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Research article

Some novel existence and uniqueness results for the Hilfer fractional integro-differential equations with non-instantaneous impulsive multi-point boundary conditions and their application

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Abstract: In this article, we discuss conditions that are sufficient for the existence of solutions for some ψ-Hilfer fractional integro-differential equations with non-instantaneous impulsive multi-point boundary conditions. By applying Krasnoselskii's and Banach's fixed point theorems, we investigate the existence and uniqueness of these solutions. Moreover, we have proved its boundedness of the method. We extend some earlier results by introducing and including the ψ-Hilfer fractional derivative, nonlinear integral terms and non-instantaneous impulsive conditions. Finally, we offer an application to explain the consistency of our theoretical results.

Keywords: ψ-Hilfer fractional derivative; non-instantaneous impulsive; fixed point theory
Mathematics Subject Classification: 26A33, 34A12, 34K20

Abbreviations: FDE: Fractional differential equation; IDE: Impulsive differential equation; BVP: Boundary value problem; FIDE: Fractional integro-differential equation; R-L: Riemann-Liouville fractional integral

1. Introduction

FDEs, which provide a very important class of DEs for describing many processes in the real world, differ from ODEs. FDEs can be found in a variety of areas, including control theory, physics, et cetera. In the literature, many authors focused on R-L and Caputo type derivatives in investigating fractional differential equations. A generalization of derivatives of both R-L and Caputo was given by Hilfer in [1], the known as the Hilfer fractional derivative of order α and a type $\beta \in [0, 1]$, which interpolates between the R-L and Caputo derivative, respectively. This justify the utilization of the Hilfer fractional operator and their generalization in integro-differential equations. In recent years, many researchers have studied the existence, uniqueness and stability of different boundary value problems via Hilfer operators and their generalization.

Asawasamrit et al. [2] studied the ψ-Caputo (or, more appropriately, ψ-Liouville-Caputo) fractional derivative and non-instantaneous impulsive BVPs. Abdo et al. [3] discussed the ψ-Hilfer fractional derivative involving boundary conditions. Ali et al. in [4] found solution of fractional Volterra-Fredholm integro-differential equations under mixed boundary conditions by using the HOBW method. Anguraj et al. in [5] established new existence results for FIDEs with impulsive and integral conditions. Agarwal et al. in [6] investigated non-instantaneous impulses in Caputo FDEs. Abdo et al. in [7] considered fractional BVP with ψ-Caputo fractional derivative. Kailasavalli et al. in [8] derived existence of solutions for fractional BVPs involving integro-differential equations in Banach spaces. Karthikeyan et al. in [9] investigated existence results for fractional impulsive integro differential equations with integral conditions of Katugampola type. Nuchpong et al. in [10] considered BVPs of Hilfer-type FIDEs and inclusions with nonlocal integro-multipoint boundary conditions. Kilbas et al. in [11] give some basic theory and applications of FDEs. Podlubny in [12] investigated some FDEs. Srivastava in [13] overview recent developments of fractional-order derivatives and integrals. Srivastava in [14] considered some parametric and argument variations of the operators of fractional calculus and related special functions, and integral transformations. Srivastava in [15] give an introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions.

Recent theories regarding IDEs arise in many fields like, biology, physics, engineering and medicine, where objects change their state rapidly at certain points, see [16–20]. Hernandez et al. in [21] introduced non-instantaneous IDE. Practical problems in the area of psychology related to non-instantaneous impulses can be found in [22–27]. Asawasamrit et al. [28] considered the nonlocal BVPs for Hilfer FDEs. Mahmudov et al. [29] investigated the fractional-order BVPs with the so-called Katugampola (or, equivalently, the Erdélyi-Kober type) fractional integral conditions. da Costa Sousa et al. [30] studied a Gronwall inequality via the ψ-Hilfer operator. Phuangthong et al. [31] investigated the nonlocal sequential BVPs for Hilfer type FIDEs and inclusions. Sitho et al. in [32], studied the BVPs regarding ψ-Hilfer type sequential FDEs. Sudsutad et al. in [33], investigated the existence and stability results for ψ-Hilfer FIDE. Subashini et al. [34] obtained some results of fractional order regarding Hilfer integro-differential equations. Wang et al. [35] studied the existence results for FDEs with integral and multipoint boundary conditions. Yu [36] investigated β-Ulam-Hyers stability for a special class of FDEs. Zhang et al. [37] studied the FDEs with not instantaneous impulses. ψ-Hilfer FDEs with impulsive conditions were studied in [38, 39].

Abbas [16] studied the following proportional fractional derivatives:

$$\begin{aligned}
&{}_{a_1}\mathcal{D}^{p,q,g}\hbar(\jmath) = \mathcal{Y}(\jmath, \hbar(\jmath), {}_{a_1}\mathcal{I}^{r,q,g}\hbar(\jmath)), \ \jmath \in (r_k, \jmath_{k+1}],\\
&\hbar(\jmath) = \psi_k(\jmath, \hbar(t_k^+)), \ \jmath \in (\jmath_k, r_k], \ k = 1, \cdots, \varrho,\\
&\mathcal{I}^{1-p,q,g}\hbar(a_1) = \hbar_0 \in R,
\end{aligned}$$

where ${}_{a_1}\mathcal{D}^{p,q,g}$ and ${}_{a_1}\mathcal{I}^{r,q,g}$ denote the proportional fractional derivative and the proportional fractional integral and the function $\mathcal{Y}$ is continuous.

Nuchpong et al. [10] discussed the Hilfer fractional derivative with non-local boundary conditions of the form given by

$$\begin{aligned}
{}^{H}\mathcal{D}^{p,q}\hbar(\jmath) &= \mathcal{Y}(\jmath, \hbar(\jmath), \mathcal{I}^{\delta}\hbar(\jmath)), \ \ \jmath \in [a_1, a_2],\\
\hbar(a_1) = 0, \ \ \wp &+ \int_{a_1}^{a_2} \hbar(\imath) d\imath = \sum_{k=1}^{\varrho-2} \varsigma_k \hbar(\vartheta_k),
\end{aligned}$$

where we have used the ${}^{H}\mathcal{D}^{p,q}$-Hilfer fractional derivative and the $\mathcal{I}^{\delta}$-R-L, and the function $\mathcal{Y}$ is continuous.

Salim et al. [23] studied the BVP for implicit fractional-order generalized Hilfer-type fractional derivative with non-instantaneous impulses of the form:

$$\begin{aligned}
&\left({}^{\alpha}\mathcal{D}^{p,q}_{\tau^+}\hbar\right)(\jmath) = \mathcal{Y}(\jmath, \hbar(\jmath), ({}^{\alpha}\mathcal{D}^{p,q}\hbar)(\jmath)), \ \ \jmath \in \mathcal{J}_k,\\
&\hbar(\jmath) = \mathcal{H}_k(\jmath, \hbar(\jmath)), \ \ \jmath \in (\jmath_k, r_k], \ \ k = 1, \cdots, \varrho,\\
&\varphi_1 \left({}^{\alpha}\mathcal{I}^{1-\epsilon}_{a_1^+}\right)(a_1) + \varphi_2 \left({}^{\alpha}\mathcal{I}^{1-\epsilon}_{\tau^+}\right)(a_2) = \varphi_3,
\end{aligned}$$

where ${}^{\alpha}\mathcal{D}^{p,q}_{\tau^+}$ and ${}^{\alpha}\mathcal{I}^{1-\epsilon}_{a_1^+}$ are the generalized Hilfer-type fractional derivative and fractional integral and the function $\mathcal{Y}$ is continuous.

