Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(2): 3185–3203.
DOI: 10.3934/math.2023163
Received: 17 September 2022
Revised: 18 October 2022
Accepted: 26 October 2022
Published: 16 November 2022

Research article

State feedback stabilization problem of stochastic high-order and low-order nonlinear systems with time-delay

Yanghe Cao[1], **Junsheng Zhao**[1,*] **and Zongyao Sun**[2]

[1] School of Mathematical Science, Liaocheng University, Liaocheng 252059, China
[2] Institute of Automation, Qufu Normal University, Qufu 273100, China

* **Correspondence:** Email: zhaojunsheng@lcu.edu.cn; Tel: +8606358238282.

Abstract: This manuscript considers the state feedback stabilization problem for a class of stochastic high-order and low-order nonlinear systems with time-delay. Compared with the previous results, a distinctive feature to be studied is that the considered systems involve high-order, low-order, intricate stochastic diffusion terms and time-delay simultaneously. First, the homogeneous domination approach and suitable coordinate transformations are introduced to obtain the updating laws. Then, a state feedback controller is devised to make the closed-loop systems globally asymptotically stable in probability. Finally, a simulation example is shown to prove the proposed approach powerfully.

Keywords: homogeneous domination approach; high-order and low-order; state feedback; stochastic nonlinear systems with time-delay
Mathematics Subject Classification: 93D12, 93B52,93E15

1. Introduction

Recently, the theory of nonlinear systems with time-delay has been a hot topic, due to its wide application in practical problems, such as physical engineering, biological systems and economic processes. Among these, the Lyapunov-Krasovskii methodology plays a crucial role in dealing with time-delay systems. Based on the above method, Pepe [1] addressed the input state stability of nonlinear systems with time-delay. Zhang [2] designed a stabilized controller for time-delay feed-forward nonlinear systems to achieve system stability. In order to address the stabilization problem of high-order nonlinear systems with time-delay, some researchers try to find new ways to design corresponding controllers. Yang and Sun [3] investigated the state feedback stabilization problem of controlled systems with high-order or/and time-delay via the homogeneous domination idea. With the help of the saturation function technique, homogeneous domination idea and Lyapunov approach, Song [4] studied the stabilization problem of high order feed-forward time-delay nonlinear systems. In addition to the above works, many results in [5–10] have established and improved the concept framework of nonlinear systems with time-delay.

Ever since the stochastic stability theory was founded and enriched by Deng and Zhu [11, 12], great progress has been made on the global stabilization of stochastic nonlinear systems [13–16]. Subsequently, Florchinger [17] extended the theory of control with the Lyapunov-Krasovskii functional. With the stochastic stability theory in mind, it is still important and meaningful to address high-order stochastic nonlinear systems with time-delay. Zha [18] investigated the issue of output feedback stabilization. Liu [19] studied the output feedback stabilization problem for time-delay stochastic feed-forward systems. By using a power integrator approach, the work in [20–23] also considered the state-feedback stabilization problems. However, the state feedback stabilization problem for stochastic high-order and low-order nonlinear systems with time-delay has not been well addressed, which leads us to take the interesting problem into account.

How to deal with the state feedback stabilization problem for high-order and low-order nonlinear systems with time-delay? By using a power integrator approach, Liu & Sun [24] constructed a time-delay independent controller for the aforementioned systems to relax the growth condition and the power order limitations. However, to the best of our knowledge, research on the corresponding stochastic version is limited with scarcely a few convincing results. The main difficulties are explained from two aspects. On one hand, the *Itô* formula brings the gradient terms and the Hessian terms in the Lyapunov analysis. On the other hand, the particularity of its structure has made many traditional methods inapplicable. Therefore, we need to give a new way to consider stochastic nonlinear systems. Inspired by a large number of results in [25–29], stochastic high-order and low-order nonlinear systems with time-delay will be considered as follows:

$$\begin{cases} dx_i(t) = x_{i+1}^{p_i}(t)dt + f_i(\bar{x}_i(t), \bar{x}_i(t-\tau), t)dt \\ \qquad\qquad + g_i^T(\bar{x}_i(t), \bar{x}_i(t-\tau), t)d\omega(t), \\ dx_n(t) = u^{p_n}(t)dt + f_n(x(t), x(t-\tau), t)dt \\ \qquad\qquad + g_n^T(x(t), x(t-\tau), t)d\omega(t), \end{cases} \tag{1.1}$$

where $x(t) = [x_1(t), \dots, x_n(t)]^T \in R^n$ is state, and $u(t) \in R$ is input; the nonnegative real number τ is the time-delay of the states. $\omega(t) = [\omega_1(t), \dots, \omega_r(t)]^T$. The high-order can be revealed by $p_i \in R_{odd}^{>1} =: \{\frac{p}{q} | p \geq q > 0 \ and \ p, q \ are \ odd \ integers\}$. The drift terms $f_i : R^i \times R^i \times R_+ \longrightarrow R$ and the diffusion terms $g_i : R^i \times R^i \times R_+ \longrightarrow R^r, \ i = 1, \dots, n$ are considered as locally Lipschitz with $f_i(0, 0, t) = 0$ and $g_i(0, 0, t) = 0$.

The contributions are highlighted in the following:

(*i*) Systems considered are more general. Systems in [24] only solve the control issues for deterministic cases. It is more complex to consider the stochastic disturbance. By using the homogeneous domination idea, one can give a novel perspective to generalize the control strategy for deterministic systems to the corresponding stochastic cases.

(*ii*) The result extends the works [30–32] by relaxing the growth condition and the power order limitations. The low order of the nonlinear terms is successfully relaxed to the high-order and low-order of the nonlinear terms. Based on the above situations, we use a proper Lyapunov-Krasovskii functional to handle the stabilization problem under the weaker assumptions.

Notations: $R_+ \triangleq \{x | x \geq 0, x \in R\}$, $R^n \triangleq \{x^n | x \geq 0\}$. For a given vector/matrix D, D^T denotes its transpose, $Tr\{D\}$ is the trace when D is square, and the Euclidean norm of a vector $|D|$. C^i is composed of continuous and ith partial derivable functions. $\mathcal{K}$ is composed of continuous functions and strictly increasing; $\mathcal{K}_\infty$ is composed of functions with $\mathcal{K}$. One sometimes denotes $X(t)$ by X to simplify the procedure.

2. Problem statement and preliminaries

2.1. Problem statement

Now, the time-delay stochastic nonlinear systems are addressed as follows:

$$dx(t) = f(\bar{x}_i(t), \bar{x}_i(t-\tau), t)dt + g^T(\bar{x}_i(t), \bar{x}_i(t-\tau), t)d\omega(t). \tag{2.1}$$

$\{x(s)\colon -d \leq s \leq 0\}= z\in C^b_{\mathcal{F}_0}([-d, 0]; R^n)$ is an initial data, and $\omega(t)$ denotes a Brownian motion with dimension r defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

The following assumptions are needed:

Assumption 1. *For $i = 1, \ldots, n$, there exist two constants $a_1 > 0$ and $a_2 > 0$ such that*

$$\begin{aligned}
|f_i(\bar{x}_i(t), \bar{x}_i(t-\tau), t)| &\leq a_1 \sum_{j=1}^{i}(|x_j(t)|^{\frac{r_i+\theta}{r_j}} + |x_j(t-\tau)|^{\frac{r_i+\theta}{r_j}}) \\
&\quad + a_1 \sum_{j=1}^{i-1}(|x_j(t)|^{\frac{1}{p_j \cdots p_{i-1}}} + |x_j(t-\tau)|^{\frac{1}{p_j \cdots p_{i-1}}}) + a_1(|x_i(t)| + |x_i(t-\tau)|), \\
\|g_i(\bar{x}_i(t), \bar{x}_i(t-\tau), t)\| &\leq a_2 \sum_{j=1}^{i}(|x_j(t)|^{\frac{2r_i+\theta}{2r_j}} + |x_j(t-\tau)|^{\frac{2r_i+\theta}{2r_j}}) \qquad (2.2)\\
&\quad + a_2 \sum_{j=1}^{i-1}(|x_j(t)|^{\frac{1}{2p_j \cdots p_{i-1}}} + |x_j(t-\tau)|^{\frac{1}{2p_j \cdots p_{i-1}}}) + a_2(|x_i(t)| + |x_i(t-\tau)|),
\end{aligned}$$

in which $\theta= \frac{m}{n} \geq 0$, n is an odd integer, m is an even integer, and $r_i's$ have the following definitions:

$$r_1 = 1,\ r_i = \frac{r_{i-1} + \theta}{p_{i-1}},\ i = 2,\ 3, \ldots,\ n+1. \tag{2.3}$$

Remark 1. *Assumption 1 encompasses and extends high-order and/or low-order results. We discuss this point from two cases.*

Case I: Condition (2.2), when $\tau = 0$ it reduces to high-order growth condition with $\theta \geq 0$,

$$\begin{aligned}
|f_i(\bar{x}_i(t), t)| &\leq a_1 \sum_{j=1}^{i}(|x_j(t)|^{\frac{r_i+\theta}{r_j}} + |x_j(t)|^{\frac{1}{p_j \cdots p_{i-1}}} + a_1|x_i(t)|, \\
\|g_i(\bar{x}_i(t), t)\| &\leq a_2 \sum_{j=1}^{i}(|x_j(t)|^{\frac{2r_i+\theta}{2r_j}} + |x_j(t)|^{\frac{1}{2p_j \cdots p_{i-1}}} + a_2(|x_i(t)|,
\end{aligned}$$

and low-order growth condition with $\theta = 0$,

$$|f_i(\bar{x}_i(t), t)| \leq a_1 \sum_{j=1}^{i}|x_j(t)|^{\frac{1}{p_j \cdots p_{i-1}}} + a_1|x_i(t)|,$$

$$\|g_i(\bar{x}_i(t),t)\| \ \leq \ a_2\sum_{j=1}^{i}|x_j(t)|^{\frac{1}{2p_j\cdots p_{i-1}}} + a_2(|x_i(t)|.$$

We further discuss its significance from value ranges of both low-order and high-order. From $\theta \in (-\frac{1}{p_j\cdots p_{i-1}}, 0]$*, it is easy to see that* $0 < \frac{r_i+\theta}{r_j} \leq \frac{1}{p_j\cdots p_{i-1}}$*, which implies that both low-order and high-order in Assumption 1 can take any value in* $(0, \frac{1}{p_j\cdots p_{i-1}}], [\frac{1}{p_j\cdots p_{i-1}}, +\infty)$*, respectively.*

Case II: When $\tau \neq 0$*, several new results [18–22] have been achieved on feedback stabilization of high-order nonlinear time-delay systems. The nonlinearities in [18–22] only have high-order terms. The nonlinearities in [24] include linear and nonlinear parts, and their nonlinear parts only allow low-order* $\frac{1}{p_j\cdots p_{i-1}}$ *and high-order* $\frac{r_i+\theta}{r_j}$ *with* $\theta \geq 0$.

