Research article

Note on a new class of operators between some spaces of holomorphic functions

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Abstract: The boundedness and compactness of a new class of linear operators from the weighted Bergman space to the weighted-type spaces on the unit ball are characterized.

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1. Introduction

By $\mathbb{B}$ we denote the open unit ball in $\mathbb{C}^n$, $S$ is the unit sphere in $\mathbb{C}^n$, $B(z, r)$ is the open ball centered at $z$ and with radius $r$, $d\sigma$ is the normalized rotation invariant measure on $S$, $dV(z)$ is the Lebesgue measure, and $dV_\alpha(z) := c_{\alpha,n}(1 - |z|^2)^\alpha dV(z)$, $\alpha > -1$, where $c_{\alpha,n}$ is the normalization constant such that $V_\alpha(\mathbb{B}) = 1$. The linear space of holomorphic functions on $\mathbb{B}$ we denote by $H(\mathbb{B})$, whereas $S(\mathbb{B})$ denotes the class of holomorphic self-maps of $\mathbb{B}$. The standard inner product between the vectors $z, w \in \mathbb{C}^n$ is denoted by $\langle z, w \rangle$, whereas $|z| = \sqrt{\langle z, z \rangle}$ is the Euclidean norm in $\mathbb{C}^n$. Many classical results on functions in $H(\mathbb{B})$ can be found in [1]. If $f \in C(\mathbb{B})$ is a positive function, then we call it a weight function, and the class of functions is denoted by $W(\mathbb{B})$. If $p, q \in \mathbb{N}_0$, $p \leq q$, then the notation $j = \overline{p,q}$ is an abbreviation for the notation $j = p, p + 1, \ldots, q$. If $X$ is a Banach space, then by $B_X$ we denote the unit ball in $X$.

Each $\varphi \in S(\mathbb{B})$ induces the composition operator $C_\varphi f(z) = f(\varphi(z))$, whereas each $u \in H(\mathbb{B})$ induces the multiplication operator $M_u f(z) = u(z)f(z)$. The radial derivative of $f \in H(\mathbb{B})$ is defined by

$$\Re f(z) = \sum_{j=1}^{n} z_j D_j f(z),$$
Lemma 2.1. Assume $p \geq 1$, $\alpha > -1$, $\mu \in W(\mathbb{B})$, $u_j \in H(\mathbb{B})$, $j = \bar{0}, \bar{m}$, $m \in \mathbb{N}$, $\varphi \in S(\mathbb{B})$, and that the operator $\mathcal{E}_{\alpha, \varphi}^{m} : A^p_{\alpha} \to H^\infty_{\mu}$ (or $H^\infty_{\mu, 0}$), where $p \geq 1$ and $\alpha > -1$.

By $C$ we denote some positive constants independent of essential variables and functions which may differ from line to line, whereas $a \leq b$ (resp. $a \geq b$) means that there is $C > 0$ such that $a \leq Cb$ (resp. $a \geq Cb$). If $a \leq b$ and $b \leq a$, then we use the notation $a \asymp b$.

2. Auxiliary results

The first result is a standard Schwartz-type lemma [38].

Lemma 2.1. Assume $p \geq 1$, $\alpha > -1$, $\mu \in W(\mathbb{B})$, $u_j \in H(\mathbb{B})$, $j = \bar{0}, \bar{m}$, $m \in \mathbb{N}$, $\varphi \in S(\mathbb{B})$, and that the operator $\mathcal{E}_{\alpha, \varphi}^{m} : A^p_{\alpha} \to H^\infty_{\mu}$ is bounded. Then, the operator is compact if and only if for every bounded sequence $(f_k)_{k \in \mathbb{N}} \subset A^p_{\alpha}$ uniformly converging to zero on compacts of $\mathbb{B}$, we have

$$\lim_{k \to +\infty} \|\mathcal{E}_{\alpha, \varphi}^{m} f_k\|_{H^\infty_{\mu}} = 0.$$
The following lemma was essentially proved in [39], so we omit the proof.

**Lemma 2.2.** A closed set $K$ in $H^{\infty}_{\mu,0}$ is compact if and only if it is bounded and

$$\lim_{|z|\to 1} \sup_{f \in K} |f(z)| = 0.$$  

The following lemma is well known (see [29]; for a less precise version see also [1]).

**Lemma 2.3.** Assume $p \in (0, \infty)$, $\alpha > -1$, and $f \in A^p_\alpha(\mathbb{B})$; Then,

$$|f(z)| \leq \frac{\|f\|_{A^p_\alpha}}{(1 - |z|^2)^{\frac{n+1}{p} + \frac{\alpha+1}{p}}}, \quad z \in \mathbb{B}.$$  

(2.1)

**Lemma 2.4.** Assume $p \in (0, \infty)$, $\alpha > -1$, and $m \in \mathbb{N}$. Then,

$$|\mathcal{R}^m f(z)| \leq \frac{|z|^{\frac{n+1}{p} + m}}{(1 - |z|^2)^{\frac{n+1}{p} + \frac{\alpha+1}{p} + m}} \|f\|_{A^p_\alpha},$$  

(2.2)

for every $f \in A^p_\alpha$ and $z \in \mathbb{B}$.