Inspired by the above works, we study here new important class of FIDEs namely ψ-Hilfer FIDEs with non-instantaneous impulsive multi-point boundary conditions of the form given by

$$ {}^{H}\mathcal{D}^{p,q;\psi}\hbar(\jmath) = \mathcal{Y}(\jmath, \hbar(\jmath), \psi\hbar(\jmath)), \ \jmath \in (r_k, \jmath_{k+1}], \tag{1.1}$$

$$ \hbar(\jmath) = \mathcal{H}_k(\jmath, \hbar(\jmath)), \ \ \jmath \in (\jmath_k, r_k], \ \ k = 1, \cdots, \varrho, \tag{1.2}$$

$$ \hbar(0) = 0, \ \ \hbar(T_\star) = \sum_{k=1}^{\varrho} \nu_k \mathcal{I}^{\varsigma_k}\hbar(\upsilon_k), \ \ \nu_k \in R, \ \ \upsilon_k \in [0, T_\star], \tag{1.3}$$

where the order $p \in (1, 2)$ and with the parameters $q \in [0, 1]$, $\nu_k \in R$, $\upsilon_k \in [0, T_\star]$, and $\mathcal{I}^{\varsigma_k}$-is ψ-R-L of order $\varsigma_k > 0$, and $0 = r_0 < \jmath_1 \leqq \jmath_2 < \cdots < \jmath_\varrho \leqq r_\varrho \leqq r_{\varrho+1} = T_\star$, which is pre-fixed, $\mathcal{Y} : [0, T_\star] \times R \times R \longrightarrow R$ with $\mathcal{H}_k : [\jmath_k, r_k] \times R \longrightarrow R$ that are continuous. Moreover, $\psi\hbar(\jmath) = \int_0^{\jmath} k(\jmath, \imath)\hbar(\imath)d\imath$ and $k \in C(D, R^+)$ with domain $D := \{(\jmath, r) \in R^2 : 0 \leqq r \leqq \jmath \leqq T_\star\}$.

Motivated from above results, we introduce ψ-Hilfer FIDEs class with multi-point boundary conditions via the ψ-Hilfer fractional derivative. Moreover, we investigate via Krasnoselskii's and

Banach's fixed point theorems, the existence and uniqueness of solutions of the problem given by the Eqs (1.1)–(1.3). Also, we extend the results studied in [28] by including the ψ-Hilfer fractional derivative, nonlinear integral terms and non-instantaneous impulsive conditions.

This paper is organized as follows: In Section 2, we recall several known results. In Section 3, we use the suitable conditions for existence and uniqueness of solution for the problem given by the Eqs (1.1)–(1.3). Moreover, we prove its boundedness of the method. In Section 4, we consider an application to explain the consistency of our theoretical results.

2. Definitions and preliminaries

Let the space $\mathrm{PC}([0,\mathrm{T}_\star],\mathrm{R}) := \{\hbar : [0,\mathrm{T}_\star] \to \mathrm{R} : \hbar \in C(\jmath_{\mathrm{k}}, \jmath_{\mathrm{k}+1}], \mathrm{R}\}$ be continuous. Suppose that there exists $\hbar(\jmath_{\mathrm{k}}^-)$ and $\hbar(\jmath_{\mathrm{k}}^+)$, where $\hbar(\jmath_{\mathrm{k}}^-) = \hbar(\jmath_{\mathrm{k}}^+)$ is equipped with the norm given by $\|\hbar\|_{\mathrm{PC}} := \sup\{|\hbar(\jmath)| : 0 \leqq \jmath \leqq \mathrm{T}_\star\}$. Set

$$\mathbb{PC}^\infty([0,\mathrm{T}_\star],\mathrm{R}) := \{\hbar \in \mathrm{PC}([0,\mathrm{T}_\star],\mathrm{R}) : \hbar' \in \mathrm{PC}([0,\mathrm{T}_\star],\mathrm{R})\}$$

with norm $\|\hbar\|_{\mathbb{PC}^\infty} := \max\{\|\hbar\|_{\mathrm{PC}}, \|\hbar'\|_{\mathrm{PC}}\}$. Clearly, $\mathbb{PC}^\infty([0,\mathrm{T}_\star],\mathrm{R})$ equipped with $\|.\|_{\mathbb{PC}^\infty}$.

Definition 2.1. [11–15] *The R-L derivative of $\mathcal{Y}$ with order* $\mathrm{p} > 0$ *is defined by*

$$\mathcal{D}^{\mathrm{p}}_{0^+}\mathcal{Y}(\jmath) = \frac{1}{\Gamma(\sigma-\mathrm{p})}\left(\frac{\mathrm{d}}{\mathrm{d}\jmath}\right)^{\sigma}\int_0^{\jmath}(\jmath-\imath)^{\sigma-\mathrm{p}-1}\mathcal{Y}(\imath)\mathrm{d}\imath,\ \sigma-1<\mathrm{p}<\sigma.$$

Definition 2.2. [11–15] *The R-L integral of $\mathcal{Y}$ with order* $\mathrm{p} > 0$ *is given as follows:*

$$\mathcal{I}^{\mathrm{p}}\mathcal{Y}(\jmath) = \frac{1}{\Gamma(\mathrm{p})}\int_0^{\jmath}(\jmath-\imath)^{\mathrm{p}-1}\mathcal{Y}(\imath)\mathrm{d}\imath,$$

with $\Gamma(\mathrm{p}) = \int_0^\infty \exp(-\imath)\imath^{\mathrm{p}-1}\mathrm{d}\imath$.

Definition 2.3. [30] *The R-L integrals and derivatives of $\mathcal{Y}$ with regard to another function ψ are defined by*

$$\mathcal{I}^{\mathrm{p};\psi}\mathcal{Y}(\jmath) = \frac{1}{\Gamma(\mathrm{p})}\int_0^{\jmath}\psi'(\imath)(\psi(\jmath)-\psi(\imath))^{\mathrm{p}-1}\mathcal{Y}(\imath)\mathrm{d}\imath$$

and

$$\mathrm{D}^{\mathrm{p};\psi}\mathcal{Y}(\jmath) = \left(\frac{1}{\psi'(\jmath)}\frac{\mathrm{d}}{\mathrm{d}\jmath}\right)^{\sigma}\mathcal{I}^{\sigma-\mathrm{p};\psi}\mathcal{Y}(\jmath) = \frac{1}{\Gamma(\sigma-\mathrm{p})}\left(\frac{1}{\psi'(\jmath)}\frac{\mathrm{d}}{\mathrm{d}\jmath}\right)^{\sigma}\int_0^{\jmath}\psi'(\imath)(\psi(\jmath)-\psi(\imath))^{\sigma-\mathrm{p}-1}\mathcal{Y}(\imath)\mathrm{d}\imath,$$

respectively.

