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Research article

On fixed points, their geometry and application to satellite web coupling problem in $\mathcal{S}-$metric spaces

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Abstract: We introduce an $\mathcal{M}-$class function in an $\mathcal{S}-$metric space which is a viable, productive, and powerful technique for finding the existence of a fixed point and fixed circle. Our conclusions unify, improve, extend, and generalize numerous results to a widespread class of discontinuous maps. Next, we introduce notions of a fixed ellipse (elliptic disc) in an $\mathcal{S}-$metric space to investigate the geometry of the collection of fixed points and prove fixed ellipse (elliptic disc) theorems. In the sequel, we validate these conclusions with illustrative examples. We explore some conditions which eliminate the possibility of the identity map in the existence of an ellipse (elliptic disc). Some remarks, propositions, and examples to exhibit the feasibility of the results are presented. The paper is concluded with a discussion of activation functions that are discontinuous in nature and, consequently, utilized in a neural network for increasing the storage capacity. Towards the end, we solve the satellite web coupling problem and propose two open problems.

Keywords: continuity; fixed ellipse; fixed elliptic disc; $\mathcal{M}-$class function; $\mathcal{S}-$Caristi map
Mathematics Subject Classification: 47H10, 54H25, 55M20, 37E10

1. Introduction

Many real-world problems have a need to find a distance connecting two or more items that may not be easy to measure accurately. Consequently, to model distinct problems of practical nature, we require an appropriate metric. There exist several approaches to measure the distance more precisely which are being utilized to widen the extent of the investigation of fixed point theory. Non-unique and unique fixed point conclusions have been widely investigated with a different outlook via distinct metrics in the theory of fixed points (for instance, [1, 2, 7–10, 24, 36, 42–46] and so on). Recently, the geometry of the collection of fixed points has been considered in various forms, such as the fixed circle problem, fixed disc problem, fixed ellipse problem, fixed elliptic disc problem and so on. The most general form of these problems is the "fixed figure problem."

In the current work, we introduce an $\mathcal{M}-$class function to establish a unique fixed point and fixed circle via the $\mathcal{S}-$metric introduced by Sedghi et al. [37]. Further, we investigate the notion of a fixed ellipse (elliptic disc) in an $\mathcal{S}-$metric space to conclude that the set of fixed points incorporates an ellipse (elliptic disc) under appropriate conditions and verify this by illustrative examples. In the sequel, we explore some conditions which eliminate the possibility of the identity map in the existence of an ellipse (elliptic disc) in an $\mathcal{S}-$metric space. Further, we give propositions for the existence of a self-map that fixes the given ellipse (elliptic disc) and demonstrate that an ellipse (elliptic disc) contains all the points of space except its foci. It is fascinating to mention that the uniqueness of a fixed ellipse, as well as the existence of the greatest fixed elliptic disc, may be established using celebrated contractive conditions like Banach contraction [4], Ćirić contraction [8], Quasi contraction [9], Rhoades contraction [35] and so on. These fixed ellipse (elliptic disc) conclusions encourage further investigations and implementations in $\mathcal{S}-$metric spaces. It is significant to mention that the collection of fixed points carries out a significant role in the theory of fixed points and may form some geometrical shapes like circles, discs, ellipses or elliptic discs. In particular, the ellipse has numerous applications in Physics, Astronomy, Neural Networks, Biology, Artificial Intelligence, Economics and so on.

2. Preliminaries

Definition 2.1. *[37] An $\mathcal{S}-$metric on a non-empty set $\mathcal{U}$ is a function $\mathcal{S}:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\longrightarrow\mathbb{R}^+$ so that*

($\mathcal{S}_1$) $\mathcal{S}(\omega,\upsilon,u)\geq 0$;
($\mathcal{S}_2$) $\mathcal{S}(\omega,\upsilon,u)=0$ *iff* $\omega=\upsilon=u$;
($\mathcal{S}_3$) $\mathcal{S}(\omega,\upsilon,u)\leq\mathcal{S}(\omega,\omega,\alpha)+\mathcal{S}(\upsilon,\upsilon,\alpha)+\mathcal{S}(u,u,\alpha),\ \ \omega,\upsilon,u,\alpha\in\mathcal{U}.$

Geometrically, we connect three points ω, υ and u to get a triangle, and if α is a point mediating this triangle, then ($\mathcal{S}_3$) holds.

Remark 2.1. *[37] In an $\mathcal{S}-$metric space $\mathcal{S}(\omega,\omega,\upsilon)=\mathcal{S}(\upsilon,\upsilon,\omega)$.*

Definition 2.2. *[37] Let $\{\omega_n\}$ be a sequence in an $\mathcal{S}-$metric space $(\mathcal{U},\mathcal{S})$. Then,*

(1) $\{\omega_n\}$ *is convergent to* $\omega\in\mathcal{U}$ *if* $\lim_{n\to\infty}\mathcal{S}(\omega_n,\omega_n,\omega)=0$;
(2) $\{\omega_n\}$ *is a Cauchy sequence if* $\lim_{n,m\to\infty}\mathcal{S}(\omega_n,\omega_n,\omega_m)=0,\ n,m>N$;
(3) $(\mathcal{S},\mathcal{U})$ *is complete if every Cauchy sequence in* $\mathcal{U}$ *converges to a point in* $\mathcal{U}$.

Definition 2.3. *[26] Let $(\mathcal{S},\mathcal{U})$ be an $\mathcal{S}-$metric space, and $\mathcal{C}(\omega_0,r)=\{\omega\in\mathcal{U}:\mathcal{S}(\omega,\omega,\omega_0)=r,r>0\}$ is a circle centered on ω_0 with a radius r. For a self-map $\mathcal{A}:\mathcal{U}\to\mathcal{U}$, if $\mathcal{A}\omega=\omega,\omega\in\mathcal{C}(\omega_0,r)$, then $\mathcal{C}(\omega_0,r)$ is called a fixed circle of $\mathcal{A}$.*

3. Main results

First, we introduce an $\mathcal{M}-$class function in an $\mathcal{S}-$metric space which may be used as a tool to find a fixed point of contraction maps as well as explore its geometry. These functions also give the assurance of a fixed point, fixed circle (disc) and fixed ellipse (elliptic disc), and they unify, improve, extend and generalize numerous existing conclusions in the literature to $\mathcal{S}-$metric spaces.

We denote a set of continuous functions $\mathfrak{f}:[0,\infty)^5 \to [0,\infty)$ by $\mathcal{M}$, satisfying the following:

($\mathfrak{f}_1$) $\mathfrak{f}(1,1,0,3,1) \in [0,1)$;

($\mathfrak{f}_2$) $\mathfrak{f}$ is a linear homogeneous function, that is,

$$\mathfrak{f}(\lambda\omega) = \lambda\mathfrak{f}(\omega) \text{ or } \mathfrak{f}(\lambda\omega_1,\lambda\omega_2,\lambda\omega_3,\lambda\omega_4,\lambda\omega_5) = \lambda\mathfrak{f}(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5),$$

where $\omega = (\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) \in [0,\infty)^5$, $\lambda \geq 0$;

($\mathfrak{f}_3$) $\mathfrak{f}$ is a non-decreasing function, that is,

$$\omega \leq \nu \Rightarrow \mathfrak{f}\omega \leq \mathfrak{f}\nu \text{ or } \omega_i \leq \nu_i,\ i = 1,2,\ldots,5 \Rightarrow \mathfrak{f}(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) \leq \mathfrak{f}(\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,\upsilon_5),$$

where $\omega = (\omega_1,\omega_2,\omega_3,\omega_4,\omega_5)$ and $\upsilon = (\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4,\upsilon_5) \in [0,\infty)^5$.

Then, function $\mathfrak{f}$ is said to be an $\mathcal{M}$-class function.

Example 3.1. *Define* $\mathfrak{f}_1 : [0,\infty)^5 \to [0,\infty)$ *by* $\mathfrak{f}_1(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) = \alpha\omega_1,\ \alpha \in [0,1)$. *Then,* $\mathfrak{f}_1 \in \mathcal{M}$.

Example 3.2. *Define* $\mathfrak{f}_2 : [0,\infty)^5 \to [0,\infty)$ *by* $\mathfrak{f}_2(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) = \alpha(\omega_2+\omega_5),\ \alpha \in [0,\frac{1}{3})$. *Then,* $\mathfrak{f}_2 \in \mathcal{M}$.

Example 3.3. *Define* $\mathfrak{f}_3 : [0,\infty)^5 \to [0,\infty)$ *by* $\mathfrak{f}_3(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) = \alpha\max\{\omega_2,\omega_5\},\ \alpha \in [0,\frac{1}{3})$. *Then,* $\mathfrak{f}_3 \in \mathcal{M}$.

Example 3.4. *Define* $\mathfrak{f}_4 : [0,\infty)^5 \to [0,\infty)$ *by* $\mathfrak{f}_4(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) = \alpha\omega_1+\beta\omega_2+\gamma\omega_5,\ \alpha+\beta+\gamma \in [0,1)$. *Then,* $\mathfrak{f}_4 \in \mathcal{M}$.

Example 3.5. *Define* $\mathfrak{f}_5 : [0,\infty)^5 \to [0,\infty)$ *by* $\mathfrak{f}_4(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) = \max\{\omega_1,\omega_2,\omega_5\}$. *Then,* $\mathfrak{f}_5 \in \mathcal{M}$.

Example 3.6. *Define* $\mathfrak{f}_6 : [0,\infty)^5 \to [0,\infty)$ *by* $\mathfrak{f}_4(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) = \alpha(\omega_3+\omega_4),\ \alpha \in [0,\frac{1}{3})$. *Then,* $\mathfrak{f}_6 \in \mathcal{M}$.

Example 3.7. *Define* $\mathfrak{f}_7 : [0,\infty)^5 \to [0,\infty)$ *by* $\mathfrak{f}_7(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) = \alpha\max\{\omega_3,\omega_4\},\ \alpha \in [0,\frac{1}{3})$. *Then,* $\mathfrak{f}_7 \in \mathcal{M}$.

Example 3.8. *Define* $\mathfrak{f}_8 : [0,\infty)^5 \to [0,\infty)$ *by*

$$\mathfrak{f}_8(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) = \alpha\omega_1+\beta\omega_2+\gamma(\omega_3+\omega_4)+\delta\omega_5,\ \alpha+\beta+3\gamma+\delta \in [0,1).$$

Then, $\mathfrak{f}_8 \in \mathcal{M}$.

Example 3.9. *Define* $\mathfrak{f}_9 : [0,\infty)^5 \to [0,\infty)$ *by*

$$\mathfrak{f}_9(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) = \alpha\max\{\omega_1,\omega_2,\omega_3,\omega_4,\omega_5\},\ \alpha \in [0,\frac{1}{3}).$$

Then, $\mathfrak{f}_9 \in \mathcal{M}$.

Lemma 3.10. *If* $\mathfrak{f} \in \mathcal{M}$ *and* $\omega, \upsilon \in [0,\infty)$ *are such that*

$$\begin{aligned} \omega \leq &\max\{\mathfrak{f}(\upsilon,\omega,0,\upsilon+2\omega,\omega),\mathfrak{f}(\upsilon+2\omega,\omega,0,\upsilon,\upsilon), \\ &\mathfrak{f}(\upsilon,\upsilon+2\omega,0,\upsilon,\omega),\mathfrak{f}(\omega,\upsilon,0,\omega,\upsilon+2\omega)\}, \end{aligned} \tag{3.1}$$

then $\omega \leq \eta\upsilon$, *where* $\eta = \mathfrak{f}(1,1,0,3,1) \in [0,1)$.

Proof. We may presume without loss of generality that $\omega \leq \mathfrak{f}(\upsilon,\omega,0,\upsilon+2\omega,\omega)$. If $\upsilon < \omega$,

$$\omega \leq \mathfrak{f}(\upsilon,\omega,0,\upsilon+2\omega,\omega) < \mathfrak{f}(\omega,\omega,0,3\omega,\omega) = \omega\mathfrak{f}(1,1,0,3,1) \leq \omega,$$

a contradiction. Thus, $\omega \leq \upsilon$. Also,

$$\omega \leq \mathfrak{f}(\upsilon,\omega,0,\upsilon+2\omega,\omega) < \mathfrak{f}(\upsilon,\upsilon,0,3\upsilon,\upsilon) = \upsilon\mathfrak{f}(1,1,0,3,1) = \eta\upsilon.$$

□

Theorem 3.11. *Let* $(\mathcal{U},\mathcal{S})$ *be a complete* $\mathcal{S}-$*metric space. For all* $\omega \neq \upsilon \in \mathcal{U}$, $\mathfrak{f} \in \mathcal{M}$ *and* $\mathcal{A}:\mathcal{U}\to\mathcal{U}$, *if* $\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\upsilon) > 0$ *implies*

$$\begin{aligned} \mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\upsilon) \leq &\mathfrak{f}\big(\mathcal{S}(\omega,\omega,\upsilon),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\upsilon), \\ &\mathcal{S}(\mathcal{A}\upsilon,\mathcal{A}\upsilon,\omega),\mathcal{S}(\mathcal{A}\upsilon,\mathcal{A}\upsilon,\upsilon)\big), \end{aligned} \tag{3.2}$$

then $\mathcal{A}$ *has a unique fixed point.*

Proof. For arbitrary $\omega_0 \in \mathcal{U}$, let $\omega_1 = \mathcal{A}\omega_0$. If $\omega_1 = \mathcal{A}\omega_1$, ω_1 is a fixed point of $\mathcal{A}$, and the proof is concluded. So, consider $\omega_1 \neq \mathcal{A}\omega_1$ and $\omega_2 = \mathcal{A}\omega_1$. Consequently, $\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\mathcal{A}\omega_1) > 0$, and then, using inequality (3.2) for $\omega = \omega_0$, $\upsilon = \omega_1$ and properties of $\mathcal{M}-$class function,

$$\begin{aligned} \mathcal{S}(\omega_1,\omega_1,\omega_2) &= \mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\mathcal{A}\omega_1) \\ &\leq \mathfrak{f}(\mathcal{S}(\omega_0,\omega_0,\omega_1),\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0),\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_1),\mathcal{S}(\mathcal{A}\omega_1,\mathcal{A}\omega_1,\omega_0),\mathcal{S}(\mathcal{A}\omega_1,\mathcal{A}\omega_1,\omega_1)) \\ &= \mathfrak{f}(\mathcal{S}(\omega_0,\omega_0,\omega_1),\mathcal{S}(\omega_1,\omega_1,\omega_0),\mathcal{S}(\omega_1,\omega_1,\omega_1),\mathcal{S}(\omega_2,\omega_2,\omega_0),\mathcal{S}(\omega_2,\omega_2,\omega_1)) \\ &= \mathfrak{f}(\mathcal{S}(\omega_0,\omega_0,\omega_1),\mathcal{S}(\omega_0,\omega_0,\omega_1),0,\mathcal{S}(\omega_0,\omega_0,\omega_2),\mathcal{S}(\omega_1,\omega_1,\omega_2)) \\ &\leq \mathfrak{f}(\mathcal{S}(\omega_0,\omega_0,\omega_1),\mathcal{S}(\omega_0,\omega_0,\omega_1),0,2\mathcal{S}(\omega_0,\omega_0,\omega_1)+\mathcal{S}(\omega_1,\omega_1,\omega_2),\mathcal{S}(\omega_1,\omega_1,\omega_2)). \end{aligned}$$

Now, using Lemma 3.10, we have

$$\mathcal{S}(\omega_1,\omega_1,\omega_2) \leq \eta\mathcal{S}(\omega_0,\omega_0,\omega_1). \tag{3.3}$$

Repeatedly, we get $\mathcal{S}(\omega_2,\omega_2,\omega_3) \leq \eta\mathcal{S}(\omega_1,\omega_1,\omega_2) \leq \eta^2\mathcal{S}(\omega_0,\omega_0,\omega_1)$.