**Proof.** Note that it is enough to prove that for all $f \in A^p_\alpha$ and $z \in \mathbb{B}$,

$$|\mathcal{R}^m f(z)| \leq \frac{|z|^{\frac{n+1}{p} + m}}{(1 - |z|^2)^{\frac{n+1}{p} + \frac{\alpha+1}{p} + m}} \|f\|_{A^p_\alpha},$$  

(2.3)

Let $r \in (0, 1)$ be fixed. Then, the Cauchy-Schwartz and Cauchy inequalities imply

$$|\mathcal{R} f(z)| \leq |z| \sup_{w \in B(z,r(1-|z|))} \frac{|f(w)|}{1 - |z|}, \quad z \in \mathbb{B}, \; f \in H(\mathbb{B}).$$  

(2.4)

Inequality (2.1) implies that

$$\sup_{w \in B(z,r(1-|z|))} |f(w)| \leq \frac{\|f\|_{A^p_\alpha}}{(1 - |z|^2)^{\frac{n+1}{p} + \frac{\alpha+1}{p}}}.$$  

(2.5)

Since $r$ is fixed, by (2.4) and (2.5) we get

$$|\mathcal{R} f(z)| \leq \frac{|z|^{\frac{n+1}{p} + \frac{\alpha+1}{p} + m}}{(1 - |z|^2)^{\frac{n+1}{p} + \frac{\alpha+1}{p} + m}} \|f\|_{A^p_\alpha},$$  

(2.6)

that is, (2.3) holds when $m = 1$.

Assume that for a $k \in \mathbb{N} \setminus \{1\}$ and all $f \in A^p_\alpha$ and $z \in \mathbb{B}$ holds,

$$|\mathcal{R}^{k-1} f(z)| \leq \frac{|z|^{\frac{n+1}{p} + k - 1}}{(1 - |z|^2)^{\frac{n+1}{p} + k - 1}} \|f\|_{A^p_\alpha}.$$  

(2.7)

Then, since $w \in B(z, r(1-|z|))$ we have $(1 - r)^{\frac{n+1}{p} + k - 1}(1 - |z|)^{\frac{n+1}{p} + k - 1} \leq (1 - |w|)^{\frac{n+1}{p} + k - 1}$, from (2.7) we have

$$\sup_{w \in B(z,r(1-|z|))} |\mathcal{R}^{k-1} f(w)| \leq \frac{1}{(1 - |z|^2)^{\frac{n+1}{p} + k - 1}} \|f\|_{A^p_\alpha}.$$  

(2.8)
If in (2.4) we replace \( f \) by \( \Re k^{-1} f \), we get

\[
|\Re f(z)| \leq |z| \sup_{w \in B(z, r(z))} |\Re k^{-1} f(w)| \frac{1}{1 - |z|}.
\]  

(2.9)

Combining (2.8) and (2.9), we have

\[
|\Re f(z)| \leq \sup_{w \in B(z, r(z))} |\Re k^{-1} f(w)| \frac{1}{1 - |z|}. 
\]

Thus, (2.3) holds for each \( m \in \mathbb{N} \), implying (2.2). □

The following lemma is well known.

**Lemma 2.5.** Let \( p \geq 1 \) and \( \alpha > -1 \). Then, for any \( t \geq 0 \) and \( w \in \mathbb{B} \),

\[
f_{w,t}(z) := \frac{(1 - |w|^2)^{t+1}}{(1 - \langle z, w \rangle)^{\frac{\alpha + t + 1}{p}}},
\]

belongs to \( A_p^\alpha \) and \( \sup_{w \in \mathbb{B}} \|f_{w,t}\|_{A_p^\alpha} \leq 1 \).

The following lemma is from [34] and [35].

**Lemma 2.6.** Let \( s \geq 0 \), \( w \in \mathbb{B} \) and \( g_{w,s}(z) = (1 - \langle z, w \rangle)^{-s} \). Then,

\[
\Re g_{w,s}(z) = s \frac{P_k(\langle z, w \rangle)}{(1 - \langle z, w \rangle)^{s+k}},
\]

where \( P_k(w) = s^{k-1}w^k + p_{k-1}^{(k)}(s)w^{k-1} + \cdots + p_1^{(k)}(s)w + p_0^{(k)}(s) = \frac{1}{k!}, \) are nonnegative polynomials for \( s > 0 \);

\[
\Re g_{w,s}(z) = \sum_{t=1}^{k} a_t^{(k)} \left( \sum_{j=0}^{s} \binom{s}{j} \langle z, w \rangle^j \right) \frac{(1 - \langle z, w \rangle)^{t+s+k}}{(1 - \langle z, w \rangle)^{s+t}},
\]

where \( (a_t^{(k)}) \) is defined as

\[
a_1^{(k)} = 1, \quad a_k^{(k)} = k \in \mathbb{N};
\]

and for \( 2 \leq t \leq k - 1, k \geq 3 \),

\[
a_t^{(k)} = ta_{t-1}^{(k-1)} + a_{t-1}^{(k-1)}.
\]

**Lemma 2.7.** Assume \( p \geq 1 \), \( \alpha > -1 \), \( m \in \mathbb{N} \), \( w \in \mathbb{B} \), \( f_{w,t} \) is defined in (2.10), and \( (a_t^{(k)})_{t=1}^{m} \), \( k \in \mathbb{N}, \) are defined in (2.13) and (2.14). Then,

(a) for each \( l \in \{1, \ldots, m\} \), there is

\[
h_w^{(l)}(z) = \sum_{k=0}^{m} c_k^{(l)} f_{w,k}(z),
\]

where

\[
c_k^{(l)} = \sum_{j=0}^{m} c_k^{(l)} f_{w,k}(z),
\]

where \( (c_k^{(l)}) \) is defined in (2.13) and (2.14). Then,
(b) there is

\[ h^{(0)}_w(z) = \sum_{k=0}^{m} c^{(0)}_k f_{w,k}(z), \]

where \( c^{(0)}_k \), \( k = 0, \ldots, m \), are numbers, such that

\[ \mathbb{R}^j h^{(0)}_w(w) = 0, \quad 0 \leq j < l, \]

\[ \mathbb{R}^j h^{(0)}_w(w) = a^{(j)}_j \frac{|w|^{2j}}{(1 - |w|^2)^{\frac{n+1}{p} + j}}, \quad l \leq j \leq m, \]

hold. Moreover, we have \( \sup_{w \in \mathbb{B}} |h^{(0)}_w|_{\mathcal{A}^p} < +\infty \).

