leq 1$. Thus, from (2.15) and Assumption (A_2), we get

$$\begin{aligned} c_2\|\Gamma(m_k)\|^2 &\leq -\Gamma(m_k)^T t_k \\ &= (\Gamma(m_k + \tilde{\Lambda}_k t_k) - \Gamma(m_k))^T t_k - \langle \Gamma(m_k + \tilde{\Lambda}_k t_k), t_k \rangle \\ &\leq \|\Gamma(m_k + \tilde{\Lambda}_k t_k) - \Gamma(m_k)\|\|t_k\| - \langle \Gamma(m_k + \tilde{\Lambda}_k t_k), t_k \rangle \\ &\leq L\|m_k + \tilde{\Lambda}_k t_k - m_k\|\|t_k\| + \sigma \tilde{\Lambda}_k \|t_k\|^2 \min\{1, \|\Gamma(m_k + \varsigma^j t_k)\|^{\frac{1}{c}}\} \\ &\leq L\|m_k + \tilde{\Lambda}_k t_k - m_k\|\|t_k\| + \sigma \tilde{\Lambda}_k \|t_k\|^2 \\ &\leq \tilde{\Lambda}_k L \|t_k\|^2 + \sigma \tilde{\Lambda}_k \|t_k\|^2 \\ &\leq \tilde{\Lambda}_k (L+\sigma)\|t_k\|^2. \end{aligned}$$

Therefore,

$$\tilde{\Lambda}_k \geq \frac{c_2\|\Gamma(m_k)\|^2}{(L+\sigma)\|t_k\|^2}. \tag{2.40}$$

Substituting $\tilde{\Lambda}_k = \Lambda_k \varsigma^{-1}$ in (2.40) and solving for Λ_k, we get

$$\Lambda_k \geq \frac{c_2\varsigma\|\Gamma(m_k)\|^2}{(L+\sigma)\|t_k\|^2}. \tag{2.41}$$

Thus, we have

$$\Lambda_k \geq \max\left\{\kappa, \frac{c_2 \varsigma \|\Gamma(m_k)\|^2}{(L+\sigma)\|t_k\|^2}\right\}.$$

□

Theorem 2.7. *If the Assumptions ($A_1 - A_3$) hold, and the sequence $\{m_k\}$ is produced by Algorithm 1, then*

$$\liminf_{k\to\infty} \|\Gamma(m_k)\| = 0. \tag{2.42}$$

Proof. We show the proof by contradiction. Suppose (2.42) does not hold, and then there exists $s > 0$ such that $\forall k \geq 0$,

$$\|\Gamma(m_k)\| \geq s. \tag{2.43}$$

From Eqs (2.15) and (2.43), we get $\forall k \geq 0$,

$$\|t_k\| \geq sc_2. \tag{2.44}$$

Multiplying $\|t_k\|$ on both sides of (2.39), and using (2.14) and (2.43), we obtain

$$\begin{aligned}
\Lambda_k\|t_k\| &\geq \max\left\{\kappa, \frac{c_2\varsigma\|\Gamma(m_k)\|^2}{(L+\sigma)\|t_k\|^2}\right\}\|t_k\| \\
&\geq \max\left\{\kappa, \frac{c_2\varsigma c_3^2}{(L+\sigma)c_2^2}\right\}c_2 s \\
&= \max\left\{\kappa c_2 s, \frac{\varsigma c_3^2 s}{(L+\sigma)}\right\}.
\end{aligned} \tag{2.45}$$

Taking the limit as $k \to \infty$ on both sides gives

$$\lim_{k\to\infty} \Lambda_k\|t_k\| > 0, \tag{2.46}$$

which contradicts (2.16). Hence, $\liminf_{k\to\infty} \|\Gamma(m_k)\| = 0$. □

3. Numerical experiments

In this section, we present the numerical experiments performed by solving a set of test problems taken from the literature. To depict the efficiency of the iSDFM algorithm, we perform numerical comparison with two other existing methods. The first one is the DAIS1 algorithm proposed in [22], which is an inertial-based algorithm for solving a system of nonlinear equations. The second algorithm is the MSGPALG proposed by Yu et al. in [21] based on the convex combination of the modified BB long and short parameters. As noted in Remark 2.4, both DAIS1 and MSGPALG can be viewed as special cases of the proposed iSDFM algorithm. We test the performances of all these three algorithms on seven test problems with eight different initial points (see Table 1) and five dimensions (1000, 5000, 10000, 50000, and 100000), thus making the total number of the test problems 280. These three algorithms are coded on MATLAB R2019b which runs on a PC of corei3-4005U processor with 4 GB RAM and 1.70 GHz CPU. The choice of parameters for MSGPALG and DAIS1 are maintained as reported in their respective references [21, 22]. In the iSDFM algorithm, we choose $\varsigma = 0.47$, $\alpha_k = 1/(k+1)^2$, $\eta = 1.79$, $\mu = 0.5$, $\sigma = 0.01$, $r = 0.001$, $c = 2$ and $\kappa = 1$. The stopping criterion is set to be $\|\Gamma(m_k)\| < 10^{-6}$.

Table 1. Initial guesses used for the problems.

Initial guess	Value
m_1	$(1, 1, 1, \ldots, 1)^T$
m_2	$(0.1, 0.1, 0.1, \cdots, 0.1)^T$
m_3	$(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n})^T$
m_4	$(1 - \frac{1}{n}, 1 - \frac{2}{n}, 1 - \frac{3}{n}, \ldots, 0)^T$
m_5	$(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n})^T$
m_6	$(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n})^T$
m_7	$(\frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n}, \ldots, 0)^T$
m_8	$(\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, 0)^T$

We consider the following test problems, where $\Gamma(m) = (g_1(m), g_2(m), \ldots, g_n(m))^T$:

Problem 1. [29]:

$$g_1(m) = e^{m_1} - 1, \;\; g_j(m) \;\; = e^{m_j} + m_j - 1, \text{ for } j = 2, ..., n, \text{ and } C = \mathbb{R}^n_+.$$

Problem 2. [28] Modified logarithmic function:

$$g_j(m) = \ln(m_j + 1) - \frac{m_j}{n}, \text{ for } \; j = 1, 2, 3, ..., n,$$
$$\text{and } C = \{m \in \mathbb{R}^n : \sum_{j=1}^{n} m_j \leq n, \; m_j > -1, \; j = 1, 2, \ldots, n\}.$$

Problem 3. [30] Nonsmooth function:

$$g_j(m) = 2m_j - \sin|m_j|, \; j = 1, 2, 3, ..., n,$$
$$\text{and } C = \{m \in \mathbb{R}^n : \sum_{j=1}^{n} m_j \leq n, \; m_j \geq 0, \; j = 1, 2, \ldots, n\}.$$

Problem 4. [31] Strictly convex function:

$$g_j(m) = e^{m_j} - 1, \text{ for } j = 1, 2, ..., n, \;\text{ and } \;\; C = \mathbb{R}^n_+.$$

Problem 5. [20] Nonsmooth function:

$$g_j(m) = m_j - \sin|m_j - 1|, \; j = 1, 2, 3, ..., n,$$
$$\text{and } C = \{m \in \mathbb{R}^n : \sum_{j=1}^{n} m_i \leq n, \; m_j \geq -1, \; j = 1, 2, \ldots, n\}.$$

