On some dynamic inequalities of Hilbert’s-type on time scales

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Abstract: In this article, we will prove some new conformable fractional Hilbert-type dynamic inequalities on time scales. These inequalities generalize some known dynamic inequalities on time scales, unify and extend some continuous inequalities and their corresponding discrete analogues. Our results will be proved by using some algebraic inequalities, conformable fractional Hölder inequalities, and conformable fractional Jensen’s inequalities on time scales.

Keywords: Hilbert’s inequality; dynamic inequality; time scales

Mathematics Subject Classification: 26D10, 26D15, 26E70, 34A40

1. Introduction

The celebrated Hardy-Hilbert’s integral inequality with powers \(p\) and \(q\) [1] is

\[
\int_0^{\infty} \int_0^{\infty} \frac{F(\vartheta)g(\varsigma)}{\vartheta + \varsigma} \, d\vartheta \, d\varsigma \leq \pi \frac{1}{\sin \frac{\pi}{p}} \left[ \int_0^{\infty} F^p(\vartheta) \, d\vartheta \right]^{\frac{1}{p}} \left[ \int_0^{\infty} g^q(\varsigma) \, d\varsigma \right]^{\frac{1}{q}},
\]

where \(p > 1\). Putting \(p = q = 2\), we get:

\[
\int_0^{\infty} \int_0^{\infty} \frac{F(\vartheta)g(\varsigma)}{\vartheta + \varsigma} \, d\vartheta \, d\varsigma \leq \pi \left[ \int_0^{\infty} F^2(\vartheta) \, d\vartheta \right]^{\frac{1}{2}} \left[ \int_0^{\infty} g^2(\varsigma) \, d\varsigma \right]^{\frac{1}{2}},
\]
where the coefficient $\pi$ is best possible.

Pachpatte [2] proved the following two inequalities:

\[
\sum_{m=1}^{k} \sum_{n=1}^{r} \frac{\Phi(a_m)\Psi(b_n)}{m+n} \leq M(k, r) \left( \sum_{m=1}^{k} (k - m + 1) \left( p_m \Phi \left( \frac{\nabla a_m}{p_m} \right) \right)^{\frac{1}{2}} \right) \times \left( \sum_{n=1}^{r} (r - n + 1) \left( q_n \Psi \left( \frac{\nabla b_n}{q_n} \right) \right)^{\frac{1}{2}} \right),
\]

where

\[
M(k, r) = \frac{1}{2} \left( \sum_{m=1}^{k} \left( \frac{\Phi(P_m)}{P_m} \right)^{\frac{1}{2}} \right) \left( \sum_{n=1}^{r} \left( \frac{\Psi(Q_n)}{Q_n} \right)^{\frac{1}{2}} \right)
\]

\[
\int_{0}^{\vartheta} \int_{0}^{\psi} \frac{\Phi(F(s))\Psi(g(\vartheta))}{s + \vartheta} dsd\vartheta \leq L(\vartheta, \psi) \left( \int_{0}^{\vartheta} (\vartheta - s) \Phi \left( \frac{F'(s)}{F(s)} \right) ds \right)^{\frac{1}{2}} \times \left( \int_{0}^{\psi} (\psi - \vartheta) \Phi \left( \frac{\psi'(\vartheta)}{\psi(\vartheta)} \right) d\vartheta \right)^{\frac{1}{2}}.
\]

where

\[
L(\vartheta, \psi) = \frac{1}{2} \left( \int_{0}^{\vartheta} \left( \frac{\Phi(P(s))}{P(s)} \right)^{2} ds \right)^{\frac{1}{2}} \left( \int_{0}^{\psi} \left( \frac{\Psi(Q(\vartheta))}{Q(\vartheta)} \right)^{2} d\vartheta \right)^{\frac{1}{2}}.
\]

Handley et al. [3], extended (1.3) and (1.4) as follows:

\[
\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \prod_{\ell=1}^{n} \Phi(\ell, m_\ell) \leq M(k_1, \ldots, k_n) \prod_{\ell=1}^{n} \left( \sum_{m_\ell=1}^{k_\ell} (k_\ell - m_\ell + 1) \left( p_{\ell, m_\ell} \Phi \left( \frac{\nabla a_{\ell, m_\ell}}{p_{\ell, m_\ell}} \right) \right)^{\frac{1}{2}} \right),
\]

where

\[
M(k_1, \ldots, k_n) = \frac{1}{(\gamma')^n} \prod_{\ell=1}^{n} \left( \sum_{m_\ell=1}^{k_\ell} \left( \frac{\Phi(\ell, m_\ell)}{P_{\ell, m_\ell}} \right)^{\frac{1}{2}} \right)^{\gamma_\ell},
\]

and

\[
\int_{0}^{\vartheta_1} \cdots \int_{0}^{\vartheta_n} \prod_{\ell=1}^{n} \Phi(\ell, F(s_\ell)) \left( \sum_{m_\ell=1}^{k_\ell} \gamma'_\ell s_\ell \right) ds_1 \cdots ds_n \leq L(\vartheta_1, \ldots, \vartheta_n) \prod_{\ell=1}^{n} \left( \int_{0}^{\vartheta_\ell} (\vartheta_\ell - s_\ell) \left( p_{\ell, s_\ell} \Phi \left( \frac{F'(s_\ell)}{F(s_\ell)} \right) \right)^{\frac{1}{2}} ds_\ell \right)^{\gamma_\ell},
\]

where

\[
L(\vartheta_1, \ldots, \vartheta_n) = \frac{1}{(\gamma')^n} \prod_{\ell=1}^{n} \left( \int_{0}^{\vartheta_\ell} \left( \Phi(\ell, F(s_\ell)) \right)^{\frac{1}{2}} ds_\ell \right)^{\gamma_\ell}.
\]
In 2006, Zhao and Cheung [4] gave the reverse versions of the above inequalities, which are more extensive results for this type of inequalities.

\[ \int_{\theta}^{\bar{\theta}} \int_{s_i}^{s_i+1} \ldots \int_{s_n}^{s_n+1} \prod_{\ell=1}^{n} \left( \frac{\Phi_{\ell}(F_{\ell}(s_{\ell}, \mathfrak{I}_{\ell}))}{\frac{1}{\gamma} \sum_{\ell=1}^{n} \gamma_{\ell}(s_{\ell}, \mathfrak{I}_{\ell})} \right)^{\frac{1}{\gamma}} d s_{\ell} d \mathfrak{I}_{\ell} \]  

(1.7) 

where 

\[ G(\theta_1 s_1, \ldots, \theta_n y_n) = \prod_{\ell=1}^{n} \left( \int_{\theta}^{\bar{\theta}} \int_{s_1}^{s_1+1} \ldots \int_{s_n}^{s_n+1} \left( \Phi_{\ell}(P_{\ell}(s_{\ell}, \mathfrak{I}_{\ell})) \right)^{\frac{1}{\gamma}} d s_{\ell} d \mathfrak{I}_{\ell} \right)^{\frac{1}{\gamma}} \]

and 

\[ P_{\ell}(s_{\ell}, \mathfrak{I}_{\ell}) = \int_{\theta}^{\bar{\theta}} \int_{s_1}^{s_1+1} \ldots \int_{s_n}^{s_n+1} p_{\ell}(\xi_{\ell}) q_{\ell}(\tau_{\ell}) d \xi_{\ell} d \tau_{\ell}. \]  

(1.8)

In [5], Pachpattte established the following Hilbert type integral inequalities.

\[ \int_{0}^{\theta} \int_{0}^{s} \frac{F^h(s)G^l(\mathfrak{J})}{s+\mathfrak{I}} d s d \mathfrak{I} \leq \frac{1}{2} h h(x,y) \left( \int_{0}^{\theta} (\theta-s) \left( F^{h-1}(s)F(s) \right)^2 d s \right)^{\frac{1}{2}} \times \left( \int_{0}^{s} (s-\mathfrak{I}) \left( G^{l-1}g(\mathfrak{J}) \right)^2 d \mathfrak{I} \right)^{\frac{1}{2}}, \]  

(1.9) 

and 

\[ \int_{0}^{\theta} \int_{0}^{s} \Phi(F(s)) \Psi(G(\mathfrak{J}))(s+\mathfrak{I}) d s d \mathfrak{I} \leq L(\theta, s) \left( \int_{0}^{\theta} (\theta-s) \left( p(s) \Phi(F(s)) \right)^2 d s \right)^{\frac{1}{2}} \times \left( \int_{0}^{s} (s-\mathfrak{I}) \left( q(\mathfrak{J}) \Psi(g(\mathfrak{J})) \right)^2 d \mathfrak{I} \right)^{\frac{1}{2}}, \]  

(1.10) 

where 

\[ L(\theta, s) = \frac{1}{2} \left( \int_{0}^{\theta} \left( \Phi(P(s)) \right)^2 d s \right)^{\frac{1}{2}} \left( \int_{0}^{s} \left( \frac{\Psi(Q(\mathfrak{J}))}{Q(\mathfrak{J})} \right)^2 d \mathfrak{I} \right)^{\frac{1}{2}} \]

and 

\[ \int_{0}^{\theta} \int_{0}^{s} \frac{P(s)Q(\mathfrak{J})\Phi(F(s))\Psi(G(\mathfrak{J}))}{s+\mathfrak{I}} d s d \mathfrak{I} \leq \frac{1}{2} (x,y) \left( \int_{0}^{\theta} (\theta-s) \left( p(s) \Phi(F(s)) \right)^2 d s \right)^{\frac{1}{2}} \times \left( \int_{0}^{s} (s-\mathfrak{I}) \left( q(\mathfrak{J}) \Psi(g(\mathfrak{J})) \right)^2 d \mathfrak{I} \right)^{\frac{1}{2}}. \]  

(1.11) 

\[ \int_{0}^{\theta_1} \int_{0}^{s_1} \ldots \int_{0}^{s_n} \prod_{\ell=1}^{n} \Phi_{\ell}(F_{\ell}(s_{\ell}, \mathfrak{I}_{\ell})) \left( \gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_{\ell}} (s_{\ell}, \mathfrak{I}_{\ell}) \right)^{\frac{1}{\gamma}} d s_{\ell} d \mathfrak{I}_{\ell} \]  

(1.12)
\[ L(\vartheta_1 \varsigma_1, \ldots, \vartheta_n \nu_n) \times \prod_{\ell=1}^{n} \left( \int_{0}^{\vartheta_\ell} \int_{0}^{\varsigma_\ell} (\vartheta_\ell - s_\ell)(\varsigma_\ell - \varphi_\ell) \left( \frac{p_\ell(s_\ell, \varphi_\ell)}{p_\ell(s_\ell, \varphi_\ell)} \right)^{\gamma_\ell} ds_\ell d\varphi_\ell \right)^{\frac{1}{\gamma_\ell}} \]

where

\[ L(\vartheta_1 \varsigma_1, \ldots, \vartheta_n \nu_n) = \prod_{\ell=1}^{n} \left( \int_{0}^{\vartheta_\ell} \int_{0}^{\varsigma_\ell} \left( \frac{\Phi_\ell(p_\ell(s_\ell, \varphi_\ell))}{p_\ell(s_\ell, \varphi_\ell)} \right)^{\gamma_\ell} ds_\ell d\varphi_\ell \right)^{\frac{1}{\gamma_\ell}}. \]

Over the past decade, a great number of dynamic Hilbert type inequalities on time scales has been established by many researchers who were motivated by some applications, see the papers [6–15, 24–27, 40–43], see also, [16, 18, 22, 23, 28–32]. For more details on time scales calculus see [33].

In this paper, we extend some generalizations of the integral Hardy-Hilbert inequality to a general time scale using conformable fractional. As special cases of our results, we will recover some integral and discrete inequalities known in the literature. This article is arranged as follows: In Section 2, some basic concepts of the calculus on time scales and useful lemmas are introduced. In Section 3, we state and prove the main results. In Section 4, we state the conclusion.

2. Preliminary

A time scale \( T \) is an arbitrary nonempty closed subset of the real numbers. We define jump operators forward and backward \( \sigma : T \to T \) and \( \rho : T :\to T \) respectively by

\[ \sigma(\varphi) := \inf\{s \in T : s > \varphi\}, \quad \varphi \in T, \]
\[ \rho(\varphi) := \sup\{s \in T : s < \varphi\}, \quad \varphi \in T. \]

In the preceding two definitions, we set \( \inf \emptyset = \sup T \) (i.e., if \( \tau \) is the maximum of \( T \), then \( \sigma(\tau) = \tau \)) and \( \sup \emptyset = \inf T \) (i.e., if \( \tau \) is the minimum of \( T \), then \( \rho(\tau) = \tau \)), where \( \emptyset \) denotes the empty set.