Proof. (a) Let \( d_k = \frac{n+1}{p} + k + 1, \quad k \in \mathbb{N}_0 \). Replace the constants \( c^{(l)}_k \) in (2.15) by \( c_k \). Then, from (2.12) we get

\[ h^{(0)}_w(w) = \frac{c_0 + c_1 + \cdots + c_m}{(1 - |w|^2)^{\frac{n+1}{p}}}, \]

\[ \mathbb{R}^j h^{(0)}_w(w) = \frac{(d_0c_0 + d_1c_1 + \cdots + d_mc_m)|w|^{2j}}{(1 - |w|^2)^{\frac{n+1}{p} + j}}, \]

\[ \vdots \]

\[ \mathbb{R}^m h^{(0)}_w(w) = d^{(m)}_1 \frac{(d_0c_0 + d_1c_1 + \cdots + d_mc_m)|w|^{2m}}{(1 - |w|^2)^{\frac{n+1}{p} + m}} + \cdots \]

Lemma 2.5 in [11] shows that the determinant of the system,

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & \prod_{k=0}^{l} d_k & \prod_{k=0}^{l} d_{k+1} & \cdots & \prod_{k=0}^{l} d_{m+k} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \prod_{k=0}^{m-1} d_k & \prod_{k=0}^{m-1} d_{k+1} & \cdots & \prod_{k=0}^{m-1} d_{m+k} \\
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_m
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix},
\]

is

\[
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_m
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]
is different from zero (on the right-hand side of (2.20), the unit is in the \((l + 1)\)th position). Thus, there is a unique solution \(c_k = c_k^0\), \(k = \overline{0, m}\), to (2.20). For these \(c_k\)-s, function (2.15) satisfies (2.16) and (2.17). By Lemma 2.5 we have \(\sup_{w \in \mathbb{B}} \|h_w^{(0)}\|_{\mathcal{A}_w^\mu} < +\infty.

(b) The proof is similar, so it is omitted. \(\square\)

3. Main results

Our main results are formulated and proved in this section.

**Theorem 3.1.** Let \(p \geq 1\), \(\alpha > -1\), \(k \in \mathbb{N}\), \(u \in H(\mathbb{B})\), \(\varphi \in S(\mathbb{B})\) and \(\mu \in W(\mathbb{B})\). Then, the operator \(\mathcal{R}_{u, \varphi}^k : A^\alpha_{\mu} \to H^\alpha_{\mu}\) is bounded if and only if

\[
J_k := \sup_{z \in \mathbb{B}} \frac{\mu(z)|u(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha + 1}{p} + k}} < +\infty, \tag{3.1}
\]

and if it is bounded, then we have

\[
\|\mathcal{R}_{u, \varphi}^k\|_{\mathcal{A}_u^\alpha \to H_{\mu}^\alpha} \approx J_k. \tag{3.2}
\]

**Proof.** Assume \(\mathcal{R}_{u, \varphi}^k : A^\alpha_{\mu} \to H^\alpha_{\mu}\) is bounded. Let \(g_u(z) = f_{\varphi(u), 1}(z)\). By Lemma 2.6 the coefficients of the polynomial \(P_k\) therein are nonnegative, so we have

\[
\|g_u\|_{\mathcal{A}_u^\alpha} < +\infty, \tag{3.3}
\]

The boundedness, (3.3) and the fact \(\sup_{w \in \mathbb{B}} \|g_w\|_{\mathcal{A}_w^\alpha} < +\infty\), imply

\[
\sup_{|\varphi(z)| \leq 1/2} \frac{\mu(z)|u(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha + 1}{p} + k}} \leq \|\mathcal{R}_{u, \varphi}^k\|_{\mathcal{A}_u^\alpha \to H_{\mu}^\alpha}. \tag{3.4}
\]

Further, the fact \(f_j(z) = z_j \in A^\alpha_{\mu, j}, j = \overline{1, n}\), implies \(\mathcal{R}_{u, \varphi}^k f_j \in H^\alpha_{\mu, j}, j = \overline{1, n}\), from which, together with \(\mathcal{R} f_j = f_j, j = \overline{1, n}\), we get

\[
\sup_{z \in \mathbb{B}} \mu(z)|u(z)||\varphi_j(z)| = \|\mathcal{R}_{u, \varphi}^k f_j\|_{H^\alpha_{\mu, j}} \leq \|\mathcal{R}_{u, \varphi}^k\|_{\mathcal{A}_u^\alpha \to H^\alpha_{\mu}} \|z_j\|_{A^\alpha_{\mu, j}}, j = \overline{1, n},
\]

from which we get

\[
\sup_{z \in \mathbb{B}} \mu(z)|u(z)||\varphi_j(z)| \leq \|\mathcal{R}_{u, \varphi}^k\|_{\mathcal{A}_u^\alpha \to H^\alpha_{\mu}}. \tag{3.5}
\]

Inequality (3.5) together with

\[
\sup_{|\varphi(z)| \leq 1/2} \frac{\mu(z)|u(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha + 1}{p} + k}} \leq \sup_{|\varphi(z)| \leq 1/2} \mu(z)|u(z)||\varphi(z)|,
\]

implies

\[
\sup_{|\varphi(z)| \leq 1/2} \frac{\mu(z)|u(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha + 1}{p} + k}} \leq \|\mathcal{R}_{u, \varphi}^k\|_{\mathcal{A}_u^\alpha \to H^\alpha_{\mu}}. \tag{3.6}
\]
Proof. Assume (3.1) holds. Then, Lemma 2.4 implies that for any \( f \in A^p_\alpha(\mathbb{B}) \) and \( z \in \mathbb{B} \),

\[
\mu(z) |\mathfrak{R}^k_{u,\varphi} f(z)| \leq \frac{\mu(z) |u(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha + 1}{p} + k}} |f|_{A^p_\alpha}. \tag{3.7}
\]