Definition 2.4. [3] *Let* $\sigma-1<\mathrm{p}<\sigma$, *where* $\sigma\in\mathbb{N}$ *and* $\mathcal{Y},\psi\in\mathrm{PC}([\mathrm{a}_1,\mathrm{a}_2],\mathrm{R})$ *such that ψ is increasing, and $\psi'(\jmath)\neq 0$ for all $\jmath\in[\mathrm{a}_1,\mathrm{a}_2]$. The ψ-Hilfer fractional derivative* ${}^{\mathrm{H}}\mathcal{D}^{\mathrm{p},\mathrm{q};\psi}(.)$ *of function $\mathcal{Y}$ with order* p *and parameter* $0\leqq\mathrm{q}\leqq 1$ *is given by*

$${}^{\mathrm{H}}\mathcal{D}^{\mathrm{p},\mathrm{q};\psi}\mathcal{Y}(\jmath) = \mathcal{I}^{\mathrm{q}(\sigma-\mathrm{p});\psi}\left(\frac{1}{\psi'(\jmath)}\frac{\mathrm{d}}{\mathrm{d}\jmath}\right)^{\sigma}\mathcal{I}^{(1-\mathrm{q})(\sigma-\mathrm{p});\psi}\mathcal{Y}(\jmath),$$

where $\sigma = [\mathrm{p}]+1$, $[\mathrm{p}]$ *represents the integer part of the real number* p.

Lemma 2.1. [3] *Let* $\mathrm{p}, \iota > 0$ *and* $\delta > 0$. *Then*

(1) $\mathcal{I}^{\mathrm{p};\psi}\mathcal{I}^{\iota;\psi}\hbar(\jmath) = \mathcal{I}^{\mathrm{p}+\iota;\psi}\hbar(\jmath)$, *(semigroup property);*
(2) $\mathcal{I}^{\mathrm{p};\psi}(\psi(\jmath) - \psi(0))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\mathrm{p}+\delta)}(\psi(\jmath) - \psi(0))^{\mathrm{p}+\delta-1}$.

Note: ${}^{\mathrm{H}}\mathcal{D}^{\mathrm{p,q};\psi}(\psi(\jmath) - \psi(0))^{\theta-1} = 0$.

Lemma 2.2. [3] *Let* $\mathcal{Y} \in \mathcal{L}(\mathrm{a}_1, \mathrm{a}_2)$, $\sigma - 1 < \mathrm{p} \leqq \sigma$, $\sigma \in \mathbb{N}$ *with* $\theta = \mathrm{p} + \sigma\mathrm{q} - \mathrm{pq}$, *and* $\mathcal{I}^{(\sigma-\mathrm{p})(1-\mathrm{q})}\mathcal{Y} \in \mathcal{AC}^{\mathrm{k}}[\mathrm{a}_1, \mathrm{a}_2]$. *Then*

$$(\mathcal{I}^{\mathrm{p};\psi}; \psi^{\mathrm{H}}\mathcal{D}^{\mathrm{p,q};\psi}\mathcal{Y})(\jmath) = \mathcal{Y}(\jmath) - \sum_{\mathrm{k}=1}^{\sigma} -\frac{(\psi(\jmath) - \psi(0)}{\Gamma(\theta - \mathrm{k} + 1)}\mathcal{Y}_{\psi}^{[\sigma-\mathrm{k}]} \lim_{\jmath \longrightarrow \mathrm{a}_1{}^{+}} (\mathcal{I}^{(\sigma-\mathrm{p})(1-\mathrm{q});\psi}\mathcal{Y})(\jmath),$$

where $\mathcal{Y}_{\psi}^{[\sigma-\mathrm{k}]} = \left(\frac{1}{\psi'(\jmath)}\frac{\mathrm{d}}{\mathrm{d}\jmath}\right)^{\sigma-\mathrm{k}}\mathcal{Y}(\jmath)$.

Assume that $\epsilon > 0$ be a real number. Let $\sigma - 1 < \mathrm{p} < \sigma$, where $\sigma \in \mathbb{N}$ and $\mathcal{Y}, \psi \in \mathrm{PC}([\mathrm{a}_1, \mathrm{a}_2], \mathrm{R})$ such that ψ is increasing, and $\psi'(\jmath) \neq 0$ for all $\jmath \in [\mathrm{a}_1, \mathrm{a}_2]$, where the parameter $0 \leqq \mathrm{q} \leqq 1$.

We consider the following inequality:

$$\left|{}^{\mathrm{H}}\mathcal{D}^{\mathrm{p,q};\psi}\hbar(\jmath) - \mathcal{Y}(\jmath, \hbar(\jmath), \psi\hbar(\jmath))\right| \leq \epsilon. \tag{2.1}$$

Definition 2.5. [33, 40] *The problem given by the Eqs* (1.1)–(1.3) *is said to be Ulam-Hyers stable (see [41]), if there exists a real number* $\mathcal{M}_{\mathcal{Y}} > 0$ *such that for every* $\epsilon > 0$ *and for each solution* $\hbar \in \mathrm{PC}([\mathrm{a}_1, \mathrm{a}_2], \mathrm{R})$ *of the inequality* (2.1)*, there exists a solution* $\hbar_1 \in \mathrm{PC}([\mathrm{a}_1, \mathrm{a}_2], \mathrm{R})$ *of the problem given by the Eqs* (1.1)–(1.3) *with*

$$|\hbar(\jmath) - \hbar_1(\jmath)| \leq \mathcal{M}_{\mathcal{Y}}\,\epsilon, \quad \forall\, \jmath \in [\mathrm{a}_1, \mathrm{a}_2]. \tag{2.2}$$

Fixed point theorems play a major role in establishing the existence theory for the problem given by the Eqs (1.1)–(1.3). The following two well-known fixed point theorems will be used in the sequel.

Theorem 2.1. (Banach's Fixed Point Theorem [42]) *Let* $C([0, \mathrm{T}_\star], \mathrm{R})$ *be a Banach space and let* $\mathcal{N} : \mathrm{R} \to \mathrm{R}$ *be a contraction mapping. If* C *is a nonempty closed subset of* $C([0, \mathrm{T}_\star], \mathrm{R})$*, then* $\mathcal{N}$ *has a unique fixed point.*

Theorem 2.2. (Krasnoselskii's Fixed Point Theorem [42]) *Let* U *be a Banach space and* E *be a closed convex, bounded and nonempty subset of* U. *Suppose that* $\mathcal{Q}$ *and* $\mathcal{R}$ *are two operators that satisfy the following conditions*:

(1) $\mathcal{Q}\mathrm{x}_1 + \mathcal{R}\mathrm{x}_2 \in \mathrm{E}$, $\forall\, \mathrm{x}_1, \mathrm{x}_2 \in \mathrm{E}$*;*
(2) $\mathcal{Q}$ *is completely continuous operator;*
(3) $\mathcal{Q}$ *is contraction operator.*

Then there exists at least one fixed point $\mathrm{z}_1 \in \mathrm{E}$ *such that* $\mathrm{z}_1 = \mathcal{Q}\mathrm{z}_1 + \mathcal{R}\mathrm{z}_1$.