Recently, depending just on the basic limit definition of the derivative, Khalil et al. [34] proposed the conformable derivative \( T_{a}(f)(\xi) \) \( (\alpha \in (0, 1)) \) of a function \( f : \mathbb{R}^+ \to \mathbb{R} \)

\[ T_{a}(f)(\xi) = \lim_{\epsilon \to 0} \frac{f(\xi + \epsilon \xi^{1-a}) - f(\xi)}{\epsilon}, \]

for all \( \xi > 0, \alpha \in (0, 1) \). The researchers in [34] also suggested a definition for the \( \alpha \)-conformable integral of a function \( \eta \) as follows:

\[ \int_{a}^{b} \eta(\xi) d_{a}\xi = \int_{a}^{b} \eta(\xi) \xi^{a-1} d\xi. \]

After that, Abdeljawad [35] studied extensive research of the newly introduced conformable calculus. In his work, he introduced a generalization of the conformable derivative \( T_{a}(f)(\xi) \) definition. For \( \xi > a \in \mathbb{R}^+ \) as \( f : \mathbb{R}^+ \to \mathbb{R} \)

\[ T_{a}(f)(\xi) = \lim_{\epsilon \to 0} \frac{f(\xi + \epsilon(\xi - a)^{1-a}) - f(\xi)}{\epsilon}. \]
Benkhettou et al. [36] introduced a conformable calculus on an arbitrary time scale, which is a natural extension of the conformable calculus.

However, in the last few decades, many authors pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various real materials, e.g., polymers. Fractional derivatives provides an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantages of fractional derivatives in comparison with classical integer-order models.

In [37], the authors studied a version of the nabla conformable fractional derivative on arbitrary time scales.

**Definition 2.1.** Let $\xi : \mathbb{T} \rightarrow \mathbb{R}$, $\tau \in \mathbb{T}^k$, and $\alpha \in (0, 1]$. For $\tau > 0$, we define $T^\Delta_\alpha(\xi)(\tau)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a $\delta$- neighborhood $U_\tau \subset \mathbb{T}$ of $\tau$, $\delta > 0$, such that

$$\|[\xi(\sigma(\tau)) - \xi(s)] \tau^{1-\alpha} - T^\Delta_\alpha(\xi)(\tau)[\sigma(\tau) - s]\| \leq \epsilon|\sigma(\tau) - s|,$$

for all $s \in U_\tau$. We call $T^\Delta_\alpha(\xi)(\tau)$ the conformable derivative of $\xi$ of order $\alpha$ at $\tau$, and we define conformable derivative on $\mathbb{T}$ at 0, as $T^\Delta_\alpha(\xi)(0) = \lim_{\tau \rightarrow 0}, T^\Delta_\alpha(\xi)(\tau)$.

**Remark 2.1.** If $\alpha = 1$ then we obtain from Definition 2.1 the delta derivative of time scales. The conformable derivative of order zero is defined by the identity operator: $T^\Delta_0(\xi) = \xi$.

**Remark 2.2.** Along the work, we also use the notation $(\xi)^{\Delta}(\tau) = T^\Delta_\alpha(\xi)(\tau)$.

**Theorem 2.1.** Let $\alpha \in (0, 1]$ and $\mathbb{T}$ be a time scale. Assume $\xi : \mathbb{T} \rightarrow \mathbb{R}$ and $\tau \in \mathbb{T}^k$. The following properties hold.

(i) If $\xi$ is conformal differentiable of order $\alpha$ at $\tau > 0$, then $\xi$ is continuous at $\tau$.

(ii) If $\xi$ is continuous at $\tau$ and $\tau$ is right-scattered, then $\xi$ is conformable differentiable of order $\alpha$ at $\tau$ with

$$T^\Delta_\alpha(\xi)(\tau) = \frac{\xi(\sigma(\tau)) - \xi(\tau)}{\mu(\tau)} \tau^{1-\alpha}.$$  

(iii) If $\tau$ is right-dense, then $\xi$ is conformable differentiable of order $\alpha$ at $\tau$ if and only if the limit

$$\lim_{s \rightarrow \tau} \frac{\xi(\tau) - \xi(s)}{\tau - s} \tau^{1-\alpha}$$

exists as a finite number. In this case,

$$T^\Delta_\alpha(\xi)(\tau) = \lim_{s \rightarrow \tau} \frac{\xi(\tau) - \xi(s)}{\tau - s} \tau^{1-\alpha}.$$  

(iv) If $\xi$ is differentiable of order $\alpha$ at $\tau$, then

$$\xi(\sigma(\tau)) = \xi(\tau) + \mu(\tau)\tau^{\alpha-1}T^\Delta_\alpha(\xi)(\tau).$$

**Theorem 2.2.** Assume $\xi, \sigma : \mathbb{T} \rightarrow \mathbb{R}$ are conformable differentiable of order $\alpha \in (0, 1]$, then following properties are hold:

(i) The sum $\xi + \sigma : \mathbb{T} \rightarrow \mathbb{R}$ is conformable differentiable with

$$T^\Delta_\alpha(\xi + \sigma) = T^\Delta_\alpha(\xi) + T^\Delta_\alpha(\sigma).$$

(ii) For any $k \in \mathbb{R}$, $k\xi : \mathbb{T} \rightarrow \mathbb{R}$ is conformable differentiable with

$$T^\Delta_\alpha(k\xi) = kT^\Delta_\alpha(\xi).$$
(iii) If \( \xi \) and \( \varpi \) are continuous, then the product \( \xi \varpi : \mathbb{T} \rightarrow \mathbb{R} \) is conformable differentiable with

\[
T^\Delta_{a}(\xi \varpi) = T_{a}^\Delta(\xi)\varpi + \xi^\varpi T_{a}^\Delta(\varpi) = T_{a}^\Delta(\xi)\varpi^\varpi + \xi T_{a}^\Delta(\varpi).
\]

(iv) If \( \xi \) is continuous, then \( 1/\xi \) is conformable differentiable with

\[
T^\Delta_{a}\left(\frac{1}{\xi}\right) = \frac{-T_{a}^\Delta(\xi)}{\xi(\xi \circ \varpi)},
\]
valid at all points \( \tau \in \mathbb{T}^k \) for which \( \xi(\xi \circ \varpi) \neq 0 \).

(v) If \( \xi \) and \( \varpi \) are continuous, then \( \xi / \varpi \) is conformable differentiable with

\[
T^\Delta_{a}\left(\frac{\xi}{\varpi}\right) = \frac{T_{a}^\Delta(\xi)\varpi - \xi T_{a}^\Delta(\varpi)}{\varpi^\varpi},
\]
valid \( \forall \tau \in \mathbb{T}^k \), for which \( \varpi \varpi^\varpi \neq 0 \).

**Definition 2.2.** Let \( \xi : \mathbb{T} \rightarrow \mathbb{R} \) be regulated function. Then the \( \alpha \)-conformable integral of \( \xi \), \( 0 < \alpha \leq 1 \), is defined by

\[
\int_{a}^{b} \xi(\tau)\Delta_{\alpha} \tau = \int_{a}^{b} \xi(\tau)\tau^{\alpha-1}\Delta \tau.
\]

**Definition 2.3.** Suppose \( \xi : \mathbb{T} \rightarrow \mathbb{R} \) is a regulated function. Denote the indefinite \( \alpha \)-conformable integral of \( \xi \) of order \( \alpha, \alpha \in (0,1] \), as follows: \( F_{a}(\tau) = \int_{a}^{\tau} \xi(\tau)\Delta_{\alpha} \tau \). Then, for all \( a, b \in \mathbb{T} \) we define the Cauchy \( \alpha \)-conformable integral by

\[
\int_{a}^{b} \xi(\tau)\Delta_{\alpha} \tau = F_{a}(b) - F_{a}(a).
\]

**Theorem 2.3.** Let \( \alpha \in (0,1] \). Then, for any rd-continuous function \( \xi : \mathbb{T} \rightarrow \mathbb{R} \), there exists a function \( F_{a} : \mathbb{T} \rightarrow \mathbb{R} \) such that \( T_{a}^\Delta(F_{a})(\tau) = \xi(\tau) \) for all \( \tau \in \mathbb{T}^k \). Function \( F_{a} \) is said to be an \( \alpha \)-antiderivative of \( \xi \).

The conformable integral satisfying the next properties

**Theorem 2.4.** Let \( \alpha \in (0,1] \), \( a, b, c \in \mathbb{T} \), and \( \omega \in \mathbb{R} \), \( \xi, \varpi \) be two rd-continuous functions. Then

(i) \( \int_{a}^{b}[\xi(\tau) + \varpi(\tau)]\Delta_{\alpha} \tau = \int_{a}^{b} \xi(\tau)\Delta_{\alpha} \tau + \int_{a}^{b} \varpi(\tau)\Delta_{\alpha} \tau \).

(ii) \( \int_{a}^{b} \omega \xi(\tau)\Delta_{\alpha} \tau = \omega \int_{a}^{b} \xi(\tau)\Delta_{\alpha} \tau \).

(iii) \( \int_{a}^{b} \xi(\tau)\Delta_{\alpha} \tau = -\int_{a}^{b} \xi(\tau)\Delta_{\alpha} \tau \).

(iv) \( \int_{a}^{b} \varpi(\tau)\Delta_{\alpha} \tau = \int_{a}^{b} \xi(\tau)\Delta_{\alpha} \tau + \int_{c}^{b} \xi(\tau)\Delta_{\alpha} \tau \).

(v) \( \int_{a}^{b} \xi(\tau)\Delta_{\alpha} \tau = 0 \).

(vi) If there exists \( \xi : \mathbb{T} \rightarrow \mathbb{R} \) with \( |\xi(\tau)| \leq |\xi(\tau)| \) for all \( \tau \in [a,b] \), then \( \int_{a}^{b} \xi(\tau)\Delta_{\alpha} \tau \leq \int_{a}^{b} |\xi(\tau)|\Delta_{\alpha} \tau \).

(vii) If \( \xi > 0 \) for all \( \tau \in [a,b] \), then \( \int_{a}^{b} \xi(\tau)\Delta_{\alpha} \tau \geq 0 \).

We use the following crucial relations between calculus on time scales \( \mathbb{T} \) and differential calculus on \( \mathbb{R} \) and difference calculus on \( \mathbb{Z} \). Note that:

(i) for any time scales \( \mathbb{T} \), we have

\[
(\xi)^{\Delta_{\alpha}}(\tau) = (\xi)^{\Delta}(\tau)^{1-\alpha}, \quad \int_{a}^{b} \xi(\tau)\Delta_{\alpha} \tau = \int_{a}^{b} \xi(\tau)^{\alpha-1}\Delta \tau.
\]
Lemma 2.2. If $T = R$, then
\[ \sigma(\tau) = \tau, \quad \mu(\tau) = 0, \quad f^A(\tau) = f'(\tau), \quad \int_a^b f(\tau) d\tau = \int_a^b f(\tau) d\tau. \quad (2.1) \]

(iii) If $T = Z$, then
\[ \sigma(\tau) = \tau + 1, \quad \mu(\tau) = 1, \quad f^A(\tau) = \Delta f(\tau), \quad \int_a^b f(\tau) \Delta \tau = \sum_{\tau=a}^{b-1} f(\tau). \quad (2.2) \]

Next, we write Hölder’s inequality and Jensen’s inequality on time scales.

Lemma 2.1. Suppose $u, v \in T$ with $u < v$. Assume $F, g \in CC^1_{rd}([u,v]_T \times [u,v]_T, R)$ be integrable functions and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$ then
\[ \int_u^v \int_u^v |F^*(r^*, \mathcal{J}^*) g^*(r^*, \mathcal{J}^*)|^{\alpha^p \Delta r^* \Delta \mathcal{J}^*} \leq \left[ \int_u^v \int_u^v |F^*(r^*, \mathcal{J}^*)|^{\alpha^p \Delta r^* \Delta \mathcal{J}^*} \right]^\frac{1}{p} \times \left[ \int_u^v \int_u^v |g^*(r^*, \mathcal{J}^*)|^{\alpha^q \Delta r^* \Delta \mathcal{J}^*} \right]^\frac{1}{q}. \quad (2.3) \]

This inequality is reversed if $0 < p < 1$ and if $p < 0$ or $q < 0$.

Lemma 2.2. Let $r^*, \mathcal{J}^* \in R$ and $-\infty \leq m^*, n^* \leq \infty$. If $F \in CC^1_{rd}(R, (m^*, n^*))$, and $\Phi : (m^*, n^*) \rightarrow R$ is convex then
\[ \Phi \left( \int_u^v \int_u^v F^*(r^*, \mathcal{J}^*) \Delta r^* \Delta \mathcal{J}^* \right) \leq \left( \int_u^v \int_u^v \Phi(F^*(r^*, \mathcal{J}^*)) \Delta r^* \Delta \mathcal{J}^* \right) \frac{1}{\int_u^v \int_u^v \Delta r^* \Delta \mathcal{J}^*}. \quad (2.4) \]

This inequality is reversed if $\Phi \in C_{rd}((c, d), R)$ is concave.

Theorem 2.5. (Chain rule on time scales [33]) Let $g : R \rightarrow R$ is continuous, $g : T \rightarrow R$ is $\Delta^p$-differentiable on $\mathcal{J}^*$, and $F : R \rightarrow R$ is continuously differentiable. Then there exists $c \in [\mathcal{J}, \sigma(\mathcal{J})]$ with
\[ (F \circ g)^{\Delta^p}(\mathcal{J}) = F'(g(c))(g) \Delta^p(\mathcal{J}). \quad (2.5) \]

Definition 2.4. $\Phi$ is called a supermultiplicative function on $[0, \infty)$ if
\[ \Phi(\vartheta \varsigma) \geq \Phi(\vartheta)\Phi(\varsigma), \quad \text{for all } \vartheta, \varsigma \geq 0. \quad (2.6) \]

Next, we write Fubini’s theorem on time scales.