Taking the supremum in (3.7) over \( B_{\alpha}^j \), and employing (3.1), the boundedness of \( \mathfrak{R}^k_{u,\varphi} : A^p_\alpha \to H^{\infty}_\mu \) and the relation \( \|\mathfrak{R}^k_{u,\varphi}\|_{A^p_\alpha \to H^{\infty}_\mu} \leq J_k \) follow, implying (3.2). \( \square \)

The following result is known. For a more general result, see [31].

**Theorem 3.2.** Let \( p \geq 1, \alpha > -1, \mu \in W(\mathbb{B}), u \in H(\mathbb{B}) \) and \( \varphi \in S(\mathbb{B}) \). Then, the operator \( \mathfrak{R}^0_{u,\varphi} : A^p_\alpha \to H^{\infty}_\mu \) is bounded if and only if

\[
J_0 = \sup_{\varphi \in \mathbb{B}} \frac{\mu(z) |u(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha + 1}{p}}} < +\infty, \tag{3.8}
\]

and if it is bounded, then \( \|\mathfrak{R}^0_{u,\varphi}\|_{A^p_\alpha \to H^{\infty}_\mu} = J_0 \).

**Theorem 3.3.** Let \( p \geq 1, \alpha > -1, m \in \mathbb{N}, u_j \in H(\mathbb{B}), j = 0, m, \varphi \in S(\mathbb{B}) \) and \( \mu \in W(\mathbb{B}) \). Then, the operators \( \mathfrak{R}^j_{u,\varphi} : A^p_\alpha \to H^{\infty}_\mu \), \( j = 0, m \), are bounded and if only if \( \mathfrak{Z}^m_{u,\varphi} : A^p_\alpha \to H^{\infty}_\mu \) is bounded and

\[
\sup_{\varphi \in \mathbb{B}} \mu(z) |u_j(z)||\varphi(z)| < +\infty, \quad j = 1, m. \tag{3.9}
\]

**Proof.** Assume \( \mathfrak{Z}^m_{u,\varphi} : A^p_\alpha \to H^{\infty}_\mu \) is bounded and (3.9) holds. We need to prove

\[
I_j = \sup_{\varphi \in \mathbb{B}} \frac{\mu(z) |u_j(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha + 1}{p} + j}} < +\infty, \quad j = 1, m, \tag{3.10}
\]

and

\[
I_0 = \sup_{\varphi \in \mathbb{B}} \frac{\mu(z) |u_0(z)|}{(1 - |\varphi(z)|^2)^{\frac{\alpha + 1}{p}}} < +\infty. \tag{3.11}
\]

If \( \varphi(w) \neq 0 \), then there is \( h^{(m)}_{\varphi(w)} \in A^p_\alpha \) such that

\[
\mathfrak{R}^j_{h^{(m)}_{\varphi(w)}}(\varphi(w)) = 0, \quad 0 \leq j < m, \quad 0 \leq j < m, \quad \mathfrak{R}^m_{h^{(m)}_{\varphi(w)}}(\varphi(w)) = \frac{|\varphi(w)|^{2m}}{(1 - |\varphi(w)|^2)^{\frac{\alpha + 1}{p} + m}},
\]

and \( \sup_{w \in \mathbb{B}} \|h^{(m)}_{\varphi(w)}\|_{A^p_\alpha} < +\infty \) (see Lemma 2.7 (a)). This, together with the boundedness of \( \mathfrak{Z}^m_{u,\varphi} : A^p_\alpha \to H^{\infty}_\mu \) implies

\[
\|\mathfrak{Z}^m_{u,\varphi}\|_{A^p_\alpha \to H^{\infty}_\mu} \geq \|\mathfrak{Z}^m_{u,\varphi} h^{(m)}_{\varphi(w)}\|_{H^{\infty}_\mu} \geq \mu(w) \sum_{j=0}^{m} u_j(w) \mathfrak{R}^j_{h^{(m)}_{\varphi(w)}}(\varphi(w)) \geq \frac{\mu(w) |u_m(w)||\varphi(w)|^{2m}}{(1 - |\varphi(w)|^2)^{\frac{\alpha + 1}{p} + m}}, \tag{3.12}
\]

\[
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\]
from which it follows that
\[
\sup_{|\varphi(z)| \geq 1/2} \frac{\mu(z)|u_m(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{m+n}{p}+m}} \leq \|\sum_{m}^{n} \varphi(z)\|_{A_\mu^p \rightarrow H_\mu^s},
\]
and along with
\[
\sup_{|\varphi(z)| \leq 1/2} \frac{\mu(z)|u_m(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{m+n}{p}+m}} \leq \sup_{\varphi \in \mathcal{B}} \mu(z)|u_m(z)||\varphi(z)| < +\infty,
\]
implies \( I_m < +\infty \).

