Research article

Z₃-connectivity of graphs with independence number at most 3

Chunyan Qin¹, Xiaoxia Zhang² and Liangchen Li¹,*

¹ Department of Mathematics, Luoyang Normal University, Luoyang, China
² Department of Mathematics, Xinyang Normal University, Xinyang, China

* Correspondence: Email: liangchen_li@163.com.

Abstract: It was conjectured by Jaeger et al. that all 5-edge-connected graphs are Z₃-connected. In this paper, we confirm this conjecture for all 5-edge-connected graphs with independence number at most 3.

Keywords: nowhere-zero flows; Z₃-connectivity; independence number

Mathematics Subject Classification: 05C21

1. Introduction

Graphs considered in this paper are finite, loopless and multiple edges are allowed. Terminologies and notations not defined here can be found in [1].

Let Γ be a graph. For vertex subsets U,W ⊆ V(Γ), denote by eΓ(U,W) the number of edges with one end in U and the other in W. For convenience, we write eΓ(U) and eΓ(Γ \ U) respectively. For a graph Γ, let α(Γ) denote the independence number of Γ.

Let D be an orientation of Γ. Let d = xy be an edge in Γ directed from x to y. Then we call x and y the tail and head of d, respectively. For a vertex x ∈ V(Γ), denote E°(x) = {e| x is tail of e} and E°H(x) = {e|x is head of e}.

Let Zₖ be the cyclic group with order k and Zₖ⁺ = Zₖ \ {0}. Denote M(Γ, Zₖ) = {g|g : E(Γ) → Zₖ} and M*(Γ, Zₖ) = {g|g : E(Γ) → Zₖ⁺}. Given a mapping g ∈ M(Γ, Zₖ), for each vertex x ∈ V(Γ), define

$$\partial g(x) = \sum_{e \in E^°(x)} g(e) - \sum_{e \in E^°H(x)} g(e).$$

The value \(\partial g(x)\) is said to be the outflow at x of g.

Suppose that Γ is a graph and β is a mapping from V(Γ) to Zₖ. If \(\sum_{x \in V(Γ)} β(x) = 0\), then β is said to be a zero-sum mapping. Set \(O(Γ, Zₖ) = \{β|β \text{ is zero-sum}\}\). Given a mapping β of \(O(Γ, Zₖ)\), a mapping g ∈ M*(Γ, Zₖ) is called nowhere-zero (Zₖ, β)-flow if \(\partial g = β\) under some orientation of Γ. When β = 0,
Lemma 2.4. \[12\]

The resulting graph is denoted by \( Z \).

Lemma 2.3. \[5\]

Lemma 2.2. \[2\]

Lemma 2.1. \[2\]

main theorem.

Theorem 1.2. Let \( \Gamma \) be a 5-edge-connected graph with independence number at most 3. Then \( \Gamma \) is \( Z_3 \)-connected.

2. Preliminaries

In this section, we will introduce some lemmas and theorems that will be needed in the proof of our main theorem.

Lemma 2.1. \[2\] Let \( k \) and \( n \) be positive integers. Then we have the following:

(1) if \( n \geq 5 \), then \( K_n \) and \( K_n^\prime \) are \( Z_3 \)-connected.

(2) \( C_n \) is \( Z_k \)-connected if and only if \( k > n \).

(3) \( W_{2k} \) is \( Z_3 \)-connected and \( W_{2k+1} \) is not \( Z_3 \)-connected.

Lemma 2.2. \[5\] Suppose that \( \Gamma \) have a subgraph \( K \) and \( x \) is a vertex in \( V(\Gamma) \setminus V(K) \) with \( e_1(x, V(K)) \geq 2 \). If \( K \) is \( Z_3 \)-connected, then the subgraph induced by \( V(K) \cup \{x\} \) is \( Z_3 \)-connected.

Lemma 2.3. \[2, 3\] Suppose that \( K \) is a subgraph of \( \Gamma \). Then \( \Gamma \) is \( Z_3 \)-connected if both \( \Gamma \) and \( \Gamma/K \) are \( Z_3 \)-connected.