Lemma 2.3. (Fubini’s Theorem, see [38]) Assume that $(\vartheta, \Sigma_1, \mu_\vartheta)$ and $(\varsigma, \Sigma_2, \nu_\varsigma)$ are two finite-dimensional time scales measure spaces. Moreover, suppose that $F : \vartheta \times \varsigma \rightarrow R$ is a delta integrable function and define the functions
\[ \phi(\varsigma) = \int_\vartheta F(\vartheta, \varsigma) d\mu_\vartheta(\vartheta), \quad \varsigma \in \varsigma, \]

and
\[ \psi(\theta) = \int_\varsigma F(\theta, \varsigma) d\nu_\varsigma(\varsigma), \quad \theta \in \theta. \]

Then $\phi$ is delta integrable on $\varsigma$ and $\psi$ is delta integrable on $\theta$ and
\[ \int_\vartheta d\mu_\vartheta(\vartheta) \int_\varsigma F(\theta, \varsigma) d\nu_\varsigma(\varsigma) = \int_\varsigma d\nu_\varsigma(\varsigma) \int_\theta F(\theta, \varsigma) d\mu_\vartheta(\vartheta). \]

Now we are ready to state and prove our main results.
3. Main results

First, we enlist the following assumptions for the proofs of our main results:

(S1) \( T \) be time scales with \( S_0, \theta_\ell, \varsigma_\ell, s_\ell, \mathcal{I}_\ell \in T, (\ell = 1, \ldots, n) \).
(S2) \( F_\ell(s_\ell, \mathcal{I}_\ell) \) are nonnegative, right-dense continuous functions defined on \([S_0, \theta_\ell)_T \times [S_0, \varsigma_\ell)_T (\ell = 1, \ldots, n) \).
(S3) \( F_\ell(s_\ell, \mathcal{I}_\ell) \) have a partial \( \Delta^\ell \)-derivatives \( F_\ell^{(\Delta^\ell)}(s_\ell, \mathcal{I}_\ell) \) and \( F_\ell^{(\Delta^\ell)}(s_\ell, \mathcal{I}_\ell) \) with respect \( s_\ell \) and \( \mathcal{I}_\ell \) respectively.
(S4) \( F_\ell(s_\ell, \mathcal{I}_\ell) \in C^2_{\gamma}(S_0, \theta_\ell)_T \times [S_0, \varsigma_\ell)_T, [0, \infty) (\ell = 1, \ldots, n) \) are increasing.
(S5) \( F_\ell(s_\ell, \mathcal{I}_\ell) \in C^2_{\gamma}(S_0, \theta_\ell)_T \times [S_0, \varsigma_\ell)_T, [0, \infty) (\ell = 1, \ldots, n) \).
(S6) \( p_\ell(\xi_\ell, \tau_\ell) \) are \( n \) positive right-dense continuous functions defined for \( \xi_\ell \in (S_0, s_\ell)_T, \tau_\ell \in (S_0, \mathcal{I}_\ell)_T \).
(S7) \( p_\ell(\xi_\ell) \) and \( q_\ell(\tau_\ell) \) are positive right-dense continuous functions defined for \( \xi_\ell \in (S_0, s_\ell)_T, \tau_\ell \in (S_0, \mathcal{I}_\ell)_T \).
(S8) \( \Phi_\ell (\ell = 1, \ldots, n) \) are \( n \) real-valued nonnegative concave and supermultiplicative functions defined on \((0, \infty) \).
(S9) \( \theta_\ell \) and \( \varsigma_\ell \) are positive real numbers.
(S10) \( s_\ell \in [S_0, \theta_\ell)_T \) and \( \mathcal{I}_\ell \in [S_0, \varsigma_\ell)_T \).
(S11) \( F_\ell(s_\ell, \mathcal{I}_\ell) = F_\ell(s_\ell, \mathcal{I}_\ell) = 0, (\ell = 1, \ldots, n) \).
(S12) \( F_\ell^{(\Delta^\ell)}(s_\ell, \mathcal{I}_\ell) = F_\ell^{(\Delta^\ell)}(s_\ell, \mathcal{I}_\ell) \).
(S13) \( P_\ell(s_\ell, \mathcal{I}_\ell) = \int_{S_0}^{s_\ell} \int_{S_0}^{\mathcal{I}_\ell} p_\ell(\xi_\ell, \tau_\ell) \Delta^\ell \xi_\ell \Delta^\ell \tau_\ell \).
(S14) \( \bar{F}_\ell(s_\ell, \mathcal{I}_\ell) = \int_{S_0}^{s_\ell} \int_{S_0}^{\mathcal{I}_\ell} F_\ell(\xi_\ell, \tau_\ell) \Delta^\ell \xi_\ell \Delta^\ell \tau_\ell \).
(S15) \( P_\ell(s_\ell, \mathcal{I}_\ell) = \int_{S_0}^{s_\ell} \int_{S_0}^{\mathcal{I}_\ell} p_\ell(\xi_\ell, \tau_\ell) \Delta^\ell \xi_\ell \Delta^\ell \tau_\ell \).
(S16) \( \bar{F}_\ell(s_\ell, \mathcal{I}_\ell) = \int_{S_0}^{s_\ell} \int_{S_0}^{\mathcal{I}_\ell} F_\ell(\xi_\ell, \tau_\ell) \Delta^\ell \xi_\ell \Delta^\ell \tau_\ell \).
(S17) \( \gamma_\ell \in (1, \infty), \gamma_\ell = 1 - \gamma_\ell, \gamma = \sum_{\ell=1}^{n} \gamma_\ell, \text{ and } \gamma' = \sum_{\ell=1}^{n} \gamma'_\ell = n - \gamma, (\ell = 1, \ldots, n) \).
(S18) \( 0 < \beta_\ell < 1 \).
(S19) \( h_\ell \geq 2 \).
(S20) \( \sum_{\ell=1}^{n} \frac{1}{\gamma'_\ell} = 1 \).
(S21) \( h_\ell \geq 1 \).
(S22) \( F_\ell(\xi_\ell) \in C^1_{\gamma}(S_0, \theta_\ell)_T, (\ell = 1, \ldots, n) \).
(S23) \( \theta_\ell \) is positive real number.
(S24) \( \bar{F}_\ell(s_\ell) = \int_{S_0}^{s_\ell} F_\ell(\xi_\ell) \Delta^\ell \xi_\ell \).
(S25) \( s_\ell \in [S_0, \theta_\ell)_T \).
(S26) \( p_\ell(\xi_\ell) \) are \( n \) positive functions.
(S27) \( P_\ell(s_\ell) = \int_{S_0}^{s_\ell} p_\ell(\xi_\ell) \Delta^\ell \xi_\ell \).
(S28) \( \bar{F}_\ell(s_\ell) = \frac{1}{p_\ell(s_\ell)} \int_{S_0}^{s_\ell} p_\ell(\xi_\ell) F_\ell(\xi_\ell) \Delta^\ell \xi_\ell \).
(S29) \( F_\ell(\mathcal{I}_\ell) = 0 \).

Now, we are ready to state and prove the main results that extend several results in the literature.

**Theorem 3.1.** Let \( S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9 \), and \( S_{17} \) be satisfied. Then for \( S_{10} \) we have that

\[
\int_{S_0}^{s_1} \int_{S_0}^{s_2} \cdots \int_{S_0}^{s_n} \left( \frac{\Phi_\ell(F_\ell(s_\ell, \mathcal{I}_\ell))}{\gamma'_{\ell=1} \gamma'_\ell (s_\ell - S_0) (\mathcal{I}_\ell - S_0)} \right)^{\frac{\Delta^\ell_{s_1} \Delta^\ell_{s_2} \cdots \Delta^\ell_{s_n} \gamma'}{\gamma'}} D^\ell_{s_1} \Delta^\ell_{s_2} \cdots \Delta^\ell_{s_n} \mathcal{I}_n
\]
Using the following inequality on the term $obtain

By using inverse Jensen’s dynamic inequality, we get that

We obtain that

Proof. From the hypotheses of Theorem 3.1, we obtain

From (3.2) and $S_8$, it is easy to observe that

By using inverse Jensen’s dynamic inequality, we get that

Applying inverse Hölder’s inequality on the right hand side of (3.4) with indices $1/\gamma_\ell$ and $1/\gamma_\ell'$, we obtain

Using the following inequality on the term $[(s_\ell - \mathcal{I}_0)(\mathcal{I}_\ell - \mathcal{I}_0)]^{\gamma_\ell'}$ where $\gamma_\ell' < 0$ and $\lambda_\ell > 0$.

We obtain that

\[ \prod_{\ell=1}^n \Phi_\ell(F_\ell(s_\ell, \mathcal{I}_\ell)) \geq \prod_{\ell=1}^n \frac{\Phi_\ell(P_\ell(s_\ell, \mathcal{I}_\ell))}{P_\ell(s_\ell, \mathcal{I}_\ell)} \left( \frac{1}{\gamma_\ell'} \sum_{\ell=1}^n \gamma_\ell(s_\ell - \mathcal{I}_0)(\mathcal{I}_\ell - \mathcal{I}_0) \right)^{\gamma_\ell'}. \]
From (3.7), we have that
\[ \prod_{\ell=1}^{n} \frac{\Phi_{\ell}(F_{\ell}(s_{\ell}, \mathcal{J}_{\ell}))}{(1/\gamma) \sum_{\ell=1}^{n} \gamma_{\ell}'(s_{\ell} - \mathcal{J}_{\ell})(\mathcal{J}_{\ell} - \mathcal{J}_{0})} \geq \prod_{\ell=1}^{n} \frac{\Phi_{\ell}(P_{\ell}(s_{\ell}, \mathcal{J}_{\ell})))}{(1/\gamma) \sum_{\ell=1}^{n} \gamma_{\ell}'(s_{\ell} - \mathcal{J}_{\ell})(\mathcal{J}_{\ell} - \mathcal{J}_{0})} \Delta^{a} s_{1} \Delta^{a} \mathcal{J}_{1} \ldots \Delta^{a} s_{n} \Delta^{a} \mathcal{J}_{n} \]
(3.8)

Integrating both sides of (3.8) over $s_{\ell}, \mathcal{J}_{\ell}$ from $\mathcal{J}_{0}$ to $\vartheta_{\ell}, \varsigma_{\ell}$ ($\ell = 1, \ldots, n$), we get that
\[ \int_{\mathcal{J}_{0}}^{\vartheta_{1}} \int_{\mathcal{J}_{0}}^{\varsigma_{1}} \cdots \int_{\mathcal{J}_{0}}^{\vartheta_{n}} \int_{\mathcal{J}_{0}}^{\varsigma_{n}} \prod_{\ell=1}^{n} \frac{\Phi_{\ell}(F_{\ell}(s_{\ell}, \mathcal{J}_{\ell}))}{(1/\gamma) \sum_{\ell=1}^{n} \gamma_{\ell}'(s_{\ell} - \mathcal{J}_{\ell})(\mathcal{J}_{\ell} - \mathcal{J}_{0})} \Delta^{a} s_{1} \Delta^{a} \mathcal{J}_{1} \ldots \Delta^{a} s_{n} \Delta^{a} \mathcal{J}_{n} \]
(3.9)

By applying inverse Hölder’s inequality on the right hand side of (3.9) with indices $1/\gamma_{\ell}'$ and $1/\gamma'_{\ell}$, we obtain
\[ \int_{\mathcal{J}_{0}}^{\vartheta_{1}} \int_{\mathcal{J}_{0}}^{\varsigma_{1}} \cdots \int_{\mathcal{J}_{0}}^{\vartheta_{n}} \int_{\mathcal{J}_{0}}^{\varsigma_{n}} \prod_{\ell=1}^{n} \frac{\Phi_{\ell}(F_{\ell}(s_{\ell}, \mathcal{J}_{\ell}))}{(1/\gamma) \sum_{\ell=1}^{n} \gamma_{\ell}'(s_{\ell} - \mathcal{J}_{\ell})(\mathcal{J}_{\ell} - \mathcal{J}_{0})} \Delta^{a} s_{1} \Delta^{a} \mathcal{J}_{1} \ldots \Delta^{a} s_{n} \Delta^{a} \mathcal{J}_{n} \]
(3.10)

By using Fubini’s theorem, we observe that
\[ \int_{\mathcal{J}_{0}}^{\vartheta_{1}} \int_{\mathcal{J}_{0}}^{\varsigma_{1}} \cdots \int_{\mathcal{J}_{0}}^{\vartheta_{n}} \int_{\mathcal{J}_{0}}^{\varsigma_{n}} \prod_{\ell=1}^{n} \frac{\Phi_{\ell}(F_{\ell}(s_{\ell}, \mathcal{J}_{\ell}))}{(1/\gamma) \sum_{\ell=1}^{n} \gamma_{\ell}'(s_{\ell} - \mathcal{J}_{\ell})(\mathcal{J}_{\ell} - \mathcal{J}_{0})} \Delta^{a} s_{1} \Delta^{a} \mathcal{J}_{1} \ldots \Delta^{a} s_{n} \Delta^{a} \mathcal{J}_{n} \]
(3.11)