While in this paper, (2.2) not only includes time-delays but relaxes the intervals of low-order and high-order.

Remark 2. *When* $p_i = 1, i = 1, 2, \ldots, n-1$*, and* $\tau = 0$*, equation (1) reduces to the well-known form, for which the feedback control problem has been well developed in recent years [16, 24, 26].*

Proposition 1. *For* $r_1, \ldots, r_n$ *and* $\sigma = p_1\ldots p_n r_{n+1}$ *having the following properties:*

- $r_k \in R^{\geq 1}_{odd}$, $\frac{\sigma}{r_k} \in R^{\geq 1}_{odd}$, $\sigma \in R^{\geq 1}_{odd}$, $\frac{\sigma}{r_k p_{k-1}\ldots p_1} \in R^{\geq 1}_{odd}$.
- $\sigma \geq max_{1\leq k\leq n}\{r_k + \theta\}$.
- *There hold*

$$4 \leq 4 - \frac{1}{p_1\ldots p_{k-1}} + \frac{r_{k+1}p_k}{r_k p_{k-1}\ldots p_1}, \frac{4\sigma - r_{k+1}p_k}{r_k p_{k-1}\ldots p_1} + \frac{1}{p_1\ldots p_{k-1}} \leq \frac{4\sigma}{r_k p_{k-1}\ldots p_1};$$

$$4 \leq 4 - \frac{1}{p_1\ldots p_{k-1}} + \frac{r_{k+1}p_k}{2r_k p_{k-1}\ldots p_1}, \frac{4\sigma - r_{k+1}p_k}{2r_k p_{k-1}\ldots p_1} + \frac{1}{2p_1\ldots p_{k-1}} \leq \frac{4\sigma}{r_k p_{k-1}\ldots p_1}.$$

- *For* $i = 1, \ldots, k-1$*, one has*

$$4 \leq \frac{4r_{k+1}p_k\ldots p_1}{r_i p_{i-1}\ldots p_1}, \ \frac{4\sigma}{r_i p_{i-1}\ldots p_1} \leq \frac{4\sigma}{r_i p_{i-1}\ldots p_1}.$$

Remark 3. *It is not difficult to see that system (1.1) is a class of high-order and low-order stochastic nonlinear systems with time-delay satisfying Assumption 1. Compared with [30], it is significant to point out that system (1.1) addressed here is more general. The systems can be composed by time-delay and the coupling of the high-order and low-order terms. Moreover, if* $g = 0$*, Assumption 1 will generate the same assumption as in [24]. When* $p_i > 3$*, the state feedback stabilization problem under constraint* $p_i = p$ *can give similar results as [19]. Under Assumption 1 with* $\tau = 0$*, we can obtain the same results with [30], if there are no low-order nonlinearities.*

Remark 4. *For the case of* $\tau = 0$ *in system (1.1), with the help of adding a power integrator, fruitful results have been achieved over the past years. However, for the case of* $\tau \neq 0$*, some essential difficulties will inevitably be encountered in constructing the desired controller. For instance, the time-delay effect will make the common assumption on the high-order system nonlinearities infeasible, and what conditions should be placed to the nonlinearities remains unanswered. Second, due to the higher power, time-delay and assumptions on the nonlinearities, it is more complicated to find a Lyapunov-Krasovskii functional which can be behaved well in theoretical analysis.*

2.2. Useful definitions and lemmas

For ease of the controller design, some helpful definitions are presented.

Definition 1. *[19] Consider the stochastic system $dx(t) = f(x,t)dt + g(x,t)d\omega$. For any given C^2 function $V(x,t)$, the differential operator $\mathcal{L}$ is defined as follows:*

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) + \frac{1}{2} Tr\{g^T \frac{\partial^2 V}{\partial^2 t} g\},$$

where $\frac{1}{2}Tr\{g^T \frac{\partial^2 V}{\partial^2 t} g\}$ is called the Hessian term of $\mathcal{L}$.

Definition 2. *[25] There exists coordinate $(x_1, \ldots, x_n) \in R^n$, $h_i > 0$, $i = 1, \ldots, n$, for arbitrarily $\varepsilon > 0$.*

- *The dilation $\Delta_\varepsilon(x) = (\varepsilon^{h_1} x_1, \ldots, \varepsilon^{h_n} x_n)$, and h_i is referred to as the weights. And one defines dilation weight as $\triangle = (h_1, \ldots, h_n)$.*
- *A function $U \in C(R^n, R)$ is considered as homogeneous of degree μ, if $\mu \in R$, then $U(\Delta_\varepsilon(x)) = \varepsilon^\mu U(x_1, \ldots, x_n)$, for arbitrarily $x \in R^n \setminus \{0\}$.*
- *A vector field $f_i \in C(R^n, R)$ is considered as homogeneous of degree μ, if $\mu \in R$, then $f_i(\Delta_\varepsilon(x)) = \varepsilon^{\mu + h_i} f_i(x)$, for arbitrarily $x \in R^n \setminus \{0\}$, $i = 1, \ldots, n$.*
- *A homogeneous γ-norm is considered as $\|x\|_{\triangle,\gamma} = (\sum_{i=1}^n |x_i|^{\frac{\gamma}{h_i}})^{\frac{1}{\gamma}}$, for any $x \in R^n$, where $\gamma \geq 1$. We use $\|x\|_\triangle$ or $\|x\|_{\triangle,2}$ to a exhibit 2-norm.*

With the above definitions, we give some lemmas which will be crucial for controller design.

Lemma 1. *[13] For $m \in R^{\geq 1}_{odd}$, $\forall a \in R$ and $\forall b \in R$, there hold*

$$(|a| + |b|)^{\frac{1}{m}} \leq |a|^{\frac{1}{m}} + |b|^{\frac{1}{m}} \leq 2^{\frac{m-1}{m}} (|a| + |b|)^{\frac{1}{m}},$$

$$|a - b|^m \leq 2^{m-1} |a^m - b^m|.$$

Lemma 2. *[13] For given $a, b \geq 0$ and a given positive function $f(x,y)$, there exists a positive function $g(x,y)$, such that*

$$|f(x,y) x^a y^b| \leq g(x,y)|x|^{a+b} + \frac{b}{a+b} (\frac{a}{(a+b)g(x,y)})^{\frac{a}{b}} |f(x,y)|^{\frac{a+b}{b}} |y|^{a+b}, \ \forall x, \ y \in R.$$

Lemma 3. *[13] For a continuous function g, if it is monotone, and $g(s) = 0$, then*

$$|\int_s^t g(x)dx| \leq |g(t)| \cdot |t - s|.$$

Lemma 4. *[19] Given $\tau_i \in R$, $i = 1, \ldots, n$ satisfying $0 \leq \tau_1 \leq \ldots \leq \tau_n$ and for given nonnegative functions $a_i(x,y)$, $i = 1, \ldots, n$, there holds*

$$a_1(x,y)|x|^{\tau_1} + a_n(x,y)|x|^{\tau_n} \leq \sum_{j=1}^n a_j(x,y)|x|^{\tau_j} \leq (|x|^{\tau_1} + |x|^{\tau_n}) \sum_{j=1}^n a_j(x,y), \forall x, y \in R.$$

3. Design procedures

3.1. Control design procedures

Consider the stochastic high-order and low-order nonlinear systems with time-delay as follows:

$$\begin{cases} dx_i = (x_{i+1}^{p_i} + f_i)dt + g_i d\omega(t), \ i = 1, \dots, n-1, \\ dx_n = u^{p_n} dt. \end{cases} \tag{3.1}$$

Step 0: To begin with, introducing the complete form of the controller,

$$\begin{cases} z_i(t) = x_i^{p_1 \cdots p_{i-1}}(t) - \alpha_{i-1}^{p_1 \cdots p_{i-1}}(t), \ i = 1, \dots, n, \\ \alpha_i(t) = -\varrho_i^{\frac{1}{p_1 \cdots p_i}} (z_i(t) + z_i^{\frac{r_{i+1} p_i}{r_i}}(t))^{\frac{1}{p_1 \cdots p_i}}, \ i = 1, \dots, n, \\ u(t) = \alpha_n(t). \end{cases} \tag{3.2}$$

The purpose of this work is to construct a state controller to render system (1.1) globally asymptotically stable in probability. To achieve this goal, propositions are presented as follows.