Problem 6. [29]:

$$g_j(m) = e^{{m_j}^2} + 1.5\sin(2m_j) - 1, \text{ for } j = 1, 2, ..., n, \;\text{ and } \;\; C = \mathbb{R}^n_+.$$

Problem 7. [32]:

$$g_1(m) = \frac{5}{2}m_1 + m_2 - 1,$$
$$g_j(m) = m_{j-1} + \frac{5}{2}m_j + m_{j+1} - 1, \text{ for } j = 2, 3, ..., n-1,$$
$$g_n(m) = m_{n-1} + \frac{5}{2}m_n - 1 \text{ and } C = \mathbb{R}^n_+.$$

Based on these settings, the results of the experiments are tabulated in Tables 2–8 with ITER, FVAL and TIME denoting the number of iterations, number of function evaluations and CPU time, respectively. Based on these metrics, it can be observed that the performances of the three algorithms varies in terms of the ITER, FVAL and TIME. However, taking the whole results of the experiment into consideration, it can be seen that the proposed iSDFM algorithm outperformed the DAIS1 and the MSGPALG algorithms in most instances. By outperforming we mean the new iSDFM recorded the least ITER, FVAL and TIME in most cases of the experiment.

With the help of the Dolan and Moré performance profile [33], we present the information in Tables 2–8 graphically for better and easier visualization of each algorithm's performance. These graphs are plotted in Figures 1–3. It can be clearly observed from Figure 1 that the iSDFM algorithm solved about 72% of the problems with the least ITER, as compared to the DAIS1 and MSGPALG with around 30% and 12%, respectively. Figure 2 shows that the iSDFM outperformed the other two methods by solving around 70% of the problems with the least FVAL. In terms of TIME, iSDFM algorithm competes favorably with the MSGPALG algorithm. In general, our proposed iSDFM algorithm shows better efficiency as compared to the DAIS1 and MSGPALG algorithms. This might not be unconnected with taking the convex combination of the BB-like parameters incorporated with the inertial technique.

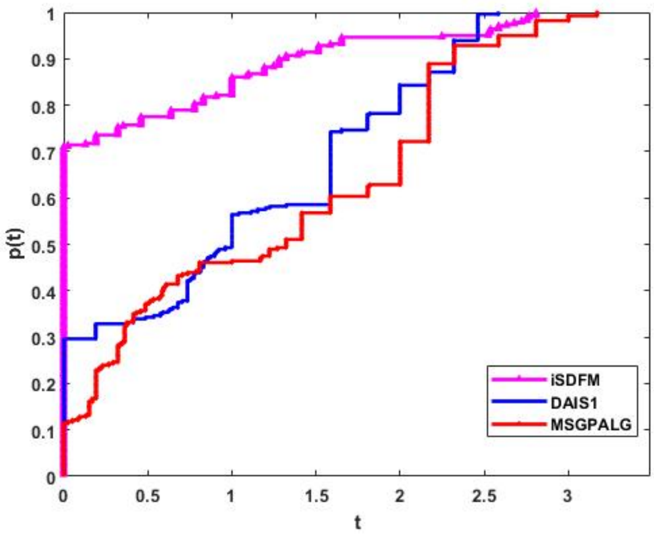


Figure 1. Performance profile on number of iterations.

Table 2. Numerical results of the three algorithms on Problem 1.

		iSDFM				DAIS1				MSGPALG			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	m_1	1	2	0.0747	0.00E+00	5	7	0.1166	9.24E-08	8	9	0.1666	2.76E-07
	m_2	1	2	0.0054	0.00E+00	5	7	0.0519	5.76E-08	7	8	0.0395	1.06E-07
	m_3	2	3	0.0186	9.93E-16	10	11	0.0344	8.05E-08	10	11	0.0240	2.07E-07
	m_4	2	3	0.0052	0.00E+00	6	8	0.0253	2.30E-07	8	9	0.0112	2.52E-07
	m_5	2	3	0.0078	0.00E+00	6	8	0.0385	1.10E-07	8	9	0.0213	2.46E-07
	m_6	3	4	0.0064	0.00E+00	9	11	0.0278	7.28E-08	9	10	0.0057	1.98E-07
	m_7	2	3	0.0064	0.00E+00	6	8	0.0118	2.30E-07	8	9	0.0094	2.52E-07
	m_8	2	3	0.0041	0.00E+00	6	8	0.0101	1.11E-07	8	9	0.0100	2.47E-07
5000	m_1	1	2	0.1996	0.00E+00	5	7	0.2617	2.37E-07	8	9	0.0676	6.05E-07
	m_2	1	2	0.0113	0.00E+00	4	6	0.0242	8.65E-08	7	8	0.0182	1.65E-07
	m_3	2	3	0.0138	9.93E-16	10	11	0.0423	8.05E-08	10	11	0.0224	2.07E-07
	m_4	2	3	0.0145	0.00E+00	6	8	1.1522	2.91E-07	8	9	0.1283	5.54E-07
	m_5	2	3	0.0556	0.00E+00	6	8	0.0308	2.55E-07	8	9	0.0286	5.52E-07
	m_6	3	4	0.1059	0.00E+00	9	11	0.0446	7.33E-08	9	10	0.0253	1.95E-07
	m_7	2	3	0.2689	0.00E+00	6	8	0.0281	2.91E-07	8	9	0.0196	5.54E-07
	m_8	2	3	0.0683	0.00E+00	6	8	0.0291	2.55E-07	8	9	1.4239	5.52E-07
10000	m_1	1	2	0.0335	0.00E+00	5	7	0.9254	3.37E-07	8	9	0.0582	8.53E-07
	m_2	1	2	0.0139	0.00E+00	4	6	0.0311	1.06E-07	7	8	0.3954	2.21E-07
	m_3	2	3	1.7858	9.93E-16	10	11	0.1457	8.05E-08	10	11	0.1475	2.07E-07
	m_4	2	3	0.0597	0.00E+00	6	8	0.0676	3.85E-07	8	9	0.0401	7.82E-07
	m_5	2	3	0.0287	0.00E+00	6	8	0.0427	3.60E-07	8	9	0.0347	7.81E-07
	m_6	3	4	0.0428	0.00E+00	9	11	0.4798	7.33E-08	9	10	0.0475	1.94E-07
	m_7	2	3	0.0353	0.00E+00	6	8	0.0518	3.85E-07	8	9	0.1343	7.82E-07
	m_8	2	3	0.5467	0.00E+00	6	8	0.1948	3.61E-07	8	9	2.2142	7.81E-07
50000	m_1	1	2	0.1128	0.00E+00	5	7	0.2563	7.55E-07	9	10	0.2087	9.07E-08
	m_2	1	2	0.5707	0.00E+00	4	6	2.7546	1.93E-07	7	8	0.1446	4.74E-07
	m_3	2	3	0.1359	9.93E-16	10	11	1.7252	8.05E-08	10	11	0.1304	2.07E-07
	m_4	2	3	0.1764	0.00E+00	6	8	0.1702	8.17E-07	9	10	0.1625	8.32E-08
	m_5	2	3	0.0968	0.00E+00	6	8	0.5717	8.07E-07	9	10	1.3341	8.31E-08
	m_6	3	4	1.1641	0.00E+00	9	11	0.4882	7.34E-08	9	10	0.1229	1.94E-07
	m_7	2	3	0.1440	0.00E+00	6	8	0.2247	8.17E-07	9	10	0.1264	8.32E-08
	m_8	2	3	0.0931	0.00E+00	6	8	1.1037	8.07E-07	9	10	0.1459	8.31E-08
100000	m_1	1	2	0.1883	0.00E+00	6	8	0.4077	5.32E-09	9	10	0.9471	1.28E-07
	m_2	1	2	0.6308	0.00E+00	4	6	0.2373	2.65E-07	7	8	0.2009	6.66E-07
	m_3	2	3	0.2199	9.93E-16	10	11	4.2777	8.05E-08	10	11	0.2548	2.07E-07
	m_4	2	3	0.1781	0.00E+00	7	9	0.4334	5.71E-09	9	10	0.2946	1.18E-07
	m_5	2	3	0.8688	0.00E+00	7	9	1.8487	5.68E-09	9	10	0.2558	1.18E-07
	m_6	3	4	0.1510	0.00E+00	9	11	0.6243	7.34E-08	9	10	2.5348	1.94E-07
	m_7	2	3	0.3463	0.00E+00	7	9	0.3045	5.71E-09	9	10	0.2682	1.18E-07
	m_8	2	3	0.3872	0.00E+00	7	9	3.9198	5.68E-09	9	10	0.3627	1.18E-07