Continuing like this,

$$\mathcal{S}(\omega_n,\omega_n,\omega_{n+1}) \leq \eta\mathcal{S}(\omega_{n-1},\omega_{n-1},\omega_n) \leq \cdots \leq \eta^n\mathcal{S}(\omega_0,\omega_0,\omega_1).$$

Now, define a Picard sequence $\omega_{n+1} = \mathcal{A}\omega_n$, $n \in \mathbb{N}\cup\{0\}$, with initial point $\omega_o \in \mathcal{U}$. If for some $n \in \mathbb{N}$, $\omega_n = \omega_{n+1} = \mathcal{A}\omega_n$, then ω_n is a fixed point of $\mathcal{A}$, and the proof is complete. So, presume that $\omega_n \neq \omega_{n+1}$, for all $n \in \mathbb{N}\cup\{0\}$. By using $(\mathcal{S}_3)$,

$$\begin{aligned}\mathcal{S}(\omega_n,\omega_n,\omega_{n+m}) &\le 2\mathcal{S}(\omega_n,\omega_n,\omega_{n+1})+\mathcal{S}(\omega_{n+m},\omega_{n+m},\omega_{n+1})\\&=2\mathcal{S}(\omega_n,\omega_n,\omega_{n+1})+\mathcal{S}(\omega_{n+1},\omega_{n+1},\omega_{n+m})\\&\le 2\mathcal{S}(\omega_n,\omega_n,\omega_{n+1})+2\mathcal{S}(\omega_{n+1},\omega_{n+1},\omega_{n+2})+\mathcal{S}(\omega_{n+m},\omega_{n+m},\omega_{n+2})\\&\le 2\mathcal{S}(\omega_n,\omega_n,\omega_{n+1})+2\mathcal{S}(\omega_{n+1},\omega_{n+1},\omega_{n+2})+\cdots+\mathcal{S}(\omega_{n+m},\omega_{n+m},\omega_{n+m-1})\\&\le 2[\mathcal{S}(\omega_n,\omega_n,\omega_{n+1})+\mathcal{S}(\omega_{n+1},\omega_{n+1},\omega_{n+2})+\cdots+\mathcal{S}(\omega_{n+m-1},\omega_{n+m-1},\omega_{n+m})]\\&\le 2\big(\eta^n+\eta^{n+1}+\cdots+\eta^{n+m-1}\big)\mathcal{S}(\omega_0,\omega_0,\omega_1)\\&=2\frac{\eta^n(1-\eta^m)}{1-\eta}\mathcal{S}(\omega_0,\omega_0,\omega_1)\to 0,\ \text{as } n\to\infty,\end{aligned}$$

and hence $\{\omega_n\}$ is a Cauchy sequence. Since $(\mathcal{S},\mathcal{U})$ is complete, $\{\omega_n\}$ converges to $\omega\in\mathcal{U}$, that is, $\lim_{n\to\infty}\mathcal{S}(\omega_n,\omega_n,\omega)=0$. We assert that ω is a fixed point of $\mathcal{A}$. If not, $\omega\neq\mathcal{A}\omega$, that is, $\mathcal{S}(\omega,\omega,\mathcal{A}\omega)>0$. Now,

$$\begin{aligned}&\mathcal{S}(\omega,\omega,\mathcal{A}\omega)\le 2\mathcal{S}(\omega,\omega,\omega_{n+1})+\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega_{n+1})\\&=2\mathcal{S}(\omega,\omega,\omega_{n+1})+\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\omega_n)\\&\le 2\mathcal{S}(\omega,\omega,\omega_{n+1})+\mathfrak{f}(\mathcal{S}(\omega,\omega,\omega_n),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega_n),\mathcal{S}(\mathcal{A}\omega_n,\mathcal{A}\omega_n,\omega),\mathcal{S}(\mathcal{A}\omega_n,\mathcal{A}\omega_n,\omega_n))\\&\le 2\mathcal{S}(\omega,\omega,\omega_{n+1})+\mathfrak{f}\big(\mathcal{S}(\omega,\omega,\omega_n),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega)+\mathcal{S}(\omega_n,\omega_n,\omega),\mathcal{S}(\mathcal{A}\omega_n,\mathcal{A}\omega_n,\omega),\\&\quad\mathcal{S}(\mathcal{A}\omega_n,\mathcal{A}\omega_n,\omega_n)\big)\\&=2\mathcal{S}(\omega,\omega,\omega_{n+1})+\mathfrak{f}\big(\mathcal{S}(\omega,\omega,\omega_n),\mathcal{S}(\omega,\omega,\mathcal{A}\omega),\mathcal{S}(\omega,\omega,\mathcal{A}\omega)+\mathcal{S}(\omega_n,\omega_n,\omega),\mathcal{S}(\omega_{n+1},\omega_{n+1},\omega),\\&\quad\mathcal{S}(\omega_{n+1},\omega_{n+1},\omega_n)\big).\end{aligned}$$

Since $\mathfrak{f}$ is continuous, as $n\to\infty$, using Lemma 3.10, we obtain

$$\mathcal{S}(\omega,\omega,\mathcal{A}\omega)\le\mathfrak{f}(0,\mathcal{S}(\omega,\omega,\mathcal{A}\omega),\mathcal{S}(\omega,\omega,\mathcal{A}\omega)+0,0,0).$$

Consequently, $\mathcal{S}(\omega,\omega,\mathcal{A}\omega)=0$, that is, $\mathcal{A}\omega=\omega$.

Now, suppose ω^* is another fixed point of $\mathcal{A}$, that is, $\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\omega^*)>0$, and

$$\begin{aligned}\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\omega^*)&\le\mathfrak{f}\big(\mathcal{S}(\omega,\omega,\omega^*),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega^*),\mathcal{S}(\mathcal{A}\omega^*,\mathcal{A}\omega^*,\omega),\mathcal{S}(\mathcal{A}\omega^*,\mathcal{A}\omega^*,\omega^*)\big)\\\mathcal{S}(\omega,\omega,\omega^*)&\le\mathfrak{f}\big(\mathcal{S}(\omega,\omega,\omega^*),\mathcal{S}(\omega,\omega,\omega),\mathcal{S}(\omega,\omega,\omega^*),\mathcal{S}(\omega^*,\omega^*,\omega),\mathcal{S}(\omega^*,\omega^*,\omega^*)\big)\\\mathcal{S}(\omega,\omega,\omega^*)&\le\mathfrak{f}\big(\mathcal{S}(\omega,\omega,\omega^*),0,\mathcal{S}(\omega,\omega,\omega^*),\mathcal{S}(\omega,\omega,\omega^*),0\big).\end{aligned}$$

Again by Lemma 3.10, $\mathcal{S}(\omega,\omega,\omega^*)=0$, that is, $\omega=\omega^*$. Hence, $\mathcal{A}$ has a unique fixed point in $\mathcal{U}$. □

Example 3.12. *Let $\mathcal{U}=[0,1]$ and an $\mathcal{S}$ metric $\mathcal{S}:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\to\mathbb{R}^+$ be*

$$\mathcal{S}(\omega,\upsilon,u)=|\omega-\upsilon|+|\omega+\upsilon-2u|,\omega,\upsilon,u\in\mathcal{U}.$$

Then, $(\mathcal{S},\mathcal{U})$ is a complete $\mathcal{S}-$metric space. Define maps $\mathfrak{f}:[0,\infty)^5\to[0,\infty)$ and $\mathcal{A}:\mathcal{U}\to\mathcal{U}$ as

$$\mathfrak{f}(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5)=\alpha(\omega_1+\omega_2+\omega_5),\ \alpha\in[0,\frac{1}{3})$$

and $\mathcal{A}\omega = \frac{\omega}{4\omega+1}$, *respectively. Then,* $\mathfrak{f} \in \mathcal{M}$. *Since*

$$\mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \mathcal{A}\upsilon) = 2|\mathcal{A}\omega - \mathcal{A}\upsilon| > 0,$$

$$\begin{aligned}
\mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \mathcal{A}\upsilon) &= 2\Big|\frac{\omega}{1+4\omega} - \frac{\upsilon}{1+4\upsilon}\Big| \\
&= 2\Big|\frac{\omega - \upsilon}{(1+4\omega)(1+4\upsilon)}\Big| \\
&\leq \frac{1}{4}\Big(|\omega - \upsilon| + 8\frac{\omega^2}{4\omega+1} + 8\frac{\upsilon^2}{4\upsilon+1}\Big) \\
&= \alpha(\mathcal{S}(\omega, \omega, \upsilon) + \mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \omega) + \mathcal{S}(\mathcal{A}\upsilon, \mathcal{A}\upsilon, \upsilon)), \ \alpha = \frac{1}{4},
\end{aligned}$$

that is, $\mathcal{A}$ *satisfies contraction condition (3.2), and* 0 *is a unique fixed point of* $\mathcal{A}$.

Corollary 3.13. *Theorem 3.11 also continues to be true if (3.2) is replaced by*

$$\mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \mathcal{A}\upsilon) \leq \alpha\mathcal{S}(\omega, \omega, \upsilon).$$

Proof. Define $\mathfrak{f} : [0,\infty)^5 \to [0,\infty)$ by $\mathfrak{f}(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) = \alpha\omega_1, \ \alpha \in [0,1)$. Then, $\mathfrak{f} \in \mathcal{M}$ and the proof complies with Theorem 3.11. □

Remark 3.1. *Corollary 3.13 is an enhancement of the Banach contraction theorem [4] in* $\mathcal{S}-$*metric space, which is the result of Sedghi et al. [37].*

Corollary 3.14. *Theorem 3.11 also continues to be true if (3.2) is replaced by*

$$\mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \mathcal{A}\upsilon) \leq \alpha\big(\mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \omega) + \mathcal{S}(\mathcal{A}\upsilon, \mathcal{A}\upsilon, \upsilon)\big), \ \alpha \in [0, \frac{1}{3}).$$

Proof. Define $\mathfrak{f} : [0,\infty)^5 \to [0,\infty)$ by $\mathfrak{f}(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) = \alpha(\omega_2 + \omega_5), \ \alpha \in [0, \frac{1}{3})$. Then, $\mathfrak{f} \in \mathcal{M}$, and the proof complies with Theorem 3.11. □

Remark 3.2. *Corollary 3.14 is an enhancement of the Kannan contraction theorem [20] in* $\mathcal{S}-$*metric space, which is the result of Phaneendra [33].*

Corollary 3.15. *Theorem 3.11 also continues to be true if (3.2) is replaced by*

$$\mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \mathcal{A}\upsilon) \leq \alpha\big(\mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \upsilon) + \mathcal{S}(\mathcal{A}\upsilon, \mathcal{A}\upsilon, \omega)\big), \ \alpha \in [0, \frac{1}{3}).$$

Proof. Define $\mathfrak{f} : [0,\infty)^5 \to [0,\infty)$ by $\mathfrak{f}(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) = \alpha(\omega_3 + \omega_4), \ \alpha \in [0, \frac{1}{3})$. Then, $\mathfrak{f} \in \mathcal{M}$, and the proof complies with Theorem 3.11. □

Remark 3.3. *Corollary 3.15 is an enhancement of the Chatterjee contraction theorem [7] in* $\mathcal{S}-$*metric space, which is the result of Phaneendra and Swamy [31].*

Corollary 3.16. *Theorem 3.11 also continues to be true if (3.2) is replaced by*

$$\begin{aligned}
\mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \mathcal{A}\upsilon) \leq& \alpha \max\{\mathcal{S}(\omega, \omega, \upsilon), \mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \omega), \mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \upsilon), \\
& \mathcal{S}(\mathcal{A}\upsilon, \mathcal{A}\upsilon, \omega), \mathcal{S}(\mathcal{A}\upsilon, \mathcal{A}\upsilon, \upsilon)\}, \ \alpha \in [0, \frac{1}{3}).
\end{aligned}$$

Proof. Define $\mathfrak{f}:[0,\infty)^5 \to [0,\infty)$ by $\mathfrak{f}(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) = \alpha\{\omega_1,\omega_2,\omega_3,\omega_4,\omega_5\}\ \alpha \in [0,\frac{1}{3})$. Then, $\mathfrak{f} \in \mathcal{M}$, and the proof complies with Theorem 3.11. □

Remark 3.4. *Corollary 3.16 is an enhancement of the Ćirić type contraction theorem [9] in $\mathcal{S}-$metric space, which is the result of Phaneendra and Swamy [31].*

Corollary 3.17. *Theorem 3.11 also continues to be true if (3.2) is replaced by*

$$\begin{aligned}\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\upsilon) \leq& \alpha\mathcal{S}(\omega,\omega,\upsilon)+\beta\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega)+\gamma(\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\upsilon)\\&+\mathcal{S}(\mathcal{A}\upsilon,\mathcal{A}\upsilon,\omega)+\delta\mathcal{S}(\mathcal{A}\upsilon,\mathcal{A}\upsilon,\upsilon)\},\end{aligned}$$

where $\alpha+\beta+3\gamma+\delta \in [0,1)$.

Proof. Define $\mathfrak{f}:[0,\infty)^5 \to [0,\infty)$ by

$$\mathfrak{f}(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) = \alpha\omega_1+\beta\omega_2+\gamma(\omega_3+\omega_4)+\delta\omega_5,\ \alpha+\beta+3\gamma+\delta \in [0,1).$$

Then, $\mathfrak{f} \in \mathcal{M}$, and the proof complies with Theorem 3.11. □

Remark 3.5. *Corollary 3.17 is an enhancement of the Hardy-Roger type contraction theorem [11] in $\mathcal{S}-$metric space.*

Corollary 3.18. *Theorem 3.11 also continues to be true if (3.2) is replaced by*

$$\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\upsilon) \leq \alpha\mathcal{S}(\omega,\omega,\upsilon)+\beta\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega)+\delta\mathcal{S}(\mathcal{A}\upsilon,\mathcal{A}\upsilon,\upsilon)\},\textit{where } \alpha+\beta+\delta \in [0,1).$$

Proof. Define $\mathfrak{f}:[0,\infty)^5 \to [0,\infty)$ by $\mathfrak{f}(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5) = \alpha\omega_1+\beta\omega_2+\delta\omega_5,\ \alpha+\beta+\delta \in [0,1)$. Then, $\mathfrak{f} \in \mathcal{M}$, and the proof complies with Theorem 3.11. □

Remark 3.6. *Corollary 3.18 is an enhancement of the Reich-type contraction theorem [34] in $\mathcal{S}-$metric space.*

On suitably varying the elements of an $\mathcal{M}-$class function, distinct existing well-known conclusions in the literature may be deduced. In all the above results, we have generalized, extended, unified and improved some well-known results wherein the fixed point is always unique. However, there may arise some situations where the fixed point is not unique, and the collection of fixed points may include some geometrical shape. So, now we review the geometry of the collection of fixed points in $\mathcal{S}-$metric space via an $\mathcal{M}-$class function. It is significant to notice that the $\mathcal{S}-$metric is not in general created by any metric.

Now, define a set

($\mathfrak{f}_4$) $\mathcal{M}^* = \{\mathfrak{f} \in \mathcal{M} : \mathfrak{f}(0,1,1,1,1) \in [0,1)\}$.
Clearly, $\mathcal{M}^* \subseteq \mathcal{M}$.

Theorem 3.19. *Let $(\mathcal{S},\mathcal{U})$ be an $\mathcal{S}-$metric space, $\mathfrak{f} \in \mathcal{M}^*$. For self-map $\mathcal{A}:\mathcal{U}\to\mathcal{U}$, if $\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega) > 0$ implies*

$$\begin{aligned}\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega) \leq& \mathfrak{f}\big(\mathcal{S}(\omega,\omega,\omega_0),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega),\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0),\\&\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega_0),\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega)\big),\end{aligned} \tag{3.4}$$

then, $\mathcal{C}(\omega_0,r)$ is a fixed circle of $\mathcal{A}$ centered on ω_0 with radius $r=\inf\{\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega):\mathcal{A}\omega\neq\omega,\omega\in\mathcal{U}\}$.

Proof. Let $\mathcal{C}(\omega_0,r)$ be any circle centered on ω_0 with radius r and $\mathcal{A}\omega_0 \neq \omega_0$, that is, $\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0) > 0$. So, by using inequality (3.4), ($\mathfrak{f}_2$) and ($\mathfrak{f}_4$), we have

$$\begin{aligned}
\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0) \leq & \mathfrak{f}\big(\mathcal{S}(\omega_0,\omega_0,\omega_0),\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0),\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0),\\
&\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0),\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0)\big)\\
&= \mathfrak{f}\big(0,\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0),\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0),\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0),\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0)\big)\\
&= \mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0)\mathfrak{f}(0,1,1,1,1)\\
&< \mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0),
\end{aligned}$$

a contradiction, which implies $\mathcal{A}\omega_0 = \omega_0$.

Now, consider $\omega \in \mathcal{C}(\omega_0,r)$ and $\mathcal{A}\omega \neq \omega$, that is, $\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega) > 0$. Also, by definition of r, $\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega) \geq r$.

Now, by using inequality (3.4), properties of the $\mathcal{M}-$class function and Lemma 3.10,

$$\begin{aligned}
\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega) &\leq \mathfrak{f}\big(\mathcal{S}(\omega,\omega,\omega_0),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega),\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega_0),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega_0),\mathcal{S}(\mathcal{A}\omega_0,\mathcal{A}\omega_0,\omega)\big)\\
&= \mathfrak{f}\big(\mathcal{S}(\omega,\omega,\omega_0),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega),\mathcal{S}(\omega_0,\omega_0,\omega_0),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega_0),\mathcal{S}(\omega_0,\omega_0,\omega)\big)\\
&\leq \mathfrak{f}\big(\mathcal{S}(\omega,\omega,\omega_0),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega),0,2\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega)+\mathcal{S}(\omega_0,\omega_0,\omega),\mathcal{S}(\omega,\omega,\omega_0)\big)\\
&= \mathfrak{f}(r,\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega),0,2\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega)+r,r)\\
&\leq \mathfrak{f}(\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega),0,3\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega),\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega))\\
&= \mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega)\mathfrak{f}(1,1,0,3,1)\\
&\leq \eta\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\omega) < \mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\omega),
\end{aligned}$$

a contradiction. Thus, $\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\omega) = 0$, that is, $\mathcal{A}\omega = \omega$. Hence, $\mathcal{C}(\omega_0,r)$ is a fixed circle of $\mathcal{A}$. □

Example 3.20. *Let an $\mathcal{S}-$metric $\mathcal{S}: \mathcal{U}\times\mathcal{U}\times\mathcal{U} \longrightarrow \mathbb{R}^+$ be*

$$\mathcal{S}(\omega,\omega,\upsilon) = |w_1-u_1|+|w_1+u_1-2v_1|+|w_2-u_2|+|w_2+u_2-2v_2|+|w_3-u_3|+|w_3+u_3-2v_3|,$$

where $\omega = (w_1,w_2,w_3), \upsilon = (v_1,v_2,v_3),\ u = (u_1,u_2,u_3) \in \mathcal{U} = \mathbb{R}^3$. Then,

$$\mathcal{C}(\omega_0,8) = \{\omega \in \mathcal{U} : \mathcal{S}(\omega_0,\omega_0,\omega) = 8\},$$

where $\omega_0 = (1,2,3) \in \mathcal{U}$ and $r = 8$, that is, a circle centered at $(1,2,3)$ with radius 8 is given by

$$\begin{aligned}
&|1-w_1|+|1+w_1-2|+|2-w_2|+|2+w_2-4|+|3-w_3|+|3+w_3-6| = 8\\
&|1-w_1|+|2-w_2|+|3-w_3| = 4.
\end{aligned} \tag{3.5}$$

Define $\mathcal{A}:\mathcal{U}\longrightarrow\mathcal{U}$ as $\mathcal{A}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \begin{cases}(\mathfrak{a},\mathfrak{b},\mathfrak{c}), & (\mathfrak{a},\mathfrak{b},\mathfrak{c}) \in \mathcal{C}(\omega_0,8)\\ (1,0,2), & \textit{otherwise}\end{cases}.$

Then, map $\mathcal{A}$ validates all the hypotheses of Theorem 3.19 and fixes the unique circle $\mathcal{C}(\omega_0,8)$, that is, the set of fixed points of a self-map $\mathcal{A}$ contains a unique circle $\mathcal{C}(\omega_0,8)$ (see Figure 1).

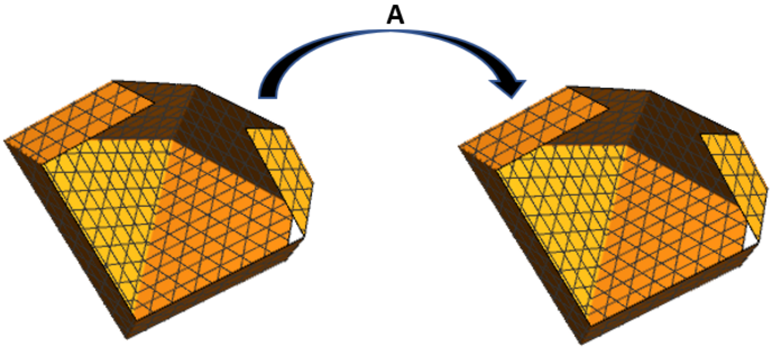

Figure 1. Demonstrates that the circle (3.5) is fixed by the function $\mathcal{A}$.

