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Research article

Assisting the decision making-A generalization of choice models to handle the binary choices

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Abstract: This research fundamentally aims at providing a generalized framework to assist the launch of paired comparison models while dealing with discrete binary choices. The purpose is served by exploiting the fundaments of the exponential family of distributions. The proposed generalization is proved to cater to seven paired comparison models as members of this newly developed mechanism. The legitimacy of the devised scheme is demonstrated through rigorous simulation-based investigation as well as keenly persuaded empirical evaluations. A detailed analysis, covering a wide range of parametric settings, through the launch of Gibbs Sampler—a notable extension of Markov Chain Monte Carlo methods, is conducted under the Bayesian paradigm. The outcomes of this research substantiate the legitimacy of the devised general structure by not only successfully retaining the preference ordering but also by staying consistent with the established theoretical framework of comparative models.

Keywords: choice behaviors; comparative models; exponential family of distributions; paired comparison
Mathematics Subject Classification: 03E25

1. Introduction

The utility of paired comparison (PC) models in analyzing choice behaviors is well appreciated in various fields of research. For example, [1,2] demonstrated the applicability of PC schemes in health surveillance. Similarly, [3,4] applied PC methods to study food preferences and quality characteristics. Further, [5,6] persuaded the PC approach in the exploring the socio-political behaviors of the voters. Moreover, [7,8] employed PC models to conduct sports analysis. Recently, [9,10] elucidated the applicability of PC models in public health administration while facilitating the arduous task of project prioritization. For the account of more applications, one may see [11,12] in field of sensory analysis, [13,14] in engineering and reliability and [15,16] for measurement systems.

The PC models usually arise by considering a latent point-scoring process while conducting a pair-wise comparison among streams of objects, strategies or treatments [17]. Avoiding the literary jargon, a selector is requested to answer a simple query in "yes" or "no" fashion "do you prefer item i over item j?" while pairwise comparing a string of competing items. Table 1 below summarizes the hypothetical choice matrix comprehending the binary responses of a single selector resulting from the above inquiry while comparing m rival items. Each cell of the table documents the comparative choice of the decision maker while comparing a pair of objects specified by a certain row and column of the table. The choice strings then follow Binomial distribution where the likelihood of preferences remains estimable as a function of worth parameters defining the relative utility of competing objects.

Table 1. Choice matrix involving single decision maker and m competing items, Y = yes and N = No.

Items	*1*	*2*	*3*	*4*	*5*	-	m
1	-	Y	Y	Y	N	-	Y
2		-	N	N	N	-	N
3			-	N	Y	-	Y
4				-	Y	-	Y
5					-	-	N
-						-	-
m							-

It is trivial to extend the afore-mentioned scenario for k selectors or judges. Despite the simplistic formation, the capability of above documented contingency in facilitating the optimization of complex decision making by inter-relating non-linear functionals is well established [10,18–20].

Inspired by the subtle nature of the pre-describe design, this research aims at the proposition of a generalized framework encapsulating a broad range of choice or comparative models in a single comprehensive expression. The devised generalization is argued to be advantageous especially due to its capability to entertain various probabilistic structures governing the utility functionals as latent phenomena. The objectives are achieved by exploiting the fundaments of the exponential family of distributions. The choice of the exponential family of distribution in this regard is mainly motivated by three facts. Firstly, the family of distributions provides the fundaments of linear models and generalized linear models [21] and therefore is anticipated to offer natural support to the modeling of the binary choice data [22]. Secondly, the ability of the exponential family in encompassing of complex linear and non-linear functions is well cherished [23,24]. Lastly, the involvement of the exponential function in the estimation procedure usually results in more precise estimates [25]. The legitimacy of the proposed scheme is established through meticulously launched methodological and simulation-based operations using the Bayesian paradigm. Moreover, the inferential aspects of the suggested generalization are explored in order to derive a statistically sound and mathematically workable line of actions to attain an optimal decision-making strategy. Furthermore, the applicability of the targeted generalization is advocated by studying the water brand choice data.

This article is mainly divided into five parts. Section 2 delineates the mathematical foundations of the proposed generalization whereas section 3 reports simulation-based outcomes advocating the legitimacy of the devised scheme. Section 4 is dedicated to the empirical evaluation and lastly, section 5 summarizes the main findings in a compact manner.