By using the fact $\vartheta_{\ell} \geq \rho(\vartheta_{\ell})$, and $\varsigma_{\ell} \geq \rho(\varsigma_{\ell})$, we get that
\[ \int_{\mathcal{J}_{0}}^{\vartheta_{1}} \int_{\mathcal{J}_{0}}^{\varsigma_{1}} \cdots \int_{\mathcal{J}_{0}}^{\vartheta_{n}} \int_{\mathcal{J}_{0}}^{\varsigma_{n}} \prod_{\ell=1}^{n} \frac{\Phi_{\ell}(F_{\ell}(s_{\ell}, \mathcal{J}_{\ell}))}{(1/\gamma) \sum_{\ell=1}^{n} \gamma_{\ell}'(s_{\ell} - \mathcal{J}_{\ell})(\mathcal{J}_{\ell} - \mathcal{J}_{0})} \Delta^{a} s_{1} \Delta^{a} \mathcal{J}_{1} \ldots \Delta^{a} s_{n} \Delta^{a} \mathcal{J}_{n} \]
\[ G(\theta_1 \varsigma_1, \ldots, \theta_n \varsigma_n) \times \prod_{\ell=1}^{n} \left( \int_{3_\ell}^{\delta_\ell} \left( \rho(\theta_\ell) - s_\ell \right) \left( \sigma(\varsigma_\ell) - \varsigma_\ell \right) \left( p_\ell(s_\ell) q_\ell(\varsigma_\ell) \right) \Phi_\ell \left( F^{p_\ell, q_\ell}_{\ell}(s_\ell, \varsigma_\ell) \right) \right)^{\gamma_\ell} \Delta^\alpha s_\ell \Delta^\alpha \varsigma_\ell. \]

This completes the proof. \[\square\]

**Remark 3.1.** In Theorem 3.1, if \( T = \mathbb{Z}, \alpha = 1 \) we get the result due to Zhao et al. [4, Theorem 1.5].

**Remark 3.2.** In Theorem 3.1, if we take \( T = \mathbb{R}, \alpha = 1 \) we get equality 1.7.

**Remark 3.3.** Let \( S_1, S_2, S_9, S_{11}, S_7, S_{13}, S_3 \) and \( S_{12} \) be satisfied and let \( \Phi_\ell, \gamma_\ell, \gamma'_\ell, \gamma, \) and \( \gamma' \) be as in inequality 1.7. Similar to proof of Theorem 3.1, we have

\[ \int_{3_\ell}^{\delta_\ell} \left( \sigma(\theta_\ell) - s_\ell \right) \left( \sigma(\varsigma_\ell) - \varsigma_\ell \right) \left( p_\ell(s_\ell) q_\ell(\varsigma_\ell) \right) \Phi_\ell \left( F^{p_\ell, q_\ell}_{\ell}(s_\ell, \varsigma_\ell) \right) \left( \frac{F^{p_\ell, q_\ell}_{\ell}(s_\ell, \varsigma_\ell)}{p_\ell(s_\ell) q_\ell(\varsigma_\ell)} \right)^{\gamma_\ell} \Delta^\alpha s_\ell \Delta^\alpha \varsigma_\ell. \]

This is an inverse form of the inequality (3.1).

**Corollary 2.1.** Let \( S_2, S_{23}, S_2, S_{26}, S_{27}, S_9, S_{17} \) and \( S_8 \) be satisfied. Then we have that

\[ \int_{3_\ell}^{\delta_\ell} \Phi_\ell \left( F^{p_\ell, q_\ell}_{\ell}(s_\ell, \varsigma_\ell) \right) \left( \frac{F^{p_\ell, q_\ell}_{\ell}(s_\ell, \varsigma_\ell)}{p_\ell(s_\ell) q_\ell(\varsigma_\ell)} \right)^{\gamma_\ell} \Delta^\alpha s_\ell \Delta^\alpha \varsigma_\ell. \]

This is an inverse form of the inequality (3.1).
where
\[ D(\vartheta_1, \vartheta_2) = 4 \left( \int_{s_0}^{g_1} \left( \Phi_1(P_1(s_1)) \right)^{-1} \Delta^\alpha s_1 \right) \left( \int_{s_0}^{g_1} \left( \Phi_2(P_2(s_2)) \right)^{-1} \Delta^\alpha s_1 \right)^{-1}. \]

**Remark 3.7.** If we take \( T = \mathbb{Z}, \alpha = 1 \) the inequality (3.13) is an inverse of inequality of (1.3), which was given by Pachpatte.

**Remark 3.8.** If we take \( T = \mathbb{R}, \alpha = 1 \) the inequality (3.13) is an inverse of inequality of (1.4), which was given by Pachpatte.

**Theorem 3.2.** Let \( S_1, S_4, S_9, \) and \( S_{14} \) be satisfied. Then for \( S_{10}, S_{18}, S_{19} \) and \( S_{20} \) we have that
\[
\int_{s_0}^{g_1} \cdots \int_{s_0}^{g_1} \prod_{\gamma=1}^n P_{\ell}^{h_{\ell}}(s, \mathcal{G}_\ell) \Delta^\alpha s_1 \Delta^\alpha s_2 \cdots \Delta^\alpha s_n \]
\[
\geq \prod_{\ell=1}^n \left[ h_t [(\vartheta_\ell - \mathcal{G}_0)(\varphi_\ell - \mathcal{G}_0)] \right]^{\gamma_{\ell}} 
\times \left\{ \int_{s_0}^{g_1} \int_{s_0}^{g_1} (\rho(\vartheta_\ell) - s_\ell)(\rho(\varphi_\ell) - s_\ell) \left( H(h_\ell, s_\ell, \mathcal{G}_\ell) + F_{\ell}^{h_{\ell}^{-1}}(s_\ell, \sigma(\mathcal{G}_\ell))F_{\ell}(s_\ell, \mathcal{G}_\ell) \right)^{\beta_{\ell}} \right\}^{\frac{1}{\beta_{\ell}}}

where
\[
H(h_\ell, \xi_\ell, \tau_\ell) = (h_\ell - 1) F_{\ell}^{h_{\ell}^{-1}}(\xi_\ell, \tau_\ell) \frac{\partial F_{\ell}}{\Delta^\alpha \tau_\ell} (\xi_\ell, \tau_\ell) \frac{\partial F_{\ell}}{\Delta^\alpha \xi_\ell} (\xi_\ell, \tau_\ell).
\]

**Proof.** From the hypotheses and by using the chain rule on time scales, we have that
\[
F_{\ell}^{h_{\ell}}(s_\ell, \mathcal{G}_\ell) \geq \int_{s_0}^{g_1} \int_{s_0}^{g_1} h_t \left( H(h_\ell, \xi_\ell, \tau_\ell) + F_{\ell}^{h_{\ell}^{-1}}(\xi_\ell, \sigma(\mathcal{G}_\ell)) \right) \frac{\partial^2 F_{\ell}}{\Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell} (\xi_\ell, \tau_\ell) \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell
\]
\[
= h_t \int_{s_0}^{g_1} \int_{s_0}^{g_1} \left( H(h_\ell, \xi_\ell, \tau_\ell) + F_{\ell}^{h_{\ell}^{-1}}(\xi_\ell, \sigma(\mathcal{G}_\ell))F_{\ell}(\xi_\ell, \mathcal{G}_\ell) \right) \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell
\]
where
\[
H(h_\ell, \xi_\ell, \tau_\ell) = (h_\ell - 1) F_{\ell}^{h_{\ell}^{-1}}(\xi_\ell, \tau_\ell) \frac{\partial F_{\ell}}{\Delta^\alpha \tau_\ell} (\xi_\ell, \tau_\ell) \frac{\partial F_{\ell}}{\Delta^\alpha \xi_\ell} (\xi_\ell, \tau_\ell).
\]
Applying inverse Hölder’s inequality on the right hand side of (3.14) with indices \( \gamma_{\ell} \) and \( \beta_{\ell} \), it is easy to observe that
\[
F_{\ell}^{h_{\ell}}(s_\ell, \mathcal{G}_\ell) \geq h_t [(s_\ell - \mathcal{G}_0)(\mathcal{G}_\ell - \mathcal{G}_0)]^{\frac{1}{\gamma_{\ell}}}
\times \left\{ \int_{s_0}^{g_1} \int_{s_0}^{g_1} \left( H(h_\ell, \xi_\ell, \tau_\ell) + F_{\ell}^{h_{\ell}^{-1}}(\xi_\ell, \sigma(\mathcal{G}_\ell))F_{\ell}(\xi_\ell, \mathcal{G}_\ell) \right)^{\beta_{\ell}} \right\}^{\frac{1}{\beta_{\ell}}}
\]
Let us note the following means inequality
\[ \prod_{\ell=1}^{n} m_{\ell}^{\frac{1}{q_{\ell}}} \geq \left( \gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_{\ell}} m_{\ell} \right)^{\frac{1}{q}} \] (3.16)

we obtain that
\[
\prod_{\ell=1}^{n} F_{\ell}^{h_{\ell}}(s_{\ell}, \mathcal{G}_{\ell}) \\
\left( \gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_{\ell}} (s_{\ell} - \mathcal{G}_{0})(\mathcal{G}_{\ell} - \mathcal{G}_{0}) \right)^{\frac{1}{q}} \\
\geq \prod_{\ell=1}^{n} h_{\ell} \left( \int_{\mathcal{G}_{0}}^{\mathcal{G}_{\ell}} \int_{\mathcal{G}_{0}}^{\mathcal{G}_{\ell}} \left( H(h_{\ell}, \xi_{\ell}, \tau_{\ell}) + F_{\ell}^{h_{\ell}}(s_{\ell}, \sigma(\tau_{\ell}))F_{\ell}(\xi_{\ell}, \tau_{\ell}) \right)^{\beta_{\ell}} \Delta^{\alpha_{\xi_{\ell}}} \Delta^{\alpha_{\tau_{\ell}}} \right)^{\frac{1}{q_{\ell}}}. 
\] (3.17)

Integrating both sides of (3.17) over \( s_{\ell}, \mathcal{G}_{\ell} \) from \( \mathcal{G}_{0} \) to \( \vartheta_{\ell}, \varsigma_{\ell} (\ell = 1, \ldots, n) \), we get that
\[
\int_{\mathcal{G}_{0}}^{\vartheta_{\ell}} \int_{\mathcal{G}_{0}}^{\varsigma_{\ell}} \cdots \int_{\mathcal{G}_{0}}^{\vartheta_{n}} \int_{\mathcal{G}_{0}}^{\varsigma_{n}} \frac{\prod_{\ell=1}^{n} F_{\ell}^{h_{\ell}}(s_{\ell}, \mathcal{G}_{\ell})}{\left( \gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_{\ell}} (s_{\ell} - \mathcal{G}_{0})(\mathcal{G}_{\ell} - \mathcal{G}_{0}) \right)^{\frac{1}{q}} \Delta^{\alpha_{s_{1}}} \Delta^{\alpha_{s_{n}}} \Delta^{\alpha_{n}}} \\
\Delta^{\alpha_{s_{1}}} \Delta^{\alpha_{s_{n}}} \Delta^{\alpha_{n}} \Delta^{\alpha_{\mathcal{G}_{n}}} \\
\geq \prod_{\ell=1}^{n} h_{\ell} \left( \vartheta_{\ell} - \mathcal{G}_{0} \right) \left( \varsigma_{\ell} - \mathcal{G}_{0} \right) \left( \vartheta_{n} - \mathcal{G}_{0} \right) \left( \varsigma_{n} - \mathcal{G}_{0} \right) \right)^{\frac{1}{q}} \\
\times \left\{ \int_{\mathcal{G}_{0}}^{\vartheta_{\ell}} \int_{\mathcal{G}_{0}}^{\varsigma_{\ell}} \cdots \int_{\mathcal{G}_{0}}^{\vartheta_{n}} \int_{\mathcal{G}_{0}}^{\varsigma_{n}} \\
H(h_{\ell}, \xi_{\ell}, \tau_{\ell}) + F_{\ell}^{h_{\ell}}(s_{\ell}, \sigma(\tau_{\ell}))F_{\ell}(\xi_{\ell}, \tau_{\ell}) \right)^{\beta_{\ell}} \Delta^{\alpha_{\xi_{\ell}}} \Delta^{\alpha_{\tau_{\ell}}} \right\}^{\frac{1}{q_{\ell}}} \\
\Delta^{\alpha_{s_{1}}} \Delta^{\alpha_{s_{n}}} \Delta^{\alpha_{n}} \Delta^{\alpha_{\mathcal{G}_{n}}} \\
\Delta^{\alpha_{\mathcal{G}_{n}}} . 
\] (3.18)