Table 3. Numerical results of the three algorithms on Problem 2.

		iSDFM				**DAIS1**				**MSGPALG**			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	m_1	4	6	0.0300	2.13E-07	8	10	0.0332	6.00E-08	9	10	0.0283	3.85E-07
	m_2	19	21	0.0497	9.06E-07	4	6	0.0119	1.25E-07	6	7	0.0145	5.14E-07
	m_3	7	9	0.0239	4.09E-07	7	9	0.0146	2.78E-08	7	8	0.0096	5.83E-07
	m_4	7	9	0.0172	9.09E-07	8	10	0.0195	2.49E-07	9	10	0.0096	4.50E-07
	m_5	7	9	0.0333	9.09E-07	8	10	0.0177	2.49E-07	9	10	0.0115	4.50E-07
	m_6	8	10	0.0192	4.95E-08	8	10	0.0181	1.95E-08	7	8	0.0120	7.79E-07
	m_7	7	9	0.0145	9.09E-07	8	10	0.0181	2.49E-07	9	10	0.0185	4.50E-07
	m_8	7	9	0.0165	8.88E-07	8	10	0.0203	2.50E-07	9	10	0.0107	4.54E-07
5000	m_1	4	6	0.4667	9.03E-07	8	10	1.2040	1.34E-07	9	10	0.7974	8.59E-07
	m_2	23	25	3.0068	7.92E-07	4	6	0.8070	2.75E-07	7	8	0.4587	1.03E-07
	m_3	7	9	0.1027	4.14E-07	7	9	0.0807	2.75E-08	7	8	0.1973	5.62E-07
	m_4	10	12	0.0980	8.56E-07	8	10	2.7056	5.64E-07	10	11	0.1100	9.17E-08
	m_5	10	12	0.0644	8.56E-07	8	10	0.4899	5.64E-07	10	11	0.1570	9.17E-08
	m_6	8	10	0.0563	1.62E-07	8	10	0.0748	2.00E-08	7	8	0.0203	7.77E-07
	m_7	10	12	1.7635	8.56E-07	8	10	0.5620	5.64E-07	10	11	1.6032	9.17E-08
	m_8	10	12	2.0989	8.51E-07	8	10	0.3175	5.64E-07	10	11	0.0398	9.18E-08
10000	m_1	6	8	0.0674	8.99E-07	8	10	3.0765	1.90E-07	10	11	0.0635	1.10E-07
	m_2	24	26	4.3594	8.95E-07	4	6	0.8040	3.88E-07	7	8	0.4238	1.45E-07
	m_3	7	9	0.0568	4.14E-07	7	9	0.1828	2.75E-08	7	8	0.1624	5.59E-07
	m_4	11	13	0.1394	9.00E-07	8	10	0.2587	7.98E-07	10	11	1.5964	1.30E-07
	m_5	11	13	1.8668	9.00E-07	8	10	0.1017	7.98E-07	10	11	0.7227	1.30E-07
	m_6	8	10	0.0638	1.75E-07	8	10	0.1647	2.02E-08	7	8	0.0366	7.77E-07
	m_7	11	13	0.1010	9.00E-07	8	10	2.4034	7.98E-07	10	11	0.1130	1.30E-07
	m_8	11	13	0.7089	8.97E-07	8	10	0.2677	7.99E-07	10	11	0.0738	1.30E-07
50000	m_1	10	12	0.3468	8.09E-07	8	10	0.6028	4.24E-07	10	11	4.0162	2.47E-07
	m_2	27	29	2.2526	9.72E-07	4	6	2.6259	8.66E-07	7	8	0.3940	3.23E-07
	m_3	7	9	0.2112	4.15E-07	7	9	0.5050	2.75E-08	7	8	0.6980	5.57E-07
	m_4	14	16	1.2757	9.82E-07	9	11	1.2621	1.77E-08	10	11	0.2290	2.90E-07
	m_5	14	16	0.4675	9.82E-07	9	11	4.5622	1.77E-08	10	11	0.5405	2.90E-07
	m_6	8	10	1.9899	1.84E-07	8	10	1.0180	2.04E-08	7	8	0.3657	7.77E-07
	m_7	14	16	0.4672	9.82E-07	9	11	1.4871	1.77E-08	10	11	2.3145	2.90E-07
	m_8	14	16	1.4642	9.82E-07	9	11	0.8096	1.77E-08	10	11	1.8791	2.90E-07
100000	m_1	11	13	4.5167	9.03E-07	8	10	1.1287	5.99E-07	10	11	1.8709	3.49E-07
	m_2	29	31	2.7946	8.55E-07	5	7	0.3214	1.21E-08	7	8	0.3972	4.56E-07
	m_3	7	9	1.2771	4.15E-07	7	9	1.3914	2.75E-08	7	8	0.5613	5.57E-07
	m_4	16	18	2.8902	8.59E-07	9	11	0.8460	2.50E-08	10	11	1.2346	4.10E-07
	m_5	16	18	2.5125	8.59E-07	9	11	2.1172	2.50E-08	10	11	0.5491	4.10E-07
	m_6	8	10	1.8713	1.85E-07	8	10	0.5126	2.04E-08	7	8	0.4579	7.77E-07
	m_7	16	18	2.3005	8.59E-07	9	11	1.5732	2.50E-08	10	11	1.2220	4.10E-07
	m_8	16	18	2.3307	8.58E-07	9	11	1.8067	2.50E-08	10	11	0.5572	4.10E-07

Table 4. Numerical results of the three algorithms on Problem 3.