Proposition 2. *For $c_1 > 0$, $c_2 > 0$, $i = 1, \dots, n$, there hold*

$$\begin{aligned} |f_i(t, \bar{x}_i(t), \bar{x}_i(t-\tau))| \le & c_1 \sum_{j=1}^{i} (|z_j(t)|^{\frac{1}{p_1 \cdots p_{i-1}}} + |z_j(t)|^{\frac{r_i + \theta}{r_j p_1 \cdots p_{i-1}}}) \\ & + c_1 \sum_{j=1}^{i} (|z_j(t-\tau)|^{\frac{1}{p_1 \cdots p_{i-1}}} + |z_j(t-\tau)|^{\frac{r_i+\theta}{r_j p_1 \cdots p_{i-1}}}) \\ \|g_i(t, \bar{x}_i(t), \bar{x}_i(t-\tau))\| \le & c_2 \sum_{j=1}^{i} (|z_j(t)|^{\frac{1}{2p_1 \cdots p_{i-1}}} + |z_j(t)|^{\frac{2r_i+\theta}{2r_j p_1 \cdots p_{i-1}}}) \\ & + c_2 \sum_{j=1}^{i} (|z_j(t-\tau)|^{\frac{1}{2p_1 \cdots p_{i-1}}} + |z_j(t-\tau)|^{\frac{2r_i+\theta}{2r_j p_1 \cdots p_{i-1}}}). \end{aligned} \tag{3.3}$$

Step 1. First, we will construct a Lyapunov-candidate-function $V_1 = \int_0^{x_1} s^3 ds + \int_0^{x_1} s^{\frac{4\sigma - r_2 p_1}{r_1}} ds + n \int_{t-\tau}^{t} (z_1^4(l) + z_1^{\frac{4\sigma}{r_1}}(l)) dl$. Along the solution of (3.1) , one has

$$\begin{aligned} \mathcal{L}V_1 = & x_1^3 (x_2^{p_1} + f_1) + x_1^{\frac{4\sigma - r_2 p_1}{r_1}} (x_2^{p_1} + f_1) + n(z_1^4(t) + z_1^{\frac{4\sigma}{r_1}}(t)) \\ & - n(z_1^4(t-\tau) + z_1^{\frac{4\sigma}{r_1}}(t-\tau)) + \Psi_1, \end{aligned}$$

where $\Psi_1 = \frac{1}{2} Tr\{g_1^T \frac{\partial^2 V_1}{\partial x_1^2} g_1\}$, which leads to

$$\begin{aligned} \mathcal{L}V_1 = & (z_1^3 + z_1^{\frac{4\sigma - r_2 p_1}{r_1}})(x_2^{p_1} - \alpha_1^{p_1}) + (z_1^3 + z_1^{\frac{4\sigma - r_2 p_1}{r_1}})\alpha_1^{p_1} \\ & + (z_1^3 + z_1^{\frac{4\sigma - r_2 p_1}{r_1}}) f_1 + n(z_1^4(t) + z_1^{\frac{4\sigma}{r_1}}(t)) \\ & - n(z_1^4(t-\tau) + z_1^{\frac{4\sigma}{r_1}}(t-\tau)) + \Psi_1. \end{aligned} \tag{3.4}$$

With Proposition 2, Lemma 1 and Lemma 2 in mind, one has

$$\begin{aligned}(z_1^3+z_1^{\frac{4\sigma-r_2p_1}{r_1}})f_1 \le& c_1(|z_1|^3+|z_1|^{\frac{4\sigma-r_2p_1}{r_1}})(|z_1|+|z_1|^{\frac{r_2p_1}{r_1}}\\ &+|z_1(t-\tau)|+|z_1(t-\tau)|^{\frac{r_2p_1}{r_1}})\\ \le& c_1(|z_1|^4+|z_1|^{\frac{4\sigma-r_2p_1}{r_1}}z_1+|z_1|^{\frac{r_2p_1}{r_1}}z_1^3+|z_1|^{\frac{4\sigma}{r_1}})\\ &+c_1(|z_1|^3|z_1(t-\tau)|+|z_1|^3|z_1(t-\tau)|^{\frac{r_2p_1}{r_1}}+|z_1|^{\frac{4\sigma-r_2p_1}{r_1}}z_1|z_1(t-\tau)|+|z_1|^{\frac{4\sigma-r_2p_1}{r_1}|z_1(t-\tau)|^{\frac{r_2p_1}{r_1}}});\end{aligned} \tag{3.5}$$

with the help of Lemma 4, we can see it satisfies $|z_1|^{\frac{r_2p_1}{r_1}}z_1^3 \le |z_1|^{\frac{r_2+\theta+3r_1}{r_1}} \le |z_1|^{4+\theta} \le z_1^4+z_1^{\frac{4\sigma}{r_1}}$ when $4\le 4+\theta\le \frac{2r_2p_1}{\sigma}+3\le\frac{4\sigma}{r_1}$. Similarly, one can obtain

$$(z_1^3+z_1^{\frac{4\sigma-r_2p_1}{r_1}})f_1 \le \beta_1(z_1^4+z_1^{\frac{4\sigma}{r_1}})+(z_1(t-\tau)^4+z_1(t-\tau)^{\frac{4\sigma}{r_1}}), \tag{3.6}$$

where $\beta_1=4c_1+2c_1^2+\frac{4\sigma-r_2p_1}{\sigma}(\frac{2r_2p_1}{\sigma})^{\frac{r_2p_1}{4\sigma-r_2p_1}}c_1^{\frac{2\sigma}{4\sigma-r_2p_1}}$. Now, one designs the virtual controller α_1 as

$$\alpha_1^{p_1}(x_1)=-(2n+\beta_1)(z_1+z_1^{\frac{r_2p_1}{r_1}})=-\varrho_1(z_1+z_1^{\frac{r_2p_1}{r_1}}), \tag{3.7}$$

where $\varrho_1>1$ is a positive constant. Noticing that

$$-\varrho_1 z_1^{1+\frac{4\sigma-r_2p_1}{r_1}}\le 0,\ \ -\varrho_1 z_1^{1+\frac{r_2p_1}{r_1}}\le 0,$$

and using (4.1) and (3.7) with (3.4) after complex calculations, one finally obtains

$$\begin{aligned}\mathcal{L}V_1\le& -n(z_1^4+z_1^{\frac{4\sigma}{r_1}})+(z_1^3+z_1^{\frac{4\sigma-r_2p_1}{r_1}})(x_2^{p_1}-\alpha_1^{p_1})\\ &-(n-1)(z_1^4(t-\tau)+z_1^{\frac{4\sigma}{r_1}}(t-\tau))+\Psi_1.\end{aligned} \tag{3.8}$$

To complete the induction, at the kth step, we now define

$$\begin{aligned}W_{Lk}&=\int_{\alpha_{k-1}}^{x_k}(s^{p_1\ldots p_{k-1}}-\alpha_{k-1}^{p_1\ldots p_{i-1}})^{4-\frac{1}{p_1\cdots p_{i-1}}}ds\\ W_{Hk}&=\int_{\alpha_{k-1}}^{x_k}(s^{p_1\ldots p_{k-1}}-\alpha_{k-1}^{p_1\ldots p_{i-1}})^{\frac{4\sigma-r_{k+1}p_k}{r_kp_1\cdots p_{i-1}}}ds\\ W_{Dk}&=(n-k+1)\int_{t-\tau}^{t}(z_k^4(l)+z_k^{\frac{4\sigma}{r_kp_1\cdots p_{i-1}}}(l))dl.\end{aligned}$$

Lyapunov function $V_k=V_{k-1}+W_{Lk}+W_{Hk}+W_{Dk}$ is C^2, proper and positive definite. Moreover, for $i=1,\ldots,k-1$, $W_{Lk}(\cdot)$, $W_{Hk}(\cdot)$, $W_{Dk}(\cdot)$ satisfy

$$\begin{aligned}&\frac{\partial W_{Lk}}{\partial x_k}=z_k^{4-\frac{1}{p_1\cdots p_{i-1}}},\ \frac{\partial W_{Hk}}{\partial x_k}=z_k^{\frac{4\sigma-r_{k+1}p_k}{r_kp_1\cdots p_{i-1}}},\\ &frac\partial^2W_{Lk}\partial x_k^2=(4-\frac{1}{p_1\ldots p_{i-1}})z_k^{3-\frac{1}{p_1\cdots p_{i-1}}}(p_1\ldots p_{k-1})x^{p_1\ldots p_{k-1}-1}\end{aligned} \tag{3.9}$$

$$\frac{\partial^2 W_{Hk}}{\partial x_k^2} = (\frac{4\sigma - r_{k+1}p_k}{r_k p_1 \ldots p_{i-1}}) z_k^{\frac{4\sigma - r_{k+1}p_k}{r_k p_1 \cdots p_{i-1}} - 1} (p_1 \ldots p_{k-1}) x^{p_1 \cdots p_{k-1} - 1}$$

$$\frac{\partial W_{Lk}}{\partial x_i} = -(4 - \frac{1}{p_1 \ldots p_{i-1}}) \int_{\alpha_{k-1}}^{x_k} (s^{p_1 \cdots p_{k-1}} - \alpha_{k-1}^{p_1 \cdots p_{i-1}})^{3 - \frac{1}{p_1 \cdots p_{i-1} \cdots p_{i-1}}} ds \frac{\partial \alpha_{k-1}^{p_1 \cdots p_{i-1}}}{\partial x_i}$$

$$\frac{\partial^2 W_{Lk}}{\partial x_k x_i} = -(4 - \frac{1}{p_1 \ldots p_{i-1}}) z_k^{3 - \frac{1}{p_1 \cdots p_{i-1}}} \frac{\partial \alpha_{k-1}^{p_1 \cdots p_{i-1}}}{\partial x_i}, \frac{\partial^2 W_{Hk}}{\partial x_k x_i} = -(\frac{4\sigma - r_{k+1}p_k}{r_k p_1 \ldots p_{i-1}}) z_k^{\frac{4\sigma - r_{k+1}p_k}{r_k p_1 \cdots p_{i-1}} - 1} \frac{\partial \alpha_{k-1}^{p_1 \cdots p_{i-1}}}{\partial x_i}$$