2. Materials and methods

2.1. Proposed generalization

Let us say that a pairwise comparison is persuaded among m objects by n judges, where the pair of stimuli elicits a continuous discriminal process. The latent preferences of competing object i and object j are then thought to follow exponential family of distributions over the consistent support in the population, such as;

$$f(x_i;\theta_i) = a(\theta_i)b(x_i)e^{g(x_i)h(\theta_i)},\ \ c < x_i < d,$$

and

$$f(x_j;\theta_j) = a(\theta_j)b(x_j)e^{g(x_j)h(\theta_j)},\ \ c < x_j < d,$$

where, θ_i and θ_j are worth parameters highlighting the utility associated with respective objects. The interest lies in the deduction of precipitated preferences, such as $p_{ij} = P(X_i > X_j)$ and $p_{ji} = P(X_j > X_i)$, as a function of estimated worth parameters. We proceed by defining a general functional facilitating the estimation of preference probabilities such as;

$$P(X_i > X_j) = \int_c^d \int_{x_j}^d f(x_j;\theta_j)\ f(x_i;\theta_i)dx_i dx_j, \tag{1}$$

where, $\int_{x_j}^d f(x_i;\theta_i)dx_i = F(d;\theta_i) - F(x_j;\theta_i)$. By using this expression in Eq (1), we obtained:

$$P(X_i > X_j) = \int_c^d f(x_j;\theta_j)\, F(d;\theta_i)dx_j - \int_c^d f(x_j;\theta_j)\, F(x_j;\theta_i)dx_j. \tag{2}$$

For further simplification, let us denote,

$$A = \int_c^d f(x_j;\theta_j)\, F(d;\theta_i)dx_j,$$

and

$$B = \int_c^d f(x_j;\theta_j)\, F(x_j;\theta_i)dx_j.$$

It remain verifiable that on solving, we get

$$A = F(d;\theta_i)[F(d;\theta_j) - F(c;\theta_j)].$$

$$B = F(d;\theta_i)\, F(d;\theta_j) - F(d;\theta_i)\, F(c;\theta_j) - [F(d;\theta_i) - F(c;\theta_i)][F(d;\theta_j) - F(c;\theta_j)].$$

The Eq (2) now becomes,

$$P(X_i > X_j) = F(d;\theta_i)[F(d;\theta_j) - F(c;\theta_j)] - F(d;\theta_i)\, F(d;\theta_j) + F(d;\theta_i)\, F(c;\theta_j) +$$

$$[F(d;\theta_i) - F(c;\theta_i)][F(d;\theta_j) - F(c;\theta_j)],$$

which on further simplification reduces to,

$$P(X_i > X_j) = F(d;\theta_i)F(d;\theta_j) - F(d;\theta_i)F(c;\theta_j) - F(c;\theta_i)F(d;\theta_j) + F(c;\theta_i)F(c;\theta_j),$$

where, $F(d;\theta_i) = [1 - F(c;\theta_i)]$ and $F(d;\theta_j) = [1 - F(c;\theta_j)]$. Using these specifications, we finally achieve the general expression confirming the preference of object i over object j, as under

$$P(X_i > X_j) = 1 - 2F(c;\theta_i) - 2F(c;\theta_j) + 4\,F(c;\theta_i)F(c;\theta_j). \quad (3)$$

The two features of the devised generalized formation given in Eq (3) remain immediately noticeable. Firstly, it remains verifiable that for any permissible value of lower limit of the support, c, the above given functional reduces to 1, ensuring the ability of the general scheme in establishing the true preferences. Secondly, the preference probabilities remain estimable as a function of worth parameters ensuring the desirable character of decision making that is utility based choices. Both realizations are consistent with classic and eminent luce's choice axiom. One may also notice that the above given functional can also be derived for $P(X_j > X_i)$. Table 2, presents seven PC models based on more prominent exponential family of distributions' member which stay as special cases of aforementioned general scheme. It is to be noted that we are only considering these seven cases for demonstration purposes, in fact every PC model which arises due to the assumption that latent point process follows exponential family of distributions can be easily seen as sub-case of the proposition.

Table 2. Some of the members of proposed generalization.