		iSDFM				DAIS1				MSGPALG			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	m_1	2	3	0.0489	0.00E+00	7	9	0.0312	2.58E-08	9	10	0.0271	2.19E-07
	m_2	2	3	0.0036	0.00E+00	5	7	0.0089	8.35E-07	7	8	0.0061	9.15E-07
	m_3	2	3	0.0068	0.00E+00	6	8	0.0109	1.27E-08	7	8	0.0068	2.49E-07
	m_4	2	3	0.0039	0.00E+00	6	8	0.0106	7.79E-07	9	10	0.0130	1.09E-07
	m_5	2	3	0.0046	0.00E+00	6	8	0.0139	7.79E-07	9	10	0.0106	1.09E-07
	m_6	2	3	0.0046	0.00E+00	6	8	0.0091	5.53E-08	8	9	0.0098	1.00E-07
	m_7	2	3	0.0047	0.00E+00	6	8	0.0115	7.79E-07	9	10	0.0087	1.09E-07
	m_8	2	3	0.0041	0.00E+00	6	8	0.0122	7.78E-07	9	10	0.0075	1.10E-07
5000	m_1	2	3	0.6745	0.00E+00	7	9	0.0553	5.78E-08	9	10	0.5417	4.89E-07
	m_2	2	3	0.0132	0.00E+00	6	8	0.0326	1.85E-08	8	9	0.0784	1.86E-07
	m_3	2	3	0.0482	0.00E+00	6	8	0.0719	1.27E-08	7	8	1.2708	2.49E-07
	m_4	2	3	0.0132	0.00E+00	8	10	0.9730	5.57E-07	9	10	1.0767	2.45E-07
	m_5	2	3	0.0159	0.00E+00	8	10	0.2707	5.57E-07	9	10	0.0275	2.45E-07
	m_6	2	3	0.4898	0.00E+00	6	8	0.0905	5.53E-08	8	9	0.2965	1.00E-07
	m_7	2	3	0.0259	0.00E+00	8	10	0.2239	5.57E-07	9	10	0.0474	2.45E-07
	m_8	2	3	0.0089	0.00E+00	8	10	0.0748	5.57E-07	9	10	0.0289	2.45E-07
10000	m_1	2	3	0.6468	0.00E+00	7	9	0.0893	8.17E-08	9	10	1.1107	6.91E-07
	m_2	2	3	0.0174	0.00E+00	6	8	2.0953	2.61E-08	8	9	0.0422	2.63E-07
	m_3	2	3	0.0241	0.00E+00	6	8	0.1933	1.27E-08	7	8	0.0554	2.49E-07
	m_4	2	3	0.0180	0.00E+00	8	10	0.5469	7.88E-07	9	10	0.7111	3.46E-07
	m_5	2	3	1.5830	0.00E+00	8	10	2.3199	7.88E-07	9	10	0.0512	3.46E-07
	m_6	2	3	0.0905	0.00E+00	6	8	0.1884	5.53E-08	8	9	1.9455	1.00E-07
	m_7	2	3	0.0225	0.00E+00	8	10	0.0539	7.88E-07	9	10	0.0370	3.46E-07
	m_8	2	3	0.4293	0.00E+00	8	10	0.5833	7.88E-07	9	10	0.2404	3.47E-07
50000	m_1	2	3	0.1251	0.00E+00	7	9	0.2378	1.83E-07	10	11	0.1597	1.41E-07
	m_2	2	3	0.2099	0.00E+00	6	8	0.3338	5.84E-08	8	9	0.3668	5.88E-07
	m_3	2	3	0.2718	0.00E+00	6	8	0.1467	1.27E-08	7	8	0.1518	2.49E-07
	m_4	2	3	0.0892	0.00E+00	9	11	1.8976	1.74E-08	9	10	3.4018	7.75E-07
	m_5	2	3	0.0738	0.00E+00	9	11	0.8494	1.74E-08	9	10	0.5303	7.75E-07
	m_6	2	3	0.4336	0.00E+00	6	8	0.2686	5.53E-08	8	9	0.3395	1.00E-07
	m_7	2	3	0.1016	0.00E+00	9	11	3.6617	1.74E-08	9	10	0.1285	7.75E-07
	m_8	2	3	0.4282	0.00E+00	9	11	0.2094	1.74E-08	9	10	0.3406	7.75E-07
100000	m_1	2	3	0.1608	0.00E+00	7	9	0.3985	2.58E-07	10	11	2.8327	1.99E-07
	m_2	2	3	0.4025	0.00E+00	6	8	2.4220	8.27E-08	8	9	0.5146	8.32E-07
	m_3	2	3	0.4997	0.00E+00	6	8	0.4001	1.27E-08	7	8	0.4550	2.49E-07
	m_4	2	3	0.2278	0.00E+00	9	11	1.2758	2.47E-08	10	11	0.6024	9.96E-08
	m_5	2	3	1.4800	0.00E+00	9	11	1.1169	2.47E-08	10	11	0.6103	9.96E-08
	m_6	2	3	0.1908	0.00E+00	6	8	1.1200	5.53E-08	8	9	0.3045	1.00E-07
	m_7	2	3	2.2168	0.00E+00	9	11	0.4203	2.47E-08	10	11	2.8192	9.96E-08
	m_8	2	3	0.5285	0.00E+00	9	11	1.6755	2.47E-08	10	11	0.5757	9.96E-08

Table 5. Numerical results of the three algorithms on Problem 4.