Other recently published papers related fixed point results can be found in [43–46].

Lemma 2.3 below is our first result.

Lemma 2.3. *A function* $\hbar \in \mathrm{PC}([0,\mathrm{T}_\star],\mathrm{R})$ *given by*

$$\hbar(\jmath) := \begin{cases} \mathcal{H}_\mathrm{k}(\mathrm{r}_\varrho) + \frac{1}{\Gamma(\mathrm{p})}\int_{\mathrm{a}_1}^{\jmath} \psi'(\imath)(\psi(\jmath)-\psi(\imath))^{\mathrm{p}-1}\omega(\imath)\mathrm{d}\imath \\ +\frac{(\psi(\jmath)-\psi(0))^{\theta-1}}{\Delta\Gamma(\mathrm{p})}\left[\sum_{\mathrm{k}=1}^{\varrho}\nu_\mathrm{k}\int_0^{\upsilon_\mathrm{k}}\psi'(\jmath)(\psi(\upsilon_k)-\psi(\imath))^{\mathrm{p}-1}\omega(\imath)\mathrm{d}\imath\right], \quad \jmath \in [0,\jmath_1], \\ \mathcal{H}_\mathrm{k}(\jmath), \ \jmath \in (\jmath_\mathrm{k},\mathrm{r}_\mathrm{k}], \ \mathrm{k}=1,2,\cdots,\varrho, \\ \mathcal{H}_\mathrm{k}(\mathrm{r}_\mathrm{k}) + \frac{1}{\Gamma(\mathrm{p})}\int_0^{\jmath}\psi'(\imath)(\psi(\jmath)-\psi(\imath))^{\mathrm{p}-1}\omega(\imath)\mathrm{d}\imath \\ -\frac{1}{\Gamma(\mathrm{p})}\int_0^{\mathrm{r}_\mathrm{k}}\psi'(\imath)(\psi(\mathrm{r}_\mathrm{k})-\psi(\mathrm{r}))^{\mathrm{p}-1}\omega(\imath)\mathrm{d}\imath, \quad \jmath \in (\mathrm{r}_\mathrm{k},\jmath_{\mathrm{k}+1}], \ \mathrm{k}=1,2,\cdots,\varrho \end{cases} \tag{2.3}$$

is a solution of the following system:

$$\begin{aligned} &{}^{\mathrm{H}}\mathcal{D}^{\mathrm{p},\mathrm{q};\psi}\hbar(\jmath) = \omega(\jmath), \ \jmath \in (\mathrm{r}_\mathrm{k},\jmath_{\mathrm{k}+1}] \subset [0,\mathrm{T}_\star], \ 0<\mathrm{p}<1, \\ &\hbar(\jmath) = \mathcal{H}_\mathrm{k}(\jmath), \ \jmath \in (\jmath_\mathrm{k},\mathrm{r}_\mathrm{k}], \ \mathrm{k}=1,\cdots,\varrho, \\ &\hbar(0)=0, \ \hbar(\mathrm{T}_\star) = \sum_{\mathrm{k}=1}^{\varrho}\nu_\mathrm{k}\mathcal{I}^{\varsigma_k}\hbar(\upsilon_\mathrm{k}), \end{aligned} \tag{2.4}$$

where

$$\Delta := (\psi(\jmath)-\psi(0))^{\theta-1}\sum_{\mathrm{k}=1}^{\varrho}\nu_\mathrm{k}(\psi(\upsilon_\mathrm{k})-\psi(0))^{\theta-1} \neq 0.$$

Proof. Assume that $\hbar(\jmath)$ is satisfies for Eq (2.4). Integrating the first equation of (2.4) for $\jmath \in [0,\jmath_1]$, we have

$$\hbar(\jmath) = \hbar(\mathrm{T}_\star) + \frac{1}{\Gamma(\mathrm{p})}\int_0^{\jmath}\psi'(\imath)(\psi(\jmath)-\psi(\imath))^{\mathrm{p}-1}\omega(\imath)\mathrm{d}\imath. \tag{2.5}$$

On other hand, if $\jmath \in (\mathrm{r}_\mathrm{k}, \ \jmath_{\mathrm{k}+1}], \ \mathrm{k}=1,2,\cdots,\varrho$, after integrating again (2.4), we get

$$\hbar(\jmath) = \hbar(\imath_\mathrm{k}) + \frac{1}{\Gamma(\mathrm{p})}\int_{\mathrm{r}_\mathrm{k}}^{\jmath}\psi'(\imath)(\psi(\jmath)-\psi(\imath))^{\mathrm{p}-1}\omega(\imath)\mathrm{d}\imath. \tag{2.6}$$

Applying impulsive condition, $\hbar(\jmath) = \mathcal{H}_\mathrm{k}(\jmath), \ \jmath \in (\jmath_\mathrm{k},\mathrm{r}_\mathrm{k}]$, we obtain

$$\hbar(\imath_\mathrm{k}) = \mathcal{H}_\mathrm{k}(\mathrm{r}_\mathrm{k}). \tag{2.7}$$

Consequently, from (2.6) and (2.7), we get

$$\hbar(\jmath) = \mathcal{H}_\mathrm{k}(\mathrm{r}_\mathrm{k}) + \frac{1}{\Gamma(\mathrm{p})}\int_0^{\jmath}\psi'(\imath)(\psi(\jmath)-\psi(\imath))^{\mathrm{p}-1}\omega(\imath)\mathrm{d}\imath, \tag{2.8}$$

and

$$\begin{aligned} \hbar(\jmath) = {} & \mathcal{H}_\mathrm{k}(\mathrm{r}_\mathrm{k}) + \frac{1}{\Gamma(\mathrm{p})}\int_0^{\jmath}\psi'(\imath)(\psi(\jmath)-\psi(\imath))^{\mathrm{p}-1}\omega(\imath)\mathrm{d}\imath \\ & - \frac{1}{\Gamma(\mathrm{p})}\int_0^{\mathrm{r}_k}(\psi'(\imath)\psi(\mathrm{r}_\mathrm{k})-\psi(\mathrm{r}))^{\mathrm{p}-1}\omega(\imath)\mathrm{d}\imath. \end{aligned} \tag{2.9}$$

Now, we prove that $\hbar$ satisfies the boundary conditions (2.4). Obviously, $\hbar(0) = 0$.

$$\begin{aligned}\sum_{k=1}^{\varrho}\nu_k\mathcal{I}^{\varphi_k}\hbar(\upsilon_k) &= \sum_{k=1}^{\varrho}\nu_k\frac{(\psi(\jmath)-\psi(0))^{p-1}}{\Delta\Gamma(\theta)}\Big[\sum_{k=1}^{\varrho}\nu_k\mathcal{I}^{p+\varphi_k;\psi}\omega(\upsilon_k)-\mathcal{I}^{\alpha;\psi}\omega(a_2)\Big]+\sum_{k=1}^{\varrho}\nu_k\mathcal{I}^{\alpha+\varphi_k}\omega(\upsilon_k)\\ &= \frac{(\psi(\jmath)-\psi(0))^{\theta-1}}{\Delta}\Big[\sum_{k=1}^{\varrho}\nu_k\mathcal{I}^{p+\varphi_k;\psi}\omega(\upsilon_k)\Big]+\mathcal{I}^{p;\psi}\omega(T_\star) \qquad (2.10)\\ &= \hbar(T_\star).\end{aligned}$$

Now, it's clear that (2.5), (2.9) and (2.10) $\Rightarrow$ (2.3), which completes the proof. □

3. Main results

First main result is Theorem 3.1 below.