Assume (3.10) holds for \( j = s + 1, m \), for an \( s \in \{1, 2, \ldots, m-1\} \). Let \( h^{(s)}_{\varphi(w)}(z) \) be as in Lemma 2.7 (a). Then, \( \sup_{w \in \mathbb{B}} \|h^{(s)}_{\varphi(w)}\|_{A_\mu^p} < +\infty \), and
\[
\mu(w) \left| \sum_{j=s}^{m} d_j(z) u_j(w) \frac{|\varphi(z)|^{2s}}{(1 - |\varphi(z)|^2)^{\frac{m+n}{p}+s}} \right| \leq \sup_{\varphi \in \mathcal{B}} \mu(z) \left| \sum_{j=0}^{m} u_j(z) R^{j} h^{(s)}_{\varphi(w)}(z) \right| \\
\leq \|\sum_{m}^{n} \varphi(z)\|_{A_\mu^p \rightarrow H_\mu^s},
\]
from which we easily get
\[
\mu(w)|u_s(w)||\varphi(w)|^{2s} \frac{(1 - |\varphi(z)|^2)^{\frac{m+n}{p}+s}}{(1 - |\varphi(w)|^2)^{\frac{m+n}{p}+s}} \leq \|\sum_{m}^{n} \varphi(z)\|_{A_\mu^p \rightarrow H_\mu^s} + \sum_{j=s+1}^{m} \mu(w)|u_j(w)||\varphi(w)|^{2s} \frac{(1 - |\varphi(z)|^2)^{\frac{m+n}{p}+s}}{(1 - |\varphi(w)|^2)^{\frac{m+n}{p}+s}}.
\]
(3.13)

From (3.13) and the fact \( s \geq 1 \), we have
\[
\sup_{|\varphi(z)| \geq 1/2} \frac{\mu(z)|u_s(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{m+n}{p}+s}} \leq \|\sum_{m}^{n} \varphi(z)\|_{A_\mu^p \rightarrow H_\mu^s} + \sum_{j=s+1}^{m} \sup_{|\varphi(z)| \geq 1/2} \frac{\mu(z)|u_j(z)||\varphi(z)|^{2s}}{(1 - |\varphi(z)|^2)^{\frac{m+n}{p}+s}} \\
\leq \|\sum_{m}^{n} \varphi(z)\|_{A_\mu^p \rightarrow H_\mu^s} + \sum_{j=s+1}^{m} I_j.
\]
This, together with the fact
\[
\sup_{|\varphi(z)| \leq 1/2} \frac{\mu(z)|u_s(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{m+n}{p}+s}} \leq \sup_{\varphi \in \mathcal{B}} \mu(z)|u_s(z)||\varphi(z)| < +\infty,
\]
implies (3.10) for \( j = s \). Thus, (3.10) holds for any \( j \in \{1, \ldots, m\} \).

For any \( w \in \mathbb{B} \), there is \( h^{(0)}_{\varphi(w)} \in A_\mu^p \) such that
\[
h^{(0)}_{\varphi(w)}(\varphi(w)) = \frac{1}{\sum_{m}^{n} \varphi(z)\|_{A_\mu^p \rightarrow H_\mu^s}} \frac{R^{j} h^{(0)}_{\varphi(w)}(\varphi(w))}{\varphi(w)} = 0, \quad j = 1, m,
\]
and \( \sup_{w \in \mathbb{B}} \|h^{(0)}_{\varphi(w)}\|_{A_\mu^p} < +\infty \) (see Lemma 2.7 (b)).

This together with the boundedness of \( \sum_{m}^{n} : A_\mu^p \rightarrow H_\mu^\infty \) implies
\[
\frac{\mu(w)|u_0(w)|}{(1 - |\varphi(w)|^2)^{\frac{m+n}{p}}} \leq \|\sum_{m}^{n} h^{(0)}_{\varphi(w)}\|_{H_\mu^\infty} \leq \|\sum_{m}^{n} \varphi(z)\|_{A_\mu^p \rightarrow H_\mu^\infty},
\]
(3.14)
from which (3.11) follows, as claimed.

Assume \( \sum_{j=0}^{m} \mu(z)|u_j(z)||\varphi(z)|^{2s} \frac{(1 - |\varphi(z)|^2)^{\frac{m+n}{p}+s}}{(1 - |\varphi(w)|^2)^{\frac{m+n}{p}+s}} \leq \|\sum_{m}^{n} \varphi(z)\|_{A_\mu^p \rightarrow H_\mu^\infty}, \)
(3.11)

Theorem 3.4. Let $p \geq 1$, $\alpha > -1$, $k \in \mathbb{N}$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operator $\mathcal{R}^k_{u,\varphi} : A^p_{\alpha} \to H^\infty_{\mu}$ is compact if and only if it is bounded and

$$
\lim_{|\varphi(z)| \to 1} \frac{|\mu(z)||u(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1}{p} + k}} = 0.
$$

(3.15)

Proof. If $\mathcal{R}^k_{u,\varphi} : A^p_{\alpha} \to H^\infty_{\mu}$ is compact, it is also bounded. If $||\varphi||_\infty < 1$, (3.15) automatically/vacuously holds. If $||\varphi||_\infty = 1$ and $(z_j)_{j \in \mathbb{N}} \subset \mathbb{B}$ is such that $|\varphi(z_j)| \to 1$ as $j \to +\infty$, and $h_j(z) = f_{\varphi(z_j)}(z)$, then $\sup_{j \in \mathbb{N}} \|h_j\|_{A^p_{\alpha}} < +\infty$. From $\lim_{j \to +\infty} (1 - |\varphi(z_j)|^2)^{j+1} = 0$, we have $h_j \to 0$ as $j \to +\infty$, uniformly on compacta of $\mathbb{B}$. Using Lemma 2.1, it follows that $\lim_{j \to +\infty} \|\mathcal{R}^k_{u,\varphi} h_j\|_{H^\infty_{\mu}} = 0$, from which, along with the consequence of (3.3),

$$
\frac{\mu(z_j)|u(z_j)||\varphi(z_j)|}{(1 - |\varphi(z_j)|^2)^{\frac{n+1}{p} + k}} \leq C\|\mathcal{R}^k_{u,\varphi} h_j\|_{H^\infty_{\mu}},
$$

which holds for sufficiently large $j$, and we easily get (3.15).