Lemma 2.4. \[12\] Suppose that \( \Gamma \) is a 2-connected simple graph. If \( \delta(\Gamma) \geq 4 \) and \( \alpha(\Gamma) \leq 2 \), then \( \Gamma \) is \( Z_3 \)-connected.

Let \( v, v_1, v_2 \in V(\Gamma) \) and \( vv_1, vv_2 \in E(\Gamma) \). Removing edges \( vv_1, vv_2 \) and adding new edge \( v_1v_2 \) in \( \Gamma \), the resulting graph is denoted by \( \Gamma_{[vv_1,vv_2]} \). Obviously \( \Gamma_{[vv_1,vv_2]} = \Gamma \cup \{v_1v_2\} - \{vv_1, vv_2\} \).
Lemma 2.5. [2] Suppose that $\Gamma$ is a graph and $v \in V(\Gamma)$ with $d_\Gamma(v) \geq 4$. Then $\Gamma$ is $Z_3$-connected if $\Gamma_{[v_1,v_2]}$ is $Z_3$-connected, where $v_1, v_2$ are two neighbors of $v$.

Lemma 2.6. [6] Suppose that $K$ is a $Z_3$-reduced graph with $\alpha(K) \leq 3$. Then the order of $K$ is at most 14. Furthermore, $K$ is 5-edge-connected and contains a $K_4$ if $|V(K)| = 14$.

Lemma 2.7. [6] Suppose that $\Gamma$ is a $Z_3$-reduction of a connected graph. If $|V(\Gamma)| \leq 15$ and $\delta(\Gamma) > 4$, then $\Gamma$ is essentially 8-edge-connected and 5-edge-connected.

Lemma 2.8. [6] Suppose that $W_{2k+1}$ is a proper subgraph of the graph $\Gamma$ and $U, W$ are two subsets of $V(W_{2k+1})$ with $U \cup W = V(W_{2k+1})$. Denote by $\Gamma_{[U,W]}$ the graph obtained from $\Gamma$ by contracting $U$ and $W$ into $u$ and $w$, respectively, and then deleting the loops and replacing the edges between $u$ and $w$ by one edge $uw$. Then $\Gamma$ is $Z_3$-connected if $\Gamma_{[U,W]}$ is $Z_3$-connected.

A simple graph $\Gamma$ is said to satisfy the Ore-condition if for every pair of nonadjacent vertices $x$ and $y$ in $\Gamma$, $d_\Gamma(x) + d_\Gamma(y) \geq |V(\Gamma)|$.

Theorem 2.9. [9] Suppose that $\Gamma$ is a simple graph satisfying Ore-condition. If $|V(\Gamma)| > 6$, then $\Gamma$ is $Z_3$-connected.

Theorem 2.10. [7] A graph is $Z_3$-connected if it is 6-edge-connected.

3. Proof of Theorem 1.2

The proof of Theorem 1.2 will be given in this section.

Proof of Theorem 1.2. Suppose that $\Gamma$ is a 5-edged-connected graph with $\alpha(\Gamma) \leq 3$. Suppose that $K$ is a $Z_3$-reduction of $\Gamma$. If $K = K_1$, then we have done. Thus in the following, we assume that $K \neq K_1$. Hence $K$ is not $Z_3$-connected and is a 5-edged-connected graph with $\alpha(K) \leq 3$.

Claim 1. $\Gamma$ is simple. Thus $\delta(K) \geq 5$ and $\alpha(K) = 3$.

Proof of Claim 1. By the definition of the reduction, it is clear that $K$ is simple and $\delta(K) \geq 5$. If $\alpha(K) \leq 2$, then by Lemma 2.4 we get that $K$ is $Z_3$-connected. That contradicts our assumption. Therefore $\alpha(K) = 3$.

Claim 2. $11 \leq |V(K)| \leq 14$ and $K$ contains a triangle as subgraph.