Applying inverse Hölder’s inequality on the right hand side of (3.18) with indices \( \gamma_{\ell} \) and \( \beta_{\ell} \), it is easy to observe that
\[
\int_{\mathcal{G}_{0}}^{\vartheta_{\ell}} \int_{\mathcal{G}_{0}}^{\varsigma_{\ell}} \cdots \int_{\mathcal{G}_{0}}^{\vartheta_{n}} \int_{\mathcal{G}_{0}}^{\varsigma_{n}} \frac{\prod_{\ell=1}^{n} F_{\ell}^{h_{\ell}}(s_{\ell}, \mathcal{G}_{\ell})}{\left( \gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_{\ell}} (s_{\ell} - \mathcal{G}_{0})(\mathcal{G}_{\ell} - \mathcal{G}_{0}) \right)^{\frac{1}{q}} \Delta^{\alpha_{s_{1}}} \Delta^{\alpha_{s_{n}}} \Delta^{\alpha_{n}}} \\
\Delta^{\alpha_{s_{1}}} \Delta^{\alpha_{s_{n}}} \Delta^{\alpha_{n}} \Delta^{\alpha_{\mathcal{G}_{n}}} \\
\geq \prod_{\ell=1}^{n} h_{\ell} \left( \vartheta_{\ell} - \mathcal{G}_{0} \right) \left( \varsigma_{\ell} - \mathcal{G}_{0} \right) \left( \vartheta_{n} - \mathcal{G}_{0} \right) \left( \varsigma_{n} - \mathcal{G}_{0} \right) \right)^{\frac{1}{q}} \\
\times \left\{ \int_{\mathcal{G}_{0}}^{\vartheta_{\ell}} \int_{\mathcal{G}_{0}}^{\varsigma_{\ell}} \cdots \int_{\mathcal{G}_{0}}^{\vartheta_{n}} \int_{\mathcal{G}_{0}}^{\varsigma_{n}} \\
H(h_{\ell}, \xi_{\ell}, \tau_{\ell}) + F_{\ell}^{h_{\ell}}(s_{\ell}, \sigma(\tau_{\ell}))F_{\ell}(\xi_{\ell}, \tau_{\ell}) \right)^{\beta_{\ell}} \Delta^{\alpha_{\xi_{\ell}}} \Delta^{\alpha_{\tau_{\ell}}} \right\}^{\frac{1}{q_{\ell}}} \\
\Delta^{\alpha_{s_{1}}} \Delta^{\alpha_{s_{n}}} \Delta^{\alpha_{n}} \Delta^{\alpha_{\mathcal{G}_{n}}} \\
\Delta^{\alpha_{\mathcal{G}_{n}}} . 
\]
By using the fact \( \vartheta_\ell \geq \rho(\vartheta_\ell) \), and \( \zeta_\ell \geq \rho(\zeta_\ell) \), we get that
\[
\int_{\vartheta_0}^{\vartheta_1} \int_{\vartheta_0}^{\vartheta_2} \cdots \int_{\vartheta_0}^{\vartheta_n} \prod_{\ell=1}^{n} F_{\ell}(s_{\ell}, \vartheta_{\ell}) \Delta^a s_1 \Delta^a s_1 \cdots \Delta^a s_{\ell} \Delta^a \vartheta_1 \cdots \Delta^a \vartheta_n \left( \gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_{\ell}} (s_{\ell} - \vartheta_0) (\vartheta_{\ell} - \vartheta_0) \right)^{1/\gamma} 
\geq \prod_{\ell=1}^{n} h_{\ell} \left( (\vartheta_\ell - \vartheta_0)(\zeta_\ell - \vartheta_0) \right)^{1/\gamma_\ell} 
\times \left\{ \int_{\vartheta_0}^{\vartheta_1} \int_{\vartheta_0}^{\vartheta_2} \cdots \int_{\vartheta_0}^{\vartheta_n} (\rho(\vartheta_\ell) - s_{\ell})(\rho(\zeta_\ell) - \vartheta_{\ell}) \left( H(h_{\ell}, s_{\ell}, \vartheta_{\ell}) + F_{\ell}^{h_{\ell}-1}(s_{\ell}, \sigma(\vartheta_{\ell})) F_{\ell}(s_{\ell}, \vartheta_{\ell}) \right)^{1/\gamma_\ell} d s_{\ell} d \vartheta_{\ell} \right\}^{1/\gamma_\ell}.
\]
This completes the proof. \( \square \)

As a special case of Theorem 3.2, when \( T = \mathbb{R}, \alpha = 1 \) we have \( \rho(n) = n \), we get the following result.

**Corollary 3.2.** Let \( F_{\ell}(\xi_{\ell}, \tau_{\ell}) \in C^2[(0, \vartheta_0) \times (0, \zeta_0), (0, \infty)], \ell = 1, \ldots, n \), \( \vartheta_0, \zeta_0 \) are positive real numbers and define \( F(s_{\ell}, \vartheta_{\ell}) = \int_{\vartheta_0}^{s_{\ell}} \int_{0}^{\infty} F_{0}(s_{\ell}, \vartheta_{\ell}) d \xi_{\ell} d \tau_{\ell} \), for \( s_{\ell} \in (0, \vartheta_0) \), \( \vartheta_{\ell} \in (0, \zeta_0) \). Then
\[
\int_{\vartheta_0}^{\vartheta_1} \int_{\vartheta_0}^{\vartheta_2} \cdots \int_{\vartheta_0}^{\vartheta_n} \prod_{\ell=1}^{n} F_{\ell}^{h_{\ell}}(s_{\ell}, \vartheta_{\ell}) \Delta^a s_1 \Delta^a s_1 \cdots \Delta^a s_{\ell} \Delta^a \vartheta_1 \cdots \Delta^a \vartheta_n \left( \gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_{\ell}} (s_{\ell} - \vartheta_0) (\vartheta_{\ell} - \vartheta_0) \right)^{1/\gamma_\ell} 
\geq \prod_{\ell=1}^{n} h_{\ell} \left( \vartheta_0 \vartheta_{\ell} \right)^{1/\gamma_\ell} 
\times \left\{ \int_{\vartheta_0}^{\vartheta_1} \int_{\vartheta_0}^{\vartheta_2} \cdots \int_{\vartheta_0}^{\vartheta_n} (\vartheta_\ell - s_{\ell})(\zeta_\ell - \vartheta_{\ell}) \left( H(h_{\ell}, s_{\ell}, \vartheta_{\ell}) + F_{\ell}^{h_{\ell}-1}(s_{\ell}, \sigma(\vartheta_{\ell})) F_{\ell}(s_{\ell}, \vartheta_{\ell}) \right)^{1/\gamma_{\ell}} d s_{\ell} d \vartheta_{\ell} \right\}^{1/\gamma_{\ell}}
\]
where
\[
H(h_{\ell}, \xi_{\ell}, \tau_{\ell}) = (h_{\ell} - 1) F_{\ell}^{h_{\ell}-2}(\xi_{\ell}, \tau_{\ell}) \frac{\partial F_{\ell}(\xi_{\ell}, \tau_{\ell})}{\partial \xi_{\ell}} \frac{\partial F_{\ell}(\xi_{\ell}, \tau_{\ell})}{\partial \tau_{\ell}}.
\]

As a special case of Theorem 3.2, when \( T = \mathbb{Z}, \alpha = 1 \) we have \( \rho(n) = n - 1 \), we get the following result.

**Corollary 3.3.** Let \( \{ a_{s_{\ell}, \vartheta_{\ell}, m_{s_{\ell}}, m_{\vartheta_{\ell}}} \} (\ell = 1, \ldots, n) \) be \( n \) sequences of nonnegative numbers defined for \( m_{s_{\ell}} = 1, \ldots, k_{s_{\ell}} \), and \( m_{\vartheta_{\ell}} = 1, \ldots, k_{\vartheta_{\ell}} \), and define
\[
A_{s_{\ell}, \vartheta_{\ell}, m_{s_{\ell}}, m_{\vartheta_{\ell}}} = \sum_{m_{s_{\ell}}}^{m_{s_{\ell}}} \sum_{m_{\vartheta_{\ell}}}^{m_{\vartheta_{\ell}}} a_{s_{\ell}, \vartheta_{\ell}, m_{s_{\ell}}, m_{\vartheta_{\ell}}}.
\]
Then
\[
\sum_{m_{s_{1}}}^{k_{s_{1}}} \sum_{m_{s_{2}}}^{k_{s_{2}}} \cdots \sum_{m_{s_{n}}}^{k_{s_{n}}} \prod_{\ell=1}^{n} A_{s_{\ell}, \vartheta_{\ell}, m_{s_{\ell}}, m_{\vartheta_{\ell}}} \left( \gamma \sum_{\ell=1}^{n} \frac{1}{\gamma_{\ell}} (m_{s_{\ell}} - 1)(m_{\vartheta_{\ell}} - 1) \right)^{1/\gamma_{\ell}} 
\geq h_{0} \left( m_{s_{1}} m_{\vartheta_{1}} \right)^{1/\gamma_{1}} 
\times \prod_{\ell=1}^{n} \left( \sum_{m_{s_{\ell}}}^{k_{s_{\ell}}} \sum_{m_{\vartheta_{\ell}}}^{k_{\vartheta_{\ell}}} (k_{s_{\ell}} - (m_{s_{\ell}} - 1))(k_{\vartheta_{\ell}} - (m_{\vartheta_{\ell}} - 1)) \left( H_{s_{\ell}, \vartheta_{\ell}} + A_{s_{\ell}, \vartheta_{\ell}, m_{s_{\ell}}, m_{\vartheta_{\ell}}} \right)^{1/\gamma_{\ell}} d s_{\ell} d \vartheta_{\ell} \right)^{1/\gamma_{\ell}}
\]
where
\[
H_{s_{\ell}, \vartheta_{\ell}} = (h_{\ell} - 1) A_{s_{\ell}, \vartheta_{\ell}, m_{s_{\ell}}, m_{\vartheta_{\ell}}} \nabla_{1} A_{s_{\ell}, \vartheta_{\ell}, m_{s_{\ell}}, m_{\vartheta_{\ell}}} \nabla_{2} A_{s_{\ell}, \vartheta_{\ell}, m_{s_{\ell}}, m_{\vartheta_{\ell}}}.
\]
\[ \nabla_1 A_{\tau, m_1 m_2} = A_{\tau, m_1 m_2} - A_{\tau, m_1 - 1 m_2}, \]
\[ \nabla_2 A_{\tau, m_1 m_2} = A_{\tau, m_1 m_2} - A_{\tau, m_1 m_2 - 1}. \]

**Remark 3.9.** Let \( F_\ell(\xi_\ell, \tau_\ell) \) and \( F_\ell(s_\ell, \mathcal{G}_\ell) = \int_{\mathcal{G}_\ell} \int_{\mathcal{G}_\ell} F_\ell(\xi_\ell, \tau_\ell) \Delta^\alpha_s \xi_\ell \Delta^\alpha_s \tau_\ell \) change to \( F_\ell(\xi_\ell) \) and \( F_\ell(s_\ell) = \int_{\mathcal{G}_\ell} F(\xi_\ell) \Delta^\alpha_s \xi_\ell \), respectively and with suitable changes, we have

**Corollary 3.4.** Let \( S_{21}, S_{22}, S_{23} \) and \( S_{24} \) be satisfied. Then \( S_{25}, S_{18} \) and \( S_{20} \) we have that

\[ \int_{\mathcal{G}_0}^{\theta_1} \cdots \int_{\mathcal{G}_0}^{\theta_n} \frac{\prod_{\ell=1}^{n} F_{\ell}^{\alpha}(s_\ell)}{\left( \sum_{\ell=1}^{n} 1/\gamma \right)^{n}} \Delta^\alpha_s \partial_1 \cdots \Delta^\alpha_s \partial_{n} \]
\[ \geq \left[ \prod_{\ell=1}^{n} h_{\ell}(\partial_\ell - \mathcal{G}_0) \right] \frac{1}{n} \left( \int_{\mathcal{G}_0}^{\theta_1} F_{\ell}^{\alpha-1}(s_\ell) F_{\ell}(s_\ell) \right) \Delta^\alpha_s \Delta^\alpha_{s_\ell}. \] (3.19)

**Corollary 3.5.** In Corollary 3.4, if we take \( n = 2, \beta_\ell = \frac{1}{2} \) then the inequality (3.19) changes to

\[ \int_{0}^{\theta_1} \int_{0}^{\theta_2} \frac{F_{\ell}^{h_1}(s_1) F_{\ell}^{h_2}(s_2)}{(s_1 + s_2)^2} \Delta^\alpha_s \partial_1 \Delta^\alpha_s \partial_2 \geq 4h_1 h_2 (\partial_1 - \mathcal{G}_0)(\partial_2 - \mathcal{G}_0)^{-1} \]
\[ \times \left( \int_{0}^{\theta_1} (\partial_1 - s_1)(F_{\ell}^{h_1-1}(s_1) F_{\ell}(s_1)) \Delta^\alpha_s \partial_1 \right) \left( \int_{0}^{\theta_2} (\partial_2 - s_2)(F_{\ell}^{h_2-1}(s_2) F_{\ell}(s_2)) \Delta^\alpha_s \partial_2 \right) \frac{1}{2}. \] (3.20)

**Remark 3.10.** In Corollary 3.5, if we take \( \mathbb{T} = \mathbb{R} \), then the inequality (3.20) changes to

\[ \int_{0}^{\theta_1} \int_{0}^{\theta_2} \frac{F_{\ell}^{h_1}(s_1) F_{\ell}^{h_2}(s_2)}{(s_1 + s_2)^2} ds_1 ds_2 \geq 4h_1 h_2 (\partial_1 - \mathcal{G}_0)(\partial_2 - \mathcal{G}_0)^{-1} \left( \int_{0}^{\theta_1} (\partial_1 - s_1)(F_{\ell}^{h_1-1}(s_1) F_{\ell}(s_1)) ds_1 \right) \frac{1}{2} \]
\[ \times \left( \int_{0}^{\theta_2} (\partial_2 - s_2)(F_{\ell}^{h_2-1}(s_2) F_{\ell}(s_2)) ds_2 \right) \frac{1}{2}. \] (3.21)

This is an inverse of the inequality (1.9) which was proved by Pachapte [5].