If $\mathcal{R}^k_{u,\varphi} : A^p_{\alpha} \to H^\infty_{\mu}$ is bounded and (3.15) holds, then Theorem 3.1 implies $\mu(z)|u(z)||\varphi(z)| \leq J_k < +\infty$, $z \in \mathbb{B}$, and (3.15) implies that for any $\varepsilon > 0$ there is $\delta \in (0, 1)$ such that when $\delta < |\varphi(z)| < 1$,

$$
\frac{\mu(z)|u(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1}{p} + k}} < \varepsilon.
$$

(3.16)

Suppose $(f_j)_{j \in \mathbb{N}}$ is a bounded sequence in $A^p_{\alpha}$ converging to zero uniformly on compacta of $\mathbb{B}$. Let $s_\delta = \{z \in \mathbb{B} : |\varphi(z)| \leq \delta\}$. Then, Lemma 2.4, together with the fact $\sup_{z \in \mathbb{B}} \mu(z)|u(z)||\varphi(z)| < +\infty$, and (3.16), implies

$$
\|\mathcal{R}^k_{u,\varphi} f_j\|_{H^\infty_{\mu}} \leq \sup_{z \in s_\delta} \mu(z)|u(z)||\varphi(z)| + \sup_{z \in \mathbb{B} \setminus s_\delta} \mu(z)|u(z)||\varphi(z)|
$$

$$
\leq \sup_{z \in s_\delta} \mu(z)|u(z)||\varphi(z)| + \sup_{z \in \mathbb{B} \setminus s_\delta} \frac{\mu(z)|u(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1}{p} + k}}
$$

$$
\leq \sup_{|w| \leq \delta} \left|\nabla \mathcal{R}^{k-1} f_j(w)\right| + \varepsilon.
$$

(3.17)

The assumption $f_j \to 0$ on compacta along with Cauchy’s estimate implies $\lim_{j \to +\infty} |\nabla \mathcal{R}^{k-1} f_j| = 0$ uniformly on compacta of $\mathbb{B}$. The set $\{w : |w| \leq \delta\}$ is compact, so by letting $j \to +\infty$ in (3.17), it follows that $\lim \sup_{j \to +\infty} \|\mathcal{R}^k_{u,\varphi} f_j\|_{H^\infty_{\mu}} \leq \varepsilon$, from which it follows that $\lim_{j \to +\infty} \|\mathcal{R}^k_{u,\varphi} f_j\|_{H^\infty_{\mu}} = 0$. From this and Lemma 2.1, the compactness of $\mathcal{R}^k_{u,\varphi} : A^p_{\alpha} \to H^\infty_{\mu}$ follows.

The following theorem is known. For a more general result, see [31].

Theorem 3.5. Let $p \geq 1$, $\alpha > -1$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operator $\mathcal{R}^0_{u,\varphi} : A^p_{\alpha} \to H^\infty_{\mu}$ is compact if and only if it is bounded and

$$
\lim_{|\varphi(z)| \to 1} \frac{|\mu(z)||u(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1}{p}} + k} = 0.
$$

(3.18)

Theorem 3.6. Let $p \geq 1$, $\alpha > -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = 0, 1, \ldots, m$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operator $\mathcal{R}^m_{u,\varphi} : A^p_{\alpha} \to H^\infty_{\mu}$ is compact and (3.9) holds if and only if the operators $\mathcal{R}^j_{u,\varphi} : A^p_{\alpha} \to H^\infty_{\mu}$ are compact for $j = 0, m$. 
Proof. If \( \mathcal{Z}_{\mu,0}^{m} : A_{\mu}^{m} \rightarrow H_{\mu}^{\infty} \) is compact and (3.9) holds, then the operator is bounded, from which, together with Theorem 3.3, the boundedness of \( \mathcal{R}_{\mu,0}^{m} : A_{\mu}^{m} \rightarrow H_{\mu}^{\infty} \), is obtained. The previous two theorems show that it is enough to prove

\[
\lim_{|\varphi(z)| \to 1} \frac{\mu(z)|u_{j}(z)||\varphi(z)|}{(1 - |\varphi(z)|^{2})^{\frac{n+1}{p} + j}} = 0, \quad j = 1, m,
\]

and

\[
\lim_{|\varphi(z)| \to 1} \frac{\mu(z)|u_{0}(z)|}{(1 - |\varphi(z)|^{2})^{\frac{n+1}{p}}} = 0.
\]

If \( \|\varphi\|_{\infty} < 1 \), then (3.19) and (3.20) hold. Assume \( \|\varphi\|_{\infty} = 1 \). Let \( (z_{k})_{k \in \mathbb{N}} \subset \mathbb{B} \) be such that \( \lim_{k \to +\infty} |\varphi(z_{k})| = 1 \), and \( h_{k}^{(s)}(z) = h_{\varphi(z_{k})}^{(s)}(z) \) for an \( s \in \{1, \ldots, m\} \) (see (2.15)). Then, \( \sup_{k \in \mathbb{N}} ||h_{k}^{(s)}||_{\mathcal{A}} < +\infty \). The fact \( \lim_{k \to +\infty} (1 - |\varphi(z_{k})|^{2})^{+1} = 0 \), implies \( \lim_{k \to +\infty} h_{k}^{(s)} = 0 \) uniformly on any compact of \( \mathbb{B} \). So, Lemma 2.1 implies

\[
\lim_{k \to +\infty} ||\mathcal{Z}_{\mu,\varphi}^{m} h_{k}^{(s)}||_{H_{\mu}^{\infty}} = 0.
\]