Model	p.d.f.	Exponential family $f(x;\ \theta) = a(\theta)b(x)e^{g(x)h(\theta)}$	Preference probability (p_{ij})
Beta	$f(x;\ \theta) = \frac{x^{\theta-1}(1-x)}{B(\theta,\ 2)}$, $0 < x < 1$	$a(\theta) = \theta(\theta+1), b(x) = (1-x)$, $g(x) = \ln x, h(\theta) = (\theta - 1)$	$p_{ij} = \frac{\theta_i(1+\theta_i)(2+\theta_i+3\theta_j)}{(\theta_i+\theta_j)(1+\theta_i+\theta_j)(2+\theta_i+\theta_j)}$
Power	$f(x;\ \theta) = \theta x^{\theta-1}$, $0 < x < 1$	$a(\theta) = \theta, b(x) = 1$, $g(x) = \ln x, h(\theta) = (\theta - 1)$	$p_{ij} = \frac{\theta_i}{\theta_i + \theta_j}$
Exponential	$f(x;\ \theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$, $0 < x < \infty$	$a(\theta) = \frac{1}{\theta}, b(x) = 1$, $g(x) = x, h(\theta) = \frac{1}{\theta}$	$p_{ij} = \frac{\theta_i}{\theta_i + \theta_j}$
Gamma	$f(x;\ \theta) = \frac{\theta^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}x^{\frac{1}{2}-1}e^{-\theta x}$, $0 < x < \infty$	$a(\theta) = \sqrt{\theta}, b(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)\sqrt{x}}$, $g(x) = -x, h(\theta) = \theta$	$p_{ij} = \frac{2}{\pi}\tan^{-1}\sqrt{\frac{\theta_i}{\theta_j}}$
Maxwell	$f(x;\ \theta) = \sqrt{\frac{2}{\pi}}\frac{x^2}{\theta^3}e^{-\frac{x^2}{2\theta^2}}$, $0 < x < \infty$	$a(\theta) = \frac{1}{\theta^3}, b(x) = \sqrt{\frac{2}{\pi}}x^2$, $g(x) = -\frac{x^2}{2}, h(\theta) = \frac{1}{\theta^2}$	$p_{ij} = 1 + \frac{2}{\pi}\left\{\frac{\theta_i^3\theta_j - \theta_i\theta_j^3}{(\theta_i^2+\theta_j^2)^2} - \tan^{-1}\left(\frac{\theta_j}{\theta_i}\right)\right\}$
Rayleigh	$f(x;\ \theta) = \frac{x}{\theta^2}e^{-\frac{x^2}{2\theta^2}}$, $0 < x < \infty$	$a(\theta) = \frac{1}{\theta^2}, b(x) = x$, $g(x) = -\frac{x^2}{2}, h(\theta) = \frac{1}{\theta^2}$	$p_{ij} = \frac{\theta_i^2}{\theta_i^2 + \theta_j^2}$
Weibull	$f(x;\ \theta) = \frac{3x^2}{\theta^3}e^{-\frac{x^3}{\theta^3}}$, $0 < x < \infty$	$a(\theta) = \frac{1}{\theta^3}, b(x) = 3x^2$, $g(x) = -x^3, h(\theta) = \frac{1}{\theta^3}$	$p_{ij} = \frac{\theta_i^3}{\theta_i^3 + \theta_j^3}$

The likelihood function, where n judges are deemed to pairwise comparison of m objects, is written by denoting n_{ij} as the total number of times object i and object j are pairwise compared. Also, let us represent, $\underline{r} = (r_{ij}, r_{ji})$ as a vector comprises of the observed preference data in $k'th$ repetition, when $i \neq j$, $i \geq 1$ and $j \leq m$. Based on these specifications, the likelihood function encompassing the preferences of n judges pairwise comparing m objects, is written as under:

$$l(\underline{r}, \underline{\theta}) = \prod_{i<j=1}^{m} \frac{n_{ij}!}{r_{ij}!(n_{ij}-r_{ij})!} p_{ij}^{r_{ij}} p_{ji}^{n_{ij}-r_{ij}}, \tag{4}$$

where, $p_{ji} = 1 - p_{ij}$ and $\underline{r}$ represents the preference vector along with $\underline{\theta}$ denoting the vector of worth parameters. As a fact, the number of worth parameters stays equal to the number of objects to be compared, such that $\sum_{i=1}^{m} \theta_i = 1$. The imposed condition resolves the issue of non-identifiability.

2.2. Estimation of worth parameters

The estimation of worth parameters is persuaded under the Bayesian paradigm – well cherished to channelize the historic information in order to enrich the analytical environment and thus assists the estimation procedure. For demonstration purposes, we consider two prior distributions, that is Jeffreys Prior and Uniform Prior.