Following Joshi et al. [16, 17], now, we define an ellipse (elliptic disc) in an $\mathcal{S}-$metric space and discuss their shapes in different $\mathcal{S}-$metric spaces for different lengths of semi-major axes and different foci. Next, we describe a fixed ellipse (elliptic disc) in an $\mathcal{S}-$metric space and exploit an $\mathcal{S}-$metric variant of a celebrated Caristi type map [6] to conclude that the collection of fixed points incorporates an ellipse (elliptic disc). It is well known that an ellipse is the locus of a point for which the sum of the Euclidean distances from the two foci is uniform, and the circle is the ellipse of diminishing eccentricity wherein both the focal points are identical. In fact, ellipses emerge naturally in numerous areas, such as planetary orbits.

Definition 3.1. *We define an ellipse having foci at $\mathfrak{c}_1$ and $\mathfrak{c}_2$ in an $\mathcal{S}-$metric space $(\mathcal{U},\mathcal{S})$ as*

$$\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a}) = \{\omega \in \mathcal{U} : \mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega) = 2\mathfrak{a},\ \ \mathfrak{c}_1,\mathfrak{c}_2 \in \mathcal{U},\ \mathfrak{a} \in [0,\infty)\}.$$

If $\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega) \leq 2\mathfrak{a}$, then the above definition reduces to the definition of an elliptic disc, and we denote it by $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$. For the formation of ellipse, $\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathfrak{c}_2) < 2\mathfrak{a}$.

The distance $2f = \mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathfrak{c}_2)$ is the linear eccentricity. It is well known that eccentricity is the degree of the deflection of the curve from the roundness of a specific shape. The midpoint of line $\mathfrak{c}_1\mathfrak{c}_2$ is said to be a center of an ellipse (elliptic disc). The portion of length $2\mathfrak{a}$ passing through the foci $\mathfrak{c}_1$ and $\mathfrak{c}_2$ is the major axis, and the line through the center, perpendicular to the major axis, is the minor axis.

Example 3.21. *Let an $\mathcal{S}-$metric $\mathcal{S} : \mathcal{U}\times\mathcal{U}\times\mathcal{U} \longrightarrow \mathbb{R}^+$ be*

$$\mathcal{S}(\omega,\upsilon,u) = |\omega-u|+|\omega+u-2\upsilon|,\ \omega,\upsilon,u \in \mathcal{U} = \mathbb{R},$$

that is, $\mathcal{S}(-5,-5,5) = 20$. *Then,*

$$\begin{aligned}\mathcal{E}(-5,5,11) &= \{\omega \in \mathcal{U} : \mathcal{S}(-5,-5,\omega)+\mathcal{S}(5,5,\omega) = 22\}\\ &= \{\omega \in \mathcal{U} : |5+\omega|+|-5+\omega+10|+|5-\omega|+|5+\omega-10| = 22\}\\ &= \{\omega \in \mathcal{U} : 2|5+\omega|+2|5-\omega| = 22\}\\ &= \{-5.5,5.5\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_{\mathcal{D}}(-5,5,11) &= \{\omega \in \mathcal{U} : \mathcal{S}(-5,-5,\omega)+\mathcal{S}(5,5,\omega) \leq 22\}\\ &= [-5.5,5.5],\end{aligned}$$

that is, an ellipse and elliptic disc centered at the origin having foci at -5 *and* 5 *are* $\{-5.5, 5.5\}$ *and* $[-5.5, 5.5]$*, respectively.*

Example 3.22. *Let an* $\mathcal{S}-$*metric* $\mathcal{S}_1 : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R}^+$ *be* $\mathcal{S}_1(\omega, \upsilon, u) = \Sigma_{i=1}^{2}(|\omega_i - u_i| + |\omega_i + u_i - 2v_i|)$*, where* $\omega = (w_1, w_2), \upsilon = (v_1, v_2), u = (u_1, u_2) \in \mathcal{U} = \mathbb{R}^2$*, and* $\mathcal{S}_1(\mathfrak{c}_1, \mathfrak{c}_1, \mathfrak{c}_2) = 8$*, where* $c_1 = (2, 0)$ *and* $c_2 = (0, 2)$*. Then,*

$$\begin{aligned}\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, 5) =& \{\omega \in \mathcal{U} : \mathcal{S}_1(\mathfrak{c}_1, \mathfrak{c}_1, \omega) + \mathcal{S}_1(\mathfrak{c}_2, \mathfrak{c}_2, \omega) = 10\} \\ =& \{\omega \in \mathcal{U} : |2 - \omega_1| + |2 + \omega_1 - 4| + |0 - \omega_2| + |0 + \omega_2 - 0| + |0 - \omega_1| \\ & + |0 + \omega_1 - 0| + |2 - \omega_2| + |2 + \omega_2 - 4| = 10\} \\ =& \{\omega \in \mathcal{U} : 2|2 - \omega_1| + 2|\omega_2| + 2|\omega_1| + 2|2 - \omega_2| = 10\} \\ =& \{\omega \in \mathcal{U} : |2 - \omega_1| + |2 - \omega_2| + |\omega_1| + |\omega_2| = 5\},\end{aligned} \tag{3.6}$$

that is, an ellipse centered at $(1, 1)$ *with foci* $(2, 0)$ *and* $(0, 2)$ *is shown as the blue line in Figure 2.*

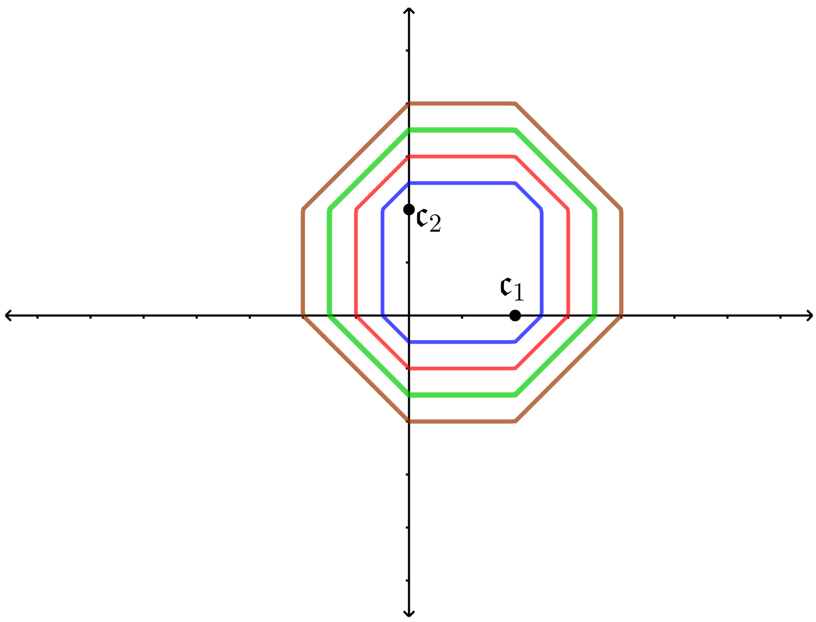

Figure 2. Ellipses corresponding to an $\mathcal{S}-$metric $\mathcal{S}_1(\omega, \upsilon, u) = \Sigma_{i=1}^{2}(|\omega_i - u_i| + |\omega_i + u_i - 2v_i|)$, centered at $(1, 1)$ with foci $(2, 0)$ and $(0, 2)$ for $\mathfrak{a} = 5, 6, 7$ and 8 are shown by the blue, the red, the green and the brown lines, respectively, in Example 3.22.

If an $\mathcal{S}-$metric $\mathcal{S}_2 : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R}^+$ is

$$\mathcal{S}_2(\omega, \upsilon, u) = \sqrt{(\omega_1 - \upsilon_1)^2 + (\omega_2 - \upsilon_2)^2 + (\upsilon_1 - u_1)^2 + (\upsilon_2 - u_2)^2 + (u_1 - \omega_1)^2 + (u_2 - \omega_2)^2},$$

$$\omega = (\omega_1, \omega_2), \upsilon = (\upsilon_1, \upsilon_2), u = (u_1, u_2) \in \mathcal{U} = \mathbb{R}^2,$$

and $\mathcal{S}_2(\mathfrak{c}_1, \mathfrak{c}_1, \mathfrak{c}_2) = 4$, then an ellipse having the same center and the same foci as above is $\sqrt{2(2 - \omega_1)^2 + 2\omega_2^2} + \sqrt{2\omega_1^2 + 2(2 - \omega_2)^2} = 6$ and is shown as the blue line in Figure 3.

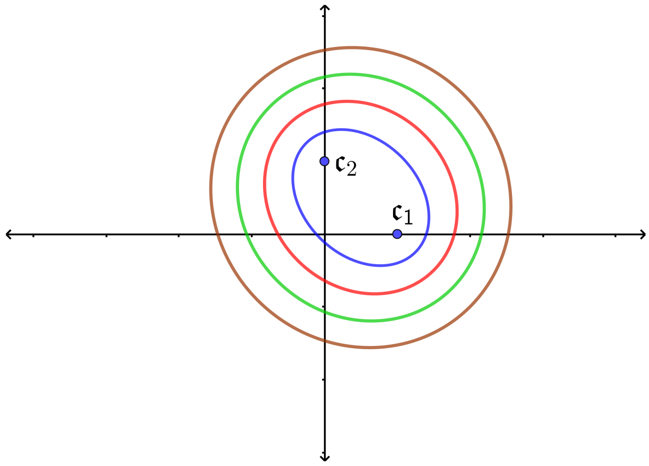

Figure 3. Ellipses corresponding to $\mathcal{S}_2(\omega, \upsilon, u) = \sqrt{(\omega_1-\upsilon_1)^2+(\omega_2-\upsilon_2)^2+(\upsilon_1-u_1)^2+(\upsilon_2-u_2)^2+(u_1-\omega_1)^2+(u_2-\omega_2)^2}$, centered at $(1,1)$ with the same foci $(2,0)$ and $(0,2)$, for $\mathfrak{a} = 3,\ 4,\ 5$ and 6 are shown by the red, the blue, the green and the brown lines, respectively.

Let $\mathcal{S}_3 : \mathcal{U}\times\mathcal{U}\times\mathcal{U} \longrightarrow \mathbb{R}^+$ be $\mathcal{S}_3(\omega, \upsilon, u) = \max\{|\omega_1-\upsilon_1|,\ |\omega_2-\upsilon_2|,\ |\upsilon_1-u_1|,\ |\upsilon_2-u_2|,\ |u_1-\omega_1|,\ |u_2-\omega_2|\},\ \omega = (\omega_1,\omega_2), \upsilon = (\upsilon_1,\upsilon_2), u = (u_1,u_2) \in \mathcal{U} = \mathbb{R}^2,$ and $\mathcal{S}_3(\mathfrak{c}_1,\mathfrak{c}_1,\mathfrak{c}_2) = 2$. Then, again, an ellipse with the same center and the same foci is $\max\{|2-w_1|,|w_2|\}+\max\{|w_1|,|2-w_2|\} = 2$ and is shown as the blue line in Figure 4.

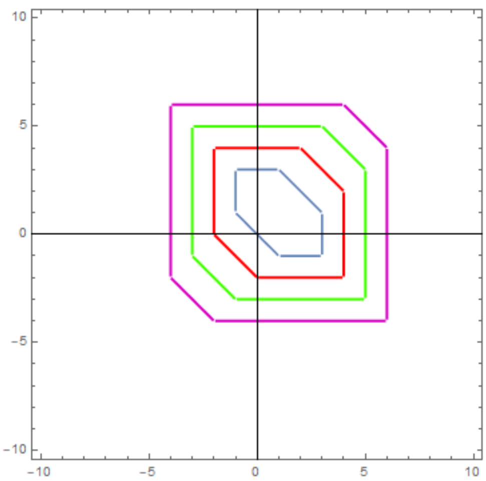

Figure 4. Ellipses corresponding to $\mathcal{S}_3(\omega, \upsilon, u) = \max\{|\omega_1-\upsilon_1|,\ |\omega_2-\upsilon_2|,\ |\upsilon_1-u_1|,\ |\upsilon_2-u_2|,\ |u_1-\omega_1|,\ |u_2-\omega_2|\}$, centered at $(1,1)$ with the same foci $(2,0)$ and $(0,2)$, for $\mathfrak{a} = 2,\ 3, 4$ and 5 are shown by the blue, the red, the green and the violet lines, respectively.

The shapes of ellipses corresponding to an $\mathcal{S}_1$ metric, as used in Figure 2, corresponding to different foci and different values of semi major axes are shown in Figure 5.

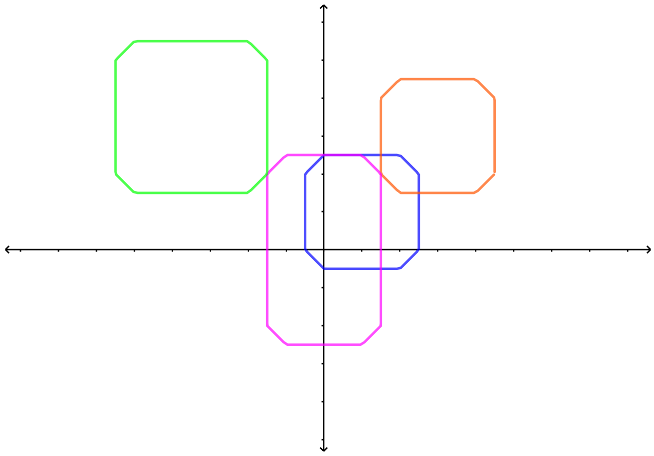

Figure 5. Ellipses corresponding to an $\mathcal{S}-$metric $\mathcal{S}_1(\omega,\upsilon,u)=\Sigma_{i=1}^{2}(|\omega_i-u_i|+|\omega_i+u_i-2v_i|)$, corresponding foci $(2,0),(0,2)$ $(\mathfrak{a}=5)$; $(2,4),(4,2)$ $(\mathfrak{a}=5)$; $(1,2),(-1,-2)$ $(\mathfrak{a}=7)$; and $(-5,5),(-2,2)$ $(\mathfrak{a}=7)$, are shown by the blue, the red, the pink and the green respectively.

Example 3.23. *Let an* $\mathcal{S}-$*metric* $\mathcal{S}_1:\mathfrak{U}\times\mathfrak{U}\times\mathfrak{U}\longrightarrow\mathbb{R}^+$ *be* $\mathcal{S}_1(\omega,\upsilon,u)=\Sigma_{i=1}^{3}(|\omega_i-\upsilon_i|+|\omega_i-u_i|)$, *where* $\omega=(\omega_1,\omega_2,\omega_3),\upsilon=(\upsilon_1,\upsilon_2,\upsilon_3),u=(u_1,u_2,u_3)\in\mathfrak{U}=\mathbb{R}^3$. *Then,*

$$\begin{aligned}\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,2)&=\{\omega\in\mathfrak{U}:\mathcal{S}_1(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}_1(\mathfrak{c}_2,\mathfrak{c}_2,\omega)=12\}\\&=\{\omega\in\mathfrak{U}:2|\omega_1|+2|\omega_2|+2|\omega_3|+2|\omega_1-1|+2|\omega_2-1|+2|\omega_3-1|=12\}\\&=\{\omega\in\mathfrak{U}:|\omega_1|+|\omega_2|+|\omega_3|+|\omega_1-1|+|\omega_2-1|+|\omega_3-1|=6,\end{aligned}\tag{3.7}$$

where $\mathfrak{c}_1=(0,0,0)$ *and* $\mathfrak{c}_2=(1,1,1)$, *that is, an ellipse centered at* $(0.5,0.5,0.5)$ *with foci* $(0,0,0)$ *and* $(1,1,1)$ *is shown in Figure 6.*

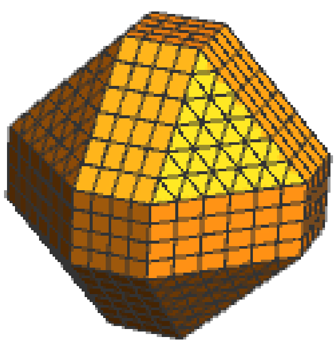

Figure 6. Ellipse corresponding to an $\mathcal{S}-$metric $\mathcal{S}_1(\omega,\upsilon,u)=\Sigma_{i=1}^{3}(|\omega_i-\upsilon_i|+|\omega_i-u_i|)$, centered at $(0.5,0.5,0.5)$ with foci $(0,0,0)$ and $(1,1,1)$, for $\mathfrak{a}=6$ is shown in Figure 6.

If an $\mathcal{S}-$metric $\mathcal{S}_2 : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R}^+$ is

$$\mathcal{S}_2(\omega, \upsilon, u) = \sqrt{\Sigma_{i=1}^{3}(\omega_i - \upsilon_i)^2} + \sqrt{\Sigma_{i=1}^{3}(\omega_i - u_i)^2},$$

$$\omega = (\omega_1, \omega_2, \omega_3), \upsilon = (\upsilon_1, \upsilon_2, \upsilon_3), u = (u_1, u_2, u_3) \in \mathcal{U} = \mathbb{R}^3,$$

then an ellipse having the same center and the same foci as above is

$$\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} + \sqrt{(\omega_1 - 1)^2 + (\omega_2 - 1)^2 + (\omega_3 - 1)^2} = 6$$

and is shown in Figure 7.

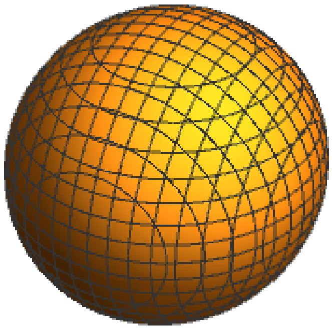

Figure 7. Ellipse corresponding to $\mathcal{S}_2(\omega, \upsilon, u) = \Sigma_{i=1}^{3} \sqrt{(\omega_i - \upsilon_i)^2} + \Sigma_{i=1}^{3} \sqrt{(\omega_i - u_i)^2}$, centered at $(0.5, 0.5, 0.5)$ with the same foci $(0,0,0)$ and $(1,1,1)$, for $\mathfrak{a} = 6$, is shown in Figure 7.