Relation (3.12) implies

\[
\frac{\mu(z_{k})|u_{m}(z_{k})||\varphi(z_{k})|}{(1 - |\varphi(z_{k})|^{2})^{\frac{n+1}{p} + m}} \leq ||\mathcal{Z}_{\mu,\varphi}^{m} h_{k}^{(m)}||_{H_{\mu}^{\infty}},
\]

for sufficiently large \( k \). From (3.22) and (3.21) with \( s = m \), relation (3.19) with \( j = m \) follows. If (3.19) holds for \( j = s + 1, m \), for a fixed \( s \in \{1, \ldots, m - 1\} \), (3.13) implies

\[
\frac{\mu(w)|u_{s}(z_{k})||\varphi(z_{k})|}{(1 - |\varphi(z_{k})|^{2})^{\frac{n+1}{p} + s}} \leq ||\mathcal{Z}_{\mu,\varphi}^{m} h_{k}^{(s)}||_{A_{\mu}^{s}} \to H_{\mu}^{\infty} + \sum_{j=s+1}^{m} \frac{\mu(w)|u_{j}(z_{k})||\varphi(z_{k})|}{(1 - |\varphi(z_{k})|^{2})^{\frac{n+1}{p} + j}},
\]

for \( k \) large, from which, along with (3.31) and the hypothesis, the relation (3.19) with \( j = s \) follows. Thus, (3.19) holds for any \( s \in \{1, \ldots, m\} \).

Let \( h_{k}^{(0)}(z) = h_{\varphi(z_{k})}^{(0)}(z) \) (see Lemma 2.7 (b)). Then, \( \sup_{k \in \mathbb{N}} ||h_{k}^{(0)}||_{\mathcal{A}} < +\infty \), and \( \lim_{k \to +\infty} h_{k}^{(0)}(z) = 0 \) uniformly on compacts of \( \mathbb{B} \). From Lemma 2.1 we have that \( \lim_{k \to +\infty} ||\mathcal{Z}_{\mu,\varphi}^{m} h_{k}^{(0)}||_{H_{\mu}^{\infty}} = 0 \), from which, along with the consequence of (3.14),

\[
\frac{\mu(z_{k})|u_{0}(z_{k})|}{(1 - |\varphi(z_{k})|^{2})^{\frac{n+1}{p}}} \leq ||\mathcal{Z}_{\mu,\varphi}^{m} h_{k}^{(0)}||_{H_{\mu}^{\infty}},
\]

(3.20) follows.

Assume \( \mathcal{R}_{\mu,\varphi}^{m} : A_{\mu}^{m} \rightarrow H_{\mu}^{\infty} \), \( j = 0, m \), are compact. Then, \( \mathcal{Z}_{\mu,\varphi}^{m} : A_{\mu}^{m} \rightarrow H_{\mu}^{\infty} \) is also compact, and by Theorem 3.3 is obtained (3.9). \( \square \)

Theorem 3.7. Let \( p \geq 1, \alpha > -1, m \in \mathbb{N}, u_{j} \in H(\mathbb{B}), j = 0, m, \varphi \in S(\mathbb{B}) \) and \( \mu \in W(\mathbb{B}) \). Then, the operator \( \mathcal{Z}_{\mu,\varphi}^{m} : A_{\mu}^{m} \rightarrow H_{\mu,0}^{\infty} \) is bounded if and only if \( \mathcal{Z}_{\mu,\varphi}^{m} : A_{\mu}^{m} \rightarrow H_{\mu}^{\infty} \) is bounded and

\[
\lim_{z \to 0} \mu(z) \left| \sum_{j=0}^{m} u_{j}(z)^{|l|} |\varphi(z)|^{|l|} = 0, \quad l \in \mathbb{N}. \quad (3.23)\]

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From (3.26) together with (3.16) we obtain (3.24) in a standard way.

Theorem 3.8. Let $p \geq 1$, $\alpha > -1$, $k \in \mathbb{N}$, $u \in H(\mathbb{B})$, $\varphi \in S(\mathbb{B})$ and $\mu \in W(\mathbb{B})$. Then, the operator $\mathcal{R}^k_{u, \varphi} : A^p_{\alpha} \rightarrow H^\infty_{\mu}$ is compact if and only if

$$
\lim_{|z| \rightarrow 1} \frac{\mu(z)|u(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n-1}{p} + k}} = 0.
$$

(3.24)

Proof: Relation (3.24) implies (3.1). Taking the supremum in (3.7) over $\mathbb{B}$ and $B_{A^p_{\alpha}}$, and employing (3.1), it follows that

$$
\sup_{f \in B_{A^p_{\alpha}}} \sup_{z \in \mathbb{B}} \mu(z)|\mathcal{R}^k_{u, \varphi} f(z)| \leq \sup_{z \in \mathbb{B}} \frac{\mu(z)|u(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n-1}{p} + k}} < +\infty.
$$

(3.25)

Hence, the set $S = \{\mathcal{R}^k_{u, \varphi} f : f \in B_{A^p_{\alpha}}\}$ is bounded in $H^\infty_{\mu}$. From (3.7) and (3.24) we easily get $\mathcal{R}^k_{u, \varphi} f \in H^\infty_{\mu, 0}$ for any $f \in B_{A^p_{\alpha}}$, i.e., $S \subset H^\infty_{\mu, 0}$. Taking the supremum in (3.7) over $B_{A^p_{\alpha}}$ and employing (3.24), it follows that

$$
\lim_{|z| \rightarrow 1} \sup_{f \in B_{A^p_{\alpha}}} \mu(z)|\mathcal{R}^k_{u, \varphi} f(z)| = 0.
$$

This fact and Lemma 2.2 imply the compactness of $\mathcal{R}^k_{u, \varphi} : A^p_{\alpha} \rightarrow H^\infty_{\mu, 0}$.