Proof of Claim 2. If $|V(K)| \leq 10$, then $K$ satisfies conditions of Theorem 2.9 since $\delta(K) \geq 5$. Then $K$ is $Z_3$-connected, a contradiction. Thus $|V(K)| \geq 11$. By Lemmas 2.6 and 2.7, we get that $K$ is a 5-edge-connected graph with $|V(K)| \leq 14$. Hence $11 \leq |V(K)| \leq 14$.

Take an arbitrary vertex, say $w$, of the graph $K$ and denote $N_K(w) = \{w_1, w_2, \ldots, w_t\}$, where $t = d_K(w) \geq 5$. Since $\alpha(K) = 3$, $K[N(w)]$ has at least two edges. Thus $K$ contains a triangle.

Claim 3. $|V(K)| \geq 12$.

Proof of Claim 3. Suppose to the contrary that $|V(K)| \leq 11$. Then by Claim 2, $|V(K)| = 11$.

We first assume that $K$ contains a $K_4$ with vertex set $\{u_1, u_2, w_1, w_2\}$. Let $U = \{u_1, u_2\}$ and $W = \{w_1, w_2\}$. Clearly $d_{K[U,W]}(v_1) + d_{K[U,W]}(v_2) \geq |V(K[U,W])|$, where $v_1, v_2$ are arbitrary nonadjacent vertices in $K[U,W]$. By Theorem 2.9, $K[U,W]$ is $Z_3$-connected. Thus, it follows that $K$ is $Z_3$-connected from Lemma 2.8. It contradicts our assumption.
Next we assume that $K$ contains no $K_4$. Let $S = uvwu$ be a triangle in the graph $K$ with $d_K(u) \geq 6$. Since $e_K(S) \geq 10$, there are two vertices $z_1, z_2$ which are adjacent to two of $u, v, w$ in $K$. Suppose that $z_1w, z_1v \in E(K)$. It follows that the graph $K_{[w,z_1v]}$ has a 2-cycle $wwv$. The resulting graph by repeatedly contracting 2-cycles in $K_{[w,z_1v]}$ is a $K_1$, which is $Z_3$-connected. Then from Lemmas 2.3 and 2.5 $K$ is $Z_3$-connected. Then we get a contradiction. Therefore $|V(K)| \geq 12$.

Claim 4. If $|V(K)| \in \{12, 13\}$, then $K$ doesn’t contain a $K_4$ or $K_4^-$.

Proof of Claim 4. Suppose to the contrary that $K$ contains a $K_4$ or $K_4^-$. We first assume that $H$ is a $K_4$ of $K$. Let $U$ and $W$ be a partition of $V(H)$ with $|U| = |W|$. Then $|V(H_{U,W})| \leq 11$ and $\delta(H_{U,W}) \geq 5$.

By Claims 2 and 3, $H_{U,W}$ is not a reduced graph. Thus $H_{U,W}$ must contain a nontrivial $Z_3$-connected graph. Then the resulting graph obtained by contracting this subgraph and repeatedly contracting 2-cycles generated in the processing is a $K_1$, which is $Z_3$-connected. By Lemmas 2.3 and 2.8, $K$ is $Z_3$-connected. This contradiction proves that $K$ doesn’t contain a $K_4$.

Now we assume that $J$ is a $K_4^-$ of $K$. Let $V(J) = \{v_1, v_2, u_1, u_2\}$ and $E(J) = \{v_1v_2, v_1u_1, v_1u_2, v_2u_1, v_2u_2\}$. We claim that $|N(v_1) \cap N(v_2)| \leq 3$. Since $K$ doesn’t contain a $K_4$, every two vertices of $N(v_1) \cap N(v_2)$ are nonadjacent. If $v_1$ and $v_2$ have at least 4 common neighbors, then $N(v_1) \cap N(v_2)$ is an independent set with at least 4 vertices. It contradicts $\alpha(K) \leq 3$. Thus, $|N(v_1) \cap N(v_2)| = 2$, or 3. We consider these two cases in the following.

Case 1. $N(v_1) \cap N(v_2) = \{u_1, u_2, u_3\}$.