If $\mathcal{S}_3 : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R}^+$ is

$$\mathcal{S}_3(\omega, \upsilon, u) = \max\{|\omega_1 - u_1|, |\omega_2 - u_2|, |\omega_3 - u_3|\} + \max\{|\upsilon_1 - u_1|, |\upsilon_2 - u_2|, |\upsilon_3 - u_3|\},$$

$$\omega = (\omega_1, \omega_2, \omega_3), \upsilon = (\upsilon_1, \upsilon_2, \upsilon_3), u = (u_1, u_2, u_3) \in \mathcal{U} = \mathbb{R}^3,$$

then again an ellipse with the same center and the same foci is

$$\max\{|\omega_1|, |\omega_2|, |\omega_3|\} + \max\{|\omega_1 - 1|, |\omega_2 - 1|, |\omega_3 - 1|\} = 6$$

and is shown in Figure 8.

It is fascinating to see that the shapes of some ellipses may change by changing the length of the semi-major axis (see Figures 1–3) or foci (see Figure 4), or involved metric (see Figures 2–4 and 6–8).

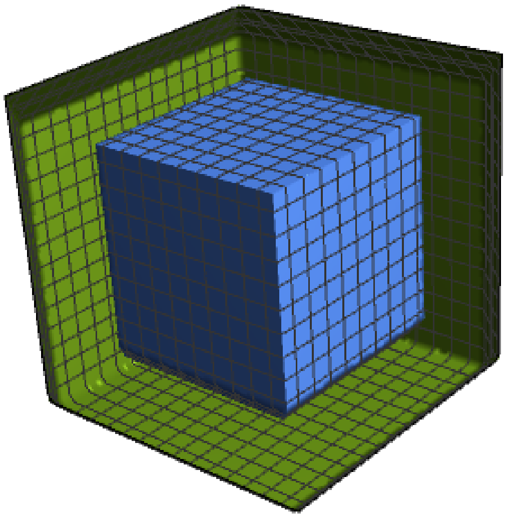

Figure 8. Ellipse corresponding to $\mathcal{S}_3(\omega,\upsilon,u) = \max\{|\omega_1 - u_1|,|\omega_2 - u_2|,|\omega_3 - u_3|\} + \max\{|\upsilon_1 - u_1|,|\upsilon_2 - u_2|,|\upsilon_3 - u_3|\}$, centered at $(0.5,0.5,0.5)$ with the same foci $(0,0,0)$ and $(1,1,1)$ for $\mathfrak{a} = 6$ is shown in Figure 8.

Remark 3.7. *The interior of Figures 2–8 is the corresponding elliptic disc with the same center, foci and length of the semi-major axis as those of the ellipse.*

Following Caristi [6], we introduce the Caristi map in an $\mathcal{S}-$metric space.

Definition 3.2. *A self-map $\mathcal{A}$ of an $\mathcal{S}-$metric space $(\mathcal{U},\mathcal{S})$ is a Caristi map on $\mathcal{U}$ if $\varsigma : \mathcal{U} \longrightarrow [0,\infty)$ is a lower semi-continuous function for $(\mathcal{U},\mathcal{S})$ and*

$$\mathcal{S}(\omega,\omega,\mathcal{A}\omega) \leq \varsigma(\omega) - \varsigma(\mathcal{A}\omega), \ \ \omega \in \mathcal{U}.$$

Following Joshi et al. [16], we introduce fixed ellipse and elliptic disc in $\mathcal{S}-$metric space to explore the geometry of the collection of non-unique fixed points.

Definition 3.3. *Let $\mathcal{A} : \mathcal{U} \longrightarrow \mathcal{U}$ be a self-map of an $\mathcal{S}-$metric space $(\mathcal{U},\mathcal{S})$. If $\mathcal{A}\omega = \omega, \ \omega \in \mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a}), \ \mathfrak{c}_1,\mathfrak{c}_2 \in \mathcal{U}, \ \mathfrak{a} \in [0,\infty)$, then $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ is said to be the fixed ellipse of $\mathcal{A}$.*

Definition 3.4. *Let $\mathcal{A} : \mathcal{U} \longrightarrow \mathcal{U}$ be a self-map of an $\mathcal{S}-$metric space $(\mathcal{U},\mathcal{S})$. If $\mathcal{A}\omega = \omega, \ \omega \in \mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a}), \ \mathfrak{c}_1,\mathfrak{c}_2 \in \mathcal{U}, \ \mathfrak{a} \in [0,\infty)$, then $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ is said to be the fixed elliptic disc of $\mathcal{A}$.*

Theorem 3.24. *Let $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ be an ellipse in an $\mathcal{S}-$metric space $(\mathcal{U},\mathcal{S})$. Define $\varsigma : \mathcal{U} \longrightarrow [0,\infty)$ as*

$$\varsigma(\omega) = \mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega) + \mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega), \ \mathfrak{c}_1,\mathfrak{c}_2,\omega \in \mathcal{U}. \tag{3.8}$$

If self-map $\mathcal{A} : \mathcal{U} \longrightarrow \mathcal{U}$ satisfies the hypotheses

$(\mathcal{E}_1)$ $\mathcal{S}(\omega,\omega,\mathcal{A}\omega) \leq \varsigma(\omega) - \varsigma(\mathcal{A}\omega)$,
$(\mathcal{E}_2)$ $\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega) + \mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega) \geq 2\mathfrak{a}$,
$(\mathcal{E}_3)$ *if* $\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\upsilon) \leq \sigma\mathcal{S}(\omega,\omega,\upsilon), \ \omega \in \mathcal{E}(\mathfrak{c}_1,c_2,\mathfrak{a}), \ \upsilon \in \mathcal{U}\backslash\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a}), \sigma \in [0,1)$,

then $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ *is a unique fixed ellipse of* $\mathcal{A}$.

Proof. Let $\omega \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ be an arbitrary point. Utilizing $(\mathcal{E}_1)$ and Eq (3.8),

$$\begin{aligned}\mathcal{S}(\omega,\omega,\mathcal{A}\omega) &\leq \mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)-\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)-\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega)\\ &= 2\mathfrak{a}-\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)-\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega)\\ &\leq 2\mathfrak{a}-2\mathfrak{a}, \ \ (\text{using } (\mathcal{E}_2))\end{aligned}$$

that is,

$$\mathcal{S}(\omega,\omega,\mathcal{A}\omega)=0 \Longrightarrow \mathcal{A}\omega=\omega, \tag{3.9}$$

that is, ω is a fixed point of $\mathcal{A}$, $\forall\, \omega \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$.

So, a self-map $\mathcal{A}$ fixes an ellipse $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$, that is, the set of fixed points of a self-map $\mathcal{A}$ contains an ellipse.

Let $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}')$ be two fixed ellipses of $\mathcal{A}$, that is, $\mathcal{A}$ satisfies both conditions $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ for each of the ellipses $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}')$. Let $\omega \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\upsilon \in \mathcal{E}(\mathfrak{c}'_1, \mathfrak{c}'_2, \mathfrak{a}')$. Using $(\mathcal{E}_3)$, $\mathcal{S}(\omega,\omega,\upsilon) = \mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\upsilon) \leq \sigma\mathcal{S}(\omega,\omega,\upsilon)$, a contradiction. Hence, $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a unique fixed ellipse of $\mathcal{A}$. □

The subsequent explanatory example with pictographic validation substantiates Theorem 3.24.

Example 3.25. *Let an* $\mathcal{S}-$*metric* $\mathcal{S}: \mathcal{U}\times\mathcal{U}\times\mathcal{U} \longrightarrow \mathbb{R}^+$ *be*

$$\mathcal{S}(\omega,\upsilon,u) = \sqrt{(w_1-v_1)^2+(w_2-v_2)^2+(\upsilon_1-u_1)^2+(\upsilon_2-u_2)^2+(\omega_1-u_1)^2+(\omega_2-u_2)^2},$$

where $\omega=(w_1,w_2), \upsilon=(v_1,v_2),\ u=(u_1,u_2)\in\mathcal{U}=\mathbb{R}^2$. *Then,*

$$\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,5)=\{\omega\in\mathcal{U}:\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_2,\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)=15\},$$

where $\mathfrak{c}_1=(4,5)$ *and* $\mathfrak{c}_2=(-2,-3)\in\mathcal{U}$, *that is, the equation of an ellipse centered at* $(1,1)$ *with foci at* $(4,5)$ *and* $(-2,-3)$ *is*

$$\sqrt{2(4-\omega_1)^2+2(5-\omega_2)^2}+\sqrt{2(2+\omega_1)^2+2(3+\omega_2)^2}=15. \tag{3.10}$$

Define $\mathcal{A}:\mathcal{U}\longrightarrow\mathcal{U}$ *as* $\mathcal{A}(\mathfrak{a},\mathfrak{b})=\begin{cases}(\mathfrak{a},\mathfrak{b}), & (\mathfrak{a},\mathfrak{b})\in\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,7.5)\\ (2.47,0), & \textit{otherwise}\end{cases}.$

Then, map $\mathcal{A}$ validates all the hypotheses of Theorem 3.24 and fixes the unique ellipse $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,5)$, that is, the set of fixed points of a self-map $\mathcal{A}$ contains a unique ellipse $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,5)$ (see Figure 9).

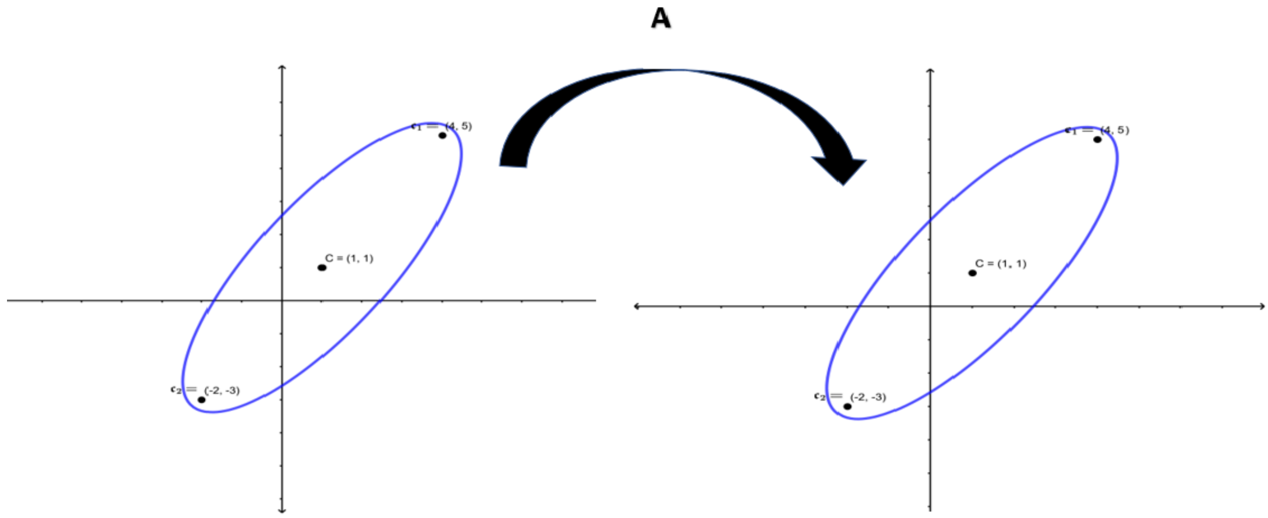

Figure 9. The blue lines demonstrate the ellipse (3.10), which is fixed by the function $\mathcal{A}$.

Geometrically, condition $(\mathcal{E}_1)$ states that $\mathcal{A}\omega$ is in the exterior of an ellipse, and condition $(\mathcal{E}_2)$ states that $\mathcal{A}\omega$ is in the interior of an ellipse.

The following examples depict the importance of hypotheses $(\mathcal{E}_1)$–$(\mathcal{E}_3)$ in the existence of a fixed ellipse or a unique fixed ellipse in Theorem 3.24.

Example 3.26. *Let an $\mathcal{S}-$metric $\mathcal{S}:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\longrightarrow\mathbb{R}^+$ be*

$$\mathcal{S}(\omega,\upsilon,u)=|\omega-\upsilon|+|\omega+u-2\upsilon|,\ \omega,\upsilon,u\in\mathcal{U}=\mathbb{R}.$$

The ellipse

$$\begin{aligned}\mathcal{E}(2,4,6)=&\{\omega\in\mathcal{U}:\mathcal{S}(2,2,\omega)+\mathcal{S}(4,4,\omega)=12\}\\ =&\{\omega\in\mathcal{U}:|2-\omega|+|2+\omega-4|+|4-\omega|+|4+\omega-8|=12\}\\ =&\{\omega\in\mathcal{U}:2|2-\omega|+2|4-\omega|=12\}\\ =&\{\omega\in\mathcal{U}:|2-\omega|+|4-\omega|=6\}\\ =&\{0\ ,\ 6\}.\end{aligned}$$

Define $\mathcal{A}:\mathcal{U}\longrightarrow\mathcal{U}$ *as* $\mathcal{A}\omega=\begin{cases}\omega, & \omega=0 \text{ or } \omega \text{ is an odd number}\\ 6, & \text{otherwise}\end{cases}.$

Then, a self-map $\mathcal{A}$ validates all the hypotheses of Theorem 3.24 except $(\mathcal{E}_3)$. Hence, a self-map $\mathcal{A}$ fixes the ellipse $\mathcal{E}(2,4,6)$. However, it is not unique, and there may exist infinitely many ellipses which are fixed by a self-map $\mathcal{A}$.

Example 3.27. *Let an $\mathcal{S}-$metric $\mathcal{S}:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\longrightarrow\mathbb{R}^+$ be*

$$\mathcal{S}(\omega,\upsilon,u)=\Sigma_{i=1}^{2}(|\omega_i-u_i|+|\omega_i+u_i-2v_i|),$$

where $\omega=(w_1,w_2),\upsilon=(v_1,v_2),\ u=(u_1,u_2)\in\mathcal{U}=\mathbb{R}^2$. Then,

$$\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,6)=\{\omega\in\mathcal{U}:\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)=12\},\tag{3.11}$$

where $\mathfrak{c}_1 = (2,0)$ *and* $\mathfrak{c}_2 = (0,2) \in \mathcal{U}$, *that is, the equation of an ellipse centered at* $(1,1)$ *with foci at* $(2,0)$ *and* $(0,2)$ *is*

$$|2-\omega_1| + |2+\omega_1-4| + |0-\omega_2| + |0+\omega_2-0|$$
$$+|0-\omega_1| + |0+\omega_2-0| + |2-\omega_2| + |2+\omega_2-4| = 12,$$

that is,

$$2[|2-\omega_1| + |\omega_2| + |\omega_1| + |2-\omega_2|] = 12,$$

that is,

$$|2-\omega_1| + |\omega_2| + |\omega_1| + |2-\omega_2| = 6.$$

Define $\mathcal{A} : \mathcal{U} \longrightarrow \mathcal{U}$ as $\mathcal{A}(\omega,\upsilon) = \begin{cases} (\omega,\upsilon), & \omega \in \{-1,0,1,2,3\} \\ (3,0), & otherwise \end{cases}.$

Then, a self-map $\mathcal{A}$ validates hypothesis $(\mathcal{E}_2)$ and does not validate hypotheses $(\mathcal{E}_1)$ and $(\mathcal{E}_3)$ of Theorem 3.24. Hence, $\mathcal{A}$ does not fix the ellipse $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,6)$ but fixes the points $(0,-1)$, $(0,3)$, $(2,-1)$, $(2,3)$, $(-1,y)$ and $(3,y) \in \mathcal{U}$, $0 \le y \le 2$ of an ellipse (3.11).

Theorem 3.28. *The conclusion of Theorem 3.24 continues to be true even if we replace* $(\mathcal{E}_1)$ *and* $(\mathcal{E}_2)$ *by*

$(\mathcal{E}_1')$ $\mathcal{S}(\omega,\omega,\mathcal{A}\omega) \le \varsigma(\omega) + \varsigma(\mathcal{A}\omega) - 4\mathfrak{a}$,
($\mathcal{E}_2'$*)* $\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega) + \mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega) \le 2\mathfrak{a}$.

Proof. Let $\omega \in \mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ be any arbitrary point. Using $(\mathcal{E}_1')$ and Equation (3.8),

$$\begin{aligned}\mathcal{S}(\omega,\omega,\mathcal{A}\omega) &\le \mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega) + \mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega) + \mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega) + \mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega) - 4\mathfrak{a} \\ &= 2\mathfrak{a} + \mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega,) + \mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega) - 4\mathfrak{a} \\ &= \mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega,) + \mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega) - 2\mathfrak{a} \\ &\le 0 \ \ (\text{using } (\mathcal{E}_2^{'})),\end{aligned}$$

a contradiction, that is, $\mathcal{S}(\omega,\omega,\mathcal{A}\omega) = 0 \Longrightarrow \mathcal{A}\omega = \omega$, $\omega \in \mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$.

The uniqueness of a fixed ellipse may be established as in Theorem 3.24. □

It is clear that geometrically the condition $(\mathcal{E}_1')$ states that $\mathcal{A}\omega$ is in the exterior of an ellipse, and the condition $(\mathcal{E}_2')$ states that $\mathcal{A}\omega$ is in the interior of an ellipse.

The relationships among the "if"s and "then"s in this theorem are unclear. Currently, it appears that the first "if" ("If self-map ...") does not have a conclusion, that is, a corresponding "then" clause. Meanwhile, hypothesis $(\mathcal{E}_{2\mathfrak{D}})$ appears to lack an expected "if" preceding its "then". Lastly, $(\mathcal{E}_{3\mathfrak{D}})$ appears to be its own complete conditional.

Theorem 3.29. *Let* $\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *be an elliptic disc in an* $\mathcal{S}-$*metric space* $(\mathcal{U},\mathcal{S})$. *Define* $\varsigma : \mathcal{U} \longrightarrow [0,\infty)$ *as in* (3.8). *If self-map* $\mathcal{A} : \mathcal{U} \longrightarrow \mathcal{U}$ *satisfies the hypotheses*

$(\mathcal{E}_{1\mathfrak{D}})$ $\mathcal{S}(\omega,\omega,\mathcal{A}\omega) \le \varsigma(\omega) + \varsigma(\mathcal{A}\omega) - 4\mathfrak{a}$,
$(\mathcal{E}_{2\mathfrak{D}})$ $\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega) + \mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega) \le 2\mathfrak{a}$, $\omega \in \mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$.

Then, $\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *is a fixed elliptic disc of* $\mathcal{A}$.