$$\frac{\partial^2 W_{Lk}}{\partial x_i^2} = \int_{\alpha_{k-1}}^{x_k} (4 - \frac{1}{p_1 \ldots p_{i-1}})(3 - \frac{1}{p_1 \ldots p_{i-1}})(s^{p_1 \cdots p_{k-1}} - \alpha_{k-1}^{p_1 \cdots p_{i-1}})^{2 - \frac{1}{p_1 \cdots p_{i-1} \cdots p_{i-1}}} ds (\frac{\partial \alpha_{k-1}^{p_1 \cdots p_{i-1}}}{\partial x_i})^2$$
$$+ \int_{\alpha_{k-1}}^{x_k} (4 - \frac{1}{p_1 \ldots p_{i-1}})(s^{p_1 \cdots p_{k-1}} - \alpha_{k-1}^{p_1 \cdots p_{i-1}})^{3 - \frac{1}{p_1 \cdots p_{i-1} \cdots p_{i-1}}} ds (\frac{\partial^2 \alpha_{k-1}^{p_1 \cdots p_{i-1}}}{\partial x_i^2})$$

$$\frac{\partial W_{Hk}}{\partial x_i} = -(\frac{4\sigma - r_{k+1}p_k}{r_k p_1 \ldots p_{i-1}}) \int_{\alpha_{k-1}}^{x_k} (s^{p_1 \cdots p_{k-1}} - \alpha_{k-1}^{p_1 \cdots p_{i-1}})^{\frac{4\sigma - r_{k+1}p_k}{r_k p_1 \cdots p_{i-1}} - 1} ds \frac{\partial \alpha_{k-1}^{p_1 \cdots p_{i-1}}}{\partial x_i}$$

$$\frac{\partial^2 W_{Hk}}{\partial x_i^2} = \int_{\alpha_{k-1}}^{x_k} (\frac{4\sigma - r_{k+1}p_k}{r_k p_1 \ldots p_{i-1}})(\frac{4\sigma - r_{k+1}p_k}{r_k p_1 \ldots p_{i-1}} - 1)(s^{p_1 \cdots p_{k-1}} - \alpha_{k-1}^{p_1 \cdots p_{i-1}})^{\frac{4\sigma - r_{k+1}p_k}{r_k p_1 \cdots p_{i-1}} - 2} ds (\frac{\partial \alpha_{k-1}^{p_1 \cdots p_{i-1}}}{\partial x_i})^2$$
$$+ \int_{\alpha_{k-1}}^{x_k} (\frac{4\sigma - r_{k+1}p_k}{r_k p_1 \ldots p_{i-1}})(s^{p_1 \cdots p_{k-1}} - \alpha_{k-1}^{p_1 \cdots p_{i-1}})^{\frac{4\sigma - r_{k+1}p_k}{r_k p_1 \cdots p_{i-1}} - 1} ds (\frac{\partial^2 \alpha_{k-1}^{p_1 \cdots p_{i-1}}}{\partial x_i^2}). \tag{3.10}$$

Step k (k=2,3,...,n): As in step k-1, there exists Lyapunov-candidate-function V_{k-1}, implying

$$\begin{aligned} \mathcal{L}V_{k-1} \leq & -(n-k+2) \sum_{i=1}^{k-1} (z_i^4 + z_i^{\frac{4\sigma}{r_i p_{i-1} \cdots p_1}}) - (n-k+1) \sum_{i=1}^{k-1} (z_i^4(t-\tau) + z_i^{\frac{4\sigma}{r_i p_{i-1} \cdots p_1}}(t-\tau)) \\ & + (z_{k-1}^{4 - \frac{1}{p_1 \cdots p_{k-2}}} + z_{k-1}^{\frac{4\sigma - r_k p_{k-1}}{r_{k-1} p_{k-2} \cdots p_1}})(x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}}) + \Psi_{k-1}, \end{aligned} \tag{3.11}$$

where $\Psi_{k-1} = \frac{1}{2} Tr\{\bar{\psi}_{k-1}^T \frac{\partial^2 V_{k-1}}{\partial \bar{x}_{k-1}^2} \bar{\psi}_{k-1}\}$, $\bar{\psi}_{k-1} = (g_1, \ldots, g_{k-1})$. Hence, one will consider $V_k = V_{k-1} + W_{Lk} + W_{Hk} + W_{Dk}$ and define an appropriate virtual controller α_k. Similar to step 1, one can obtain

$$\begin{aligned} \mathcal{L}V_k \leq & -(n-k+2) \sum_{i=1}^{k-1} (z_i^4 + z_i^{\frac{4\sigma}{r_i p_{i-1} \cdots p_1}}) \\ & -(n-k+1) \sum_{i=1}^{k-1} (z_i^4(t-\tau) + z_i^{\frac{4\sigma}{r_i p_{i-1} \cdots p_1}}(t-\tau)) \\ & + (n-k+1)(z_k^4 + z_k^{\frac{4\sigma}{r_k p_{k-1} \cdots p_1}}) + (z_k^{4 - \frac{1}{p_1 \cdots p_{k-1}}} + z_k^{\frac{4\sigma - r_k p_{k-1}}{r_k p_{k-1} \cdots p_1}})(x_{k+1}^{p_{k-1}} - \alpha_k^{p_k}) \\ & + (z_k^{4 - \frac{1}{p_1 \cdots p_{k-1}}} + z_k^{\frac{4\sigma - r_k p_{k-1}}{r_k p_{k-1} \cdots p_1}}) \alpha_k^{p_k} + (z_k^{4 - \frac{1}{p_1 \cdots p_{k-1}}} + z_k^{\frac{4\sigma - r_k p_{k-1}}{r_k p_{k-1} \cdots p_1}}) f_k \\ & + (z_{k-1}^{4 - \frac{1}{p_1 \cdots p_{k-2}}} + z_{k-1}^{\frac{4\sigma - r_k p_{k-1}}{r_{k-1} p_{k-2} \cdots p_1}})(x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}}) \\ & + \sum_{i=1}^{k-1} (\frac{\partial W_{Lk}}{\partial x_i} + \frac{\partial W_{Hk}}{\partial x_i})(x_{i+1}^{p_i} + f_i) + \Psi_k, \end{aligned} \tag{3.12}$$

where $\Psi_k = \frac{1}{2}Tr\{\bar{\psi}_k^T \frac{\partial^2 V_k}{\partial \bar{x}_k^2}\bar{\psi}_k\}$, $\bar{\psi}_k = (g_1, \ldots, g_k)$. Obviously, the virtual controller α_k is used to eliminate the last three terms of (3.12). In light of (3.2) and Lemma 1, it yields that

$$x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}} \le |(x_k^{p_1\cdots p_{k-1}})^{\frac{1}{p_1\cdots p_{k-2}}} - (\alpha_{k-1}^{p_1\cdots p_{k-1}})^{\frac{1}{p_1\cdots p_{k-2}}}| \le 2^{3-\frac{1}{p_1\cdots p_{k-2}}}|z_k|^{\frac{1}{p_1\cdots p_{k-2}}}.$$

In the case of $4\sigma - \theta \le 4\sigma$, by Lemma 2, one obtains that

$$\begin{aligned}
&(z_{k-1}^{4-\frac{1}{p_1\cdots p_{k-2}}} + z_{k-1}^{\frac{4\sigma - r_k p_{k-1}}{r_{k-1}p_{k-2}\cdots p_1}})(x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}})\\
&\le 2^{1-\frac{1}{p_1\cdots p_{k-2}}}|z_k|^{\frac{1}{p_1\cdots p_{k-2}}}(|z_{k-1}|^{4-\frac{1}{p_1\cdots p_{k-2}}} + |z_{k-1}|^{\frac{4\sigma - r_k p_{k-1}}{r_{k-1}p_{k-2}\cdots p_1}})\\
&\le \beta_{k1}(z_k^4 + z_k^{\frac{4\sigma}{r_k p_{k-1}\cdots p_1}}) + \frac{1}{3}(z_{k-1}^4 + z_{k-1}^{\frac{4\sigma}{r_{k-1}p_{k-2}\cdots p_1}}),
\end{aligned} \tag{3.13}$$

where β_{k1} denotes a positive constant. On the basis of Proposition 2 and Lemma 3, one has

$$\begin{aligned}
&(z_{k-1}^{4-\frac{1}{p_1\cdots p_{k-2}}} + z_{k-1}^{\frac{4\sigma - r_k p_{k-1}}{r_{k-1}p_{k-2}\cdots p_1}})f_k\\
\le&\frac{1}{2}\sum_{i=1}^{k-2}(z_i^4 + z_i^{\frac{4\sigma}{r_i p_{i-1}\cdots p_1}}) + \frac{1}{3}(z_{k-1}^4 + z_{k-1}^{\frac{4\sigma}{r_{k-1}p_{k-2}\cdots p_1}})\\
&+ \frac{1}{2}\sum_{i=1}^{k-1}(z_i^4(t-\tau) + z_i(t-\tau)^{\frac{4\sigma}{r_i p_{i-1}\cdots p_1}}) + z_k^4(t-\tau)\\
&+ z_k(t-\tau)^{\frac{4\sigma}{r_k p_{k-1}\cdots p_1}} + \beta_{k2}(z_k^4 + z_k^{\frac{4\sigma}{r_k p_{k-1}\cdots p_1}}),
\end{aligned} \tag{3.14}$$

where β_{k2} denotes a positive constant. In the sequel, one estimates the last term. With the help (3.2), Lemmas 2 and 4, it is not hard to achieve

$$\int_{\alpha_{k-1}}^{x_k}(s^{p_1\cdots p_{k-1}} - \alpha_{k-1}^{p_1\cdots p_{k-1}})^{3-\frac{1}{p_1\cdots p_{k-1}}}ds \le |z_k|^{3-\frac{1}{p_1\cdots p_{k-1}}}\cdot|x_k - \alpha_{k-1}| \le 2^{3-\frac{1}{p_1\cdots p_{k-1}}}|z_k|. \tag{3.15}$$