		iSDFM				DAIS1				MSGPALG			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	m_1	2	3	0.0296	0.00E+00	1	2	0.0073	0.00E+00	1	2	0.0291	0
	m_2	1	2	0.0027	0	1	2	0.0035	0.00E+00	1	2	0.0031	0
	m_3	2	3	0.0042	4.97E-16	1	2	0.0034	0.00E+00	1	2	0.0026	0.00E+00
	m_4	3	4	0.0049	0.00E+00	5	7	0.0093	2.75E-07	7	8	0.0060	6.10E-07
	m_5	3	4	0.0055	0.00E+00	5	7	0.0105	2.75E-07	7	8	0.0078	6.10E-07
	m_6	2	3	0.0034	0.00E+00	4	6	0.0067	1.70E-08	5	6	0.0046	2.61E-07
	m_7	3	4	0.0049	0.00E+00	5	7	0.0072	2.75E-07	7	8	0.0064	6.10E-07
	m_8	3	4	0.0071	0.00E+00	5	7	0.0083	2.84E-07	7	8	0.0056	6.18E-07
5000	m_1	2	3	0.0332	0.00E+00	1	2	0.0646	0.00E+00	1	2	0.0367	0
	m_2	1	2	0.0099	0	1	2	0.0213	0.00E+00	1	2	0.0174	0
	m_3	2	3	1.0121	4.97E-16	1	2	0.5463	0.00E+00	1	2	0.3020	0.00E+00
	m_4	3	4	0.0686	0.00E+00	5	7	0.6076	6.23E-07	8	9	0.2798	1.25E-07
	m_5	3	4	0.0281	0.00E+00	5	7	0.0214	6.23E-07	8	9	0.3643	1.25E-07
	m_6	2	3	0.0487	0.00E+00	4	6	0.9525	1.57E-08	5	6	0.1231	2.41E-07
	m_7	3	4	0.0160	0.00E+00	5	7	0.1311	6.23E-07	8	9	0.0235	1.25E-07
	m_8	3	4	0.3845	0.00E+00	5	7	0.0238	6.27E-07	8	9	0.0280	1.25E-07
10000	m_1	2	3	0.0432	0.00E+00	1	2	0.0152	0.00E+00	1	2	0.1671	0
	m_2	1	2	0.0130	0	1	2	0.6404	0.00E+00	1	2	0.2411	0
	m_3	2	3	0.3455	4.97E-16	1	2	0.1039	0.00E+00	1	2	0.0879	0.00E+00
	m_4	3	4	0.0962	0.00E+00	5	7	0.1264	8.83E-07	8	9	0.1373	1.76E-07
	m_5	3	4	0.0206	0.00E+00	5	7	0.0717	8.83E-07	8	9	0.0563	1.76E-07
	m_6	2	3	0.0129	0.00E+00	4	6	0.0410	1.56E-08	5	6	0.0300	2.39E-07
	m_7	3	4	2.1280	0.00E+00	5	7	2.0752	8.83E-07	8	9	0.0345	1.76E-07
	m_8	3	4	0.0435	0.00E+00	5	7	0.3836	8.85E-07	8	9	0.2295	1.77E-07
50000	m_1	2	3	0.1215	0.00E+00	1	2	0.0786	0.00E+00	1	2	0.2465	0
	m_2	1	2	0.9649	0	1	2	0.0727	0.00E+00	1	2	0.1039	0
	m_3	2	3	1.1914	4.97E-16	1	2	0.1050	0.00E+00	1	2	0.0993	0.00E+00
	m_4	3	4	0.0787	0.00E+00	6	8	0.4597	1.96E-08	8	9	0.9459	3.95E-07
	m_5	3	4	0.5360	0.00E+00	6	8	1.9008	1.96E-08	8	9	0.1311	3.95E-07
	m_6	2	3	0.0827	0.00E+00	4	6	1.4476	1.54E-08	5	6	1.6395	2.37E-07
	m_7	3	4	0.5580	0.00E+00	6	8	0.8577	1.96E-08	8	9	1.1280	3.95E-07
	m_8	3	4	0.1724	0.00E+00	6	8	0.1708	1.96E-08	8	9	0.1320	3.95E-07
100000	m_1	2	3	0.1040	0.00E+00	1	2	0.2292	0.00E+00	1	2	0.0971	0
	m_2	1	2	0.5020	0	1	2	0.3929	0.00E+00	1	2	0.0608	0
	m_3	2	3	0.6183	4.97E-16	1	2	0.5716	0.00E+00	1	2	1.2024	0.00E+00
	m_4	3	4	0.6508	0.00E+00	6	8	0.8017	2.77E-08	8	9	1.1241	5.58E-07
	m_5	3	4	2.7106	0.00E+00	6	8	0.2833	2.77E-08	8	9	0.5422	5.58E-07
	m_6	2	3	0.2422	0.00E+00	4	6	0.4995	1.54E-08	5	6	0.1569	2.37E-07
	m_7	3	4	0.4045	0.00E+00	6	8	0.8315	2.77E-08	8	9	0.6378	5.58E-07
	m_8	3	4	1.2673	0.00E+00	6	8	0.2407	2.77E-08	8	9	2.2426	5.58E-07

Table 6. Numerical results of the three algorithms on Problem 5.

		iSDFM				DAIS1				MSGPALG			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	m_1	24	26	0.0584	9.23E-07	4	6	0.0278	4.28E-07	6	7	0.0106	3.94E-07
	m_2	23	25	0.0523	9.30E-07	4	6	0.0085	2.03E-07	7	8	0.0083	3.89E-07
	m_3	12	14	0.0252	5.60E-07	6	8	0.0112	1.37E-08	7	8	0.0103	5.04E-07
	m_4	12	14	0.0248	8.98E-07	7	9	0.0125	4.92E-08	8	9	0.0077	1.11E-07
	m_5	12	14	0.0275	8.98E-07	7	9	0.0142	4.92E-08	8	9	0.0118	1.11E-07
	m_6	19	21	0.0250	8.22E-07	8	10	0.0347	8.90E-09	8	9	0.0077	2.35E-07
	m_7	12	14	0.0258	8.98E-07	7	9	0.0133	4.92E-08	8	9	0.0090	1.11E-07
	m_8	12	14	0.0186	8.96E-07	7	9	0.0120	5.06E-08	8	9	0.0098	1.12E-07
5000	m_1	28	30	0.1528	8.04E-07	4	6	0.0764	9.58E-07	6	7	0.1398	8.81E-07
	m_2	27	29	0.6379	8.10E-07	4	6	0.3117	4.53E-07	7	8	0.4172	8.70E-07
	m_3	15	17	0.0647	5.18E-07	6	8	0.2211	1.30E-08	8	9	0.0829	5.70E-08
	m_4	16	18	0.1284	7.87E-07	7	9	0.0740	1.11E-07	8	9	0.1978	2.50E-07
	m_5	16	18	1.3222	7.87E-07	7	9	0.2236	1.11E-07	8	9	0.5069	2.50E-07
	m_6	15	17	0.0576	6.47E-07	7	9	0.7438	5.65E-07	8	9	0.9363	8.56E-07
	m_7	16	18	0.5002	7.87E-07	7	9	0.5365	1.11E-07	8	9	0.7172	2.50E-07
	m_8	16	18	0.2607	7.86E-07	7	9	0.0450	1.12E-07	8	9	0.0370	2.51E-07
10000	m_1	29	31	2.2261	9.03E-07	5	7	0.5164	7.20E-09	7	8	0.0696	6.32E-08
	m_2	28	30	0.1845	9.10E-07	4	6	0.2108	6.40E-07	8	9	0.3887	6.24E-08
	m_3	15	17	0.1033	6.58E-07	6	8	0.0598	1.23E-08	8	9	1.9183	8.06E-08
	m_4	17	19	1.0910	8.89E-07	7	9	1.2817	1.58E-07	8	9	0.0453	3.54E-07
	m_5	17	19	0.1082	8.89E-07	7	9	0.0601	1.58E-07	8	9	0.2457	3.54E-07
	m_6	15	17	0.4385	7.90E-07	8	10	0.2200	1.32E-08	9	10	0.4761	5.62E-08
	m_7	17	19	0.5594	8.89E-07	7	9	0.1719	1.58E-07	8	9	0.0590	3.54E-07
	m_8	17	19	2.5253	8.89E-07	7	9	0.0791	1.58E-07	8	9	0.4212	3.54E-07
50000	m_1	32	34	2.0660	9.88E-07	5	7	0.2138	1.61E-08	7	8	0.2570	1.41E-07
	m_2	31	33	1.7560	9.95E-07	5	7	1.2728	7.61E-09	8	9	0.3223	1.39E-07
	m_3	18	20	1.1115	9.18E-07	6	8	0.6486	1.15E-08	8	9	4.0013	1.80E-07
	m_4	20	22	1.9429	9.68E-07	7	9	0.4158	3.53E-07	8	9	0.3523	7.92E-07
	m_5	20	22	0.5584	9.68E-07	7	9	0.9673	3.53E-07	8	9	0.2946	7.92E-07
	m_6	17	19	1.4478	9.79E-07	8	10	0.2156	4.69E-08	9	10	0.2300	5.16E-08
	m_7	20	22	4.0180	9.68E-07	7	9	1.8746	3.53E-07	8	9	0.6637	7.92E-07
	m_8	20	22	0.6509	9.68E-07	7	9	0.2024	3.54E-07	8	9	0.1919	7.92E-07
100000	m_1	34	36	3.6145	8.70E-07	5	7	0.9745	2.28E-08	7	8	2.1130	2.00E-07
	m_2	33	35	2.2467	8.77E-07	5	7	2.2109	1.08E-08	8	9	1.3419	1.97E-07
	m_3	16	18	1.2758	1.67E-07	6	8	0.4651	1.13E-08	8	9	0.2564	2.55E-07
	m_4	22	24	2.1591	8.54E-07	7	9	0.8911	5.00E-07	9	10	0.7061	5.68E-08
	m_5	22	24	2.1281	8.54E-07	7	9	0.3940	5.00E-07	9	10	0.7502	5.68E-08
	m_6	21	23	1.4187	9.59E-07	8	10	2.1780	5.35E-08	8	9	0.2659	9.73E-07
	m_7	22	24	2.3113	8.54E-07	7	9	0.8201	5.00E-07	9	10	2.4092	5.68E-08
	m_8	22	24	1.9841	8.54E-07	7	9	0.5536	5.00E-07	9	10	0.7111	5.68E-08