$(\mathcal{E}_{3\mathfrak{D}})$ *In addition to the above hypotheses, if* $\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\upsilon) \le \sigma\mathcal{S}(\omega,\omega,\upsilon)$, $\omega \in \mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$, $\upsilon \in \mathcal{U}\backslash\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,a), \sigma \in [0,1)$, *then* $\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *is a fixed elliptic disc of maximum semi-major axis* $\mathfrak{a}$, *that is, there is no fixed elliptic disc* $\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *of* $\mathcal{A}$ *having a semi-major axis greater than* $\mathfrak{a}$.

Proof. The existence of a fixed elliptic disc may be shown as in Theorem 3.28.

Let there exist two fixed elliptic discs $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ and $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}')$ of $\mathcal{A}$, $\mathfrak{a}<\mathfrak{a}'$, that is, $\mathcal{A}$ satisfies both the conditions $(\mathcal{E}_{1\mathcal{D}})$ and $(\mathcal{E}_{2\mathcal{D}})$ for each of the elliptic discs $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}')$. Let $\omega\in\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ and $\upsilon\in\mathcal{E}_{\mathcal{D}}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}')$. Using $(\mathcal{E}_{3\mathcal{D}})$, $\mathcal{S}(\omega,\omega,\upsilon)=\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\upsilon)\leq\sigma\mathcal{S}(\omega,\omega,\upsilon)$, a contradiction. Hence, $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ is a fixed elliptic disc of maximum semi-major axis $\mathfrak{a}$. □

The subsequent explanatory example with pictographic validation substantiates Theorems 3.28 and 3.29.

Example 3.30. *Let an* $\mathcal{S}-$*metric* $\mathcal{S}:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\longrightarrow\mathbb{R}^+$ *be*

$$\mathcal{S}(\omega,\upsilon,u)=\Sigma_{i=1}^{2}(|e^{\omega_i}-e^{u_i}|+|e^{\omega_i}+e^{u_i}-2e^{\upsilon_i}|),$$

$$\omega=(\omega_1,\omega_2),\upsilon=(\upsilon_1,\upsilon_2),u=(u_1,u_2)\in\mathcal{U}=\mathbb{R}^2.$$

Then,

$$\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})=\{\omega\in\mathcal{U}:\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)=4\},$$
$$\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,a)=\{\omega\in\mathcal{U}:\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)\leq4\},$$

where $\mathfrak{c}_1=(0,0)$ *and* $\mathfrak{c}_2=(\ln 2,0)$, *that is, the equation of an ellipse centered at* $(0.35,0)$ *with foci at* $(0,0)$ *and* $(\ln 2,0)$ *is*

$$|e^0-e^{\omega_1}|+|e^0+e^{\omega_1}-2e^0|+|e^0-e^{\omega_2}|+|e^0+e^{\omega_2}-2e^0|+|e^{\ln 2}-e^{\omega_1}|$$
$$+|e^{\ln 2}+e^{\omega_1}-2e^{\ln 2}|+|e^0-e^{\omega_2}|+|e^0+e^{\omega_2}-2e^0|=4,$$

that is,

$$2|1-e^{\omega_1}|+4|1-e^{\omega_2}|+2|2-e^{\omega_1}|=4,$$

that is,

$$|1-e^{\omega_1}|+2|1-e^{\omega_2}|+|2-e^{\omega_1}|=2 \tag{3.12}$$

is an ellipse, and

$$|1-e^{\omega_1}|+2|1-e^{\omega_2}|+|2-e^{\omega_1}|\leq2 \tag{3.13}$$

is an elliptic disc.

Define $\mathcal{A}:\mathcal{U}\longrightarrow\mathcal{U}$ as $\mathcal{A}(\mathfrak{a},\mathfrak{b})=\begin{cases}(\mathfrak{a},\mathfrak{b}), & (\mathfrak{a},\mathfrak{b})\in\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\ln 2)\\ (a-\ln 15,\ b-\ln 15), & \text{otherwise}\end{cases}$.

Then, map $\mathcal{A}$ validates all the postulates of Theorem 3.28 except $(\mathcal{E}_3)$ and all postulates of Theorem 3.29. Consequently, a self-map $\mathcal{A}$ fixes the ellipse $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,2)$ and elliptic disc $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,2)$ of maximum semi-major axis 2, that is, the set of fixed points of $\mathcal{A}$ contains an ellipse $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,2)$ as well as an elliptic disc $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,2)$ of maximum semi-major axis 2 (see Figure 10). Nevertheless, a fixed ellipse is not unique, as there exist many fixed ellipses. For example, the ellipses having foci at $(0,0)$ and $(\ln 2,0)$ with a semi-major axis of less than 2 units are the fixed ellipses of $\mathcal{A}$.

Using Eq (3.8), we give one more result for the existence of a unique fixed ellipse on an $\mathcal{S}-$metric space.

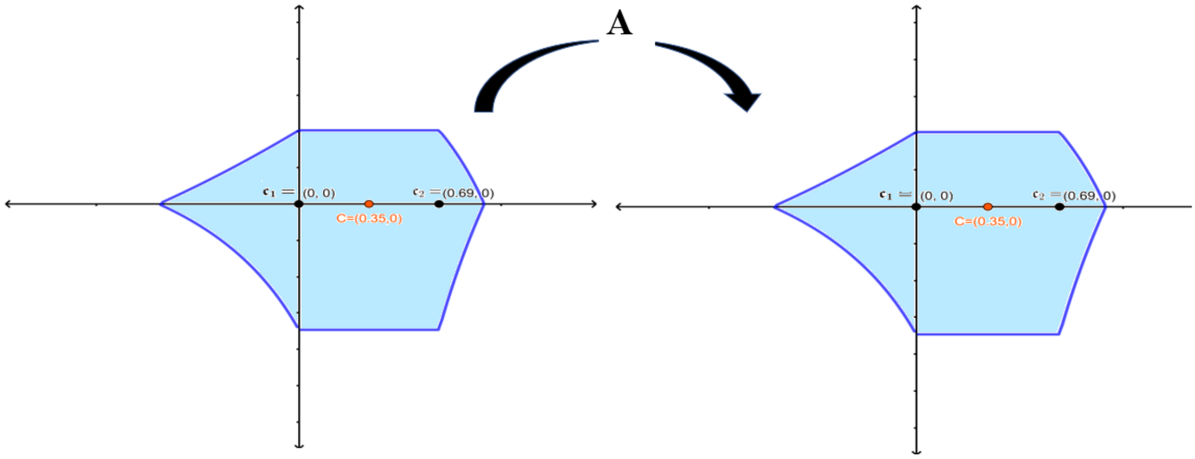

Figure 10. The boundary of the blue-shaded region demonstrates the ellipse (3.12), and its interior region indicates the greatest elliptic disc (3.13), fixed by the function $\mathcal{A}$.

Theorem 3.31. *Theorem 3.24 continues to be true even if we replace* $(\mathcal{E}_2)$ *by*

$(\mathcal{E}_2'')$ $\eta\mathcal{S}(\omega,\omega,\mathcal{A}\omega)+\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega)\geq 2\mathfrak{a}.$

Proof. Let $\omega\in\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ be any arbitrary point. Using $(\mathcal{E}_1)$ and Eq (3.8),

$$\begin{aligned}\mathcal{S}(\omega,\omega,\mathcal{A}\omega)&\leq\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)-\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)-\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega)\\&=2\mathfrak{a}-\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)-\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega)\\&\leq\eta\mathcal{S}(\omega,\omega,\mathcal{A}\omega),\ \ (\text{using }(\mathcal{E}_2'')),\end{aligned}$$

a contradiction, that is, $\mathcal{S}(\omega,\omega,\mathcal{A}\omega)=0\Longrightarrow\mathcal{A}\omega=\omega,\ \omega\in\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$.

Following the pattern of Theorem 3.24, we may establish that $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ is a unique fixed ellipse of $\mathcal{A}$. □

Theorem 3.32. *The conclusion of Theorem 3.29 continues to be true even if we replace* $(\mathcal{E}_{1\mathfrak{D}})$ *and* $(\mathcal{E}_{2\mathfrak{D}})$ *by*

$(\mathcal{E}_{1\mathfrak{D}}')$ $\mathcal{S}(\omega,\omega,\mathcal{A}\omega)\leq\varsigma(\omega)-\varsigma(\mathcal{A}\omega)$;

$(\mathcal{E}_{2\mathfrak{D}}')$ $\eta\mathcal{S}(\omega,\omega,\mathcal{A}\omega)+\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega)\geq 2\mathfrak{a}.$

Proof. Let $\omega\in\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ be any arbitrary point. Using $(\mathcal{E}_{1\mathfrak{D}}')$ and Eq (3.8),

$$\begin{aligned}\mathcal{S}(\omega,\omega,\mathcal{A}\omega)&\leq\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)-\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)-\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega)\\&\leq 2\mathfrak{a}-\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)-\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega)\\&\leq\eta\mathcal{S}(\omega,\omega,\mathcal{A}\omega),\ \ (\text{using}(\mathcal{E}_{2\mathfrak{D}}'),\end{aligned}$$

a contradiction, that is,

$$\mathcal{S}(\omega,\omega,\mathcal{A}\omega)=0\Longrightarrow\mathcal{A}\omega=\omega,\ \omega\in\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a}).$$

Following Theorem 3.29, we may establish that $\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ is a fixed elliptic disc of maximum semi-major axis $\mathfrak{a}$. □

Example 3.33. *Let an* $\mathcal{S}-$*metric* $\mathcal{S}:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\longrightarrow\mathbb{R}^+$ *be*

$$\mathcal{S}(\omega,\upsilon,u)=\Sigma_{i=1}^{2}(\left|\tan^{-1}\omega_i-\tan^{-1}u_i\right|^2+\left|\tan^{-1}\upsilon_i-\tan^{-1}u_i\right|^2),$$

$$\omega=(\omega_1,\omega_2),\upsilon=(\upsilon_1,\upsilon_2),u=(u_1,u_2)\in\mathcal{U}=\mathbb{R}^2.$$

Then

$$\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,2)=\{\omega\in\mathcal{U}:\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)=4\},$$

where $\mathfrak{c}_1=(0,3)$ *and* $\mathfrak{c}_2=(0,-3)$*, that is, the equation of an ellipse centered at* $(0,0)$ *with foci at* $(0,3)$ *and* $(0,-3)$ *is*

$$\begin{aligned}&|\tan^{-1}0-\tan^{-1}\omega_1|^2+|\tan^{-1}0-\tan^{-1}\omega_1|^2\\&+|\tan^{-1}3-\tan^{-1}\omega_2|^2+|\tan^{-1}3-\tan^{-1}\omega_2|^2+|\tan^{-1}0-\tan^{-1}\omega_1|^2\\&+|\tan^{-1}0-\tan^{-1}\omega_1|^2+|\tan^{-1}(-3)-\tan^{-1}\omega_2|^2+|\tan^{-1}(-3)-\tan^{-1}\omega_2|^2=4,\end{aligned}$$

that is,

$$|\tan^{-1}\omega_1|^2+|\tan^{-1}3-\tan^{-1}\omega_2|^2+|\tan^{-1}3+\tan^{-1}\omega_2|^2=2 \tag{3.14}$$

is an ellipse, and

$$|\tan^{-1}\omega_1|^2+|\tan^{-1}3-\tan^{-1}\omega_2|^2+|\tan^{-1}3+\tan^{-1}\omega_2|^2\leq 2 \tag{3.15}$$

is an elliptic disc.

Define $\mathcal{A}:\mathcal{U}\longrightarrow\mathcal{U}$ *as* $\mathcal{A}(\mathfrak{a},\mathfrak{b})=\begin{cases}(\mathfrak{a},\mathfrak{b}), & (\mathfrak{a},\mathfrak{b})\in\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,2)\\ \frac{1}{9}(\tan a,\tan b), & \text{otherwise}\end{cases}.$

Then, map $\mathcal{A}$ *validates all the hypotheses of Theorem 3.31 except* $(\mathcal{E}_3)$ *and all hypotheses of Theorem 3.32. Consequently, a self-map* $\mathcal{A}$ *fixes the ellipse* $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,2)$ *and elliptic disc* $\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,2)$ *of maximum semi-major axis* 2*, that is, the set of fixed points of* $\mathcal{A}$ *contains an ellipse* $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,2)$ *as well as an elliptic disc* $\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,2)$ *of maximum semi-major axis* 2 *(see Figure 11). However, a fixed ellipse is not unique, as there exist many fixed ellipses. For example, the ellipse having foci at* $(0,3)$ *and* $(0,-3)$ *with a semi-major axis less than* 2 *is also the fixed ellipse of* $\mathcal{A}$*.*

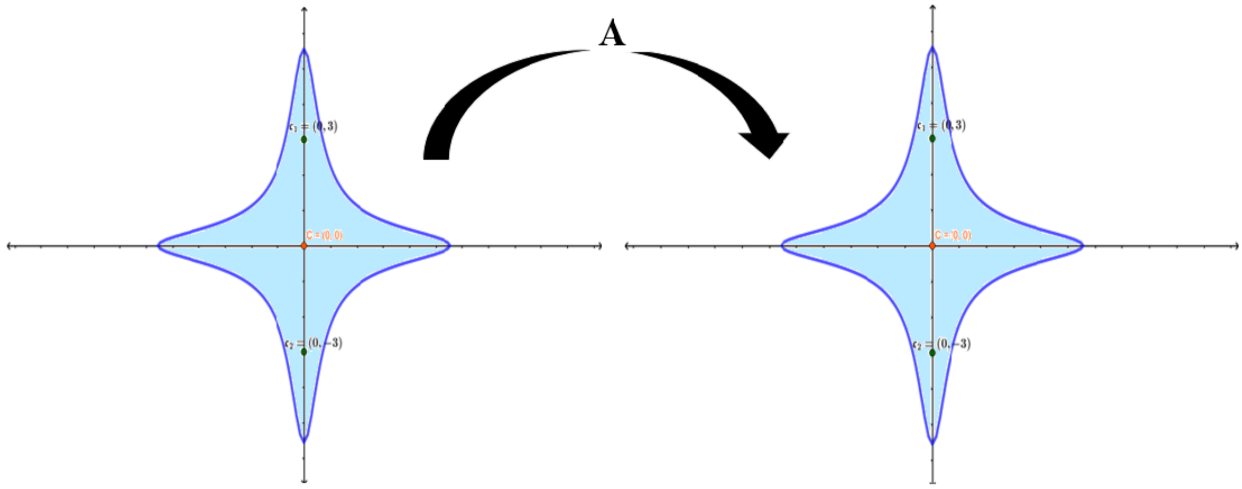

Figure 11. The boundary of blue lines demonstrates the ellipse (3.14), and its interior region indicates the greatest elliptic disc (3.15), fixed by the function $\mathcal{A}$.

Example 3.34. *Let an $\mathcal{S}-$metric $\mathcal{S}:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\longrightarrow\mathbb{R}^+$ be*

$$\mathcal{S}(\omega,\upsilon,u)=|\omega-u|+|\omega+u-2\upsilon|,\ \omega,\upsilon,u\in\mathcal{U}=\mathbb{R}.$$

The ellipse and elliptic disc centered at 3 with foci at 2 and 4 are

$$\begin{aligned}\mathcal{E}(2,4,9)&=\{\omega\in\mathcal{U}:\mathcal{S}(2,2,\omega)+\mathcal{S}(4,4,\omega)=18\}\\&=\{\omega\in\mathcal{U}:|2-\omega|+|2+\omega-4|+|4-\omega|+|4+\omega-8|=18\}\\&=\{-\frac{3}{2},\frac{15}{2}\},\end{aligned}$$

and

$$\mathcal{E}_{\mathfrak{D}}(2,4,9)=\left[-\frac{3}{2},\frac{15}{2}\right].$$

Define $\mathcal{A}:\mathcal{U}\longrightarrow\mathcal{U}$ as $\mathcal{A}\omega=\begin{cases}-\frac{3}{2}, & \omega\in(-\infty,0]\\ \frac{15}{2}, & \omega\in(0,\infty)\end{cases}.$

Then, map $\mathcal{A}$ validates all the hypotheses of Theorem 3.31 and fixes the ellipse $\mathcal{E}(2,4,9)$, that is, the set of fixed points of a self-map $\mathcal{A}$ contains unique ellipse $\mathcal{E}(2,4,9)$. The next result is proved by taking a function $\zeta_{\mathfrak{a}}$ to be discontinuous.

Theorem 3.35. *Let $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ be an ellipse in an $\mathcal{S}-$metric space $(\mathcal{U},\mathcal{S})$. Define $\zeta_{\mathfrak{a}}:[0,\infty)\longrightarrow[0,\infty)$ as*

$$\zeta_{\mathfrak{a}}(\omega)=\begin{cases}\omega+2\mathfrak{a}, & \omega>0\\ 0, & \omega=0\end{cases},\ \omega\in\mathbb{R}^+\cup\{0\},\ \mathfrak{a}\in[0,\infty). \tag{3.16}$$

If there exists $\mathcal{A}:\mathcal{M}\longrightarrow\mathcal{M}$ satisfying
$(\mathcal{E}_1''')$ $\mathcal{S}(\omega,\omega,\mathcal{A}\omega)\leq\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega)-\zeta_{\mathfrak{a}}(\mathcal{S}(\omega,\omega,\mathcal{A}\omega))$,
$(\mathcal{E}_2''')$ $\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1\mathcal{A}\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega)\leq 2\mathfrak{a}$,

$(\mathcal{E}_3^{'''})$ $\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\upsilon) \leq \sigma\mathcal{S}(\omega,\omega,\upsilon)$, $\omega,\upsilon \in \mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$, $\upsilon \in \mathcal{U} \setminus \mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$, $\sigma \in [0,1)$, then $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ is a unique fixed ellipse of $\mathcal{A}$.

Proof. Let $\omega \in \mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ such that $\mathcal{A}\omega \neq \omega$, $\omega \in \mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$. Using $(\mathcal{E}_1^{'''})$ and Eq (3.16),

$$\begin{aligned}\mathcal{S}(\omega,\omega,\mathcal{A}\omega) &\leq \mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2\mathcal{A}\omega)-\mathcal{S}(\omega,\omega,\mathcal{A}\omega)-2\mathfrak{a}\\ &\leq 2\mathfrak{a}-\mathcal{S}(\omega,\omega,\mathcal{A}\omega)-2\mathfrak{a}\end{aligned}$$

$\Longrightarrow 2\mathcal{S}(\omega,\omega,\mathcal{A}\omega) \leq 0$, a contradiction, that is, $\mathcal{A}\omega = \omega$, $\omega \in \mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$.