Similarly, one can obtain

$$\int_{\alpha_{k-1}}^{x_k}(s^{p_1\cdots p_{k-1}} - \alpha_{k-1}^{p_1\cdots p_{k-1}})^{\frac{4\sigma - r_{k+1}p_k}{r_k p_{k-1}\cdots p_1}-1}ds \le 2^{3-\frac{1}{p_1\cdots p_{k-1}}}|z_k|^{\frac{4\sigma-\theta}{r_k p_{k-1}\cdots p_1}-1}. \tag{3.16}$$

On the basis of the previous inequality, one has

$$\begin{aligned}
&(\frac{\partial W_{Lk}}{\partial x_i} + \frac{\partial W_{Hk}}{\partial x_i})(x_{i+1}^{p_i} + f_i)\\
\le&\lambda_k(|z_k| + |z_k|^{\frac{4\sigma-\theta}{r_k p_{k-1}\cdots p_1}-1})|\frac{\partial \alpha_{k-1}^{p_1\cdots p_{k-1}}}{\partial x_i}|(x_{i+1}^{p_i} + f_i)\\
\le&d_{ki}(z_k^4 + z_k^{\frac{4\sigma}{r_k p_{k-1}\cdots p_{k-1}}}) + \frac{1}{2(k-1)}(\sum_{j=1}^{k-2}(z_j^4 + z_j^{\frac{4\sigma}{r_j p_1\cdots p_{j-1}}})\\
&+ \frac{1}{3(k-1)}(z_{k-1}^4 + z_{k-1}^{\frac{4\sigma}{r_{k-1}p_1\cdots p_{k-2}}})\\
&+ \frac{1}{2(k-1)}\sum_{j=1}^{k-1}(z_j^4(t-\tau) + z_j^{\frac{4\sigma}{r_j p_1\cdots p_{j-1}}}(t-\tau)),
\end{aligned} \tag{3.17}$$

in which d_{ki} denotes a positive constant. Define $\beta_k = \beta_{k1} + \beta_{k2} + \beta_{k3}$ with $\beta_{k3} = \sum_{i=1}^{k-1} d_{ki}$ and choose the virtual controller α_k as

$$\alpha_k^{p_1 \cdots p_k}(X_k) = -\varrho_k(z_k + z_k^{\frac{r_{k+1}p_k}{r_k}}). \tag{3.18}$$

By Lemma 2, one can arrive at

$$(z_k^{4-\frac{1}{p_1 \cdots p_{k-1}}} + z_k^{\frac{4\sigma - r_{k+1}p_k}{r_k p_{k-1} \cdots p_1}})\alpha_k^{p_k} \le -(2(n-k+1) + \beta_k)(z_k^4 + z_k^{\frac{4\sigma}{r_k p_{k-1} \cdots p_1}}). \tag{3.19}$$

Substituting (3.13)–(3.18) into (3.12) yields

$$\begin{aligned}
\mathcal{L}V_k \le & -(n-k+1)\sum_{i=1}^{k}(z_i^4 + z_i^{\frac{4\sigma}{r_i p_{i-1} \cdots p_1}}) \\
& -(n-k)\sum_{i=1}^{k}(z_i^4(t-\tau) + z_i^{\frac{4\sigma}{r_i p_{i-1} \cdots p_1}}(t-\tau)) \\
& + (z_k^{4-\frac{1}{p_1 \cdots p_{k-1}}} + z_k^{\frac{4\sigma - r_{k+1}p_k}{r_k p_{k-1} \cdots p_1}})(x_{k+1}^{p_k} - \alpha_k^{p_k}) + \Psi_k.
\end{aligned}$$

It is shown that the above formula holds for $k = n$ with virtual controllers (3.18). Similarly, we choose $V_n(x) = \sum_{i=1}^{n}(W_{Li}(\cdot) + W_{Hi}(\cdot) + W_{Di}(\cdot))$. There is an actual control law

$$u(x) = -\varrho_n^{\frac{1}{p_1 \cdots p_n}}(z_n + z_n^{\frac{r_{n+1}p_n}{r_n}})^{\frac{1}{p_1 \cdots p_n}}, \tag{3.20}$$

such that

$$\mathcal{L}V_k \le -\sum_{i=1}^{n}(z_i^4 + z_i^{\frac{4\sigma}{r_i p_{i-1} \cdots p_1}}). \tag{3.21}$$

Until now, the recursive design has been completed. Under the new coordinates

$$\xi_1 = x_1,\ \xi_i = \frac{x_i}{L^{k_i}},\ \nu^p = \frac{u^p}{L^{k_n+1}}, \tag{3.22}$$

where $k_1 = 0$, $k_i = \frac{k_{i-1}+1}{p_{i-1}}$, $i = 2, \ldots, n$ and $L > 1$ is a constructed constant, system (1.1) can be rewritten in the form

$$\begin{aligned}
d\xi_i &= L\xi_{i+1}^{p_i}dt + \frac{f_i(\cdot)}{L^{k_i}}dt + \frac{g_i(\cdot)}{L^{k_i}}d\omega(t), \\
d\xi_n &= L\nu_n^{p_n}dt + \frac{f_n(\cdot)}{L^{k_n}}dt + \frac{g_n(\cdot)}{L^{k_n}}d\omega(t).
\end{aligned} \tag{3.23}$$

By (3.18) and (3.20), the system (3.1) can be integrated into the complex format

$$d\xi = LR(\xi)dt + T(t,\xi,\xi(t-\tau))dt + \psi^T(t,\xi,\xi(t-\tau))d\omega(t), \tag{3.24}$$

where $\xi= (\xi_1,\ldots,\xi_n)^T$, $R(\xi)= (\xi_2^p,\ldots,\xi_n^p,\nu^p)^T$, $T(t,\xi,\xi(t-\tau))= (f_1,\frac{f_2}{L^{k_2}},\ldots,\frac{f_n}{L^{k_n}})^T$, $\psi(t,\xi,\xi(t-\tau))= (g_1,\frac{g_2}{L^{k_2}},\ldots,\frac{g_n}{L^{k_n}})$. Introducing the dilation weight $\triangle= (r_1,r_2,\ldots,r_n)$, one gets

$$\begin{aligned} V_n(\triangle_\varepsilon(\xi)) =& \sum_{i=1}^{n}\int_{\varepsilon^{r_i}\alpha_{i-1}}^{\varepsilon^{r_i}x_i}(s^{p_1\cdots p_{i-1}}-\varepsilon^{r_ip_{i-1}\cdots p_1}\alpha_{i-1}^{p_1\cdots p_{i-1}})^{4-\frac{1}{p_1\cdots p_{i-1}}}ds \\ &+\sum_{i=1}^{n}\int_{\varepsilon^{r_i}\alpha_{i-1}}^{\varepsilon^{r_i}x_i}(s^{p_1\cdots p_{i-1}}-\varepsilon^{r_ip_{i-1}\cdots p_1}\alpha_{i-1}^{p_1\cdots p_{i-1}})^{\frac{4\sigma-r_{k+1}p_k}{r_kp_{k-1}\cdots p_1}}ds \\ &+(n-k+1)\int_{t-\tau}^{t}(z_k^4(s)+z_k^{\frac{4\sigma}{r_kp_{k-1}\cdots p_1}}(s))ds \\ =&\sum_{i=1}^{n}\int_{\alpha_{i-1}}^{x_i}(\varepsilon^{r_ip_{i-1}\cdots p_1}(\zeta^{p_1\cdots p_{i-1}}-\alpha_{i-1}^{p_1\cdots p_{i-1}}))^{4-\frac{1}{p_1\cdots p_{i-1}}}\varepsilon^{r_ip_{i-1}\cdots p_1}d\zeta \\ &+\sum_{i=1}^{n}\int_{\alpha_{i-1}}^{\varepsilon^{r_i}x_i}(s^{p_1\cdots p_{i-1}}-\varepsilon^{r_ip_{i-1}\cdots p_1}\alpha_{i-1}^{p_1\cdots p_{i-1}})^{\frac{4\sigma-r_{k+1}p_k}{r_kp_{k-1}\cdots p_1}}\varepsilon^{r_ip_{i-1}\cdots p_1}d\zeta \\ =&\ \varepsilon^{4\sigma-\theta}V_n(\xi), \end{aligned} \tag{3.25}$$

where s is defined as $s=r_i\zeta$. With the help of the above formula and Definition 2, it can be concluded that $V_n(\xi)$ is homogeneous of degree $4\sigma-\theta$.

3.2. Stability analysis

The main result of this manuscript will be stated as follows.

Theorem 1. *Suppose Assumptions 1 apply to stochastic system* (1)*, under the state feedback controller* $u^p = L^{k_n+1}\nu^p$ *and (3.20), then:*

(*i*) *There exists a unique solution on* $[-d,\infty)$*;*

(*ii*) *The equilibrium at the origin is globally asymptotically stable in probability.*

Proof. Four steps are used to verify Theorem 1.

Step 1: By the definition of $\varrho>0$, we know that $p_1\ldots p_{j-1}-1>1$, which implies that $4-\frac{1}{p_1\cdots p_{k-1}-1}>2$, $\frac{4\sigma-r_{k+1}p_k}{r_kp_{k-1}\cdots p_1}>2$. Therefore, $\frac{\partial\alpha_i^{p_1\cdots p_i}(t)}{\partial x_j(t)}$ is continuous, and $u^{p_n}=L^{k_n+1}\nu^{p_n}$ is C. As is known to all, the function is C. The closed-loop system satisfies the locally Lipschitz condition based on f_i and g_i being locally Lipschitz.