**Corollary 3.6.** In Corollary 3.4, if we take \( \beta_\ell = \frac{n-1}{n} \), the inequality (3.19) becomes

\[ \int_{\mathcal{G}_0}^{\theta_1} \cdots \int_{\mathcal{G}_0}^{\theta_n} \frac{\prod_{\ell=1}^{n} F_{\ell}^{\alpha}(s_\ell)}{\left( \sum_{\ell=1}^{n} 1/\gamma \right)^{n}} \Delta^\alpha_s \partial_1 \cdots \Delta^\alpha_s \partial_{n} \]
\[ \geq n^{\frac{n}{n-1}} \prod_{\ell=1}^{n} h_{\ell}(\partial_\ell - \mathcal{G}_0) \left( \int_{\mathcal{G}_0}^{\theta_1} F_{\ell}^{\alpha-1}(s_\ell) F_{\ell}(s_\ell) \right) \Delta^\alpha_s \partial_1 \cdots \Delta^\alpha_s \partial_{n} \frac{1}{n}. \]

**Theorem 3.3.** Let \( S_1, S_5, S_{14}, S_6, S_{15} \) and \( S_8 \) be satisfied. Then for \( S_{10}, S_{18} \) and \( S_{20} \), we have that

\[ \int_{\mathcal{G}_0}^{\theta_1} \int_{\mathcal{G}_0}^{\theta_2} \cdots \int_{\mathcal{G}_0}^{\theta_n} \frac{\prod_{\ell=1}^{n} \Phi_\ell(s_\ell, \mathcal{G}_\ell)}{\left( \sum_{\ell=1}^{n} 1/\gamma \right)^{n}} \Delta^\alpha_s \partial_1 \cdots \Delta^\alpha_s \partial_{n} \]
\[ \geq L(\partial_1 \mathcal{G}_1, \ldots, \partial_n \mathcal{G}_n) \] (3.22)
Applying inverse Hölder’s inequality on the right hand side of (3.24) with indices \( \gamma_\ell \) and \( \beta_\ell \), we obtain that

\[
\Phi_\ell(F_\ell(s_\ell, \mathfrak{J}_\ell)) \geq \Phi_\ell(P_\ell(s_\ell, \mathfrak{J}_\ell)) \left( \int_{s_\ell}^{s_{\ell+1}} \left( \int_{s_\ell}^{s_{\ell+1}} p_\ell(\xi_\ell, \tau_\ell) \Phi_\ell \left( \frac{F_\ell(\xi_\ell, \tau_\ell)}{p_\ell(\xi_\ell, \tau_\ell)} \right) \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell \right)^{\frac{1}{\gamma_\ell}} \right)^{\frac{1}{\beta_\ell}}.
\] (3.25)

By using inequality (3.16), on the term \( |(s_\ell - \mathfrak{J}_0)(\mathfrak{J}_\ell - \mathfrak{J}_0)|^{\frac{1}{\gamma}} \), we get that

\[
\prod_{\ell=1}^{n} \Phi_\ell(F_\ell(s_\ell, \mathfrak{J}_\ell)) \geq \prod_{\ell=1}^{n} \Phi_\ell(P_\ell(s_\ell, \mathfrak{J}_\ell)) \left( \int_{s_\ell}^{s_{\ell+1}} \left( \int_{s_\ell}^{s_{\ell+1}} p_\ell(\xi_\ell, \tau_\ell) \Phi_\ell \left( \frac{F_\ell(\xi_\ell, \tau_\ell)}{p_\ell(\xi_\ell, \tau_\ell)} \right) \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell \right)^{\frac{1}{\gamma_\ell}} \right)^{\frac{1}{\beta_\ell}}. \] (3.26)

Integrating both sides of (3.26) over \( s_\ell, \mathfrak{J}_\ell \) from \( \mathfrak{J}_0 \) to \( \mathfrak{J}_\ell, \mathfrak{J}_\ell (\ell = 1, \ldots, n) \), we obtain that

\[
\int_{s_\ell}^{s_\ell} \cdots \int_{s_\ell}^{s_\ell} \prod_{\ell=1}^{n} \Phi_\ell(F_\ell(s_\ell, \mathfrak{J}_\ell)) \left( \int_{s_\ell}^{s_{\ell+1}} \left( \int_{s_\ell}^{s_{\ell+1}} p_\ell(\xi_\ell, \tau_\ell) \Phi_\ell \left( \frac{F_\ell(\xi_\ell, \tau_\ell)}{p_\ell(\xi_\ell, \tau_\ell)} \right) \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell \right)^{\frac{1}{\gamma_\ell}} \right)^{\frac{1}{\beta_\ell}} \Delta^\alpha s_1 \Delta^\alpha \mathfrak{J}_1 \ldots \Delta^\alpha s_n \Delta^\alpha \mathfrak{J}_n \geq \prod_{\ell=1}^{n} \Phi_\ell(P_\ell(s_\ell, \mathfrak{J}_\ell)) \left( \int_{s_\ell}^{s_{\ell+1}} \left( \int_{s_\ell}^{s_{\ell+1}} p_\ell(\xi_\ell, \tau_\ell) \Phi_\ell \left( \frac{F_\ell(\xi_\ell, \tau_\ell)}{p_\ell(\xi_\ell, \tau_\ell)} \right) \Delta^\alpha \xi_\ell \Delta^\alpha \tau_\ell \right)^{\frac{1}{\gamma_\ell}} \right)^{\frac{1}{\beta_\ell}} \Delta^\alpha s_1 \Delta^\alpha \mathfrak{J}_1. \] (3.27)
Applying inverse Hölder’s inequality on the right hand side of (3.27) with indices \(\gamma_\ell\) and \(\beta_\ell\), it is easy to observe that

\[
\int_{\mathcal{G}_0}^{\partial_1} \cdots \int_{\mathcal{G}_0}^{\partial_n} \prod_{\ell=1}^{\partial_\ell} \Phi_\ell(F_\ell(s_\ell, \mathcal{I}_\ell)) \left( \gamma \sum_{\ell=1}^{\partial_\ell} \frac{1}{\gamma_\ell}(s_\ell - \mathcal{I}_0)(\mathcal{I}_\ell - \mathcal{I}_0) \right)^{\frac{1}{2}} \Delta^\alpha s_1 \Delta^\alpha \mathcal{I}_1 \cdots \Delta^\alpha s_n \Delta^\alpha \mathcal{I}_n \\
\geq \prod_{\ell=1}^{n} \left( \int_{\mathcal{G}_0}^{\partial_\ell} \left( \Phi_\ell(P_\ell(s_\ell, \mathcal{I}_\ell)) \right)^{\gamma_\ell} \Delta^\alpha s_\ell \Delta^\alpha \mathcal{I}_\ell \right)^{\frac{1}{\gamma_\ell}} \\
\times \prod_{\ell=1}^{n} \left( \int_{\mathcal{G}_0}^{\partial_\ell} \int_{\mathcal{G}_0}^{\gamma_\ell} \int_{\mathcal{G}_0}^{\beta_\ell} \int_{\mathcal{G}_0}^{\mathcal{I}_\ell} \left( p_\ell(s_\ell, \mathcal{I}_\ell) \Phi_\ell \left( \frac{F_\ell(s_\ell, \mathcal{I}_\ell)\ell}{p_\ell(s_\ell, \mathcal{I}_\ell)} \right) \Delta^\alpha s_\ell \Delta^\alpha \mathcal{I}_\ell \right)^{\frac{1}{\beta_\ell}} \right)^{\frac{1}{\gamma_\ell}}.
\]

Using Fubini’s theorem, we observe that

\[
\int_{\mathcal{G}_0}^{\partial_1} \cdots \int_{\mathcal{G}_0}^{\partial_n} \prod_{\ell=1}^{\partial_\ell} \Phi_\ell(F_\ell(s_\ell, \mathcal{I}_\ell)) \left( \gamma \sum_{\ell=1}^{\partial_\ell} \frac{1}{\gamma_\ell}(s_\ell - \mathcal{I}_0)(\mathcal{I}_\ell - \mathcal{I}_0) \right)^{\frac{1}{2}} \Delta^\alpha s_1 \Delta^\alpha \mathcal{I}_1 \cdots \Delta^\alpha s_n \Delta^\alpha \mathcal{I}_n \\
\geq L(\partial_1 s_1, \ldots, \partial_n s_n) \\
\times \prod_{\ell=1}^{n} \left( \int_{\mathcal{G}_0}^{\partial_\ell} (\partial_\ell - s_\ell)(s_\ell - \mathcal{I}_\ell) \left( p_\ell(s_\ell, \mathcal{I}_\ell) \Phi_\ell \left( \frac{F_\ell(s_\ell, \mathcal{I}_\ell)\ell}{p_\ell(s_\ell, \mathcal{I}_\ell)} \right) \Delta^\alpha s_\ell \Delta^\alpha \mathcal{I}_\ell \right)^{\frac{1}{\beta_\ell}} \right)^{\frac{1}{\gamma_\ell}}.
\]

By using the fact \(\partial_\ell \geq \rho(\partial_\ell)\), and \(s_\ell \geq \rho(s_\ell)\), we get that

\[
\int_{\mathcal{G}_0}^{\partial_1} \cdots \int_{\mathcal{G}_0}^{\partial_n} \prod_{\ell=1}^{\partial_\ell} \Phi_\ell(F_\ell(s_\ell, \mathcal{I}_\ell)) \left( \gamma \sum_{\ell=1}^{\partial_\ell} \frac{1}{\gamma_\ell}(s_\ell - \mathcal{I}_0)(\mathcal{I}_\ell - \mathcal{I}_0) \right)^{\frac{1}{2}} \Delta^\alpha s_1 \Delta^\alpha \mathcal{I}_1 \cdots \Delta^\alpha s_n \Delta^\alpha \mathcal{I}_n \\
\geq L(\partial_1 s_1, \ldots, \partial_n s_n) \\
\times \prod_{\ell=1}^{n} \left( \int_{\mathcal{G}_0}^{\partial_\ell} (\rho(\partial_\ell) - s_\ell)(s_\ell - \mathcal{I}_\ell) \left( p_\ell(s_\ell, \mathcal{I}_\ell) \Phi_\ell \left( \frac{F_\ell(s_\ell, \mathcal{I}_\ell)\ell}{p_\ell(s_\ell, \mathcal{I}_\ell)} \right) \Delta^\alpha s_\ell \Delta^\alpha \mathcal{I}_\ell \right)^{\frac{1}{\beta_\ell}} \right)^{\frac{1}{\gamma_\ell}}.
\]

This completes the proof. \(\square\)

**Remark 3.11.** In Theorem 3.3, if \(\mathbb{T} = \mathbb{R}, \alpha = 1\) we get the result due to Zhao et al. [39, Theorem 2].

As a special case of Theorem 3.3, when \(\mathbb{T} = \mathbb{Z}, \alpha = 1\) we have \(\rho(n) = n - 1\), we get the following result.