Following Theorem 3.24, we may establish that $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ is a unique fixed ellipse of $\mathcal{A}$. □

The subsequent example depicts the significance of condition $(\mathcal{E}_3^{'''})$ in the existence of a unique fixed ellipse in Theorem 3.35.

Example 3.36. *Let an* $\mathcal{S}-$*metric* $\mathcal{S}:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\longrightarrow\mathbb{R}^+$ *be*

$$\mathcal{S}(\omega,\upsilon,u) = |\omega-u|+|\omega+u-2\upsilon|,\ \omega,\upsilon,u \in \mathcal{U}=\mathbb{R}.$$

The ellipse

$$\begin{aligned}\mathcal{E}(2,4,4) &= \{\omega\in\mathcal{U}:\mathcal{S}(2,2,\omega)+\mathcal{S}(4,4,\omega)=8\}\\ &= \{\omega\in\mathcal{U}:2|2-\omega|+2|4-\omega|=8\}\\ &= \{\omega\in\mathcal{U}:|2-\omega|+|4-\omega|=4\}\\ &= \{1,\ 5\}.\end{aligned}$$

Define $\mathcal{A}:\mathcal{U}\longrightarrow\mathcal{U}$ as $\mathcal{A}\omega = \begin{cases} 2\frac{\omega+1}{\omega+3}, & \omega\in(-\infty,2)\\ 15\frac{\omega-3}{\omega+1}, & \omega\in[2,\infty)\end{cases}$.

Then, a self-map $\mathcal{A}$ validates the hypotheses $(\mathcal{E}_1^{'''})$ and $(\mathcal{E}_2^{'''})$ of Theorem 3.35 but does not validate the hypothesis $(\mathcal{E}_3^{'''})$. Noticeably, $\mathcal{A}$ fixes the two ellipses $\mathcal{E}(2,4,4)$ and $\mathcal{E}(2,5,11)$. It is clear that geometrically the condition $(\mathcal{E}_1^{'''})$ states that $\mathcal{A}\omega$ is in the exterior of an ellipse, and the condition $(\mathcal{E}_2^{'''})$ states that $\mathcal{A}\omega$ is in the interior of an ellipse.

The relationships among the "if"s and "then"s in this theorem are unclear, as in Theorem 3.29.

Theorem 3.37. *Let* $\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *be an elliptic disc in an* $\mathcal{S}-$*metric space* $(\mathcal{U},\mathcal{S})$*. Define* $\zeta_{\mathfrak{a}}:\mathcal{U}\longrightarrow[0,\infty)$ *as in Eq* (3.16)*. If self-map* $\mathcal{A}:\mathcal{U}\longrightarrow\mathcal{U}$ *satisfies the hypotheses*

$(\mathcal{E}_{1\mathfrak{D}}^{''})$ $\mathcal{S}(\omega,\omega,\mathcal{A}\omega) \leq \mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2\mathcal{A}\omega)-\zeta_{\mathfrak{a}}(\mathcal{S}(\omega,\omega,\mathcal{A}\omega))$,

$(\mathcal{E}_{2\mathfrak{D}})$ $\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega) \leq 2\mathfrak{a}$, $\omega\in\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$.

Then, $\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *is a fixed elliptic disc of* $\mathcal{A}$*.*

$(\mathcal{E}_{3\mathfrak{D}})$ *In addition to the above hypotheses, if* $\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\upsilon) \leq \sigma\mathcal{S}(\omega,\omega,\upsilon)$, $\omega\in\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$, $\upsilon\in\mathcal{U}\setminus\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a}),\sigma\in[0,1)$*, then* $\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *is a fixed elliptic disc of maximum semi-major axis* $\mathfrak{a}$*, that is, there is no fixed elliptic disc* $\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *of* $\mathcal{A}$ *having a semi-major axis greater than* $\mathfrak{a}$*.*

Proof. The proof is simple and follows Theorem 3.35. □

Remark 3.8 (Joshi et al. [14, 16] and Joshi and Tomar [17]). *(1) Examples 3.21–3.23, 3.26–3.36 (Examples 3.21, 3.22, 3.30–3.33) demonstrate that an ellipse (elliptic disc) in an $\mathcal{S}-$metric space may not be similar to an ellipse (elliptic disc) in a Euclidean space. Further, Examples 3.22, 3.23, 3.30 and 3.33 demonstrate the significant fact that the shape of the ellipse (elliptic disc) may alter by altering the center, the semi-major axis, the foci or the $\mathcal{S}-$metric under consideration. Also, the semi-major axis $\mathfrak{a}$ of the fixed ellipse (elliptic disc) is not dependent on a center and may not be maximal.*

(2) The fixed ellipse and fixed elliptic disc conclusions are comparable to fixed-circle and fixed disc conclusions if both the focuses coincide. Clearly, if $\mathfrak{c}_1 = \mathfrak{c}_2 = \mathfrak{u}_0$ (say), $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \mathcal{C}(\mathfrak{u}_0, \frac{\mathfrak{a}}{2})$ and $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \mathcal{D}(\mathfrak{u}_0, \frac{\mathfrak{a}}{2})$, with center $\mathfrak{u}_0$ and radius $\frac{\mathfrak{a}}{2}$. Also, $\mathcal{A}\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ does not imply that $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed ellipse of $\mathcal{A}$, and $\mathcal{A}\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ does not imply that $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ is a fixed elliptic disc of $\mathcal{A}$.

(3) It is clear from Examples 3.30 and 3.33 that if a self-map fixes an elliptic disc, then it also fixes an ellipse. However, the reverse may not hold (see Examples 3.25–3.27, 3.34, 3.36). The fixed elliptic disc is not unique, that is, all the elliptic discs inside a fixed elliptic disc of a self map in an $\mathcal{S}-$metric space are also fixed elliptic discs (see Examples 3.30 and 3.33). An elliptic disc having a maximum semi-major axis is called the greatest elliptic disc. For details on the collection of non-unique fixed points containing some geometric figures, one may refer to Aydi et al. [3], Beloul et al. [5], Joshi et al. [13–18], Joshi and Tomar [17], Mlaiki et al. [22, 23], Özgür et al. [27], Özgür and Taş [27–30], Petwal et al. [32], Taş et al. [38], Tomar and Joshi [39], Tomar et al. [40] and references therein.

Let $\mathcal{I} : \mathcal{U} \longrightarrow \mathcal{U}$ be the identity map, that is, $\mathcal{I}\omega = \omega \in \mathcal{U}$. Clearly, $\mathcal{I}$ satisfies the first two hypotheses of each of the Theorems 3.24, 3.28, 3.31 and 3.35 for an ellipse. Now, we devise conditions that preclude the possibility of the identity map $\mathcal{I}$ from Theorems 3.24, 3.28, 3.31 and 3.35.

Theorem 3.38. *Let a self-map $\mathcal{A} : \mathcal{U} \longrightarrow \mathcal{U}$ have a fixed ellipse $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$, $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{U}$, $\mathfrak{a} \in [0, \infty)$ in an $\mathcal{S}-$metric space $(\mathcal{U}, \mathcal{S})$. Let map ς be defined as in Theorems 3.24, 3.28, 3.31 and 3.35, and map $\mathcal{A}$ satisfies*

(I) $\lambda\mathcal{S}(\omega, \omega, \mathcal{A}\omega) \leq \varsigma(\omega) - 2\varsigma(\mathcal{A}\omega), \quad \omega \in \mathcal{U}, \;$ *and* $\; \lambda > 2 \Longleftrightarrow \mathcal{A} = \mathcal{I}.$

Proof. Let $\omega \in \mathcal{U}$ and $\mathcal{A}\omega \neq \omega$. Then,

$$\begin{aligned}
\lambda\mathcal{S}(\omega, \omega, \mathcal{A}\omega) &\leq \mathcal{S}(\mathfrak{c}_1, \mathfrak{c}_1, \omega) + \mathcal{S}(\mathfrak{c}_2, \mathfrak{c}_2, \omega) - 2\mathcal{S}(\mathfrak{c}_1, \mathfrak{c}_1, \mathcal{A}\omega) - 2\mathcal{S}(\mathfrak{c}_2, \mathfrak{c}_2, \mathcal{A}\omega) \\
&\leq 2\mathcal{S}(\mathfrak{c}_1, \mathfrak{c}_1, \mathcal{A}\omega) + \mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \omega) + 2\mathcal{S}(\mathfrak{c}_2, \mathfrak{c}_2, \mathcal{A}\omega) + \mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \omega) \\
&\quad 2\mathcal{S}(\mathfrak{c}_1, \mathfrak{c}_1, \mathcal{A}\omega) - 2\mathcal{S}(\mathfrak{c}_2, \mathfrak{c}_2, \mathcal{A}\omega) \\
&= 2\mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \omega) \\
&= 2\mathcal{S}(\omega, \omega, \mathcal{A}\omega) \;\; \text{(using symmetry of } \mathcal{S}\text{)},
\end{aligned}$$

$(\lambda - 2)\mathcal{S}(\omega, \omega, \mathcal{A}\omega) \leq 0$, a contradiction, since $\lambda > 2$. Hence, $\mathcal{S}(\omega, \omega, \mathcal{A}\omega) = 0$, that is, $\mathcal{A}\omega = \omega = \mathcal{I}\omega$, $\omega \in \mathcal{U}$.

Conversely, it is straightforward to validate that $\mathcal{I}$ satisfies inequality (I). □

Theorem 3.39. *The conclusion of Theorem 3.38 continues to be true if (I) is replaced by*

(I') $\mathcal{S}(\omega, \omega, \mathcal{A}\omega) \leq \lambda_1\big(\varsigma(\omega) - 2\varsigma(\mathcal{A}\omega)\big), \quad \omega \in \mathcal{U}, \;$ *and* $\; \lambda_1 < \frac{1}{2} \Longleftrightarrow \mathcal{A} = \mathcal{I}.$

Proof. Let $\omega \in \mathcal{U}$ and $\mathcal{A}\omega \neq \omega$. Then,

$$\begin{aligned}\mathcal{S}(\omega,\omega,\mathcal{A}\omega) &\leq \lambda_1\big(\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)-2\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)-2\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega)\big)\\ &\leq \lambda_1(2\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)+\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega)+2\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega)+\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega)\\ &-2\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mathcal{A}\omega)-2\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mathcal{A}\omega))\\ &=2\lambda_1\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\omega)\\ &=2\lambda_1\mathcal{S}(\omega,\omega,\mathcal{A}\omega)\ \text{(using symmetry of } \mathcal{S}),\end{aligned}$$

that is, $(1-2\lambda_1)\mathcal{S}(\omega,\omega,\mathcal{A}\omega)\leq 0$, a contradiction, since $\lambda_1<\frac{1}{2}$. Hence, $\mathcal{S}(\omega,\omega,\mathcal{A}\omega)=0$, that is, $\mathcal{A}\omega=\omega=\mathcal{I}\omega,\ \omega\in\mathcal{U}$.

Conversely, it is straightforward to validate that $\mathcal{I}$ satisfies inequality (I'). □

Theorem 3.40. *The conclusion of Theorem 3.38 continues to be true if* (I) *is replaced by*

(I'') $\mathcal{S}(\omega,\omega,\mathcal{A}\omega)\leq\zeta_{\mathfrak{a}}(\mathcal{S}(\omega,\omega,\mathcal{A}\omega))+2\mathfrak{a}$.

Proof. Let $\omega\in\mathcal{U}$ and $\mathcal{A}\omega\neq\omega$, and then $\mathcal{S}(\omega,\omega,\mathcal{A}\omega)\leq\mathcal{S}(\omega,\omega,\mathcal{A}\omega)-2\mathfrak{a}+2\mathfrak{a}=\mathcal{S}(\omega,\omega,\mathcal{A}\omega)$, a contradiction. Hence, $\mathcal{A}\omega=\omega=\mathcal{I}\omega,\ \omega\in\mathcal{U}$.

Conversely, it is straightforward to validate that $\mathcal{I}$ satisfies inequality $(I)''$. □

Remark 3.9. *Conclusions of Theorems 3.38–3.40 are also true for fixed elliptic disc* $\mathcal{E}_{\mathfrak{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$.

It is interesting to see that the fixed ellipse $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ may not be unique (see Examples 3.26, 3.30 and 3.36) except if a supplementary contraction condition is presumed. In Theorems 3.24, 3.28, 3.31, and 3.35, we have utilized the Banach contraction [4] to prove the uniqueness of a fixed ellipse. In the subsequent result, we determine the uniqueness utilizing quasi-contractive condition [9]. In the same way, we may utilize other classical contractions present in the literature to demonstrate the uniqueness of the fixed ellipse.

Theorem 3.41. *Let* $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *be a fixed ellipse of a self-map* $\mathcal{A}:\mathcal{U}\to\mathcal{U}$ *of an* $\mathcal{S}-$*metric space* $(\mathcal{U},\mathcal{S})$. *If* $\mathcal{A}$ *satisfies the first two conditions of Theorems 3.24, 3.28, 3.31 and 3.35, along with the contraction condition*

$$\begin{aligned}\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\upsilon)\leq&\eta\max\big\{\mathcal{S}(\omega,\omega,\upsilon),\ \mathcal{S}(\omega,\omega,\mathcal{A}\upsilon),\ \mathcal{S}(\upsilon,\upsilon,\mathcal{A}\omega),\\ &\mathcal{S}(\omega,\omega,\mathcal{A}\omega),\ \mathcal{S}(\upsilon,\upsilon,\mathcal{A}\upsilon))\big\},\end{aligned}\tag{3.17}$$

$\omega\in\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a}),\upsilon\in\mathcal{U}\setminus\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$, *where* $\eta\in[0,1)$, *then* $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *is a unique fixed ellipse of* $\mathcal{A}$.

Proof. Let $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}')$ be two fixed ellipses of $\mathcal{A}$, that is, $\mathcal{A}$ satisfies the first two postulates of Theorems 3.24, 3.28, 3.31 and 3.35 for both the ellipses $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}')$. Let $\omega\in\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ and $\upsilon\in\mathcal{E}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}')$. Using inequality (3.17),

$$\begin{aligned}\mathcal{S}(\omega,\omega,\upsilon)=\mathcal{S}(\mathcal{A}\omega,\mathcal{A}\omega,\mathcal{A}\upsilon)&\leq\eta\max\big\{\mathcal{S}(\omega,\omega,\upsilon),\mathcal{S}(\omega,\omega,\upsilon),\mathcal{S}(\upsilon,\upsilon,\omega),\mathcal{S}(\omega,\omega,\omega),\mathcal{S}(\upsilon,\upsilon,\upsilon)\big\}\\ &=\eta\mathcal{S}(\omega,\omega,\upsilon)\\ &<\mathcal{S}(\omega,\omega,\upsilon),\end{aligned}$$

a contradiction. Hence, $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ is a unique fixed ellipse of $\mathcal{A}$. □

Remark 3.10. *A conclusion almost identical to Theorem 3.41 may also be established for the existence of the greatest fixed elliptic disc using a quasi-contractive condition [9].*

Next, we give propositions for the existence of a self-map that fixes the given ellipses.

Proposition 3.1 Let $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}')$ be any two ellipses in an $\mathcal{S}-$metric space $(\mathcal{U},\mathcal{S})$. Then, we have more than one self-map $\mathcal{A}$ on $\mathcal{U}$ such that a self-map $\mathcal{A}$ fixes the ellipses $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}')$.

Proof. Define $\mathcal{A}:\mathcal{U}\longrightarrow\mathcal{U}$ as $\mathcal{A}\omega=\begin{cases}\omega, & \omega\in\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})\cup\mathcal{E}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}')\\ \mu, & \text{otherwise}\end{cases},\omega\in\mathcal{U},$
where μ is some constant such that $\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\mu)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\mu)\neq 2\mathfrak{a}$ and $\mathcal{S}(\mathfrak{c}'_1,\mathfrak{c}'_1,\mu)+\mathcal{S}(\mathfrak{c}'_2,\mathfrak{c}'_2,\mu)\neq 2\mathfrak{a}'$.

Now, define $\varsigma_1,\varsigma_2:\mathcal{U}\longrightarrow[0,\infty)$ as $\varsigma_1(\omega)=\mathcal{S}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)$ and $\varsigma_2(\omega)=\mathcal{S}(\mathfrak{c}'_1,\mathfrak{c}'_1,\omega)+\mathcal{S}(\mathfrak{c}'_2,\mathfrak{c}'_2,\omega)$, $\omega\in\mathcal{U}$. Then, a self-map $\mathcal{A}$ validates all the hypotheses of Theorems 3.24 and 3.28 (except $(\mathcal{E}_3)$) for the ellipses $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}')$. Hence, $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ and $\mathcal{E}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}')$ are fixed ellipses of $\mathcal{A}$. □

Following an almost identical pattern, Proposition 3.1 may be extended for *n* ellipses.

Proposition 3.2 If $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a}_1)$, $\mathcal{E}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}'_1)$, $\ldots,\mathcal{E}(\mathfrak{c}^n_1,\mathfrak{c}^n_2,\mathfrak{a}^n_1)$ are any *n* ellipses in an $\mathcal{S}-$metric space $(\mathcal{U},\mathcal{S})$, then we have more than one self-map $\mathcal{A}$ on $\mathcal{U}$ so that a self-map $\mathcal{A}$ fixes ellipses $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a}_1)$, $\mathcal{E}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}'_1)$, $\ldots,\mathcal{E}(\mathfrak{c}^n_1,\mathfrak{c}^n_2,\mathfrak{a}^n_1)$.

One may observe that the ellipses $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a}_1)$, $\mathcal{E}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}'_1)$, $\ldots,\mathcal{E}(\mathfrak{c}^n_1,\mathfrak{c}^n_2,\mathfrak{a}^n_1)$ need not be disjoint.

Remark 3.11. *Propositions similar to Propositions 3.1 and 3.2 are also true for two and n elliptic discs* $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a}_1)$, $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}'_1,\mathfrak{c}'_2,\mathfrak{a}'_1),\ldots,$ $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}^n_1,\mathfrak{c}^n_2,\mathfrak{a}^n_1)$, *respectively.*

Next, we give a proposition on an $\mathcal{S}-$metric space in which an ellipse (elliptic disc) includes all the points of space except its foci and validate it by giving an example.