Table 7. Numerical results of the three algorithms on Problem 6.

		iSDFM				DAIS1				MSGPALG			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	m_1	1	2	0.0438	0.00E+00	5	7	0.0154	3.13E-08	6	7	0.0362	5.08E-07
	m_2	2	3	0.0055	0.00E+00	4	6	0.0116	4.07E-09	6	7	0.0090	7.74E-08
	m_3	1	2	0.0059	3.14E-16	4	6	0.0074	1.95E-08	6	7	0.0126	1.35E-07
	m_4	2	3	0.0056	0.00E+00	10	12	0.0173	6.36E-08	9	10	0.0108	2.10E-08
	m_5	2	3	0.0070	0.00E+00	10	12	0.0204	6.36E-08	9	10	0.0153	2.10E-08
	m_6	2	3	0.0053	0.00E+00	6	8	0.0132	1.63E-08	8	9	0.0082	2.34E-07
	m_7	2	3	0.0056	0.00E+00	10	12	0.0232	6.36E-08	9	10	0.0091	2.10E-08
	m_8	2	3	0.0099	0.00E+00	10	12	0.0185	9.54E-08	9	10	0.0085	2.17E-08
5000	m_1	1	2	0.0374	0.00E+00	5	7	1.0327	7.00E-08	7	8	0.5082	2.23E-08
	m_2	2	3	0.0205	0.00E+00	4	6	0.0221	9.11E-09	6	7	1.3592	1.73E-07
	m_3	1	2	0.0747	3.14E-16	4	6	0.3235	1.95E-08	6	7	0.2540	1.35E-07
	m_4	2	3	0.5673	0.00E+00	11	13	0.3735	4.00E-09	9	10	0.0230	4.77E-08
	m_5	2	3	0.1877	0.00E+00	11	13	0.2003	4.00E-09	9	10	0.2913	4.77E-08
	m_6	2	3	0.0243	0.00E+00	6	8	0.5030	1.63E-08	8	9	0.0371	2.34E-07
	m_7	2	3	0.0166	0.00E+00	11	13	0.3037	4.00E-09	9	10	0.0259	4.77E-08
	m_8	2	3	0.1480	0.00E+00	11	13	0.7863	4.34E-09	9	10	1.1625	4.80E-08
10000	m_1	1	2	1.5394	0.00E+00	5	7	0.3123	9.90E-08	7	8	0.1095	3.15E-08
	m_2	2	3	0.3070	0.00E+00	4	6	0.7642	1.29E-08	6	7	0.0387	2.45E-07
	m_3	1	2	0.0315	3.14E-16	4	6	0.0600	1.95E-08	6	7	0.6297	1.35E-07
	m_4	2	3	0.2660	0.00E+00	11	13	0.4963	1.34E-08	9	10	2.2974	6.75E-08
	m_5	2	3	0.3858	0.00E+00	11	13	1.1199	1.34E-08	9	10	0.0373	6.75E-08
	m_6	2	3	0.1753	0.00E+00	6	8	0.2915	1.63E-08	8	9	2.9723	2.34E-07
	m_7	2	3	0.5146	0.00E+00	11	13	0.4325	1.34E-08	9	10	0.1870	6.75E-08
	m_8	2	3	0.2806	0.00E+00	11	13	0.1227	1.42E-08	9	10	0.0536	6.78E-08
50000	m_1	1	2	1.1916	0.00E+00	5	7	0.2469	2.21E-07	7	8	0.1870	7.04E-08
	m_2	2	3	0.9393	0.00E+00	4	6	1.1379	2.88E-08	6	7	1.6610	5.47E-07
	m_3	1	2	0.0926	3.14E-16	4	6	0.2889	1.95E-08	6	7	0.1471	1.35E-07
	m_4	2	3	1.2706	0.00E+00	11	13	1.1981	2.10E-07	9	10	1.4916	1.51E-07
	m_5	2	3	0.1308	0.00E+00	11	13	0.5316	2.10E-07	9	10	0.1756	1.51E-07
	m_6	2	3	0.8749	0.00E+00	6	8	0.7440	1.63E-08	8	9	2.3692	2.34E-07
	m_7	2	3	0.1153	0.00E+00	11	13	0.3875	2.10E-07	9	10	0.1597	1.51E-07
	m_8	2	3	1.6229	0.00E+00	11	13	1.9796	2.12E-07	9	10	0.2254	1.51E-07
100000	m_1	1	2	0.1545	0.00E+00	5	7	0.3623	3.13E-07	7	8	0.9581	9.96E-08
	m_2	2	3	0.5466	0.00E+00	4	6	0.3900	4.07E-08	6	7	0.2321	7.74E-07
	m_3	1	2	0.4549	3.14E-16	4	6	0.6795	1.95E-08	6	7	0.4348	1.35E-07
	m_4	2	3	1.1598	0.00E+00	11	13	0.5917	5.11E-07	9	10	0.9209	2.14E-07
	m_5	2	3	0.2296	0.00E+00	11	13	2.1472	5.11E-07	9	10	0.9297	2.14E-07
	m_6	2	3	0.5503	0.00E+00	6	8	0.3154	1.63E-08	8	9	1.7438	2.34E-07
	m_7	2	3	1.1678	0.00E+00	11	13	1.3542	5.11E-07	9	10	1.0237	2.14E-07
	m_8	2	3	0.2391	0.00E+00	11	13	0.6503	5.14E-07	9	10	0.3572	2.14E-07

Table 8. Numerical results of the three algorithms on Problem 7.