2.2.1. The Posterior Distribution under the Jeffreys Prior

The kernel of Jeffreys prior for $\underline{\theta}$; $(\theta_1, \theta_2, \theta_3, \ldots, \theta_m)$ is written as follows:

$$p_J(\theta_1, \theta_2, \theta_3, \ldots, \theta_m) \propto \sqrt{\det[I(\theta_1, \theta_2, \theta_3, \ldots, \theta_m)]}, \qquad 0 < \underline{\theta} < 1.$$

where, $\det[I(\underline{\theta})] = (-1)^{m-1} \begin{vmatrix} E\left[\frac{\partial^2 \ln L(.)}{\partial\theta_1^2}\right] & E\left[\frac{\partial^2 \ln L(.)}{\partial\theta_1 \partial\theta_2}\right] \ldots & E\left[\frac{\partial^2 \ln L(.)}{\partial\theta_1 \partial\theta_{m-1}}\right] \\ E\left[\frac{\partial^2 \ln L(.)}{\partial\theta_2 \partial\theta_1}\right] & E\left[\frac{\partial^2 \ln L(.)}{\partial\theta_2^2}\right] \ldots & E\left[\frac{\partial^2 \ln L(.)}{\partial\theta_2 \partial\theta_{m-1}}\right] \\ \vdots & \vdots & \vdots \\ E\left[\frac{\partial^2 \ln L(.)}{\partial\theta_{m-1} \partial\theta_1}\right] & E\left[\frac{\partial^2 \ln L(.)}{\partial\theta_{m-1} \partial\theta_2}\right] \ldots & E\left[\frac{\partial^2 \ln L(.)}{\partial\theta_{m-1}^2}\right] \end{vmatrix}$. The estimability of the worth parameter associated with $m'th$ object is conferred by ensuring that $\theta_m = 1 - \theta_1 - \theta_2 \ldots - \theta_{m-1}$. The joint posterior distribution for $\theta_1, \theta_2, \ldots, \theta_m$ is then written as:

$$p_J(\theta_1, \theta_2, \ldots, \theta_m|\mathbf{r}) = \frac{1}{k} \prod_{i<j=1}^{m} p_J(\theta_1, \theta_2, \theta_3, \ldots, \theta_m) p_{ij}^{r_{ij}} p_{ji}^{n_{ij}-r_{ij}}. \tag{5}$$

Here, $k = \int_0^1 \int_0^{1-\theta_1} \ldots \int_0^{1-\theta_1 \ldots -\theta_{m-1}} p_J(\theta_1, \theta_2, \theta_3, \ldots, \theta_m) p_{ij}^{r_{ij}} p_{ji}^{n_{ij}-r_{ij}} d\theta_{m-1} \ldots d\theta_2 d\theta_1$ and is known as normalizing constant while obliging the above given constraint, such as $\theta_m = 1 - \sum_{i=1}^{m-1} \theta_i$. Also, n_{ij} represents the frequency of pair-wise comparisons by selectors and r_{ij} denotes the frequency of referring object i over object j. The joint posterior distribution of Eq (5) is not of closed form, therefore Bayes estimates are attained by employing Gibbs sampler – a well cherished procedure of MCMC methods [26- 28]. The marginal posterior distributions (MPDs) of parameters determining the comparative worth of each object are achieved by iteratively conditioning on interim value in a continuous cycle. Let $p_J(\underline{\theta}; \underline{r})$ be the joint posterior density, where $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$, then the conditional densities are given by, $p_J(\theta_1 \mathrm{I} \theta_2, \theta_3 \ldots, \theta_m)$, $p_J(\theta_2 \mathrm{I} \theta_1, \theta_3 \ldots, \theta_m)$ $\ldots p_J(\theta_m \mathrm{I} \theta_1, \theta_2 \ldots, \theta_{m-1})$. According to the Gibbs sampler, we assume initial values such as

$\left(\theta_2{}^{(0)}, \theta_3{}^{(0)}, \ldots, \theta_m{}^{(0)}\right)$ and pursue the conditional distribution of θ_1 such that $p_J\left(\theta_1{}^{(1)} \mathrm{I} \theta_2{}^{(0)}, \theta_3{}^{(0)}, \ldots, \theta_m{}^{(0)}\right)$. The iterative procedure will continue until it converges. Here, for demonstration purposes, we provide the expression of MPD of θ_m, as follows,

$$p(\theta_m|r) = \frac{1}{k} \int_{\theta_1=0}^{1-\theta_m} \ldots \int_{\theta_m=0}^{1-\sum_{i=1}^{m-2} \theta_i - \theta_m} \prod_{i<j=1}^{m-1} p_J(\theta_1, \theta_2, \theta_3, \ldots., \theta_m)\, d\theta_{m-1} \ldots d\theta_1, \quad (6)$$

$$0 < \theta_m < 1$$

The MPDs of other parameters remain deductible in similar fashion.