Theorem 3.1. *Let the assumption below holds true*:
($\mathtt{Al}_1$)*: There exists* $\mathcal{L},\mathcal{G},\mathcal{M},\mathcal{L}_{h_k}>0$*, such that*

$$\begin{aligned}|\mathcal{Y}(\jmath,\hbar_1,\omega_1)-\mathcal{Y}(\jmath,\hbar_2,\omega_2)| &\leqq \mathcal{L}|\hbar_1-\hbar_2|+\mathcal{G}|\omega_1-\omega_2|,\ \textit{for}\ \ \jmath\in[0,T_\star],\ \hbar_1,\hbar_2,\omega_1,\omega_2\in R.\\ |k(\jmath,\iota,\vartheta)-k(\jmath,\iota,\nu)| &\leqq \mathcal{M}|\vartheta-\nu|,\ \textit{for}\ \jmath\in[\jmath_k,r_k],\ \ \vartheta,\nu\in R.\\ \Big|\mathcal{H}_k(\jmath,v_1)-\mathcal{H}_k(\jmath,v_2)\Big| &\leqq \mathcal{L}_{h_k}|v_1-v_2|,\ \textit{for}\ v_1,v_2\in R.\end{aligned}$$

If

$$\begin{aligned}\mathcal{Z}:=\max\Bigg\{&\max_{k=1,2,\cdots,\varrho}\mathcal{L}_{h_k}+\frac{(\mathcal{L}+\mathcal{G}\mathcal{M})}{\Gamma(p+1)}(\jmath_{k+1}^{p}+r_k^{p}),\\ &\mathcal{L}_{h_k}+(\mathcal{L}+\mathcal{G}\mathcal{M})\Bigg\{\frac{(\psi(\jmath)-\psi(0))^{\theta-1}}{|\Delta|\,\Gamma(\theta)}\Big[\sum_{k=1}^{\varrho}|\nu_k|\frac{(\psi(\upsilon_k)-\psi(0))^{p+\varphi_k;\psi}}{\Gamma(p+\varphi_k+1)}\Big]+\frac{(\psi(\jmath)-\psi(0))^{p}}{\Gamma(p+1)}\Bigg\}\Bigg\}<1, \qquad (3.1)\end{aligned}$$

then the problem given by (1.1) *to* (1.3) *has a unique solution on* $[0,T_\star]$.

Proof. Let expand $\mathcal{N}:PC([0,T_\star],R)\longrightarrow PC([0,T_\star],R)$ by

$$(\mathcal{N}\hbar)(\jmath):=\begin{cases}\mathcal{H}_\varrho(r_\varrho,\hbar(\iota_\varrho))+\frac{1}{\Gamma(p)}\int_0^{\jmath}\psi'(\iota)(\psi(\jmath)-\psi(\iota))^{p-1}\mathcal{Y}(\iota,\hbar(\iota),\mathcal{B}\hbar(\iota))d\iota\\ +\frac{(\psi(\jmath)-\psi(0))^{\theta-1}}{\Delta}\Big[\sum_{k=1}^{\varrho}\nu_k\int_0^{\upsilon_k}\psi'(\jmath)(\psi(\upsilon_k)-\psi(\iota))^{p-1}\mathcal{Y}(\upsilon_k,\hbar(\upsilon_k),\mathcal{B}\hbar(\upsilon_k)\Big], & \jmath\in[0,\jmath_1],\\ \mathcal{H}_k(\jmath),\quad \jmath\in(\jmath_k,r_k],\ k=1,2,\cdots,\varrho,\\ \mathcal{H}_k(r_k)+\frac{1}{\Gamma(p)}\int_0^{\jmath}\psi'(\iota)(\psi(\jmath)-\psi(\iota))^{p-1}\mathcal{Y}(\iota,\hbar(\iota),\mathcal{B}\hbar(\iota))d\iota\\ -\frac{1}{\Gamma(p)}\int_0^{r_k}\psi'(\iota)(\psi(r_k)-\psi(r))^{p-1}\mathcal{Y}(\iota,\hbar(\iota),\mathcal{B}\hbar(\iota))d\iota, & \jmath\in(r_k,\jmath_{k+1}],\ k=1,2,\cdots,\varrho.\end{cases}$$

It is evident that $\mathcal{N}$ is well-defined and $\mathcal{N}\hbar\in PC([0,T_\star],R)$. We now prove that $\mathcal{N}$ is a contraction.
Case 1. Taking $\hbar,\ \overline{\hbar}\in PC([0,T_\star],R)$ and $\jmath\in[0,\jmath_1]$, we have

$$\begin{aligned}&\Big|(\mathcal{N}\hbar)(\jmath)-(\mathcal{N}\overline{\hbar})(\jmath)\Big|\\ &\leqq \mathcal{L}_{h_k}+(\mathcal{L}+\mathcal{G}\mathcal{M})\Bigg\{\frac{(\psi(\jmath)-\psi(0))^{\theta-1}}{|\Delta|\,\Gamma(\theta)}\Big[\sum_{k=1}^{\varrho}|\nu_k|\frac{(\psi(\upsilon_k)-\psi(0))^{p+\varphi_k;\psi}}{\Gamma(p+\varphi_k+1)}\Big]+\frac{(\psi(\jmath)-\psi(0))^{p}}{\Gamma(p+1)}\Bigg\}\Big\|\hbar-\overline{\hbar}\Big\|_{PC}.\end{aligned}$$

Case 2. Choosing $\jmath \in (\jmath_k, r_k]$, we get

$$\begin{aligned}\left|(\mathcal{N}\hbar)(\jmath) - (\mathcal{N}\bar{\hbar})(\jmath)\right| &\leqq \left|\mathcal{H}_k(\jmath, \hbar(\jmath)) - \mathcal{H}_k(\jmath, \bar{\hbar}(\jmath))\right| \\ &\leqq \mathcal{L}_{h_k} \left\|\hbar - \bar{\hbar}\right\|_{PC}.\end{aligned}$$