Step 2: Consider the Lyapunov-candidate-function:

$$V(\xi)=V_n(\xi)+\sum_{i=1}^{n}\frac{h_1+h_2}{1-\gamma}\int_{t-\tau}^{t}\|\xi\|_\Delta^{4\sigma}d\eta, \tag{3.26}$$

where h_1 and h_2 are positive parameters. It is straightforward to prove that $V(\xi)$ is C^2 on ξ. Since $V_n(\xi)$ is continuous, positive definite and radially unbounded, from Lemma 1, one can have

$$\alpha_{20}(|\xi|)\le V(\xi)\le\alpha_{21}(|\xi|), \tag{3.27}$$

where α_{20} and α_{21} are $\mathcal{K}_\infty$ functions. With the help of the homogeneous theory, one finally has

$$\bar{c}_0\|\xi_i\|_\Delta^{4\sigma}\le U(\xi)\le\underline{c}_0\|\xi_i\|_\Delta^{4\sigma}, \tag{3.28}$$

in which $\bar{c}_0 > 0$, $\underline{c}_0 > 0$, and $U(\xi)$ denotes a positive definite function of the 4σ homogeneous degree. Hence, one has the formula

$$\alpha_{20}(|\xi|) \leq U(\xi) \leq \alpha_{21}(|\xi|). \tag{3.29}$$

(3.29) leads to

$$\begin{aligned} \frac{h_1 + h_2}{1-\gamma} \int_{t-\tau}^{t} \|\xi\|_{\Delta}^{4\sigma} d\eta &\leq \tilde{c} \int_{t-\tau}^{t} \bar{\alpha}_{22}(|\xi_i|) d\eta \\ &\leq \tilde{c} \int_{-\tau}^{0} \alpha_{21}(|\xi_i(t+s)|) d(t+s) \\ &\leq c \sup_{-\tau \leq s \leq 0} \bar{\alpha}_{22}(|\xi_i(s+t)|) \\ &\leq \alpha_{22}(\sup_{-\tau \leq s \leq 0} |\xi_i(s+t)|), \end{aligned} \tag{3.30}$$

where $\eta = s + t$, $\tilde{c} > 0$, $c > 0$ and α_{22} is a class $\mathcal{K}_\infty$ function. Since

$$|\xi| \leq (\sup_{-\tau \leq s \leq 0} |\xi(s+t)|), \ \alpha_{21}|\xi| \leq \alpha_{21}(\sup_{-\tau \leq s \leq 0} |\xi(s+t)|).$$

Defining $\beta_2 = \alpha_{21} + \alpha_{22}$, by (3.26)-(3.30), one gets

$$\beta_1(|\xi|) \leq V(|\xi|) \leq \beta_2(\sup_{-\tau \leq s \leq 0} |\xi(s+t)|).$$

Step 3: With the help of Lemma 1 and (3.20), c_{01} is a positive constant, and one has

$$\frac{\partial V_n(\xi)}{\partial \xi} LR(\xi) \leq -c_{01} L \|\xi\|_{\Delta}^{4\sigma}. \tag{3.31}$$

By Proposition 2 and $L > 1$, one can have

$$\begin{aligned} \left|\frac{f_i(t, \bar{\xi}(t), \bar{\xi}(t-\tau))}{L^{k_i}}\right| &\leq \bar{\delta}_1 L^{1-\gamma_{i1}} \Big(\sum_{j=1}^{i} |\xi(t)|^{\frac{r_i+\theta}{r_j}} + \sum_{j=1}^{i} |\xi(t-\tau)|^{\frac{r_i+\theta}{r_j}}\Big) \\ &\leq \delta_1 L^{1-\gamma_{i1}} (\|\xi(t)\|_{\Delta}^{r_i+\theta} + \|\xi(t-\tau)\|_{\Delta}^{r_i+\theta}), \end{aligned} \tag{3.32}$$

in which $\bar{\delta}_1, \delta_1 > 0$. With the help of Lemmas 1, 2 and (3.32), one can obtain

$$\begin{aligned} &\left|\frac{\partial V_n}{\partial \xi(t)} T(t, \xi(t), \xi(t-\tau))\right| \\ \leq &\tilde{c}_{02} L^{1-\bar{\gamma}_0} \Big(\sum_{i=1}^{n} \|\xi(t)\|_{\Delta}^{4\sigma - r_i - \theta} \|\xi(t)\|_{\Delta}^{r_i+\theta} \\ &+ \sum_{j=1}^{i} \|\xi(t-\tau)\|_{\Delta}^{4\sigma - r_i - \theta} \|\xi(t-\tau)\|_{\Delta}^{r_i+\theta}\Big) \\ \leq &L^{1-\bar{\gamma}_0} (\bar{c}_{02} \|\xi(t)\|_{\Delta}^{4\sigma} + \bar{c}_{02} \|\xi(t)\|_{\Delta}^{4\sigma} \|\xi(t-\tau)\|_{\Delta}^{4\sigma}), \end{aligned} \tag{3.33}$$

where c_{02}, $\bar{c}_{02}$, $\tilde{c}_{02}$ and $\bar{\gamma}_0 = \min_{1\le i\le n} \gamma_{i1}$ are positive constants. Similar to (3.32), we use δ_2 and $\gamma_{i2} < 1/2$ to show that

$$\begin{aligned}
&|\frac{g_i(t,\bar{\xi}(t),\bar{\xi}(t-\tau))}{L^{k_i}}| \\
&\le \frac{1}{L^{k_i}} c_2 \sum_{j=1}^{i} (|z_j(t)|^{\frac{1}{2p_1\cdots p_{i-1}}} + |z_j(t)|^{\frac{2r_i+\theta}{2r_j p_1\cdots p_{i-1}}}) \\
&+ \frac{1}{L^{k_i}} c_2 c_2 \sum_{j=1}^{i} (|z_j(t-\tau)|^{\frac{1}{2p_1\cdots p_{i-1}}} + |z_j(t-\tau)|^{\frac{2r_i+\theta}{2r_j p_1\cdots p_{i-1}}}) \\
&\le L^{\frac{1}{2}-\gamma_{i2}} (\|\xi(t)\| + \|\xi(t-\tau)\|)^{r_i+\frac{\theta}{2}} \\
&\le \delta_2 L^{\frac{1}{2}-\gamma_{i2}} (\|\xi(t)\|_{\triangle}^{r_i+\frac{\theta}{2}} + \|\xi(t-\tau)\|_{\triangle}^{r_i+\frac{\theta}{2}}).
\end{aligned}$$

Using Lemma 1, Lemma 3, Lemma 4 and (3.34), one obtains

$$\begin{aligned}
&\frac{1}{2} Tr\{\psi(t,\xi(t),\xi(t-\tau)) \frac{\partial^2 V_n}{\partial \xi^2} \cdot \psi^T(t,\xi(t),\xi(t-\tau))\} \\
&\le \frac{1}{2} r\sqrt{r} \sum_{i,j=1}^{n} |\frac{\partial^2 V_n}{\partial \xi^2}| |\psi^T(t,\xi(t),\xi(t-\tau))| |\psi(t,\xi(t),\xi(t-\tau))| \\
&\le \tilde{c}_{03} L^{1-\tilde{\gamma}_0} \sum_{i,j=1}^{n} \|\xi(t)\|_{\triangle}^{4\sigma-r_i-r_j-\theta} \times (\|\xi(t-\tau)\|_{\triangle}^{r_i+\frac{\theta}{2}} + \|\xi(t)\|_{\triangle}^{r_i+\frac{\theta}{2}}) \\
&\times (\|\xi(t)\|_{\triangle}^{r_j+\frac{\theta}{2}} + \|\xi(t-\tau)\|_{\triangle}^{r_j+\frac{\theta}{2}}) \\
&\le \tilde{c}_{03} L^{1-\tilde{\gamma}_0} (c_{03}\|\xi(t)\|_{\triangle}^{4\sigma} + \tilde{c}_{03}\bar{c}_{03}\|\xi(t)\|_{\triangle}^{4\sigma}\|\xi(t-\tau)\|_{\triangle}^{4\sigma}) \\
&\le L^{1-\tilde{\gamma}_0} (c_{03}\|\xi(t)\|_{\triangle}^{4\sigma} + \bar{c}_{03}\|\xi(t)\|_{\triangle}^{4\sigma}\|\xi(t-\tau)\|_{\triangle}^{4\sigma}),
\end{aligned} \tag{3.34}$$

in which $\tilde{\gamma}_0 = \min_{1\le i,j\le n} \{\gamma_{i2} + \gamma_{j2}\} > 0$, $c_{03} > 0$, $\bar{c}_{03} > 0$ and $\tilde{c}_{03} > 0$ are constants. Based on $L > 1$, we have

$$V(\xi) \le V_n(\xi) + \frac{h_1 + h_2}{1-\gamma} L^{1-\gamma_0} \int_{t-\tau}^{t} \|\xi\|_{\Delta}^{4\sigma} d\eta.$$

By Definition 1, (3.26), (3.31), (3.33) and (3.34), one has

$$\begin{aligned}
\mathcal{L}V \le & \frac{\partial V_n}{\partial \xi} LR(\xi) + \frac{\partial V_n(\xi)}{\partial \xi} T(t,\xi(t),\xi(t-\tau) \\
& + \frac{1}{2} Tr\{\psi^T(t,\xi(t),\xi(t-\tau)) \frac{\partial^2 V_n}{\partial \xi^2} \psi(t,\xi(t),\xi(t-\tau))\} \\
& + (h_1+h_2) L^{1-\gamma_0} \cdot (\frac{1}{1-\gamma}\|\xi(t)\|_{\triangle}^{4\sigma} - \|\xi(t-\tau)\|_{\triangle}^{4\sigma}) \\
\le & -L(c_{01} - (c_{02} + c_{03} + \frac{h_1+h_2}{1-\gamma}) L^{-\gamma_0}) \cdot \|\xi\|_{\triangle}^{4\sigma},
\end{aligned} \tag{3.35}$$

which satisfies $\gamma_0 = min\{\bar{\gamma}_0, \tilde{\gamma}_0\} < 1$. Because c_{01} is a constant independent of c_{02}, c_{03}, we choose $L > L^* = \max\{(\frac{c_{02}+c_{03}+\frac{h_1+h_2}{1-\gamma}}{c_1})^{\frac{1}{\gamma_0}}, 1\}$, and there exists a constant $B = c_{01} - (c_{02} + c_{03} + \frac{h_1+h_2}{1-\gamma})L^{-\gamma_0} > 0$, such that

$$\mathcal{L}V \leq -LB\|\xi\|_{\triangle}^{4\sigma} = -c_0\|\xi\|_{\triangle}^{4\sigma}.$$

With the help of the above formula and (3.28), one obtains

$$\mathcal{L}V(\xi(t)) \leq -(\frac{c_0}{\tilde{c}})\underline{\alpha}_{22}|(|\xi(t)|).$$

Briefly, following Steps 1–3, the system has a unique solution on $[-d, \infty]$, and $\xi(t) = 0$ is globally asymptotically stable in probability.