		iSDFM				DAIS1				MSGPALG			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	m_1	19	21	0.0328	9.46E-07	67	69	0.1959	9.10E-07	67	68	0.0613	9.89E-07
	m_2	52	53	0.0584	9.20E-07	101	103	0.1120	9.84E-07	61	62	0.0533	9.84E-07
	m_3	57	59	0.0884	9.41E-07	76	78	0.1511	9.29E-07	87	88	0.0644	9.38E-07
	m_4	55	57	0.0840	6.99E-07	83	85	0.2007	9.36E-07	83	84	0.0643	8.91E-07
	m_5	56	58	0.0663	8.51E-07	81	83	0.1548	9.41E-07	83	84	0.0635	8.91E-07
	m_6	47	49	0.0655	9.14E-07	81	83	0.1694	9.87E-07	68	69	0.0403	8.99E-07
	m_7	55	57	0.0831	9.64E-07	85	87	0.1402	9.92E-07	83	84	0.0378	8.91E-07
	m_8	54	56	0.0986	8.26E-07	83	85	0.2083	8.98E-07	82	83	0.0417	9.64E-07
5000	m_1	44	46	0.4975	9.72E-07	101	103	4.4357	9.03E-07	47	48	3.9616	7.95E-07
	m_2	41	43	2.1529	9.63E-07	91	93	0.4886	9.98E-07	93	94	0.1564	9.74E-07
	m_3	50	52	0.1946	9.94E-07	94	96	1.1680	9.55E-07	80	81	0.4414	9.74E-07
	m_4	53	55	0.3218	8.21E-07	89	91	4.2258	8.31E-07	65	66	0.7041	9.19E-07
	m_5	51	53	0.3422	9.93E-07	90	92	0.5222	8.89E-07	65	66	0.5813	9.19E-07
	m_6	56	58	1.4298	9.63E-07	73	75	0.8748	9.42E-07	56	57	0.2107	9.25E-07
	m_7	54	56	0.2158	9.01E-07	96	98	1.8010	6.68E-07	65	66	4.0542	9.19E-07
	m_8	52	54	0.5704	8.94E-07	89	91	1.7997	9.43E-07	67	68	1.2837	9.44E-07
10000	m_1	49	51	3.5965	8.84E-07	106	108	2.0185	8.15E-07	52	53	0.2338	8.65E-07
	m_2	39	41	0.3701	8.64E-07	91	93	1.3319	9.24E-07	94	95	1.4506	9.72E-07
	m_3	54	56	1.3825	8.64E-07	95	97	1.5026	9.24E-07	70	71	0.6796	8.77E-07
	m_4	56	58	3.5432	6.98E-07	96	98	2.1861	9.18E-07	57	58	4.7632	8.81E-07
	m_5	52	54	0.9342	8.00E-07	97	99	1.6761	9.85E-07	57	58	0.4527	8.81E-07
	m_6	59	60	1.5782	8.65E-07	91	93	1.8383	9.48E-07	58	59	0.5386	7.32E-07
	m_7	57	59	0.4248	7.88E-07	103	105	1.7186	7.81E-07	57	58	0.4381	8.81E-07
	m_8	58	60	1.2930	9.43E-07	99	101	2.3954	9.07E-07	53	54	4.5234	6.99E-07
50000	m_1	46	48	5.1900	8.32E-07	74	76	5.7164	9.08E-07	36	37	0.6755	6.71E-07
	m_2	55	57	2.7338	9.19E-07	96	98	14.8517	9.62E-07	95	96	2.5677	9.62E-07
	m_3	62	64	2.8549	8.89E-07	101	103	7.6005	9.46E-07	88	89	2.2433	9.13E-07
	m_4	59	61	2.7944	7.58E-07	105	107	7.4685	9.10E-07	81	82	2.4065	9.25E-07
	m_5	61	63	2.9541	7.11E-07	103	105	5.6730	9.09E-07	81	82	2.6295	9.25E-07
	m_6	54	56	2.7075	9.94E-07	102	104	5.4700	9.54E-07	56	57	1.5636	8.15E-07
	m_7	63	65	3.1767	8.45E-07	101	103	5.2194	9.21E-07	81	82	2.1452	9.25E-07
	m_8	60	62	2.8825	9.99E-07	96	98	5.2916	9.82E-07	81	82	2.2370	9.30E-07
100000	m_1	35	36	3.2271	9.76E-07	110	112	11.6739	9.71E-07	50	51	2.3679	9.55E-07
	m_2	58	60	4.4230	9.75E-07	103	105	13.3138	8.57E-07	67	68	2.9833	7.51E-07
	m_3	57	59	5.5342	8.21E-07	80	82	10.3855	9.39E-07	72	73	3.3898	9.29E-07
	m_4	55	56	5.0986	9.16E-07	88	90	10.7673	7.47E-07	88	89	4.4502	9.38E-07
	m_5	55	57	5.3401	9.05E-07	99	101	9.7451	9.95E-07	88	89	4.9388	9.38E-07
	m_6	57	59	5.4860	9.28E-07	103	105	11.7762	7.17E-07	74	75	3.1878	9.23E-07
	m_7	59	60	5.4163	8.97E-07	88	90	9.7121	8.46E-07	88	89	3.9409	9.38E-07
	m_8	54	56	5.1927	9.03E-07	99	101	10.0416	9.68E-07	88	89	3.6687	9.40E-07

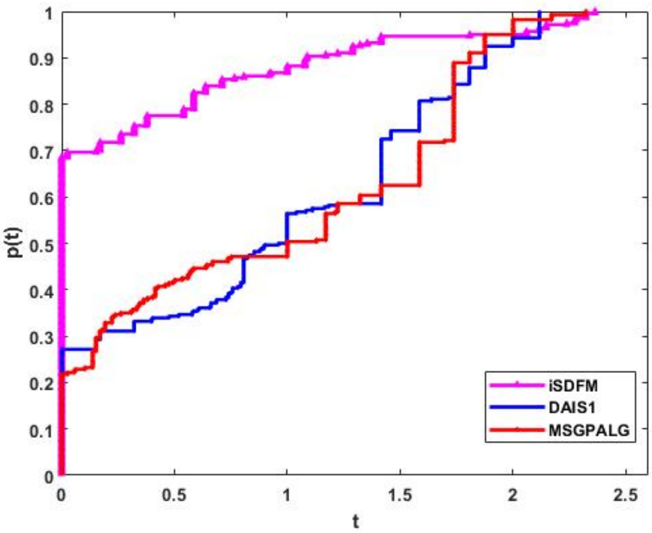


Figure 2. Performance profile on function evaluations.

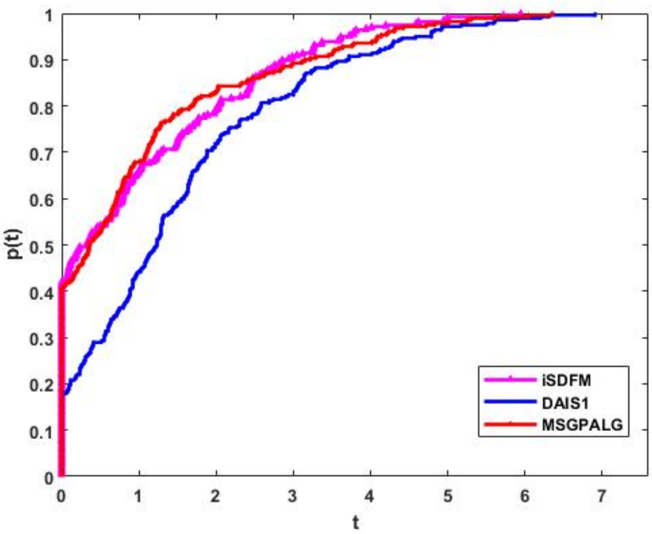


Figure 3. Performance profile on CPU time.