Case 3. Letting $\jmath \in (r_k, \jmath_{k+1}]$, we obtain

$$\begin{aligned}&\left|(\mathcal{N}\hbar)(\jmath) - (\mathcal{N}\bar{\hbar})(\jmath)\right| \\ &\leqq \left|\mathcal{H}_k(r_k, \hbar(\imath_k) - \mathcal{H}_k(\imath_k, \bar{\hbar}(r_k)\right| + \frac{1}{\Gamma(p)} \int_0^{\jmath} (\jmath - \imath)^{p-1} \left|\mathcal{Y}(\imath, \hbar(\imath), \mathcal{B}\hbar(\imath)) - \mathcal{Y}(\imath, \hbar(\imath), \mathcal{B}\hbar(\imath))\right| d\imath \\ &+ \frac{1}{\Gamma(p)} \int_0^{r_k} (r_k - r)^{p-1} \left|\mathcal{Y}(\imath, \hbar(\imath), \mathcal{B}\hbar(\imath)) - \mathcal{Y}(\imath, \hbar(\imath), \mathcal{B}\hbar(\imath))\right| d\imath \\ &\leqq \left[\mathcal{L}_{h_k} + \frac{(\mathcal{L} + \mathcal{G}\mathcal{M})}{\Gamma(p+1)} (\jmath_{k+1}^p + r_k^p)\right] \left\|\hbar - \bar{\hbar}\right\|_{PC}.\end{aligned}$$

Therefore, $\mathcal{N}$ is a contraction since

$$\mathcal{Z} = \left[\mathcal{L}_{h_k} + \frac{(\mathcal{L} + \mathcal{G}\mathcal{M})}{\Gamma(p+1)} (\jmath_{k+1}^p + r_k^p)\right] < 1.$$

Thus, clearly, the problem given by the Eqs (1.1)–(1.3) has a unique solution for each $\hbar \in PC([0, T_\star], R)$. □

Second main result is Theorem 3.2 below.

Theorem 3.2. *Let* $(\mathtt{Al}_1)$ *be satisfied and let the assumption below hold true*:
$(\mathtt{Al}_2)$: *There exists* $\mathcal{L}_{g_k} > 0$ *such that*

$$\left|\mathcal{Y}(\jmath, W_1, \omega_1)\right| \leqq \mathcal{L}_{g_k}(1 + |W_1| + |\omega_1|), \quad \jmath \in [r_k, \jmath_{k+1}], \ \forall\, W_1, \omega_1 \in R.$$

$(\mathtt{Al}_3)$: *A function* $\kappa_k(\jmath)$, $k = 1, 2, \cdots, \varrho$ *exists, with*

$$\left|\mathcal{H}_k(\jmath, W_1, \omega_1)\right| \leqq \kappa_k(\jmath), \quad \jmath \in [\jmath_k, r_k], \ \forall\, W_1, \omega_1 \in R.$$

Assume that $\mathcal{M}_k := \sup_{\jmath \in [\jmath_k, r_k]} \kappa_k(\jmath) < \infty$ *and* $\mathcal{K} := \max \mathcal{L}_{g_k} < 1$ *for all* $k = 1, 2, \cdots, \varrho$. *Then the problem given by* (1.1) *to* (1.3) *has at least one solution on* $[0, T_\star]$.

Proof. Let us set

$$\mathcal{B}_{p,r} := \{\hbar \in PC([0, T_\star], R) : \|\hbar\|_{PC} \leqq r\}.$$

Also let $\mathcal{Q}$ and $\mathcal{R}$ be two operators on $\mathcal{B}_{p,r}$ defined as follows:

$$\mathcal{Q}\hbar(\jmath) := \begin{cases} \mathcal{H}_\varrho(r_\varrho, \hbar(\imath_\varrho)), & \jmath \in [0, \jmath_1], \\ \mathcal{H}_k(\jmath, \hbar(\jmath)), & \jmath \in (\jmath_k, r_k], \ k = 1, 2, \cdots, \varrho, \\ \mathcal{H}_k(r_k, \hbar(\imath_k)), & \jmath \in (r_k, \jmath_{k+1}], \ k = 1, 2, \cdots, \varrho, \end{cases}$$

and

$$\mathcal{R}\hbar(\jmath) := \begin{cases} \frac{1}{\Gamma(\mathrm{p})}\int_{\mathrm{a}_1}^{\jmath}\psi'(\imath)(\psi(\jmath)-\psi(\imath))^{\mathrm{p}-1}\mathcal{Y}(\imath,\hbar(\imath),\mathcal{B}\hbar(\imath))\mathrm{d}\imath \\ +\frac{(\psi(\jmath)-\psi(0))^{\theta-1}}{\Delta}\Big[\sum_{\mathrm{k}=1}^{\varrho}\nu_{\mathrm{k}}\int_0^{\upsilon_{\mathrm{k}}}\psi'(\jmath)(\psi(\upsilon_{\mathrm{k}})-\psi(\imath))^{\mathrm{p}-1}\mathcal{Y}(\upsilon_{\mathrm{k}},\hbar(\upsilon_{\mathrm{k}}),\mathcal{B}\hbar(\upsilon_{\mathrm{k}}))\Big], \quad \jmath\in[0,\jmath_1], \\ 0, \ \jmath\in(\jmath_{\mathrm{k}},\mathrm{r}_{\mathrm{k}}], \ \mathrm{k}=1,2,\cdots,\varrho, \\ \frac{1}{\Gamma(\mathrm{p})}\int_0^{\jmath}\psi'(\imath)(\psi(\jmath)-\psi(\imath))^{\mathrm{p}-1}\mathcal{Y}(\imath,\hbar(\imath),\mathcal{B}\hbar(\imath))\mathrm{d}\imath \\ -\frac{1}{\Gamma(\mathrm{p})}\int_0^{\mathrm{r}_{\mathrm{k}}}\psi'(\imath)(\psi(\mathrm{r}_{\mathrm{k}})-\psi(\imath))^{\mathrm{p}-1}\mathcal{Y}(\imath,\hbar(\imath),\mathcal{B}\hbar(\imath))\mathrm{d}\imath, \quad \jmath\in(\mathrm{r}_{\mathrm{k}},\jmath_{\mathrm{k}+1}], \ \mathrm{k}=1,2,\cdots,\varrho. \end{cases}$$

Step 1. For $\hbar\in\mathcal{B}_{\mathrm{p,r}}$, we have $\mathcal{Q}\hbar+\mathcal{R}\hbar\in\mathcal{B}_{\mathrm{p,r}}$.
Case 1. For $\jmath\in[0,\jmath_1]$, we have