Step 4: Because (3.20) is an equivalent transformation, the system composed by (1) and $u^p = L^{k_{n+1}}v^p$ is similar to the systems (3.20) and (3.22). □

Remark 5. *Compared with [24], we construct a state-feedback controller independent of time delays for the stochastic nonlinear system. Compared with [30], we use the methods of adding a power integrator to relax the nonlinear growth condition to cover both high-order and low-order nonlinearities. Not only does it not need to know anything information about the unknown function, but also it can reduce burdensome computations.*

Remark 6. *The homogeneous domination method is used for the first time to solve the stabilization problem of stochastic high-order and low-order nonlinear system (1.1) with time-delay.*

Remark 7. *In this paper, it is hard to adopt a Lyapunov-Krasovskii functional. In order to solve the above the problem, a suitable Lyapunov-candidate-function is designed to guarantee good system performance, and stabilization analysis is proposed to save better resources*

Remark 8. *The construction of the controller effectively keeps away from the zero-division problem of* $\frac{\partial^2 \xi_i^{*\mu/r_i}}{\partial \xi_j^2}$. *It need be noted that the non-zero-division problem and the locally Lipschitz condition (see Step 1 in the proof of Theorem* 1*) should to be guaranteed simultaneously, which will increase more difficulties.*

4. Simulation example

Consider the following stochastic high-order and low-order nonlinear systems with time-delay:

$$\begin{cases} dx_1(t) = [x_2^3(t) + x_2^2(t)x_1(t-1)]dt + \frac{1}{4}x_1(t)\sin x_1(t-1)d\omega(t), \\ dx_2(t) = [u^3(t) + x_2(t)\cos x_2(t-1)]dt. \end{cases} \tag{4.1}$$

One can see that Assumption 1 is satisfied with $p_1 = p_2 = 3, \tau = 1, C = 1, r_1 = 1, \theta = \frac{2}{5}$. One can easily get

$$|f_1| \leq (|z_1| + |z_1|^{\frac{7}{5}} + |z_1(t-1)| + |z_1(t-1)|^{\frac{7}{5}})/5,$$

$$|g_1| \leq (|z_1| + |z_1|^{\frac{16}{5}} + |z_1(t-1)| + |z_1(t-1)|^{\frac{16}{5}})/8,$$

$$
\begin{aligned}
|f_2| \leq &(|z_1|^{\frac{1}{3}} + |z_1|^{\frac{13}{45}} + |z_2|^{\frac{1}{3}} + |z_2|^{\frac{13}{6}} + |z_1(t-1)|^{\frac{1}{3}} \\
&+ |z_1(t-1)|^{\frac{13}{45}} + |z_2(t-1)|^{\frac{1}{3}} + |z_2(t-1)|^{\frac{13}{6}})/5.
\end{aligned} \tag{4.2}
$$

In this simulation, we choose $V_1(z_1) = \frac{1}{5}z_1^5 + \frac{1}{10}z_1^{10} + 2\int_{t-1}^{t}(z_1^4 + z_1^{\frac{4\sigma}{r_1}})dl$. Several calculations lead to

$$
\begin{aligned}
\mathcal{L}V_1 \leq &-2(z_1^4 + z_1^{\frac{52}{5}}) - (z_1^4(t-1) + z_1(t-1)^{\frac{52}{5}}) \\
&+ (z_1 + z_1^9)(x_2^3 - \alpha_1^3),
\end{aligned} \tag{4.3}
$$

where $\alpha^{p_1} = -(2n+\beta_1)(z_1^3 + z_1^{\frac{7}{5}})$. By choosing $V_2(\overline{\eta}_2) = V_1(\eta_1) + W_{L2} + W_{H2} + W_{D2}$, a direct calculation leads to

$$
\mathcal{L}V_2 \leq -(z_1^4 + z_1^{\frac{52}{5}}) - (z_2^4 + z_2^{\frac{260}{7}}). \tag{4.4}
$$

From the previous manipulations, one obtains the following actual controller

$$
u(t) = -\varrho^{\frac{1}{9}}(z_2 + z_2^{\frac{13}{45}})^{\frac{1}{9}} = -306.771(z_2 + z_2^{\frac{13}{45}})^{\frac{1}{9}}. \tag{4.5}
$$

The initial condition can be given as $\xi_0(\theta) \equiv [1, -1]^T$. Figure 1 illustrates that the globally asymptotically stable in probability has been achieved and the responses of (4.5) is given in Figure 2.

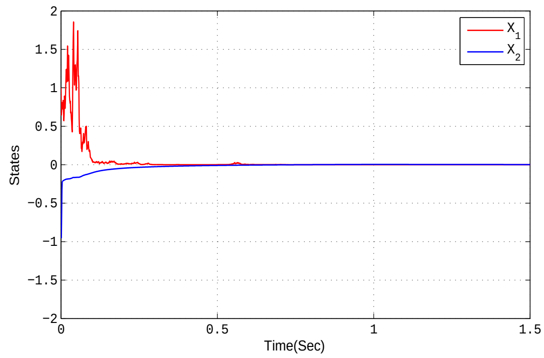

Figure 1. The trajectories of $x_1(t)$ and $x_2(t)$.

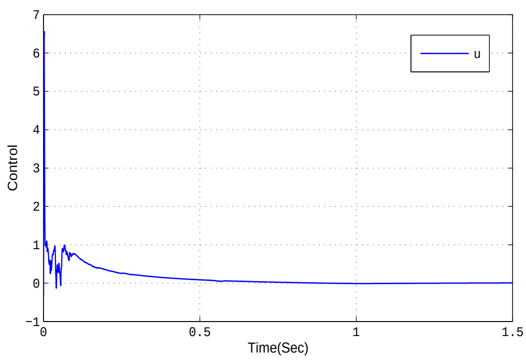

Figure 2. The trajectories of u.

5. Conclusions

In this technical note, we investigate the state feedback stabilization problem of stochastic high-order and low-order nonlinear systems with time-delay successfully. According to the homogeneous domination method and the design of integral Lyapunov functions, the control strategy is achieved with the controller design. The above results indicate that the closed-loop system is globally asymptotically stable in probability. There still remain problems to be investigated, such as how to take into account output feedback control and how to extend our results under weaker conditions.

Acknowledgments

This work is supported by National Natural Science Foundation of China under Grant 62173208, Taishan Scholar Project of Shandong Province of China under Grant tsqn202103061, Shandong Qingchuang Science and Technology Program of Universities under Grant 2019KJN036.

Conflict of interest

The authors declare no conflicts of interest.

References

1. P. Pepe, Z. Jiang, A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems, *Syst. Control Lett.*, **55** (2006), 1006–1014. http://doi.org/10.1016/j.sysconle.2006.06.013
2. X. F. Zhang, Q. R. Liu, E. Boukas, Feedback stabilization for high order feedforward onlinear time-delay systems, *Automatica*, **47** (2011), 962–967. http://doi.org/10.1016/j.automatica.2011.01.018
3. S. H. Yang, Z. Y.Sun, Z. Wang, T. Li, A new approach to global stabilization of high-order time-delay uncertain nonlinear systems via time-varying feedback and homogeneous domination, *J. Franklin I.*, **355** (2018), 6469–6492. http://doi.org/10.1016/j.jfranklin.2018.05.063
4. Z. B. Song, Z. Y. Sun, S. H. Yang, T. Li, Global stabilization via nested saturation function for high-order feedforward nonlinear systems with unknown time-varying delays, *Int. J. Robust Nonlin.*, **26** (2016), 3363–3387. http://doi.org/10.1002/rnc.3512
5. Z. Q. Zhang, J. W. Lu, S. Y. Xu, Tuning functions-based robust adaptive tracking control of a class of nonlinear systems with time delays, *Int. J. Robust Nonlin.*, **22** (2012), 1631–1646. http://doi.org/10.1002/rnc.1776
6. Z. Y. Sun, Y. Liu, Adaptive control design for a class of uncertain high-order nonlinear systems with time delay, *Asian J. Control*, **17** (2015), 535–543. http://doi.org/10.1002/asjc.895
7. Z. Y. Sun, X. H. Zhang, Continuous global stabilisation of high-order time-delay nonlinear systems, *Int. J. Control*, **86** (2013), 994–1007. http://doi.org/10.1080/00207179.2013.768776
8. Y. G. Liu, Global Asymptotic Regulation via time-varying output feedback for a class of uncertain nonlinear systems, *SIAM J. Control Optim.*, **51** (2013), 4318–4342. https://doi.org/10.1137/120862570