Let us consider the graph $K_{[v_1u_1, u_2u_3]}$. Note that $K_{[v_1u_3, u_2u_3]}$ contains the 2-cycle $v_1v_2v_1$. Since a 2-cycle is $Z_3$-connected from Lemma 2.1, $K_{[v_1u_3, u_2u_3]}$ contains a maximal $Z_3$-connected graph, say $W$, that contains $v_1v_2v_1$. Let $K^* = K_{[v_1u_1, u_3u_3]}/W$ and $W$ be contracted to the new vertex $v^*$. We can get that $V(J) \subseteq V(W)$, $e_{K^*}(u_3, V(J)) = 0$ and $d_{K^*}(u_3) \geq 3$. Thus, we have $|V(K^*)| \leq 10$.

When $|V(K^*)| = 10$, then we have $|V(K)| = 13$, $d_K(v^*) \geq 8$ and $V(W) = V(J)$. Since $e_K(u_3, V(J)) = 0$, we have $d_K(v^*) = 8$. Set $N(v^*) = \{z_1, z_2, \ldots, z_8\}$. Then $V(K^*) = N(v^*) \cup \{u_3\}$. If $d_K(u_3) \geq 5$, then $K^*$ satisfies Ore-condition. Therefore from Theorem 2.9, $K^*$ is $Z_3$-connected. It contradicts our assumption. Thus $d_K(u_3) \leq 4$. Suppose that none of $M = \{z_1, z_2, z_3, z_4\}$ is adjacent to $u_3$. By our assumption, there are $i, j \in \{1, 2, 3, 4\}$ such that $z_iz_j \notin E(K^*)$. This implies that $z_i$ and $z_j$ have at least two common neighbors in $N(v^*)$. Therefore we obtain a $W_4$ with center at $v^*$. It contradicts that $W$ is maximal.

When $|V(K^*)| = 9$, we have $|V(K)| = 12$ or 13. In the former case, $d_K(v^*) \geq 8$ and $V(W) = V(\Gamma)$. Since $e_K(u_3, V(J)) = 0$, there exists a vertex which has two neighbors in $V(W)$. That contradicts that $K^*$ is simple. The proof of the case when $|V(K)| = 13$ is similar to that of the case when $|V(K)| = 12$.

When $|V(K^*)| \leq 8$, the graph $K^*$ satisfies Ore-condition. Therefore $K^*$ is $Z_3$-connected by Theorem 2.9. It follows that $K$ is a $Z_3$-connected graph by Lemma 2.5. It contradicts our assumption.

Case 2. $N(v_1) \cap N(v_2) = \{u_1, u_2\}$.

If $v_i$ and $u_j$ have a common neighbor $z \notin \{v_1, v_2\}$ for $i, j \in \{1, 2\}$, then we can prove $K$ is $Z_3$-connected by a similar proof of the above case. This implies that $v_i$ and $u_j$ have only one common neighbor. Set $Y = N(v_1) \cup N(v_2) \setminus V(J) = \{z_1, z_2, \ldots, z_t\}$. Since $\delta(K) \geq 5$, we have $t \geq 4$. Since $K$ contains no $K_4$, there are two nonadjacent vertices, say $z_i, z_j$. It follows that $\{z_i, z_j, u_1, u_2\}$ is an independent set. That contradicts $\alpha(K) \leq 3$.

By Case 1 and Case 2, we get that if $|V(K)| \in \{12, 13\}$, then $K$ doesn’t contain a $K_4$ or $K_4^-$. 

AIMS Mathematics Volume 8, Issue 2, 3204–3209.
Claim 5. $|V(K)| = 14$.

Proof of Claim 5. Since $12 \leq |V(K)| \leq 14$, by Claim 4, we only need to show that if $K$ does not contain a $K_4^-$, then $K$ is $Z_3$-connected. Since the proofs of $|V(K)| = 12$ and $|V(K)| = 13$ are similar, we only prove the case when $|V(K)| = 13$.