2.2.2. The Posterior Distribution under the Uniform Prior

The Uniform prior for $\underline{\theta}$; $(\theta_1, \theta_2, \theta_3, \ldots., \theta_m)$, is given as,

$$p_U(\theta_1, \theta_2, \ldots, \theta_m) \propto 1, \underline{\theta} > 0$$

The joint posterior distribution given the preference data is now determined as,

$$p_U(\theta_1, \theta_2, \ldots, \theta_m|r) = \frac{1}{k} \prod_{i<j=1}^{m} p_{ij}^{r_{ij}}\, p_{ji}^{n_{ij}-r_{ij}}. \quad (7)$$

Here, the normalizing constant under the estimability condition of $\theta_m = 1 - \sum_{i=1}^{m-1} \theta_i$ takes the form such as $k = \int_0^1 \int_0^{1-\theta_1} \ldots \int_0^{1-\theta_1 \ldots - \theta_{m-1}} \prod_{i<j=1}^{m} p_{ij}^{r_{ij}}\, p_{ji}^{n_{ij}-r_{ij}}\, d\theta_{m-1} \ldots d\theta_2 d\theta_1$. Next phase provides the expression of MPD for θ_m using the above mentioned method of Gibbs sampler. The MPD is given as,

$$p_U(\theta_m|r) = \frac{1}{K} \int_{\theta_1=0}^{1-\theta_m} \ldots \int_{\theta_m=0}^{1-\sum_{i=1}^{m-2} \theta_i - \theta_m} \prod_{i<j=1}^{m-2} p_{ij}^{r_{ij}}\, p_{ji}^{n_{ij}-r_{ij}}\, d\theta_{m-1} \ldots d\theta_1, 0 < \theta_m < 1. \quad (8)$$