Proposition 3.3 For $\mathfrak{a}\in\mathbb{R}^+$, define the map $\mathcal{S}_\mathfrak{a}:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\longrightarrow[0,\infty)$ as

$$\mathcal{S}_\mathfrak{a}(\omega,\upsilon,u)=\begin{cases}0, & \omega=\upsilon=u\\ \mathfrak{a}, & \text{otherwise}\end{cases},\ \omega,\ \upsilon,\ u\in\mathcal{U}.$$

Then, the ellipse $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ $\big($elliptic disc $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})\big)$ includes all the points of space $\mathcal{U}$ except the foci $\mathfrak{c}_1,\mathfrak{c}_2\in\mathcal{U}$.

Proof. Obviously, the function $\mathcal{S}_\mathfrak{a}$ is an $\mathcal{S}-$metric on $\mathcal{U}$, and consequently, $(\mathcal{U},\mathcal{S}_\mathfrak{a})$ is an $\mathcal{S}-$metric space. Let the ellipse $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})=\{\omega\in\mathcal{U}:\mathcal{S}_\mathfrak{a}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}_\mathfrak{a}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)=2\mathfrak{a}\}$ (elliptic disc $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})=\{\omega\in\mathcal{U}:\mathcal{S}_\mathfrak{a}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}_\mathfrak{a}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)\leq 2\mathfrak{a}\}$). Clearly, ellipse $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ (elliptic disc $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$) consists of all of the points $\omega\in\mathcal{U}$ so that $\omega\notin\{\mathfrak{c}_1,\mathfrak{c}_2\}$. □

Example 3.42. *Let* $(\mathcal{U},\mathcal{S}_\mathfrak{a})$ *be an* $\mathcal{S}-$*metric space so that the* $\mathcal{S}-$*metric* $\mathcal{S}_\mathfrak{a}$ *be as in Proposition* 3.3. *Consider a set* $\mathcal{J}=\{\omega_i:1\leq i\leq n\}$, $n\in\mathbb{N}$. *Obviously, there exists an ellipse* $\mathcal{E}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *(elliptic disc* $\mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1,\mathfrak{c}_2,\mathfrak{a})$ *) consisting of the elements of* $\mathcal{J}$ *as follows:*

$$\{\omega\in\mathcal{U}:\mathcal{S}_\mathfrak{a}(\mathfrak{c}_1,\mathfrak{c}_1,\omega)+\mathcal{S}_\mathfrak{a}(\mathfrak{c}_2,\mathfrak{c}_2,\omega)=2\mathfrak{a}\}=\{\omega_1,\omega_2,\ldots,\omega_n\},\mathfrak{c}_1,\ \mathfrak{c}_2\in\mathcal{U}\setminus\mathcal{J}$$

$$(\{\omega \in \mathcal{U} : \mathcal{S}_{\mathfrak{a}}(\mathfrak{c}_1, \mathfrak{c}_1, \omega) + \mathcal{S}_{\mathfrak{a}}(\mathfrak{c}_2, \mathfrak{c}_2, \omega) \leq 2\mathfrak{a}\} = \{\omega_1, \omega_2, \ldots, \omega_n\}, \mathfrak{c}_1,\ \mathfrak{c}_2 \in \mathcal{U} \setminus \mathcal{J}),$$

that is, $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}) = \mathcal{E}_{\mathcal{D}}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$.

4. Discontinuous maps as activation functions in neural networks

First, we discuss the continuity of a self-map on a fixed ellipse in $\mathcal{S}-$metric spaces. This discussion will also be beneficial for providing the answer to the query of continuity of contractive maps (Rhoades [35]) at the fixed ellipse (elliptic disc).

Theorem 4.1. *Let* $\mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a}),\ \mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{U},\ \mathfrak{a} \in [0, \infty)$*, be a fixed ellipse of a self-map* $\mathcal{A}$ *in an* $\mathcal{S}-$*metric space* $(\mathcal{U}, \mathcal{S})$ *satisfying*

(i) $\mathcal{S}(\mathcal{A}\omega, \mathcal{A}\omega, \mathcal{A}\upsilon) \leq \eta \boldsymbol{M}(\omega, \omega, \upsilon)$*, where*

$$\boldsymbol{M}(\omega, \omega, \upsilon) = \max\{\mathcal{S}(\omega, \omega, \upsilon), \mathcal{S}(\omega, \omega, \mathcal{A}\upsilon), \mathcal{S}(\upsilon, \upsilon, \mathcal{A}\omega), \mathcal{S}(\omega, \omega, \mathcal{A}\omega), \mathcal{S}(\upsilon, \upsilon, \mathcal{A}\upsilon)\},$$

$\eta \in [0, 1),\ \omega, \upsilon \in \mathcal{U}$*;*

(ii) for $\varepsilon > 0$, *there exists a* $\delta > 0$ *so that* $\varepsilon < \boldsymbol{M}(\omega, \omega, \upsilon) < \varepsilon + \delta \Longrightarrow d(\mathcal{A}\omega, \mathcal{A}\omega, \mathcal{A}\upsilon) < \varepsilon$.

Then, a self-map $\mathcal{A}$ is continuous at $u \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ iff $\lim_{\omega_n \longrightarrow u} \mathbf{M}(\omega_n, \omega_n, u) = 0$, or in other words, $\mathcal{A}$ is discontinuous at $u \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ iff $\lim_{\omega_n \longrightarrow u} \mathbf{M}(\omega_n, \omega_n, u) \neq 0$.

Proof. Let $\mathcal{A}$ be continuous at $u \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$ and $\omega_n \longrightarrow u$. So, $\mathcal{A}\omega_n \longrightarrow \mathcal{A}u = u$.

$$\begin{aligned}\lim_{\omega_n \longrightarrow u} \mathbf{M}(\omega_n, \omega_n, u) = &\lim_{\omega_n \longrightarrow u} \max\{\mathcal{S}(\omega_n, \omega_n, u), \mathcal{S}(\omega_n, \omega_n, \mathcal{A}\omega_n), \\ &\mathcal{S}(u, u, \mathcal{A}u), \mathcal{S}(\omega_n, \omega_n, \mathcal{A}\omega_n), \mathcal{S}(u, u, \mathcal{A}u))\} \\ =&\max\{\mathcal{S}(u, u, u), \mathcal{S}(u, u, \mathcal{A}u), \mathcal{S}(u, u, \mathcal{A}u), \mathcal{S}(u, u, \mathcal{A}u), \mathcal{S}(u, u, \mathcal{A}u))\} \\ =&0.\end{aligned}$$

Conversely, if $\lim_{\omega_n \longrightarrow u} \mathbf{M}(\omega_n, \omega_n, u) = 0$, that is, $\lim_{\omega_n \longrightarrow u} \mathcal{S}(\mathcal{A}\omega_n, \mathcal{A}\omega_n, \mathcal{A}u) = 0$ as $\omega_n \longrightarrow u$, hence $\mathcal{A}\omega_n \longrightarrow \mathcal{A}u$, that is, $\mathcal{A}$ is continuous at $u \in \mathcal{E}(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{a})$. □

Remark 4.1. *Following similar steps, we may determine continuity and discontinuity at the fixed elliptic disc.*

Continuity at a fixed ellipse (elliptic disc) may be decided by making use of function $\mathbf{M}(\omega, \omega, \upsilon)$ in $\mathcal{S}-$metric spaces. Inspired by the reality that the majority of the phenomena appearing in the physical world are discontinuous, now, we discuss Mexican-hat-type [41] and Gaussian-wavelet-type [25] activation functions, which are utilized to examine non-linear properties, local stability and coexistence of several equilibrium points to the neural network.

Example 4.2. *Let an* $\mathcal{S}-$*metric* $\mathcal{S} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}^+$ *be* $\mathcal{S}(\omega, \upsilon, u) = |\omega - u| + |\omega + u - 2\upsilon|, \omega, \upsilon, u \in \mathcal{U}$. *The ellipse*

$$\begin{aligned}E(-3, 4, 9) &= \{\omega \in \mathcal{U} : \mathcal{S}(-3, -3, \omega) + \mathcal{S}(4, 4, \omega) = 18\} \\ &= \{\omega \in \mathcal{U} : 2|3 + \omega| + 2|4 - \omega| = 18\}\end{aligned}$$

$$= \{\omega \in \mathcal{U} : |3+\omega| + |4-\omega| = 9\}$$
$$= \{-4,\ 5\}.$$

The Mexican-hat-type activation function is

$$\mathcal{A}\omega = \begin{cases} -4, & -\infty < \omega < -1 \\ 2\omega - 2, & -1 \leq \omega < 1 \\ -\omega + 1, & 1 \leq \omega < 5 \\ 5, & 5 \leq \omega < \infty \end{cases}. \tag{4.1}$$

Clearly, $\mathcal{A}$ validates all the hypotheses of Theorems 3.24, 3.28, 3.31 and 3.35. Function $\mathcal{A}$ has two fixed points, -4 and 5, which are also the elements of the ellipse $\mathcal{E}(-3,4,9)$. Hence, $\mathcal{A}$ fixes the ellipse $\mathcal{E}(-3,4,9)$.

Since we have $\lim_{\omega_n \longrightarrow -4} \boldsymbol{M}(\omega_n, \omega_n, -4) = 0$, and $\lim_{\omega_n \longrightarrow 5} \boldsymbol{M}(\omega_n, \omega_n, 5)$ does not exist, hence $\mathcal{A}$ is continuous at the fixed point $u = -4 \in \mathcal{E}(-3,4,9)$ and discontinuous at $u = 5 \in \mathcal{E}(-3,4,9)$. As a result, $\mathcal{A}$ is discontinuous at the fixed ellipse $\mathcal{E}(-3,4,9)$ (see Figure 12). Noticeably, $\mathcal{A}$ does not fix any elliptic disc.

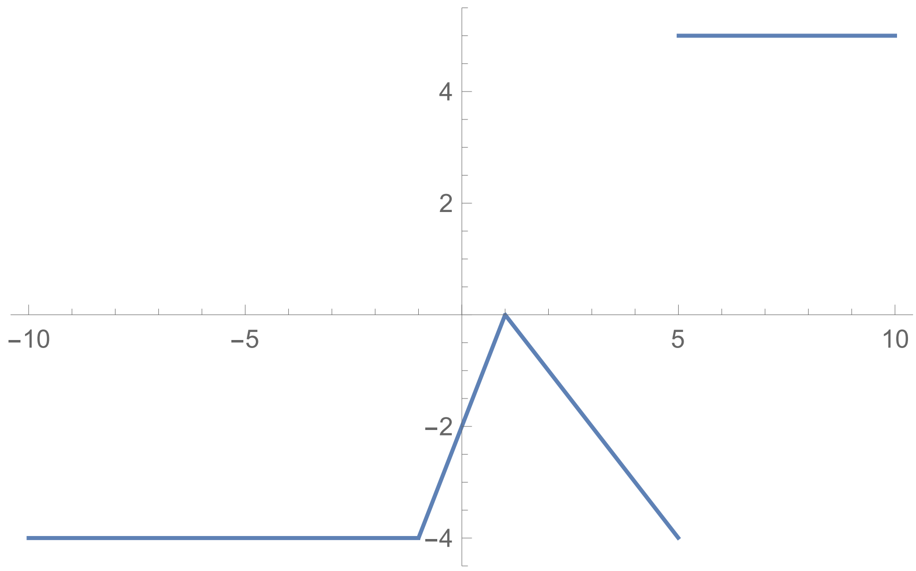

Figure 12. Mexican-hat-type discontinuous non-monotonic activation function (4.1).

Example 4.3. *Let an $\mathcal{S}-$metric $\mathcal{S} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ be $\mathcal{S}(\omega, \upsilon, u) = |\omega - u| + |\omega + u - 2\upsilon|, \omega, \upsilon, u \in \mathcal{U}$.*

The ellipse

$$\mathcal{E}(2,4,16) = \{\omega \in \mathcal{U} : \mathcal{S}(2,2,\omega) + \mathcal{S}(4,4,\omega) = 32\}$$
$$= \{\omega \in \mathcal{U} : 2|2-\omega| + 2|4-\omega| = 32\}$$
$$= \{\omega \in \mathcal{U} : |2-\omega| + |4-\omega| = 16\}$$
$$= \{-5, 11\}.$$

The Gaussian-wavelet-type activation function is

$$\mathcal{A}\omega = \begin{cases} -5, & -\infty < \omega < -3 \\ 2\omega + 4, & -3 \leq \omega \leq 3 \\ -\omega + 1, & 3 < \omega < 6 \\ 3\omega + 6, & 6 \leq \omega \leq 10 \\ 11, & 10 < \omega < \infty \end{cases} . \tag{4.2}$$

Clearly, $\mathcal{A}$ validates all the hypotheses of Theorems 3.24, 3.28, 3.31 and 3.35. The function $\mathcal{A}$ has two fixed points, -5 and 11, which are also the elements of the ellipse $\mathcal{E}(2,4,16)$, hence $\mathcal{A}$ fixes the ellipse $\mathcal{E}(2,4,16)$. Since we have $\lim_{\omega_n \longrightarrow -5} \boldsymbol{M}(\omega_n, \omega_n, -5) = 0$, and $\lim_{\omega_n \longrightarrow 11} \boldsymbol{M}(\omega_n, \omega_n, 11) = 0$, $\mathcal{A}$ is continuous at the ellipse $\mathcal{E}(2,4,16)$ (see Figure 13). Noticeably, $\mathcal{A}$ does not fix any elliptic disc.

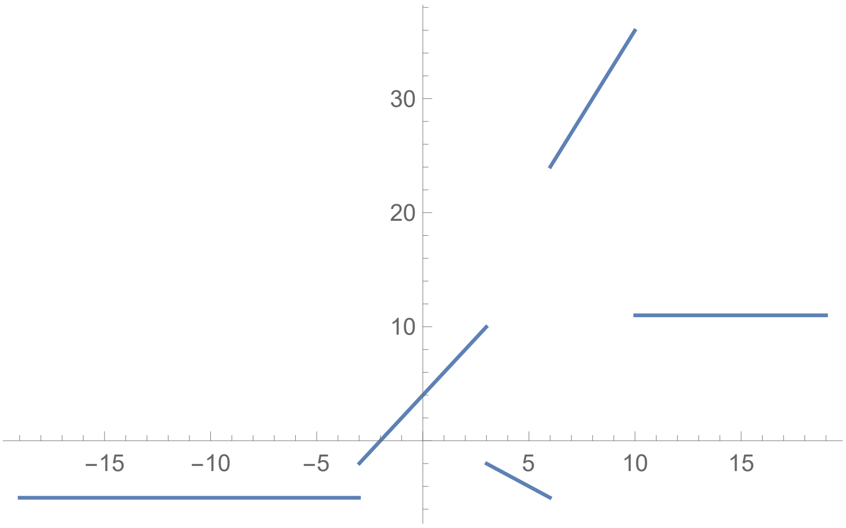

Figure 13. Gaussian-wavelet-type discontinuous non-monotonic activation function (4.2).

Example 4.4. *Let an $\mathcal{S}-$metric $\mathcal{S} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}^+$ be $\mathcal{S}(\omega, \upsilon, u) = |\omega - u| + |\omega + u - 2\upsilon|, \omega, \upsilon, u \in \mathcal{U}$.*

The ellipse

$$\begin{aligned} \mathcal{E}(-5,15,30) &= \{\omega \in \mathcal{U} : \mathcal{S}(-5,-5,\omega) + \mathcal{S}(15,15,\omega) = 60\} \\ &\{\omega \in \mathcal{U} : 2|5+\omega| + 2|15-\omega| = 60\} \\ &= \{\omega \in \mathcal{U} : |5+\omega| + |15-\omega| = 30\} \\ &= \{-10, 20\}. \end{aligned}$$

The elliptic disc

$$\begin{aligned} \mathcal{E}_{\mathcal{D}}(-8,-4,8) &= \{\omega \in \mathcal{U} : \mathcal{S}(-8,-8,\omega) + \mathcal{S}(-4,-4,\omega) \leq 16\} \\ &\{\omega \in \mathcal{U} : 2|8+\omega| + 2|4+\omega| \leq 16\} \\ &\{\omega \in \mathcal{U} : |8+\omega| + |4+\omega| \leq 8\} \end{aligned}$$

$$= [-10, -2].$$

The Mexican-hat-type activation function is

$$\mathcal{A}\omega = \begin{cases} -10, & -\infty < \omega \leq -10 \\ \omega, & -10 < \omega \leq -2 \\ -2\omega - 6, & -2 < \omega < 2 \\ 20, & 2 \leq \omega < \infty \end{cases}. \tag{4.3}$$

Clearly, $\mathcal{A}$ validates all the hypotheses of Theorems 3.29, 3.32 and 3.37. The function $\mathcal{A}$ has infinitely many fixed points in the set $[-10, -2]$ and at 20. Hence, $\mathcal{A}$ fixes the ellipse $\mathcal{E}(-5, 15, 30)$ and the elliptic disc $\mathcal{E}_{\mathcal{D}}(-8, -4, 8)$. Since we have $\lim_{\omega_n \longrightarrow \omega} \mathbf{M}(\omega_n, \omega_n, \omega) = 0, \omega \in [-10, -2]$, and $\lim_{\omega_n \longrightarrow 20} \mathbf{M}(\omega_n, \omega_n, 20)$ does not exist, hence $\mathcal{A}$ is discontinuous at the ellipse $\mathcal{E}(-5, 15, 30)$ and continuous at the elliptic disc $\mathcal{E}_{\mathcal{D}}(-8, -4, 8)$ (see Figure 14).

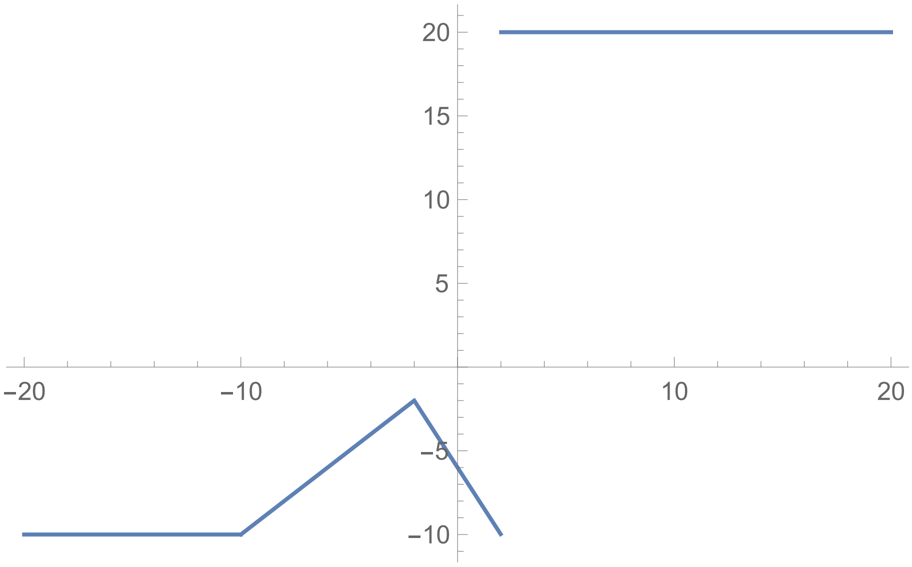

Figure 14. Mexican-hat-type discontinuous non-monotonic activation function (4.3).