4. Application in motion control

In robotics, manipulators and effectors are the aspects in which some parts of the robots interact with other objects by performing different tasks such as picking from one point and placing on another. For stability and accuracy in the robots' movement, the characteristics of motor dynamics, which are contemporarily used as actuators in $n-$link and $1-$link robot systems, need to be considered [34, 35]. This is a tracking control problem of a nonlinear system, and the motor dynamics are required to satisfy the condition that the actual output of the system can track the desired trajectory with least possible error [36].

Some of the developed methods for tracking control problems of nonlinear systems include proportional-integral-derivative (PID) control [37, 38], feedback linearization [39, 40] and optimal output tracking control by using approximation approach [41].

We present an application of our proposed algorithm in motion control of two planar robotic manipulators. Consider the following model:

$$\min_{m\in\mathbb{R}^n} f(m), \tag{4.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth and convex function. In the case of motion control, the function in (4.1) has the form $f(v_k) := \frac{1}{2}\|\upsilon_k - y_{uk}\|_2^2$. Subsequently, we minimize

$$\min_{\upsilon_k\in\mathbb{R}^2} \frac{1}{2}\|\upsilon_k - y_{uk}\|_2^2 \tag{4.2}$$

at each computational time interval $\tau_k \in [0, \tau_f]$.

As described in [42], the discrete-time kinematics equation of a two-joint planar robot manipulator at the position level is given as

$$\psi(\theta_k) = \upsilon_k. \tag{4.3}$$

The kinematics map $\psi(\cdot)$ is given as

$$\psi(\theta) = \begin{bmatrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}, \tag{4.4}$$

where l_1 and l_2 are the lengths of the rod links, and $\theta_k \in \mathbb{R}^2$ is the joint angle vector.

From (4.2), the term υ_k is controlled to track a Lissajous curve:

$$y_{uk} = \begin{bmatrix} \frac{3}{2} + \frac{1}{5}\sin(3\tau_k) \\ \frac{\sqrt{3}}{2} + \frac{1}{5}\sin(2\tau_k) \end{bmatrix}. \tag{4.5}$$

For our algorithm to fit (4.1), we present its slight modification as follows:

Algorithm 2: Modified iSDFM.

Input : Let $\alpha_k = 0$ for $k \geq 0$ in Algorithm 1. Consider the same inputs as in Algorithm 1, and let $\Gamma(m_k) = \nabla f(m_k)$. Replace Step 2 and Step 3 of the Algorithm 1 with the followings:

Step 2: Compute $\Lambda_k = \kappa\varsigma^j$, where j is the smallest non–negative integer such that

$$f(m_k + \kappa\varsigma^j t_k) - f(m_k) \leq \sigma\kappa\varsigma^j \Gamma(m_k)^T t_k. \tag{4.6}$$

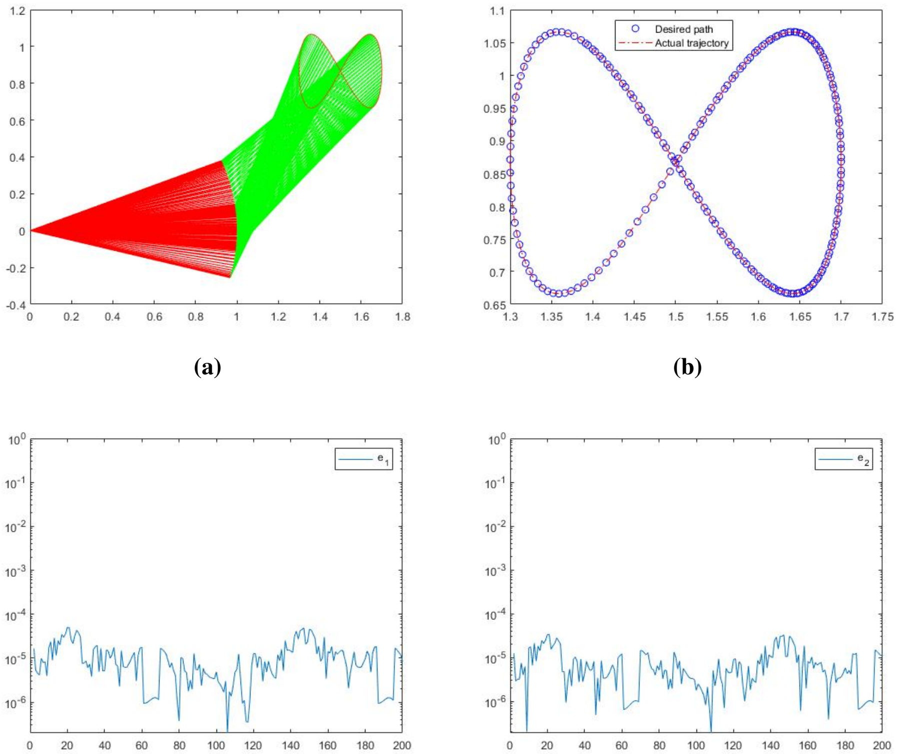
Step 3: To update the next iterate, use

$$m_{k+1} := m_k + \Lambda_k t_k. \tag{4.7}$$

Remark 4.1. *Assume that the solution to problem* (4.1) *exists and that the level set* $\{m \in \mathbb{R}^n : f(m) \leq f(m_0)\}$ *is bounded. Using Assumption* A_2 *and Theorem 2 of [43], we conclude that Algorithm 2 converges, that is,* $\liminf_{k\to\infty} \|\Gamma(m_k)\| = 0$ *holds.*

In this experiment, we chose initial joint angle vector $\theta_0 = [0, \frac{\pi}{3}]^T$, $l_1 = l_2 = 1$, $\varsigma = 0.2$ and $\sigma = 0.08$. The task duration $[0, 20]$ is subdivided into 200 equal parts. The performance of Algorithm 2 in the motion control problem is shown in the Figure 4.

Figure 4(a) represents the robot trajectories synthesized by Algorithm 2. Figure 4(b) shows the end effector and the desired path. Figure 4(c,d) shows the performance errors on x and y axes, respectively. These figures indicate that Algorithm 2 efficiently performed the assignment with an error as low as 10^{-5} on both the x-axis and y-axis.



(a) (b) (c) (d)

Figure 4. Performance generated by Algorithm 2: (a) robot trajectories; (b) end effector trajectory and desired path; (c) residual error on x-axis; (d) residual error on y-axis.

5. Conclusions

In conclusion, we have proposed an inertial-based spectral method for solving a system of nonlinear equations. The inertial-step introduced is believed to have enhanced the performance of the new method. We have discussed the global convergence of the proposed algorithm under the monotonicity and Lipschitz continuity assumptions. We have also presented some numerical experiments which depicted the efficiency of the proposed algorithm. The proposed algorithm is reported to have better numerical performance than the methods in [21, 22]. Subsequently, we have demonstrated the applicability of the new method in motion control problems arising from robotics. However, the numerical results showed that the proposed method won about 72% and 70% of the experiments in terms of ITER and FVAL. Therefore, we recommend further research in order to improve its performance to the optimum.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions. The sixth author was funded by Chiang Mai University and the NSRF via the Program Management Unit for Human Resources & Institutional Development, Research and Innovation (grant number B05F640183).

Conflict of interest

The authors declare that they have no conflicts of interest.

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