3. Simulation-based evaluation

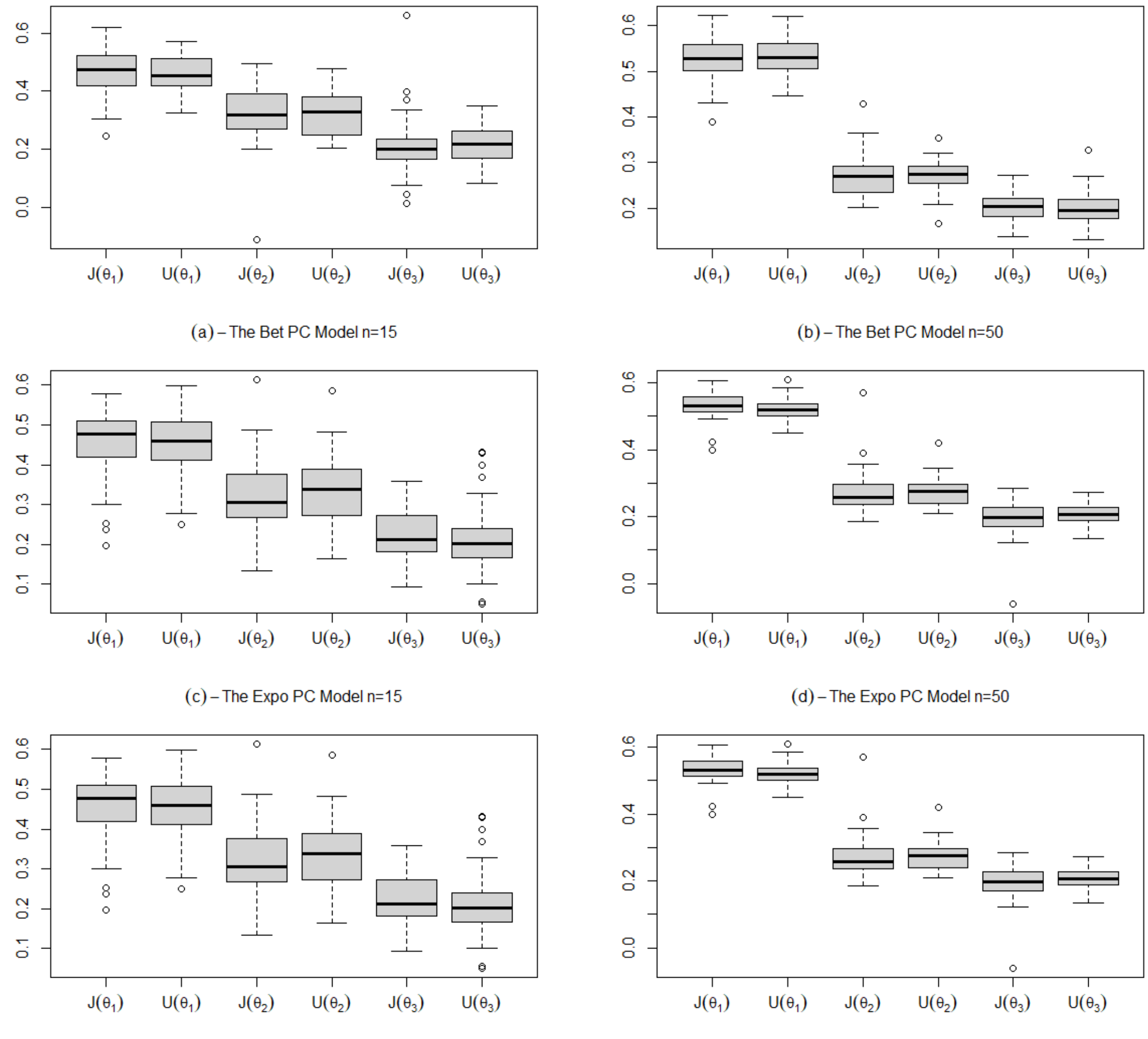
We now explore the authenticity of the proposed generalization with respect to above documented seven sub-cases. The objective is persuaded through rigorous simulation investigation mimicking wide range of experimental states. Artificial comparative data sets of two sizes $n = 15$ and 50 are generated comparing three objects, that is $i = 1,2$ and 3. The worth parameters are pre-set as $\theta_1 = 0.5$, $\theta_2 = 0.3$ and $\theta_3 = 0.2$. This setting is considered for demonstration purposes only, one can use other settings also. Table 3, presents the data under afore-mentioned settings. The Bayes estimates of worth parameters, estimated preference probabilities and Bayes factor are provided under both priors and for all considered sub-cases of the devised generalization. A detail account of the findings is documented in upcoming sections.

Table 3. Artificial data sets generated under above documented specifications.

	$n_{ij} = 15$		$n_{ij} = 50$	
(i, j)	r_{ij}	r_{ji}	r_{ij}	r_{ji}
(1, 2)	10	5	32	18
(1, 3)	9	6	38	12
(2, 3)	11	4	28	22

3.1. Discussion of MPDs

Figure 1 (a–n) presents the graphical display of the MPDs of worth parameters through side-by-side box plots. The behavior is depicted for both priors, that is Jefferys prior and Uniform priors, while covering sample of sizes $n = 15$ and $n = 50$. The delicacies of displayed outcomes are read with respect to different resultant sub-cases of the proposed family and both sample sizes. Firstly, through side-by-side box plots, it is observed that as the sample size increases more compact behavior of MPDs is observed. This is seen regardless of the considered prior distribution and sub-cases of the proposed family. Furthermore, Uniform prior is found to stand out in terms of the generation of the number of outliers. In most cases, the Uniform prior is noticed to produce a lesser extent of outliers as compared to contemporary prior model of Jefferys. It is thought that the tendency of Uniform prior to outperform Jefferys prior on this front lies in its capability of deducing the same amount of information from the data but through a more parsimonious layout.