**Corollary 3.6.** Let \(\{a_{s_\ell, \mathcal{I}_\ell, m_{s_\ell}, m_{\mathcal{I}_\ell}}\}\) and \(\{p_{s_\ell, \mathcal{I}_\ell, m_{s_\ell}, m_{\mathcal{I}_\ell}}\}\), \(\ell = 1, \ldots, n\) be \(n\) sequences of nonnegative numbers defined for \(m_{s_\ell} = 1, \ldots, k_{s_\ell}\), and \(m_{\mathcal{I}_\ell} = 1, \ldots, k_{\mathcal{I}_\ell}\), and define

\[
A_{s_\ell, \mathcal{I}_\ell, m_{s_\ell}, m_{\mathcal{I}_\ell}} = \sum_{m_{s_\ell}} \sum_{m_{\mathcal{I}_\ell}} a_{s_\ell, \mathcal{I}_\ell, m_{s_\ell}, m_{\mathcal{I}_\ell}} \\
P_{s_\ell, \mathcal{I}_\ell, m_{s_\ell}, m_{\mathcal{I}_\ell}} = \sum_{m_{s_\ell}} \sum_{m_{\mathcal{I}_\ell}} p_{s_\ell, \mathcal{I}_\ell, m_{s_\ell}, m_{\mathcal{I}_\ell}}.
\]

(3.29)
Then

$$
\sum_{m_{t_1}}^{k_1} \sum_{m_{t_2}}^{k_2} \ldots \sum_{m_{t_n}}^{k_n} \prod_{\ell=1}^{n} \Phi_\ell(A_{s_{\ell},3_{\ell},m_{s_{\ell}},m_{g_{\ell}}})
\cong C(k_1,k_2,\ldots,k_n,k_{\ell_1})
\times \prod_{\ell=1}^{n} \left( \sum_{m_{s_{\ell}}=1}^{k_{s_{\ell}}} (k_{s_{\ell}} - (m_{s_{\ell}} - 1))(k_{g_{\ell}} - (m_{g_{\ell}} - 1)) \left( P_{s_{\ell},3_{\ell},m_{s_{\ell}},m_{g_{\ell}}} \Phi_\ell \left( \frac{a_{s_{\ell},3_{\ell},m_{s_{\ell}},m_{g_{\ell}}}}{P_{s_{\ell},3_{\ell},m_{s_{\ell}},m_{g_{\ell}}}} \right) \right)^{\frac{1}{\gamma_{s_{\ell}}}} \right)
$$

where

$$
C(k_1,k_2,\ldots,k_n,k_{\ell_1}) = \prod_{\ell=1}^{n} \left( \sum_{m_{s_{\ell}}=1}^{k_{s_{\ell}}} \left( \Phi_\ell(P_{s_{\ell},3_{\ell},m_{s_{\ell}},m_{g_{\ell}}}) \right)^{\frac{1}{\gamma_{s_{\ell}}}} \right).
$$

**Remark 3.12.** Let $F_\ell(\xi_\ell,\tau_\ell)$, $p_\ell(\xi_\ell,\tau_\ell)$, $P_\ell(\xi_\ell,\tau_\ell)$, and $F_\ell(\xi_\ell,\tau_\ell)$ change to $F_\ell(\xi_\ell)$, $p_\ell(\xi_\ell)$, $P_\ell(s_\ell)$ and $F_\ell(s_\ell)$, respectively and with suitable changes, we have

**Corollary 3.7.** Let $S_{18}$, $S_{20}$, and $S_{25}$ be satisfied. Then for $S_{18}$, $S_{20}$, and $S_{25}$ we have that

$$
\int_{\mathfrak{S}_0}^{\theta_1} \ldots \int_{\mathfrak{S}_0}^{\theta_n} \prod_{\ell=1}^{n} \frac{\Phi_\ell(F_\ell(s_\ell))}{\left( \gamma \sum_{\ell=1}^{n} \int_{\mathfrak{S}_0}^{\theta_\ell} (s_\ell - \mathfrak{S}_0) \right)^{\frac{1}{\gamma}}} \Delta^\alpha s_1 \ldots \Delta^\alpha s_n
$$

(3.30)

where

$$
L^*(\theta_1,\ldots,\theta_n) = \prod_{\ell=1}^{n} \left( \int_{\mathfrak{S}_0}^{\theta_\ell} \left( \Phi_\ell(P_\ell(s_\ell)) \right)^{\frac{1}{\gamma}} \Delta^\alpha s_\ell \right)^{\frac{1}{\gamma}}.
$$

**Corollary 3.8.** In Corollary 3.7, if we take $n = 2$, $\beta_\ell = \frac{1}{2}$ then the inequality (3.30) changes to

$$
\int_{\mathfrak{S}_0}^{\theta_1} \int_{\mathfrak{S}_0}^{\theta_1} \frac{\Phi_1(F_1(s_1))\Phi_2(F_2(s_2))}{(s_1 - \mathfrak{S}_0) + (s_2 - \mathfrak{S}_0)} \Delta^\alpha s_1 \Delta^\alpha s_2 \cong L^*(\theta_1,\theta_2) \left( \int_{\mathfrak{S}_0}^{\theta_1} (\rho(\theta_1) - s_1) \left( p_1(s_1) \Phi_1(F_1(s_1)) \right)^2 \Delta^\alpha s_1 \right)^{\frac{1}{2}}
$$

$$
\times \left( \int_{\mathfrak{S}_0}^{\theta_2} (\rho(\theta_2) - s_2) \left( p_2(s_2) \Psi(F_2(s_2)) \right)^2 \Delta^\alpha s_2 \right)^{\frac{1}{2}}
$$

(3.31)

where

$$
L^*(\theta_1,\theta_2) = 4 \left( \int_{\mathfrak{S}_0}^{\theta_1} \left( \Phi_1(P_1(s_1)) \right)^{-1} \Delta^\alpha s_1 \right)^{-1} \left( \int_{\mathfrak{S}_0}^{\theta_2} \left( \Phi_2(P_2(s_2)) \right)^{-1} \Delta^\alpha s_2 \right)^{-1}
$$

**Remark 3.13.** In Corollary 3.8, if we take $\mathbb{T} = \mathbb{R}$, then the inequality (3.31) changes to

$$
\int_{0}^{\theta_1} \int_{0}^{\theta_1} \frac{\Phi_1(F_1(s_1))\Phi_2(F_2(s_2))}{(s_1 + s_2)^{-2}} ds_1 ds_2 \cong L^*(\theta_1,\theta_2) \left( \int_{0}^{\theta_1} (\theta_1 - s_1) \left( p_1(s_1) \Phi_1(F_1(s_1)) \right)^2 ds_1 \right)^{\frac{1}{2}}
$$

$$
\times \left( \int_{0}^{\theta_2} (\theta_2 - s_2) \left( p_2(s_2) \Psi(F_2(s_2)) \right)^2 ds_2 \right)^{\frac{1}{2}}
$$

(3.32)
where

\[ L^{**}(\vartheta_1, \vartheta_2) = 4 \left( \int_0^{\vartheta_1} \left( \frac{\Phi_1(P_1(s_1))}{P_1(s_1)} \right)^{-1} ds_1 \right)^{-1} \left( \int_0^{\vartheta_2} \left( \frac{\Phi_2(P_2(s_2))}{P_2(s_2)} \right)^{-1} ds_2 \right)^{-1}. \]

This is an inverse of the inequality (1.10) which was proved by Pachapte [5].

**Corollary 3.9.** In Corollary 3.7, if we take \( \beta_\ell = \frac{\alpha_1}{n} \) the inequality (3.30) becomes

\[
\int_{\mathcal{G}_0} \cdots \int_{\mathcal{G}_0} \prod_{\ell=1}^{n} \Phi_\ell(F_\ell(s_\ell)) \left( \sum_{\ell=1}^{n} (s_\ell - \mathcal{G}_0) \right) ^{-\frac{\alpha_1}{\beta_\ell}} \Delta^s_{s_1} \Delta^s_{s_2} \cdots \Delta^s_{s_n} \\
\geq L'(\vartheta_1, \ldots, \vartheta_n) \prod_{\ell=1}^{n} \left( \int_{\mathcal{G}_0} \left( \frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)} \right)^{-\left(\frac{n-1}{\beta_\ell} \right)} \Delta^s_{s_\ell} \right)^{\frac{1}{\beta_\ell}}
\]

where

\[ L'(\vartheta_1, \ldots, \vartheta_n) = n^{\frac{\alpha_1}{\beta_\ell}} \left( \int_{\mathcal{G}_0} \left( \frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)} \right)^{-\left(\frac{n-1}{\beta_\ell} \right)} \Delta^s_{s_\ell} \right)^{\frac{1}{\beta_\ell}}. \]

**Theorem 3.4.** Let \( S_1, S_s, S_6, S_9, S_{15} \), and \( S_{16} \) be satisfied. Then for \( S_{10}, S_{18} \) and \( S_{20} \) we have that

\[
\int_{\mathcal{G}_0} \int_{\mathcal{G}_0} \cdots \int_{\mathcal{G}_0} \prod_{\ell=1}^{n} P_\ell(s_\ell, \mathcal{G}_0)^{\frac{1}{\gamma}} \Delta^s_{s_1} \Delta^s_{s_2} \cdots \Delta^s_{s_n} \\
\geq \prod_{\ell=1}^{n} \left( \int_{\mathcal{G}_0} \left( \frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)} \right)^{-\left(\frac{n-1}{\beta_\ell} \right)} \Delta^s_{s_\ell} \right)^{\frac{1}{\beta_\ell}}
\]

**Proof.** From the hypotheses of Theorem 3.4, and by using inverse Jensen dynamic inequality, we have

\[ \Phi_\ell(F_\ell(s_\ell, \mathcal{G}_\ell)) = \Phi_\ell \left( \frac{1}{P_\ell(s_\ell, \mathcal{G}_\ell)} \int_{\mathcal{G}_0} \int_{\mathcal{G}_0} \int_{\mathcal{G}_0} p_\ell(\xi_\ell, \tau_\ell) F_\ell(\xi_\ell, \tau_\ell) \Delta^\sigma_{\xi_\ell} \Delta^\sigma_{\tau_\ell} \right) \]

\[
\geq \frac{1}{P_\ell(s_\ell, \mathcal{G}_\ell)} \int_{\mathcal{G}_0} \int_{\mathcal{G}_0} \int_{\mathcal{G}_0} p_\ell(\sigma_\ell, \tau_\ell) \Phi_\ell(F_\ell(\xi_\ell, \tau_\ell)) \Delta^\sigma_{\xi_\ell} \Delta^\sigma_{\tau_\ell}. \]

Applying inverse Hölder’s inequality on the right hand side of (3.34) with indices \( \gamma_\ell \) and \( \beta_\ell \), it is easy to observe that

\[ \Phi_\ell(F_\ell(s_\ell, \mathcal{G}_\ell)) \geq \frac{1}{P_\ell(s_\ell, \mathcal{G}_\ell)} \left( \int_{\mathcal{G}_0} \int_{\mathcal{G}_0} (p_\ell(\xi_\ell, \tau_\ell) \Phi_\ell(F_\ell(\xi_\ell, \tau_\ell)))^{\beta_\ell} \Delta^\sigma_{\xi_\ell} \Delta^\sigma_{\tau_\ell} \right)^{\frac{1}{\beta_\ell}}. \]

By using the inequality (3.16), on the term \( \left( \int_{\mathcal{G}_0} \int_{\mathcal{G}_0} (p_\ell(\xi_\ell, \tau_\ell) \Phi_\ell(F_\ell(\xi_\ell, \tau_\ell)))^{\beta_\ell} \Delta^\sigma_{\xi_\ell} \Delta^\sigma_{\tau_\ell} \right)^{\frac{1}{\beta_\ell}} \) we get that

\[
\frac{P_\ell(s_\ell, \mathcal{G}_\ell) \Phi_\ell(F_\ell(s_\ell, \mathcal{G}_\ell))}{\left( \sum_{\ell=1}^{n} \frac{1}{\gamma_\ell} (s_\ell - \mathcal{G}_0)(\mathcal{G}_\ell - \mathcal{G}_0) \right)^{\frac{1}{\gamma_\ell}} \left( \left( \int_{\mathcal{G}_0} \int_{\mathcal{G}_0} (p_\ell(\xi_\ell, \tau_\ell) \Phi_\ell(F_\ell(\xi_\ell, \tau_\ell)))^{\beta_\ell} \Delta^\sigma_{\xi_\ell} \Delta^\sigma_{\tau_\ell} \right)^{\frac{1}{\beta_\ell}}. \]
Integrating both sides of (3.35) over \( s_\ell \), \( \mathcal{I}_\ell \) from \( \mathcal{I}_0 \) to \( \vartheta_\ell \), \( \zeta_\ell \) (\( \ell = 1, \ldots, n \)), we get that

\[
\int_{\mathcal{I}_0}^{\vartheta_1} \int_{\mathcal{I}_0}^{\zeta_1} \cdots \int_{\mathcal{I}_0}^{\zeta_n} \left( \prod_{\ell=1}^{n} P_\ell(s_\ell, \mathcal{I}_\ell) \Phi_\ell(F_\ell(s_\ell, \mathcal{I}_\ell)) \right) \Delta^\alpha s_1 \Delta^\alpha \mathcal{I}_1 \cdots \Delta^\alpha s_n \Delta^\alpha \mathcal{I}_n \left( \sum_{\ell=1}^{n} \frac{1}{\gamma_\ell} (s_\ell - \mathcal{I}_0)(\mathcal{I}_\ell - \mathcal{I}_0) \right)^{\gamma_\ell}.
\]