$$\begin{aligned}\left|\mathcal{Q}\hbar+\mathcal{R}\bar{\hbar}\right| &\leqq \left|\mathcal{H}_{\varrho}(\mathrm{r}_{\varrho},\hbar(\imath_{\varrho}))\right| + \frac{1}{\Gamma(\mathrm{p})}\int_0^{\jmath}(\jmath-\imath)^{\mathrm{p}-1}\left|\mathcal{Y}(\imath,\hbar(\imath),\mathcal{B}\hbar(\imath))\right|\mathrm{d}\imath \\ &+ \frac{(\psi(\jmath)-\psi(0))^{\theta-1}}{\Delta}\Big[\sum_{\mathrm{k}=1}^{\varrho}\nu_{\mathrm{k}}\int_0^{\upsilon_{\mathrm{k}}}\psi'(\jmath)(\psi(\upsilon_{\mathrm{k}})-\psi(\imath))^{\mathrm{p}-1}\mathcal{Y}(\upsilon_{\mathrm{k}},\hbar(\upsilon_{\mathrm{k}}),\mathcal{B}\hbar(\upsilon_{\mathrm{k}}))\mathrm{d}\upsilon_{\mathrm{k}}\Big], \\ &\leqq \Big[\mathcal{L}_{\mathrm{h_k}} + (\mathcal{L}+\mathcal{G}\mathcal{M})\Big\{\frac{(\psi(\jmath)-\psi(0))^{\mathrm{p}}}{\Gamma(\mathrm{p}+1)} + \frac{(\psi(\jmath)-\psi(0))^{\theta-1}}{|\Delta|\,\Gamma(\theta)} \\ &\times\Big[\sum_{\mathrm{k}=1}^{\varrho}|\nu_{\mathrm{k}}|\,\frac{(\psi(\upsilon_{\mathrm{k}})-\psi(0))^{\mathrm{p}+\varphi_{\mathrm{k}};\psi}}{\Gamma(\mathrm{p}+\varphi_{\mathrm{k}}+1)}\Big]\Big\}\Big](1+\mathrm{r})\leqq \mathrm{r}.\end{aligned}$$

Case 2. For each $\jmath\in(\jmath_{\mathrm{k}},\mathrm{r}_{\mathrm{k}}]$, we have

$$\left|\mathcal{Q}\hbar+\mathcal{R}\bar{\hbar}\right| \leqq |\mathcal{H}_{\mathrm{k}}(\jmath,\mathrm{W}_1(\jmath))| \leqq \mathcal{M}_{\mathrm{k}}.$$

Case 3. For every $\jmath\in(\mathrm{r}_{\mathrm{k}},\jmath_{\mathrm{k}+1}]$,

$$\begin{aligned}\left|\mathcal{Q}\hbar+\mathcal{R}\bar{\hbar}(\jmath)\right| &\leqq |\mathcal{H}_{\mathrm{k}}(\mathrm{r}_{\mathrm{k}},\hbar(\imath_{\mathrm{k}}))| + \frac{1}{\Gamma(\mathrm{p})}\int_0^{\jmath}(\jmath-\imath)^{\mathrm{p}-1}\left|\mathcal{Y}(\imath,\hbar(\imath),\mathcal{B}\hbar(\imath))\right|\mathrm{d}\imath \\ &+ \frac{1}{\Gamma(\mathrm{p})}\int_0^{\mathrm{r}_{\mathrm{k}}}(\mathrm{r}_{\mathrm{k}}-\mathrm{r})^{\mathrm{p}-1}\left|\mathcal{Y}(\imath,\hbar(\imath),\mathcal{B}\hbar(\imath))\right|\mathrm{d}\imath, \\ &\leqq \mathcal{M}_{\mathrm{k}} + \left[\frac{\mathcal{L}_{\mathrm{g_k}}(\mathrm{r}_{\mathrm{k}}^{\mathrm{p}}+\jmath_{\mathrm{k}+1}^{\mathrm{p}})}{\Gamma(\mathrm{p}+1)}\right](1+\mathrm{r})\leqq \mathrm{r}.\end{aligned}$$

Thus

$$\mathcal{Q}\hbar+\mathcal{R}\hbar\in\mathcal{B}_{\mathrm{p,r}}.$$

Step 2. $\mathcal{Q}$ is contraction on $\mathcal{B}_{\mathrm{p,r}}$.
Case 1. Let $\hbar_1,\hbar_2\in\mathcal{B}_{\mathrm{p,r}}$. Then, by taking $\jmath\in[0,\jmath_1]$, we have

$$|\mathcal{Q}\hbar_1(\jmath)-\mathcal{Q}\hbar_2(\jmath)| \leqq \mathcal{L}_{\mathrm{g}_{\varrho}}\left|\hbar_1(\mathrm{r}_{\varrho})-\hbar_2(\mathrm{r}_{\varrho})\right| \leqq \mathcal{L}_{\mathrm{g}_{\varrho}}\|\hbar_1-\hbar_2\|_{\mathrm{PC}}.$$

Case 2. For each $\jmath\in(\jmath_{\mathrm{k}},\mathrm{r}_{\mathrm{k}}]$, $\mathrm{k}=1,2,\cdots,\varrho$, we get

$$|\mathcal{Q}\hbar_1(\jmath)-\mathcal{Q}\hbar_2(\jmath)| \leqq \mathcal{L}_{\mathrm{g_k}}\|\hbar_1-\hbar_2\|_{\mathrm{PC}}.$$

Case 3. For every $\jmath \in (\mathrm{r_k}, \jmath_{\mathrm{k}+1}]$, we obtain

$$|Q\hbar_1(\jmath) - Q\hbar_2(\jmath)| \leqq \mathcal{L}_{\mathrm{g_k}} \|\hbar_1 - \hbar_2\|_{\mathrm{PC}}.$$

Hence, we deduce the following inequality:

$$|Q\hbar_1(\jmath) - Q\hbar_2(\jmath)| \leqq \mathcal{K} \|\hbar_1 - \hbar_2\|_{\mathrm{PC}}.$$

Consequently, Q is a contraction.

Step 3. Let demonstrate that $\mathcal{R}$ be continuous.

Let $\hbar_\sigma$ be a $\ni \hbar_\sigma \to \bar{\hbar}$ sequence in $\mathrm{PC}([0, \mathrm{T}_\star], \mathrm{R})$.

Case 1. For each $\jmath \in [0, \jmath_1]$, we have

$$|Q\hbar_\sigma(\jmath) - Q\hbar(\jmath)| \leqq \left[\frac{(\psi(\jmath) - \psi(0))^{\theta-1}}{|\Delta|\,\Gamma(\theta)} \Big[\sum_{\mathrm{k}=1}^{\varrho} |\nu_{\mathrm{k}}| \frac{(\psi(\upsilon_{\mathrm{k}}) - \psi(0))^{\mathrm{p}+\varphi_{\mathrm{k}};\psi}}{\Gamma(\mathrm{p} + \varphi_{\mathrm{k}} + 1)} \Big] + \frac{(\psi(\jmath) - \psi(0))^{\mathrm{p}}}{\Gamma(\mathrm{p} + 1)} \right]$$
$$\times \|\mathcal{Y}(.,\hbar_\sigma(.),.,) - \mathcal{Y}(.,\hbar(.),.,)\|_{\mathrm{PC}}.$$

Case 2. For every $\jmath \in (\jmath_{\mathrm{k}}, \mathrm{r_k}]$, we get

$$|Q\hbar_\sigma(\jmath) - Q\hbar(\jmath)| = 0.$$

Case 3. For each $\jmath \in (\mathrm{r_k}, \jmath_{\mathrm{k}+1}],\ \mathrm{k} = 1, 2, \cdots, \varrho$, we obtain

$$|Q\hbar_\sigma(\jmath) - Q\hbar(\jmath)| \leqq \frac{(\jmath_{\mathrm{k}+1} - \mathrm{r_k})}{\Gamma(\mathrm{p} + 1)} \|\mathcal{Y}(.,\hbar_\sigma(.),.,) - \mathcal{Y}(.,\hbar(.),.,)\|_{\mathrm{PC}}.$$

We thus conclude from the above cases that $\|Q\hbar_\sigma(\jmath) - Q\hbar(\jmath)\|_{\mathrm{PC}} \longrightarrow 0$ as $\sigma \to \infty$.