(a) – The Bet PC Model n=15

(b) – The Bet PC Model n=50

(c) – The Expo PC Model n=15

(d) – The Expo PC Model n=50

(e) – The Power PC Model n=15

(f) – The Power PC Model n=50

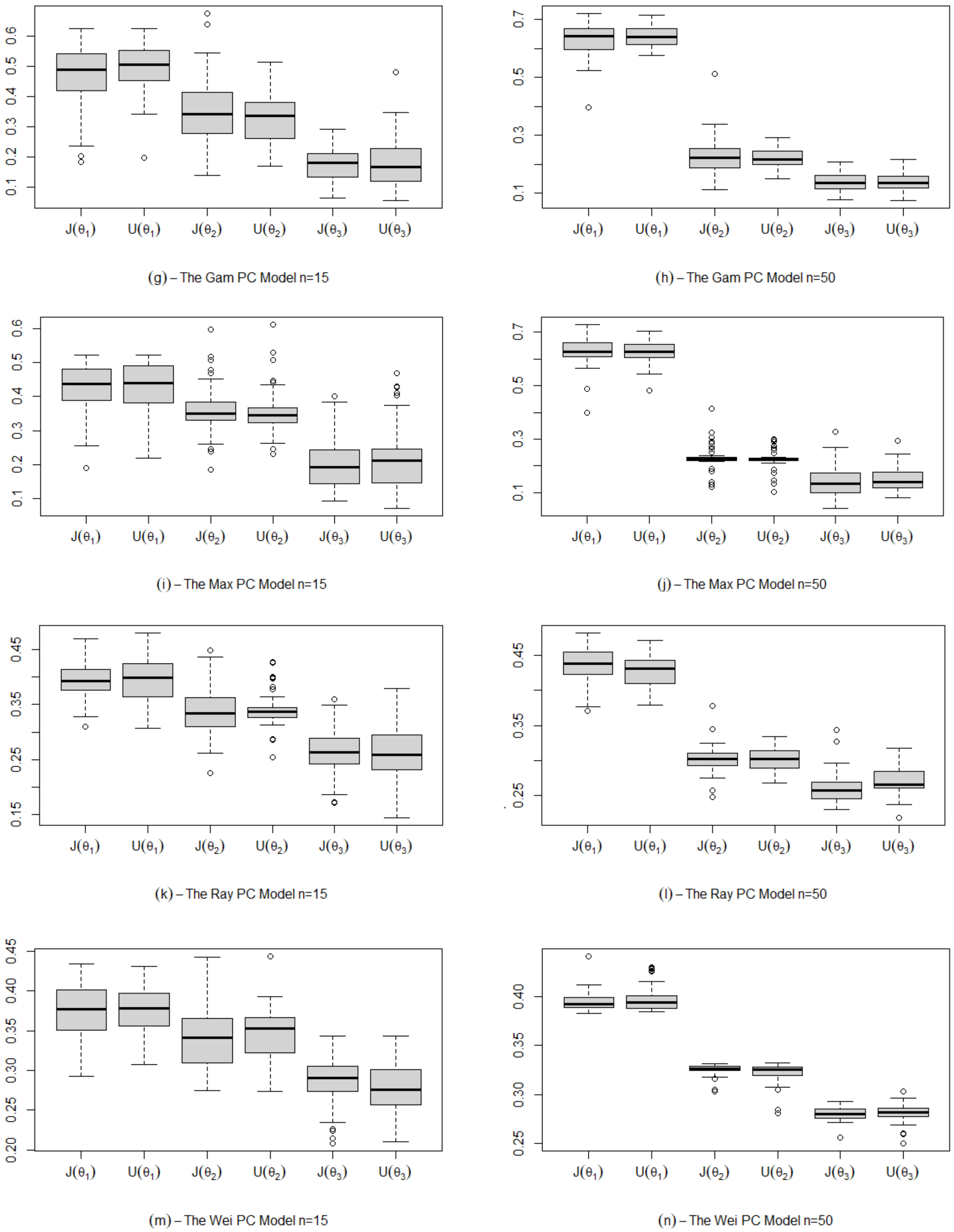


Figure 1. Display of MPDs of worth parameters under both priors and for both sample sizes. Here $J(\theta_i)$ and $U(\theta_i)$ represent MPDs under Jeffery's prior and MPDs under Uniform prior, respectively.