9. J. Wang, S. C. Huo, J. W. Xia, J. H. Park, X. Huang, H. Shen, Generalised dissipative asynchronous output feedback control for Markov jump repeated scalar non-linear systems with time-varying delay, *IET Control Theory A.*, **13** (2019), 2114–2121. http://doi.org/10.1049/IET-CTA.2018.6114

10. Y. Liu, Q. Zhu, G. Wen, Adaptive tracking control for perturbed strict-feedback nonlinear systems based on optimized backstepping technique, *IEEE T. Neur. Net. Lear.*, **33** (2020), 853–865. https://doi.org/10.1109/TNNLS.2020.3029587

11. H. Deng, M. Krstić, R. J. Williams, Stabilization of stochastic nonlinear systems driven by noise of unknown covariance, *IEEE T. Automat. Contr.*, **46** (2011), 1237–1253. https://doi.org/10.1109/9.940927

12. Q. M. Zhu, S. Billings, Parameter estimation for stochastic nonlinear rational models, *Int. J. Control*, **57** (1997), 309–333. http://doi.org/10.1080/00207179308934390

13. Z. Y. Sun, J. W. Xing, Q. H. Meng, Output feedback regulation of time-delay nonlinear systems with unknown continuous output function and unknown growth rate, *Int. J. Robust Nonlin.*, **100** (2020), 1309–1325. http://doi.org/10.1007/s11071-020-05552-3

14. Y. Liu, Decentralized output feedback stabilization for switched stochastic high-order nonlinear systems with time-varying state/input delays, *Inf. Sci.*, **603** (2022), 91–105.

15. L. Marconi, A. Isidllri, Robust global stabilization of a class of uncertain feedforward nonlinear systems, *Syst. Contml Lett.*, **41** (2006), 281–290. https://doi.org/10.1016/S0167-6911(00)00066-9

16. C. J. Qian, W. Lin, Recursive observer design, homogeneous approximation, and nonsmooth output feedback stabilization of nonlinear systems, *IEEE T. Automat. Contr.*, **51** (2006), 1457–1471. https://doi.org/10.1109/TAC.2006.880955

17. P. Florchinger, Lyapunov-like techniques for stochastic stability, *SIAM J. Control Optim.*, **33** (1995), 1151–1169. https://doi.org/10.1137/S0363012993252309

18. W. T. Zha, J. Y. Zhai, S. M. Fei, Output feedback control for a class of stochastic high-order nonlinear systems with time-varying delays, *Int. J. Robust Nonlin.*, **24** (2013), 2243–2260. http://doi.org/10.1002/rnc.2985

19. L. Liu, X. J. Xie, Output-feedback stabilization for stochastic high-order nonlinear systems with time-varying delay, *Automatica*, **47** (2011), 2772–2779. http://doi.org/10.1016/j.automatica.2011.09.014

20. Y. S. Fu, Z. Tian, S. Shi, State feedback stabilization for a class of stochastic time-delay nonlinear systems, *IEEE T. Automat. Contr.*, **2** (2003), 494–499. http://doi.org/10.1109/TAC.2002.808481

21. W. S. Chen, J. Wu, L. Jiao, State-feedback stabilization for a class of stochastic time-delay nonlinear systems, *Int. J. Robust Nonlin.*, **22** (2012), 1921–1937. https://doi.org/10.1002/rnc.1798

22. X. J. Xie, L. Liu, A homogeneous domination approach to state feedback of stochastic high-order nonlinear systems with time-varying delay, *IEEE T. Automat. Contr.*, **33** (2013), 2577–2586. https://doi.org/10.1109/TAC.2012.2208297

23. L. Liu, X. J. Xie, State-feedback stabilization for stochastic high-order nonlinear systems with SISS inverse dynamics, *Asian J. Control*, **14** (2012), 207–216. http://doi.org/10.1002/asjc.288

24. Z. Y. Sun, Z. G. Liu, X. H. Zhang, New results on global stabilization for time-delay nonlinear systems with low-order and high-order growth conditions, *Int. J. Robust Nonlin.*, **25** (2013), 878–899. http://doi.org/10.1002/rnc.3115

25. C. J. Qian, J. Li, Global output feedback stabilization of upper-triangular nonlinear systems using a homogeneous domination approach, *Int. J. Robust Nonlin.*, **16** (2006), 441–463. http://doi.org/10.1002/RNC.1074

26. F. Z. Li, T. S. Li, G. Feng, Adaptive neural control for a class of stochastic nonlinear time-delay systems with unknown dead zone using dynamic surface technique, *Int. J. Robust Nonlin.*, **26** (2016), 759–781. http://doi.org/10.1002/rnc.3336

27. Z. Y. Sun, D. Zhang, C. C. Chen, Q. H. Meng, Feedback stabilisation of time-delay nonlinear systems with continuous time-varying output function, *Int. J. Syst. Sci.*, **50** (2018), 244–255. http://doi.org/10.1080/00207721.2018.1543472

28. S. J. Liu, J.F. Zhang, Z.P. Jiang, Decentralized adaptive output-feedback stabilization for large-scale stochastic nonlinear systems, *Automatica*, **43** (2007), 238–251. http://doi.org/10.1016/j.automatica.2006.08.028

29. X. J. Xie, L. Liu, Further results on output feedback stabilization for stochastic high-order nonlinear systems with time-varying delay, *Automatica*, **48** (2012), 2577–2586. http://doi.org/10.1016/j.automatica.2012.06.061

30. G. D. Zhao, X. J. Xie, A homogeneous domination approach to state feedback of stochastic high-order nonlinear systems, *Int. J. Robust Nonlin.*, **86** (2013), 966–976. http://doi.org/10.1080/00207179.2013.767942

31. X. Y. Qin, H. F. Min, Further results on adaptive state feedback stabilization for a class of stochastic feedforward nonlinear systems with time delays, *T. Meas. Control*, **41** (2019), 127–134. https://doi.org/10.1177/0142331217752

32. Z. B. Song, J. Y. Zhai, Decentralized output feedback stabilization for switched stochastic high-order nonlinear systems with time-varying state/input delays, *ISA T.*, **90** (2019), 64–73. https://doi.org/10.1016/j.isatra.2018.12.044

Appendix

Proof of Proposition 2.

Proof. Obviously, the conclusion of $i = 1$ is easy to prove. When $i > 1$, according to (3.2) and Lemma 1, one has the inequality

$$|x_i(\theta)| \leq |z_i(\theta)|^{\frac{1}{p_1 \cdots p_{i-1}}} + \varrho_{i-1}^{\frac{1}{p_1 \cdots p_{i-1}}} |z_i(\theta)|^{\frac{1}{p_1 \cdots p_{i-1}}} + \varrho_{i-1}^{\frac{1}{p_1 \cdots p_{i-1}}} |z_{i-1}(\theta)|^{\frac{r_i+\omega}{r_{i-1} p_1 \cdots p_{i-1}}}$$

and the estimate for $j = 2, 3, \ldots, i-1$,

$$|x_j(\theta)|^{\frac{1}{p_j \cdots p_{i-1}}} \leq |z_j(\theta)|^{\frac{1}{p_1 \cdots p_{i-1}}} + \varrho_{j-1}^{\frac{1}{p_1 \cdots p_{i-1}}} |z_{j-1}(\theta)|^{\frac{1}{p_1 \cdots p_{i-1}}} + \varrho_{j-1}^{\frac{1}{p_1 \cdots p_{i-1}}} |z_{j-1}(\theta)|^{\frac{r_j p_{j-1}}{r_{j-1} p_1 \cdots p_{i-1}}},$$

in which θ denotes t *or* $t-\tau$. Similarly, with the help of (3.2) and Lemma 1 again, one arrives at

$$\begin{aligned}|x_j(\theta)|^{\frac{r_i+\omega}{r_j}} \leq \max\{1, 2^{\frac{r_i+\omega}{r_j p_{j-1}\cdots p_1}-1}\}(|z_j(\theta)|^{\frac{r_i+\omega}{r_j p_{j-1}\cdots p_1}} + \varrho_{j-1}^{\frac{r_i+\omega}{r_j p_{j-1}\cdots p_1}}|z_{j-1}(\theta)|^{\frac{1}{p_1\cdots p_{i-1}}} \\ + \varrho_{j-1}^{\frac{r_i+\omega}{r_j p_{j-1}\cdots p_1}}|z_{j-1}(\theta)|^{\frac{r_i+\omega}{r_{j-1} p_{j-2}\cdots p_1}}),\end{aligned}$$

where $j = 2, 3, \ldots, i$. From Assumption 1 and Lemma 4, one obtains

$$\begin{aligned}c_1 =& C + C \max_{1\leq j\leq i}\{2\varrho_j^{\frac{1}{p_1\cdots p_{j-1}}} + \max\{1, 2^{\frac{r_i+\omega}{r_j p_{j-1}\cdots p_1}-1}\} \\ &+ 2\varrho_j^{\frac{r_{i+1}p_i}{r_j p_{j-1}\cdots p_1}} \max\{1, 2^{\frac{r_i+\omega}{r_{j+1} p_j\cdots p_1}-1}\}\},\end{aligned}$$

and similarly we can get the parameters in $g(\cdot)$

$$\begin{aligned}c_2 =& C + C \max_{1\leq j\leq i}\{2\varrho_j^{\frac{1}{p_1\cdots p_{j-1}}} + \max\{1, 2^{\frac{2r_i+\omega}{2r_j p_{j-1}\cdots p_1}-1}\} \\ &+ 2\varrho_j^{\frac{r_{i+1}p_i}{r_j p_{j-1}\cdots p_1}} \max\{1, 2^{\frac{2r_i+\omega}{2r_{j+1} p_j\cdots p_1}-1}\}\}.\end{aligned}$$

A direct calculation leads to (3.4). □

AIMS Press