Step 4. Finally, let us prove that Q is compact.

Firstly, Q is uniformly bounded on $\mathcal{B}_{\mathrm{p,r}}$.

Since $\|Q\hbar\| \leqq \frac{\mathcal{L}_{\mathrm{g_k}}(\mathcal{T})}{\Gamma(1+\mathrm{p})} < \mathrm{r}$, therefore, we have Q is uniformly bounded on $\mathcal{B}_{\mathrm{p,r}}$.

We prove that Q maps a bounded set to a $\mathcal{B}_{\mathrm{p,r}}$ equicontinuous set.

Case 1. For interval $\jmath \in [0, \jmath_1],\ 0 \leqq \mathcal{E}_1 \leqq \mathcal{E}_2 \leqq \jmath_1, \hbar \in \mathcal{B}_{\mathrm{r}}$, we have

$$|Q\mathcal{E}_2 - Q\mathcal{E}_1| \leqq \frac{\mathcal{L}_{\mathrm{g_k}}(1 + \mathrm{r})}{\Gamma(\mathrm{p} + 1)} (\mathcal{E}_2 - \mathcal{E}_1).$$

Case 2. For each $\jmath \in (\jmath_{\mathrm{k}}, \mathrm{r_k}],\ \jmath_{\mathrm{k}} < \mathcal{E}_1 < \mathcal{E}_2 \leqq \mathrm{r_k},\ \hbar \in \mathcal{B}_{\mathrm{p,r}}$, we get

$$|Q\mathcal{E}_2 - Q\mathcal{E}_1| = 0.$$

Case 3. For every $\jmath \in (\mathrm{r_k}, \jmath_{\mathrm{k}+1}],\ \mathrm{r_k} < \mathcal{E}_1 < \mathcal{E}_2 \leqq \jmath_{\mathrm{k}+1},\ \hbar \in \mathcal{B}_{\mathrm{p,r}}$, we obtain

$$|Q\mathcal{E}_2 - Q\mathcal{E}_1| \leqq \frac{\mathcal{L}_{\mathrm{g_k}}(1 + \mathrm{r})}{\Gamma(\mathrm{p} + 1)} (\mathcal{E}_2 - \mathcal{E}_1).$$

From the above cases, we deduce that $|Q\mathcal{E}_2 - Q\mathcal{E}_1| \longrightarrow 0$ as $\mathcal{E}_2 \longrightarrow \mathcal{E}_1$ and Q is equicontinuous. As a result, $Q(\mathcal{B}_{\mathrm{p,r}})$ is relatively compact and Q is compact, by using the Ascoli–Arzela theorem. Hence, the problem given by (1.1) to (1.3) has at least one fixed point on $[0, \mathrm{T}_\star]$. □

4. Application

Let as consider the following ψ-Caputo (or, more appropriately, ψ-Liouville–Caputo) fractional boundary value problem:

$$D^{\mathrm{p},\mathrm{q};\psi}\hbar(\jmath) = \frac{\exp(-\jmath)\,|\hbar(\jmath)|}{9+\exp(-\jmath)(1+|\hbar(\jmath)|)} + \frac{1}{3}\int_0^{\jmath} \mathrm{e}^{-(\imath-\jmath)}\hbar(\imath)\mathrm{d}\imath, \quad \jmath \in (0,1], \tag{4.1}$$

$$\hbar(\jmath) = \frac{|\hbar(\jmath)|}{2(1+|\hbar(\jmath)|)}, \ \jmath \in \left(\frac{1}{2}, 1\right], \tag{4.2}$$

$$\hbar(0) = 0, \ \ \hbar(1) = \frac{1}{2}\mathcal{I}^{\frac{2}{3}}\hbar\left(\frac{7}{5}\right) + \frac{2}{3}\mathcal{I}^{\frac{4}{5}}\hbar\left(\frac{9}{5}\right) + \frac{5}{2}\mathcal{I}^{\frac{3}{4}}\hbar\left(\frac{7}{2}\right), \tag{4.3}$$

together with $\mathcal{L} = \mathcal{G} = \frac{1}{10}$, $\mathcal{M} = \frac{1}{3}$, $\mathrm{p} = \frac{5}{7}$, $\theta = \frac{2}{5}$, $\mathcal{L}_{\mathrm{h}_1} = \frac{1}{3}$, $\nu_1 = \frac{1}{2}$, $\nu_2 = \frac{2}{3}$, $\nu_3 = \frac{2}{5}$, $\upsilon_1 = \frac{2}{7}$, $\upsilon_2 = \frac{5}{9}$, $\upsilon_3 = \frac{1}{7}$, $\varphi_1 = \frac{2}{3}$, $\varphi_2 = \frac{4}{5}$, $\varphi_3 = \frac{3}{4}$. We shall check the condition (3.1), for value $\mathrm{p} \in (1,2)$. Indeed, by using Theorem 3.1, we determine that

$$\mathcal{L}_{\mathrm{h}_\mathrm{k}} + \frac{(\mathcal{L}+\mathcal{G}\mathcal{M})}{\Gamma(\mathrm{p}+1)}(\jmath_{\mathrm{k}+1}^{\mathrm{p}} + \mathrm{r}_{\mathrm{k}}^{\mathrm{p}}) \approx 0.41 < 1,$$

and

$$\mathcal{L}_{\mathrm{h}_\mathrm{k}} + (\mathcal{L}+\mathcal{G}\mathcal{M})\Big\{\frac{(\psi(\jmath)-\psi(0))^{\mathrm{p}}}{\Gamma(\mathrm{p}+1)} + \frac{(\psi(\jmath)-\psi(0))^{\theta-1}}{|\Delta|\,\Gamma(\theta)}\Big[\sum_{\mathrm{k}=1}^{\varrho}|\nu_{\mathrm{k}}|\,\frac{(\psi(\upsilon_{\mathrm{k}})-\psi(0))^{\mathrm{p}+\varphi_{\mathrm{k}};\psi}}{\Gamma(\mathrm{p}+\varphi_{\mathrm{k}}+1)}\Big]\Big\} \approx 0.485 < 1.$$

Hence, from Theorem 3.1 the problem given by (4.1) to (4.3) has a unique solution on $[0,\mathrm{T}_\star]$.

5. Conclusions

We have discussed in this paper ψ-Hilfer FIDEs class with non-instantaneous impulsive conditions and with an R-L integral boundary condition. Furthermore, the existence and uniqueness of the derived solution is investigated via two well-known fixed point theorems. Moreover, we have proved its boundedness of the method in Section 3, and hence we don't need stability analysis. Finally, the consistency of our results was demonstrated with an example. For future work, we will give the numerical algorithm for the R-L integral BVPs via different kinds of fractional derivatives. Also, interested researchers can improve our results by using the resolvents operators as well.

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Conflicts of interest

The authors declare that they have no conflicts interests.

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