Applying inverse Hölder’s inequality on the right hand side of (3.36) with indices \( \gamma_\ell \) and \( \beta_\ell \), it is easy to observe that

\[
\int_{\mathcal{I}_0}^{\vartheta_1} \int_{\mathcal{I}_0}^{\zeta_1} \cdots \int_{\mathcal{I}_0}^{\zeta_n} \left( \prod_{\ell=1}^{n} P_\ell(s_\ell, \mathcal{I}_\ell) \Phi_\ell(F_\ell(s_\ell, \mathcal{I}_\ell)) \right) \Delta^\alpha s_1 \Delta^\alpha \mathcal{I}_1 \cdots \Delta^\alpha s_n \Delta^\alpha \mathcal{I}_n \left( \sum_{\ell=1}^{n} \frac{1}{\gamma_\ell} (s_\ell - \mathcal{I}_0)(\mathcal{I}_\ell - \mathcal{I}_0) \right)^{\gamma_\ell} \geq \prod_{\ell=1}^{n} \left( \vartheta_\ell - \mathcal{I}_0 \right) \left( \zeta_\ell - \mathcal{I}_0 \right)^{\frac{1}{\gamma_\ell}} \left( \int_{\mathcal{I}_0}^{\vartheta_\ell} \int_{\mathcal{I}_0}^{\zeta_\ell} \left( \int_{\mathcal{I}_0}^{\vartheta_\ell} \int_{\mathcal{I}_0}^{\zeta_\ell} \left( p_\ell(\xi_\ell, \tau_\ell) \Phi_\ell(F_\ell(\xi_\ell, \tau_\ell)) \right) \Delta^\alpha s_\ell \Delta^\alpha \mathcal{I}_\ell \right) \right)^{\frac{1}{\beta_\ell}}.
\]

By using Fubini’s theorem, we observe that

\[
\int_{\mathcal{I}_0}^{\vartheta_1} \int_{\mathcal{I}_0}^{\zeta_1} \cdots \int_{\mathcal{I}_0}^{\zeta_n} \left( \prod_{\ell=1}^{n} P_\ell(s_\ell, \mathcal{I}_\ell) \Phi_\ell(F_\ell(s_\ell, \mathcal{I}_\ell)) \right) \Delta^\alpha s_1 \Delta^\alpha \mathcal{I}_1 \cdots \Delta^\alpha s_n \Delta^\alpha \mathcal{I}_n \left( \sum_{\ell=1}^{n} \frac{1}{\gamma_\ell} (s_\ell - \mathcal{I}_0)(\mathcal{I}_\ell - \mathcal{I}_0) \right)^{\gamma_\ell} \geq \prod_{\ell=1}^{n} \left( \vartheta_\ell - \mathcal{I}_0 \right) \left( \zeta_\ell - \mathcal{I}_0 \right)^{\frac{1}{\gamma_\ell}} \left( \int_{\mathcal{I}_0}^{\vartheta_\ell} \int_{\mathcal{I}_0}^{\zeta_\ell} \left( \int_{\mathcal{I}_0}^{\vartheta_\ell} \int_{\mathcal{I}_0}^{\zeta_\ell} \left( p_\ell(\xi_\ell, \tau_\ell) \Phi_\ell(F_\ell(\xi_\ell, \tau_\ell)) \right) \Delta^\alpha s_\ell \Delta^\alpha \mathcal{I}_\ell \right) \right)^{\frac{1}{\beta_\ell}}.
\]

By using the fact \( \vartheta_\ell \geq \rho(\vartheta_\ell) \), and \( \zeta_\ell \geq \rho(\zeta_\ell) \), we get that

\[
\int_{\mathcal{I}_0}^{\vartheta_1} \int_{\mathcal{I}_0}^{\zeta_1} \cdots \int_{\mathcal{I}_0}^{\zeta_n} \left( \prod_{\ell=1}^{n} P_\ell(s_\ell, \mathcal{I}_\ell) \Phi_\ell(F_\ell(s_\ell, \mathcal{I}_\ell)) \right) \Delta^\alpha s_1 \Delta^\alpha \mathcal{I}_1 \cdots \Delta^\alpha s_n \Delta^\alpha \mathcal{I}_n \left( \sum_{\ell=1}^{n} \frac{1}{\gamma_\ell} (s_\ell - \mathcal{I}_0)(\mathcal{I}_\ell - \mathcal{I}_0) \right)^{\gamma_\ell} \geq \prod_{\ell=1}^{n} \left( \vartheta_\ell - \mathcal{I}_0 \right) \left( \zeta_\ell - \mathcal{I}_0 \right)^{\frac{1}{\gamma_\ell}} \left( \int_{\mathcal{I}_0}^{\vartheta_\ell} \int_{\mathcal{I}_0}^{\zeta_\ell} \left( \varrho(\vartheta_\ell) - s_\ell \right) \left( \varrho(\zeta_\ell) - \mathcal{I}_\ell \right) \left( p_\ell(\xi_\ell, \tau_\ell) \Phi_\ell(F_\ell(\xi_\ell, \tau_\ell)) \right) \Delta^\alpha s_\ell \Delta^\alpha \mathcal{I}_\ell \right)^{\frac{1}{\beta_\ell}}.
\]

This completes the proof. \( \square \)

**Remark 3.14.** In Theorem 3.4, if \( \mathbb{T} = \mathbb{R} \), \( \alpha = 1 \) we get the result due to Zhao et al. [39, Theorem 3].

As a special case of Theorem 3.4, when \( \mathbb{T} = \mathbb{Z} \), \( \alpha = 1 \) we have \( \rho(n) = n - 1 \), we get the following result.

**Corollary 3.10.** Let \( \{a_{s_\ell, m_\ell, m_{s_\ell}}\} \) and \( \{p_{s_\ell, m_\ell, m_{s_\ell}}\} \), \( (\ell = 1, \ldots, n) \) be \( n \) sequences of nonnegative numbers defined for \( m_{s_\ell} = 1, \ldots, k_{s_\ell} \), and \( m_{s_\ell} = 1, \ldots, k_{s_\ell} \), and define

\[
A_{s_\ell, m_\ell, m_{s_\ell}} = \frac{1}{p_{s_\ell, m_\ell, m_{s_\ell}}} \sum_{m_{s_\ell}}^{m_{s_\ell}} a_{s_\ell, m_\ell, m_{s_\ell}} p_{s_\ell, m_\ell, m_{s_\ell}}.
\]
\[ P_{s,t,m_{x_1},m_{x_2}} = \sum_{m_{x_1}} \sum_{m_{x_2}} p_{s,t,m_{x_1},m_{x_2}}. \] (3.37)

Then
\[
\sum_{m_{x_1}} \sum_{m_{x_2}} \cdots \sum_{m_{x_n}} \prod_{\ell=1}^n P_{s,t,m_{x_1},m_{x_2}} A_{s,\ell,m_{x_1},m_{x_2}} \Phi_1(A_{s,\ell,m_{x_1},m_{x_2}}) \left( \gamma \sum_{\ell=1}^n \frac{1}{\gamma} (m_{x_1}m_{x_2}) \right) \] \[ \geq \prod_{\ell=1}^n (k_{x_1}k_{x_2})^{\frac{1}{n}} \left( \sum_{\ell=1}^n (k_{x_1} - (m_{x_1} - 1))(k_{x_2} - (m_{x_2} - 1)) \left( p_{s,t,m_{x_1},m_{x_2}} A_{s,\ell,m_{x_1},m_{x_2}} \right) \right)^{\frac{1}{\gamma}}.
\]

**Remark 3.15.** Let \( F_\ell(\xi, \tau_\ell), p_\ell(\xi, \tau_\ell), P_\ell(\xi, \tau_\ell) \) and
\[
F_\ell(s, \mathcal{G}_\ell) = \frac{1}{P_\ell(s, \mathcal{G}_\ell)} \int_0^{s_1} \int_0^{s_2} p_\ell(\xi_1, \tau_\ell) F_\ell(\xi, \tau_\ell) \Delta^a \xi_1 \Delta^a \tau_\ell
\]
changes to \( F_\ell(\xi, \tau_\ell), p_\ell(\xi, \tau_\ell), P_\ell(s_\ell) \), and
\[
F_\ell(s_\ell) = \frac{1}{P_\ell(s_\ell)} \int_0^{s_1} p_\ell(\xi, \tau_\ell) F_\ell(\xi) \Delta^a \xi_1
\]
respectively and with suitable changes, we have

**Corollary 3.11.** Let \( S_{22}, S_{23}, S_{26}, S_{27}, S_{28} \) be satisfied. Then for \( S_{18}, S_{20} \) and \( S_{25} \), we have that
\[
\int_{\mathcal{G}_0} \cdots \int_{\mathcal{G}_0} \prod_{\ell=1}^n P_\ell(s_\ell) A_{\ell,1}\cdots A_{\ell,n} \left( \gamma \sum_{\ell=1}^n \frac{1}{\gamma} (s_1 - \mathcal{G}_0) \right) \] \[ \geq \prod_{\ell=1}^n (\theta_\ell - \mathcal{G}_0)^{\frac{1}{n}} \left( \int_{\mathcal{G}_0} \left( p_\ell(s_\ell) A_{\ell,1}\cdots A_{\ell,n} \right) \right)^{\frac{1}{\gamma}}.
\] (3.38)

**Corollary 3.12.** In Corollary 3.11, if we take \( n = 2, \beta_\ell = \frac{1}{2} \) then the inequality (3.30) changes to
\[
\int_{\mathcal{G}_0} \int_{\mathcal{G}_0} \frac{P_1(s_1)P_2(s_2)A_{1,1}\cdots A_{1,n}}{(s_1 - \mathcal{G}_0) + (s_2 - \mathcal{G}_0)} \Delta^a s_1 \Delta^a s_2 \geq 4[(\theta_1 - \mathcal{G}_0)(\theta_1 - \mathcal{G}_0)]^{-1} \] \[ \times \left( \int_{\mathcal{G}_0} \left( p_1(s_1) A_{1,1}\cdots A_{1,n} \right) \right)^{\frac{1}{2}} \left( \int_{\mathcal{G}_0} \right)^{\frac{1}{2}}.
\] (3.39)

**Remark 3.16.** In Corollary 3.12, if we take \( \mathbb{T} = \mathbb{R} \), then the inequality (3.39) changes to
\[
\int_{\mathcal{G}_0} \int_{\mathcal{G}_0} \frac{P_1(s_1)P_2(s_2)A_{1,1}\cdots A_{1,n}}{(s_1 + s_2)^2} ds_1 ds_2 \geq 4[\theta_1 \theta_1]^{-1} \] \[ \times \left( \int_{\mathcal{G}_0} \right)^{\frac{1}{2}} \left( \int_{\mathcal{G}_0} \right)^{\frac{1}{2}}.
\] (3.40)
Remark 3.17. In Corollary 3.13, if we take $p_1(s_1) = p_2(s_2) = 1$, then $P_1(s_1) = s_1$, $P_2(s_2) = s_2$. Therefore the inequality (3.39) change to

$$\int_{\mathcal{S}_0} \int_{\mathcal{S}_0} \Phi_1(F_1(s_1)) \Phi_2(F_2(s_2)) (s_1, s_2)^{-2} \Delta^\alpha s_1 \Delta^\alpha s_2 \geq 4[(\vartheta_1 - \mathfrak{S}_0) (\vartheta_1 - \mathfrak{S}_0)]^{-1} \left( \int_{\mathcal{S}_0} (\varrho(\vartheta_1) - s_1) \Phi_1(F_1(s_1))^2 ds_1 \right)^{\frac{1}{2}} \left( \int_{\mathcal{S}_0} (\varrho(\vartheta_2) - s_2) \Phi_2(F_2(s_2))^2 ds_2 \right)^{\frac{1}{2}}.$$  

(3.41)

**Corollary 3.13.** In Corollary 3.12, let $p_1(s_1) = p_2(s_2) = 1$, then $P_1(s_1) = s_1$, $P_2(s_2) = s_2$. Therefore the inequality (3.39) change to

$$\int_{\mathcal{S}_0} \int_{\mathcal{S}_0} \Phi_1(F_1(s_1)) \Phi_2(F_2(s_2)) (s_1, s_2)^{-2} \Delta^\alpha s_1 \Delta^\alpha s_2 \geq 4[(\vartheta_1 - \mathfrak{S}_0) (\vartheta_1 - \mathfrak{S}_0)]^{-1} \left( \int_{\mathcal{S}_0} (\varrho(\vartheta_1) - s_1) \Phi_1(F_1(s_1))^2 ds_1 \right)^{\frac{1}{2}} \left( \int_{\mathcal{S}_0} (\varrho(\vartheta_2) - s_2) \Phi_2(F_2(s_2))^2 ds_2 \right)^{\frac{1}{2}}.$$  

This is an inverse of the inequality (1.11) which was proved by Pachaptte [5].

In this manuscript, by employing the conformable fractional Hölder inequalities and conformable fractional Jensen’s inequalities on time scales, several generalizations of the conformable fractional Hardy-Hilbert inequality on time scales are introduced. Beside that, we also apply our inequalities to discrete and continuous calculus to obtain some new inequalities as special cases.

**4. Conclusions**

In this manuscript, by employing the conformable fractional Hölder inequalities and conformable fractional Jensen’s inequalities on time scales, several generalizations of the conformable fractional Hardy-Hilbert inequality on time scales are introduced. Beside that, we also apply our inequalities to discrete and continuous calculus to obtain some new inequalities as special cases.

**Conflict of interest**

The authors declare that there is no competing interest.

**References**