Remark 4.2. *Utilizing the number $\boldsymbol{M}(\omega, \omega, \upsilon)$, we may decide on the fixed point/ fixed ellipse/ elliptic disc at which the activation function is continuous. The motivation behind using the discontinuous activation function is the fact that its storage capacity is higher than the continuous activation function, and consequently, it can handle diverse spans of input data as diverse segments.*

It is worth mentioning that a neural network is a network of neurons that is either biological, made up of real biological neurons or artificial. An artificial neural network is useful in medical diagnosis, non-linear system identification and control, sequence recognition, pattern recognition, decision-making, game-playing, financial applications, e-mail spam filtering, data mining and visualization.

5. Application to a satellite web coupling problem

Motivated by the applications of fixed point techniques in diverse real-world problems, we utilize Corollary 3.13 to solve a satellite web coupling boundary value problem [19]. A satellite web coupling may be idealized as a thin sheet connecting two cylindrical satellites. The problem of radiation from the web coupling between two satellites leads to the following non-linear boundary value problem:

$$-\frac{d^2\omega}{dt^2}=\mu\omega^4,\ 0<t<1,\ \omega(0)=\omega(1)=0, \tag{5.1}$$

where $\omega(t)$ denotes the temperature of radiation at any point $t\in[0,1]$, $\mu=\frac{2al^2K^3}{\zeta\mathfrak{h}}>0$ is a non-dimensional positive constant, K is the constant absolute temperature of both satellites, while heat is radiated from the surface of the web into space at 0 absolute temperature, l is the distance between two satellites, a is a positive constant describing the radiation properties of the surface of the web, factor 2 is required because there is radiation from both the top and bottom surfaces, ζ is thermal conductivity, and $\mathfrak{h}$ is the thickness.

The Green function

$$\mathcal{G}(t,\xi)=\begin{cases}t(1-\xi), & 0<t<\xi\\ \xi(1-t), & \xi<t<1\end{cases}.$$

Problem (5.1) is equivalent to

$$\omega(t)=1-\mu\int_0^1\mathcal{G}(t,\xi)\omega^4(\xi)d\xi.$$

Let $\mathcal{U}=\mathcal{R}[0,1]$ be a set of Riemann integrable functions on $[0,1]$. Define an $\mathcal{S}-$metric $\mathcal{S}:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\to\mathbb{R}^+$ by $\mathcal{S}(\omega,\upsilon,u)=|\omega-u|+|\upsilon-u|$. Clearly, $(\mathcal{U},\mathcal{S})$ is a complete $\mathcal{S}-$metric space, and $\|\omega\|_\infty=\sup_{t\in[0,1]}|\omega(t)|$.

Theorem 5.1. *Let $\mathcal{A}:\mathcal{U}\to\mathcal{U}$ be a self map in a complete $\mathcal{S}-$metric space $(\mathcal{U},\mathcal{S})$, satisfying*

$$\|\omega(t)-\upsilon(t)\|_\infty>0\Longrightarrow\|(\omega^2(\xi)+\upsilon^2(\xi))(\omega(\xi)+\upsilon(\xi))\|_\infty\le\frac{\kappa}{\mu},\ \kappa\in(0,8). \tag{5.2}$$

Then, the satellite web coupling boundary value problem (5.1) has a unique solution.

Proof. Define a self-map $\mathcal{A}:\mathcal{U}\to\mathcal{U}$ by

$$\mathcal{A}\omega(t)=1-\mu\int_0^1\mathcal{G}(t,\xi)\omega^4(\xi)d\xi,\ \xi\in[0,1]. \tag{5.3}$$

Clearly, a solution to the satellite web coupling problem (5.1) is a fixed point of a self map $\mathcal{A}$.

However, $\|\mathcal{A}\omega(t)-\mathcal{A}\upsilon(t)\|_\infty > 0$, so $\mathcal{S}(\mathcal{A}\omega(t),\mathcal{A}\omega(t),\mathcal{A}\upsilon(t)) > 0$. Now,

$$\begin{aligned}
\mathcal{S}(\mathcal{A}\omega(t),\mathcal{A}\omega(t),\mathcal{A}\upsilon(t)) &= 2|\mathcal{A}\omega(t)-\mathcal{A}(\upsilon)| \\
&= 2\Big|1-\mu\int_0^1 \mathcal{G}(t,\xi)\omega^4(\xi)d\xi - 1 + \mu\int_0^1 \mathcal{G}(t,\xi)\upsilon^4(\xi)d\xi\Big| \\
&= 2\mu\Big|\int_0^1 (\upsilon^4(\xi)-\omega^4(\xi))\mathcal{G}(t,\xi)d\xi\Big| \\
&= 2\mu\Big|\int_0^1 (\upsilon^2(\xi)+\omega^2(\xi))(\upsilon(\xi)+\omega(\xi))(\upsilon(\xi)-\omega(\xi))\mathcal{G}(t,\xi)d\xi\Big| \\
&= 2\mu\|\omega(t)-\upsilon(t)\|_\infty\|(\omega^2(t)+\upsilon^2(t))(\omega(t)+\upsilon(t))\|_\infty\int_0^1 \mathcal{G}(t,\xi)d\xi \\
&\le 2\kappa\|\omega(t)-\upsilon(t)\|_\infty\Big[\int_0^t \xi(1-t)d\xi + \int_t^1 t(1-\xi)d\xi\Big] \\
&\le \frac{\kappa}{8}\|\omega(t)-\upsilon(t)\|_\infty \\
&= \alpha\mathcal{S}(\omega,\omega,\upsilon),\ \alpha\in[0,1)\ (\text{since, } \kappa\in(0,8)).
\end{aligned}$$

Hence, all the postulates of Corollary 3.13 are validated. As a result, $\mathcal{A}$ has a unique fixed point, and a satellite web coupling problem (5.1) has a unique solution. □

6. Open problem

Let $\mathcal{A}:\mathcal{U}\to\mathcal{U}$ be a self-map in a complete $\mathcal{S}-$metric space $(\mathcal{U},\mathcal{S})$ and ζ be the simulation function [21].

(1) If $\mathcal{A}$ is a JS-contraction [12], does $\mathcal{A}$ have a unique fixed point / fixed ellipse/ fixed elliptic disc in $\mathcal{U}$? Does the Picard sequence $\{u_n\}$ converge to the unique fixed point u or any point u of a fixed ellipse (elliptic disc)? If not, what additional postulate(s) do we have to include?
(2) If $\mathcal{A}$ is a surjective JS-expanding map, does $\mathcal{A}$ have a unique fixed point/ fixed ellipse/ fixed elliptic disc in $\mathcal{U}$? Does the Picard sequence $\{u_n\}$ converge to the unique fixed point u or any point u of a fixed ellipse or elliptic disc? If not, what additional postulate(s) do we have to include?

7. Conclusions

We have established a unique fixed point, fixed circle, fixed ellipse and greatest fixed elliptic disc via an $\mathcal{M}-$class function while extending, generalizing, unifying and improving some popular results existing in the literature. Motivated by the reflecting property of an ellipse (elliptic disc), which is useful in Medical science, Optics, Astronomy, Whispering Galleries, and so on, we have explored a new direction to the geometry of the collection of fixed points in an $\mathcal{S}-$metric space. Furthermore, we have discussed continuity at fixed ellipses (elliptic discs) on $\mathcal{S}-$metric spaces to establish the significance of novel fixed ellipse (elliptic disc) conclusions in a neural network, which permits choosing the appropriate activation function according to the underlying problem of a neural network. In the sequel, we have presented some interesting prepositions and remarks. In this paper, investigations of fixed point and fixed figure problems in metric fixed point theory have been enriched to problems formulated in terms of $\mathcal{M}-$class contractive conditions on an $\mathcal{S}-$metric space. Consequently, more general conclusions have been established than those existing in the literature. It has been demonstrated by illustrative examples that these extensions, improvements and generalizations are genuine. Towards the end, the obtained conclusions have been applied to solve the satellite web coupling problem, which is a very significant and relevant field on its own. Our results provide a specific procedure and directions for further investigation in this recently developed space.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. A. Ali, E. Ameer, M. Arshad, H. Işık, M. Mudhesh, Fixed point results of dynamic process $\check{D}(\gamma, \mu_0)$ through F_I^C-contractions with applications, *Complexity*, **2022** (2022), 8495451. https://doi.org/10.1155/2022/8495451
2. M. Altanji, A. Santhi, V. Govindan, S. S. Santra, S. Noeiaghdam, Fixed-point results related to $b-$intuitionistic fuzzy metric space, *J. Funct. Spaces*, **2022** , (2022), 9561906. https://doi.org/10.1155/2022/9561906
3. H. Aydi, N. Taş, N. Y. Özgür, N. Mlaiki, Fixed-discs in rectangular metric spaces, *Symmetry*, **11** (2019), 294. https://doi.org/10.3390/sym11020294
4. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équation intégrales, *Fund. Math.*, **3** (1922), 133–181.
5. S. Beloul, A. Tomar, M. Joshi, On solutions to open problems and Volterra-Hammerstein non-linear integral equation, *Appl. Math. E-Notes.*, unpublished work.
6. J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, *Trans. Am. Math. Soc.*, **215** (1976), 241–251. https://doi.org/10.1090/S0002-9947-1976-0394329-4
7. S. K. Chatterjea, Fixed-point theorems, *C. R. Acad. Bulgare Sci.*, **6** (1972), 727–730.
8. L. B. Ćirić, Generalised contractions and fixed-point theorems, *Publ. Inst. Math.*, **12** (1971), 9–26.
9. L. B. Ćirić, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.*, **45** (1974), 267–273.
10. H. A. Hammad, M. Elmursi, R. A. Rashwan, H. Işık, Applying fixed point methodologies to solve a class of matrix difference equations for a new class of operators, *Adv. Contin. Discrete Models*, **2022** (2022), 1–16. https://doi.org/10.1186/s13662-022-03724-6

11. G. E. Hardy, T. D. Rogers, A generalization of a fixed point theorem of Reich, *Canad. Math. Bull.*, **16** (1973), 201–206. https://doi.org/10.4153/CMB-1973-036-0

12. M. Jleli, B. Samet, A new generalization of the Banach contraction principle, *J. Inequal. Appl.*, https://doi.org/10.1186/1029-242X-2014-38

13. M. Joshi, A. Tomar, H. A. Nabwey, R. George, On unique and non-unique fixed points and fixed circles in $\mathcal{M}_v^b-$metric space and application to cantilever beam problem, *J. Funct. Spaces*, **2021** (2021), 6681044. https://doi.org/10.1155/2021/6681044

14. M. Joshi, A. Tomar, S. K. Padaliya, On geometric properties of non-unique fixed points in $b-$metric spaces, In: *Fixed point theory and its applications to real world problem*, New York: Nova Science Publishers, 2021, 33–50.

15. M. Joshi, A. Tomar, S. K. Padaliya, Fixed point to fixed disc and application in partial metric spaces, In: *Fixed point theory and its applications to real world problem*, New York: Nova Science Publishers, 2021, 391–406.

16. M. Joshi, A. Tomar, S. K. Padaliya, Fixed point to fixed ellipse in metric spaces and discontinuous activation function, *Appl. Math. E-Notes*, **21** (2021), 225–237

17. M. Joshi, A. Tomar, On unique and non-unique fixed points in metric spaces and application to chemical sciences, *J. Funct. Spaces*, **2021** (2021), 5525472. https://doi.org/10.1155/2021/5525472

18. M. Joshi, A. Tomar, Near fixed point, near fixed interval circle and their equivalence classes in a $b-$interval metric space, *J. Nonlinear Anal. Appl.*, **13** (2022), 1999–2014. http://dx.doi.org/10.22075/ijnaa.2021.21721.2291

19. I. Stakgold, M. Hoist, *Green's functions and boundary value problems*, John Wiley & Sons, 2011.

20. R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60** (1968), 71–76.

21. F. Khojasteh, S. Shukla, S. Radenović, A new approach to the study of fixed point theory for simulation functions, *Filomat*, **29** (2015), 1189–1194. http://dx.doi.org/10.2298/fil1506189k

22. N. Mlaki, U. Çelik, N. Taş, N. Y. Özgür, A. Mukheimer, Wardowski type contractions and the fixed-circle problem on $\mathcal{S}-$metric spaces, *J. Math.*, **2018** (2018), 9127486. https://doi.org/10.1155/2018/9127486

23. N. Mlaiki, N. Y. Özgür, N. Taş, New fixed-point theorems on an $\mathcal{S}-$metric space via simulation functions, *Mathematics*, **7** (2019), 583. https://doi.org/10.3390/math7070583

24. M. Nazam, H. Işik, K. Javed, M. Naeem, M. Arshad, The existence of fixed points for a different type of contractions on partial $b-$metric spaces, *J. Math.*, **2021** (2021), 5158552. https://doi.org/10.1155/2021/5158552

25. X. Nie, J. Liang, J. Cao, Multistability analysis of competitive neural networks with Gaussian-wavelet-type activation functions and unbounded time-varying delays, *Appl. Math. Comput.*, **356** (2019), 449–468. https://doi.org/10.1016/j.amc.2019.03.026

26. N. Y. Özgür, N. Taş, Fixed-circle problem on $\mathcal{S}-$metric spaces with a geometric viewpoint, *Ser. Math. Inform.*, **34** (2019), 459–472.

27. N. Y. Özgür, N. Taş, U. Çelik, New fixed-circle results on $\mathcal{S}-$metric spaces, *Bull. Math. Anal. Appl.*, **9** (2017), 10–23.

28. N. Y. Özgür, N. Taş, Some fixed-circle theorems and discontinuity at fixed circle, *AIP Conf. Proc.*, **1926** (2018), 020048.

29. N. Y. Özgür, N. Taş, Generalization of metric spaces: from the fixed-point theory to the fixed-circle theory, In: T. Rassias, *Applications of nonlinear analysis*, Springer, **134** (2018), 847–895. https://doi.org/10.1007/978-3-319-89815-5_28

30. N. Y. Özgür, N. Taş, Some fixed-circle theorems on metric spaces, *Bull. Malays. Math. Sci. Soc.*, **42** (2019), 1433–1449.

31. T. Phaneendra, K. K. Swamy, Fixed points of Chatterjee contractions on an $\mathcal{S}-$metric space, *Int. J. Pure Appl. Math.*, **115** (2017), 361–367. https://doi.org/10.12732/ijpam.v115i2.13

32. S. Petwal, A. Tomar, M. Joshi, On unique and non-unique fixed point in parametric N_b-metric Spaces with application, *Acta Univ. Sapientiae Math.*, unpublished work.

33. T. Phaneendra, Banach and Kannan contractions on $\mathcal{S}-$metric space, *Ital. J. Pure Appl. Math.,* **39** (2018), 243–247.

34. S. Reich, Some remarks concerning contraction mappings, *Canad. Math. Bull.*, **14** (1971), 121–124. https://doi.org/10.4153/CMB-1971-024-9

35. B. E. Rhoades, Contractive definitions and continuity, In: R. F. Brown, *Fixed point theory and its applications*, Contemporary Mathematics, **72** (1988), 233–245.

36. M. Sarwar, Z. Islam, H. Ahmad, H. Işık, S. Noeiaghdam, Near-common fixed point result in cone interval $b-$metric spaces over Banach algebras, *Axioms*, **10** (2021), 251. https://doi.org/10.3390/axioms10040251

37. S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorems in $\mathcal{S}-$metric spaces, *Mat. Vesnik*, **64** (2012), 258–266.

38. N. Taş, Suzuki-Berinde type fixed-point and fixed-circle results on $\mathcal{S}-$metric spaces, *J. Linear. Topol. Algebra*, **7** (2018), 233–244.

39. A. Tomar, M. Joshi, Near fixed point, near fixed interval circle and near fixed interval disc in metric interval space, In: *Fixed point theory and its applications to real world problem*, New York: Nova Science Publishers, 2021, 131–150.

40. A. Tomar, M. Joshi, S. K. Padaliya, Fixed point to fixed circle and activation function in partial metric space, *J. Appl. Anal.*, **28** (2022), 57–66. https://doi.org/10.1515/jaa-2021-2057

41. L. Wang, T. Chen, Multistability of neural networks with Mexican-hat-type activation functions, *IEEE Trans. Neural Netw. Learn. Syst.*, **23** (2012), 1816–1826. https://doi.org/10.1109/TNNLS.2012.2210732

42. M. Zhou, X. Liu, N. Saleem, A. Fulga, N. Özgür, A new study on the fixed point sets of Proinov-type contractions via rational forms, *Symmetry*, **14** (2022), 93. https://doi.org/10.3390/sym14010093

43. M. Zhou, X. Liu, A. H. Ansari, Y. J. Cho, S. Radenović, Generalized Ulam-Hyers stability for generalized types of $(\psi - \gamma)-$Meir-Keeler mappings via fixed point theory in $\mathcal{S}-$metric spaces, *J. Comput. Anal. Appl.*, **27** (2019), 593–628.

44. M. Zhou, X. Liu, S. Radenović, $\mathcal{S} - \gamma - \Phi - \varphi-$contractive type mappings in $\mathcal{S}-$metric spaces, *J. Nonlinear Anal. Appl.*, **10** (2017), 1613–1639. http://dx.doi.org/10.22436/jnsa.010.04.27

45. M. Zhou, X. Liu, On coupled common fixed point theorem for nonlinear contractions with the mixed weakly monotone property in partially ordered $\mathcal{S}-$metric space, *J. Funct. Spaces*, **2016** (2016), 7529523. https://doi.org/10.1155/2016/7529523

46. M. Zhou, X. Liu, D. Diana, B. Damjanovic, Coupled coincidence point results for Geraghty-type contraction using monotone property in partially ordered $\mathcal{S}-$metric space, *J. Nonlinear Anal. Appl.*, **9** (2016), 5950–5969. http://dx.doi.org/10.22436/jnsa.009.12.04

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