If $\mathcal{R}^k_{u, \varphi} : A^p_{\alpha} \rightarrow H^\infty_{\mu, 0}$ is compact, then $\mathcal{R}^k_{u, \varphi} : A^p_{\alpha} \rightarrow H^\infty_{\mu}$ is also compact. From Theorem 3.4 we have that (3.15) and (3.16) hold. The fact $f_j(z) = z_j \in A^p_{\alpha}$, $j = \overline{1,n}$, implies $\mathcal{R}^k_{u, \varphi} f_j \in H^\infty_{\mu, 0}$, $j = \overline{1,n}$, from which we have $\lim_{|z| \rightarrow 1} \mu(z)|u(z)||\varphi(z)| = 0$, $j = \overline{1,n}$. Hence,

$$
\lim_{|z| \rightarrow 1} \mu(z)|u(z)||\varphi(z)| = 0.
$$

(3.26)

From (3.26) together with (3.16) we obtain (3.24) in a standard way. □
The following result is known. For a more general result, see [31].

**Theorem 3.9.** Let \( p \geq 1, \alpha > -1, u \in H(B), \varphi \in S(B) \) and \( \mu \in W(B) \). Then, the operator \( \mathcal{R}_{\alpha}^0 : A_\alpha^p \to H_{\mu,0}^\infty \) is compact if and only if

\[
\lim_{\|z\| \to 1} \frac{\mu(z)|u(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1}{p}}} = 0. \tag{3.27}
\]

**Theorem 3.10.** Let \( p \geq 1, \alpha > -1, m \in \mathbb{N}, u_j \in H(B), j = 0, m, \varphi \in S(B) \) and \( \mu \in W(B) \). Then, the operator \( \mathcal{S}_{\alpha}^m : A_\alpha^p \to H_{\mu,0}^\infty \) is compact and

\[
\lim_{\|z\| \to 1} \frac{\mu(z)|u_j(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1}{p}} j} = 0, \quad j = 1, m, \tag{3.28}
\]

if and only if \( \mathcal{R}_{\alpha}^j : A_\alpha^p \to H_{\mu,0}^\infty \) are compact for \( j = 0, m \).

**Proof.** Suppose \( \mathcal{S}_{\alpha}^m : A_\alpha^p \to H_{\mu,0}^\infty \) is compact and (3.28) holds. For the compactness of \( \mathcal{R}_{\alpha}^j : A_\alpha^p \to H_{\mu,0}^\infty \), \( j = 0, m \), it is enough to prove (see Theorems 3.8 and 3.9),

\[
\lim_{\|z\| \to 1} \frac{\mu(z)|u_j(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1}{p}}} = 0, \quad j = 1, m, \tag{3.29}
\]

and

\[
\lim_{\|z\| \to 1} \frac{\mu(z)|u_0(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1}{p}}} = 0. \tag{3.30}
\]

Note that \( \mathcal{S}_{\alpha}^m : A_\alpha^p \to H_{\mu}^\infty \) is compact, whereas (3.9) follows from (3.28). The compactness of \( \mathcal{R}_{\alpha}^j : A_\alpha^p \to H_{\mu,0}^\infty \), \( j = 0, m \), follows from Theorem 3.6. Hence, we have (3.19) and (3.20). Therefore, for every \( \varepsilon > 0 \) there is \( \delta \in (0, 1) \) such that for \( \delta < |\varphi(z)| < 1 \),

\[
\frac{\mu(z)|u_j(z)||\varphi(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1}{p}} j} < \varepsilon, \quad j = 1, m, \quad \text{and} \quad \frac{\mu(z)|u_0(z)|}{(1 - |\varphi(z)|^2)^{\frac{n+1}{p}}} < \varepsilon. \tag{3.31}
\]

From (3.28) and (3.31), (3.29) easily follows. From the fact \( f_0(z) \equiv 1 \in A_\alpha^p \) it follows that \( \mathcal{S}_{\alpha}^m 1 = u_0 \in H_{\mu,0}^\infty \), from which, together with (3.31), we similarly get (3.30).

If \( \mathcal{R}_{\alpha}^j : A_\alpha^p \to H_{\mu,0}^\infty \), \( j = 0, m \), are compact, then \( \mathcal{S}_{\alpha}^m : A_\alpha^p \to H_{\mu,0}^\infty \) is also compact. Beside this (3.26) holds when \( u \) is replaced by \( u_j \) for each \( j \in \{1, 2, \ldots, m\} \), that is, (3.28) also holds.

**Remark 3.1.** The quantities \( J_0 \) and \( J_k, k \in \mathbb{N} \), in Theorems 3.1 and 3.2, are essentially obtained by using the point evaluations in (2.1) and (2.2), respectively. Since the numerator of the right-hand side in (2.1) does not contain the term \( |z| \), the quantity \( J_0 \) does not contain the term \( |\varphi(z)| \), unlike the quantities \( J_k, k \in \mathbb{N} \). This is connected with the definition of the radial derivative operator.
4. Conclusions

Motivated, among others, by our investigations in [14–16, 35], in 2016 I came up with an idea of studying finite sums of the weighted differentiation composition operators and introduced several operators of this form acting on spaces of holomorphic functions on the unit disk or on the unit ball. One of them was the operator in (1.1). In [37] we have studied the operator from Hardy spaces to weighted-type spaces on the unit ball. Here we complement the main results therein by characterizing the boundedness and compactness of the operator from the weighted Bergman space to the weighted-type spaces on the unit ball. The methods, ideas and tricks presented here, with some modifications, can be used in some other settings, which should lead to some further investigations in the direction.

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Conflict of interest

The author declare no conflict of interest.

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