Let $Q = v_1v_2v_3v_4$ be a triangle of $K$ with $d(v_1) \geq 6$. It follows that the degree of $v_2$, $v_3$ is 5 and the degree of $v_1$ is 6 by our assumption and Theorem 2.10. Now we suppose that the intersection of $N(v_1)$ and $N(v_2)$ is $\{v_3\}$. Then $N(v_1) \setminus V(Q) = \{v_{11}, v_{12}, v_{13}, v_{14}\}$ and $N(v_2) \setminus V(Q) = \{v_{12}, v_{13}, v_{14}\}$, where $i = 2, 3$. Since $K$ contains no $K_4^-$ and $\alpha(K) = 3$, the graph induced by $N(v_1)$ contains only isolated edges. Suppose that $v_{11}v_{12}, v_{13}v_{14}, v_{21}v_{22}, v_{31}v_{32} \in E(K)$. Similarly, we get $e(v_{ij}, N(v_1)) \leq 2$, where $i = 2, 3; j = 1, 2, 3$. It follows that $e(v_{23}, N(v_3)) \geq 2$, $e(v_{33}, N(v_2)) \geq 2$, $e(v_{2j}, N(v_3)) \geq 1$, and $e(v_{3j}, N(v_2)) \geq 1$, where $j = 1, 2$. Since $K$ does not contain a $K_4^-$, we may assume $v_{23}v_{33}, v_{23}v_{32}, v_{33}v_{22} \in E(K)$. If $v_{21}v_{31} \in E(K)$, then $v_{21}v_{31}v_{32}v_{23}v_{33}v_{22}v_{21}$ is a 6-cycle. Otherwise, $v_{21}v_{32}, v_{22}v_{31} \in E(K)$. Therefore, we get a 4-cycle $v_{21}v_{32}v_{31}v_{22}v_{21}$. Contracting the 2-cycle $v_{2}v_{3}v_{2}$ in $K_{\Gamma(v_{2}v_{3}v_{1})}$, the resulting graph is denoted by $K'$. Suppose that the cycle is contracted into new vertex $v'$. Note that $v_{ij} \in N(v')$, where $i = 2, 3; j = 1, 2, 3$. Thus, $K'$ contains a 6-wheel or 4-wheel, which is $Z_3$-connected. Contracting the wheel and repeatedly contracting 2-cycles in $K'$, the resulting graph must be a $K_1$. It follows that $K$ is $Z_3$-connected from Lemmas 2.3 and 2.5. We also get a contradiction. This completes the proof of Claim 5.

The final step. By Claim 5, $|V(K)| = 14$. Thus $K$ contains a $K_4$ by Lemma 2.6. Let $V(K_4) = \{u_1, u_2, v_1, v_2\}$ and let $U = \{u_1, u_2\}$ and $W = \{v_1, v_2\}$. In this case, we consider the graph $K_{[U,W]}$. Note that the order of $K_{[U,W]}$ is 12. If $K_{[U,W]}$ contains a $Z_3$-connected subgraph, then the resulting graph obtained by contacting it and repeatedly contracting 2-cycles is a $K_1$. We know that $K_{[U,W]}$ is a $Z_3$-connected subgraph by Lemmas 2.3 and 2.5. Otherwise, $K_{[U,W]}$ is a reduced graph. Thus, $K_{[U,W]}$ is also $Z_3$-connected by Claims 4 and 5. By Lemma 2.8, $K$ is $Z_3$-connected, a contradiction. This completes our proof.

4. Conclusions

Jaeger et al. [4] constructed a 4-edge-connected graph $\Gamma$ with $\alpha(\Gamma) = 3$, which is not $Z_3$-connected. They further conjectured that each 5-edge-connected graph is $Z_3$-connected. If this conjecture is correct, then so is Tutte’s 3-Flow Conjecture. The article confirm this conjecture for all 5-edge-connected graphs with independence number at most 3.

Acknowledgments

The second author is supported by the Natural Science Foundation of China (Grant No. 11701496) and Nanhu Scholars Program for Young Scholars of XYNU. The third author is supported by the Natural Science Foundation of China (Grant No.12271438) and Basic Research Foundation of Henan Educational Committee (No. 20ZX004).

Conflict of interest

All authors declare no conflicts of interest in this paper.
References


© 2023 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)