3.2. *Estimation of worth parameters*

The estimation of worth parameters defining the utility of competing objects is facilitated by providing the posterior means of the MPDs. Table 4 comprehends the outcomes of the estimation efforts along with associated absolute differences. One may notice some interesting patterns revealed in the table. Firstly, regardless of the prior distribution, increased sample size produces more close estimates of the utility. It is important to note that the estimation performance, however, remains subject to the sub-cases of the proposed family. Moreover, the estimated worth parameters for each member remain robust towards the change in the prior distribution. This outcome remains arguable as both considered priors are non-informative and thus provide equally enriched estimation environment. As long as, model-wise estimation performance is concerned, the PC model produces the most prolific estimates and therefore is argued to be most capable in using the comparative information more rigorously. From Table 2, one may notice that the Gamma model is attributed with a more vibrant and rich utility function estimating the preference behaviors by not only involving contemporary worth parameters but also employing geometric functions. The next in line remains Exponential and Power models with equal elegancy. This outcome in fact verify the simplifications provided in the Table 2. The characterization is thought to be a result of tendency of these models to entertain the comparative behaviors by exploiting linear, product and ratio formation of the associated worth parameters through more simple manner. The Beta model is ranked third in this comparative evaluation along with Maxwell model holding the forth level in the hierarchy. Whereas, Rayleigh model and Weibull model are placed at fifth and sixth position as contestant models.

Table 4. Estimates of worth parameters and associated absolute errors.

Models	Estimators	n_{ij} =15		n_{ij} =50	
		Jeffreys	Uniform	Jeffreys	Uniform
Beta	$\widehat{\theta_1}$	0.4424	0.4424	0.5205	0.5205
	$\widehat{\theta_2}$	0.3477	0.3477	0.2783	0.2783
	$\widehat{\theta_3}$	0.2099	0.2099	0.2012	0.2012
	$\lvert\widehat{\theta_1}-\theta_1\rvert$	0.0576	0.0576	0.0205	0.0205
	$\lvert\widehat{\theta_2}-\theta_2\rvert$	0.0477	0.0477	0.0217	0.0217
	$\lvert\widehat{\theta_3}-\theta_3\rvert$	0.0099	0.0099	0.0012	0.0012
Exponential	$\widehat{\theta_1}$	0.4615	0.4465	0.5376	0.5302
	$\widehat{\theta_2}$	0.3434	0.3472	0.2710	0.2738
	$\widehat{\theta_3}$	0.1952	0.2063	0.1914	0.1960
	$\lvert\widehat{\theta_1}-\theta_1\rvert$	0.0385	0.0535	0.0376	0.0302
	$\lvert\widehat{\theta_2}-\theta_2\rvert$	0.0434	0.0472	0.0290	0.0262
	$\lvert\widehat{\theta_3}-\theta_3\rvert$	0.0048	0.0063	0.0086	0.0040
Power	$\widehat{\theta_1}$	0.4615	0.4465	0.5376	0.5302
	$\widehat{\theta_2}$	0.3434	0.3472	0.2710	0.2738
	$\widehat{\theta_3}$	0.1952	0.2063	0.1914	0.1960
	$\lvert\widehat{\theta_1}-\theta_1\rvert$	0.0385	0.0535	0.0376	0.0302
	$\lvert\widehat{\theta_2}-\theta_2\rvert$	0.0434	0.0472	0.0290	0.0262
	$\lvert\widehat{\theta_3}-\theta_3\rvert$	0.0048	0.0063	0.0086	0.0040

Continued on next page

Models	Estimators	n_{ij} =15		n_{ij} =50	
		Jeffreys	Uniform	Jeffreys	Uniform
Gamma	$\widehat{\theta_1}$	0.4872	0.4872	0.6260	0.6260
	$\widehat{\theta_2}$	0.3470	0.3470	0.2340	0.2340
	$\widehat{\theta_3}$	0.1658	0.1658	0.1400	0.1400
	$\lvert\widehat{\theta_1} - \theta_1\rvert$	0.0128	0.0128	0.1260	0.1260
	$\lvert\widehat{\theta_2} - \theta_2\rvert$	0.0470	0.0470	0.0660	0.0660
	$\lvert\widehat{\theta_3} - \theta_3\rvert$	0.0342	0.0342	0.0600	0.0600
Maxwell	$\widehat{\theta_1}$	0.4287	0.4287	0.6294	0.6294
	$\widehat{\theta_2}$	0.3748	0.3748	0.2334	0.2334
	$\widehat{\theta_3}$	0.1966	0.1966	0.1372	0.1372
	$\lvert\widehat{\theta_1} - \theta_1\rvert$	0.0713	0.0713	0.1294	0.1294
	$\lvert\widehat{\theta_2} - \theta_2\rvert$	0.0748	0.0748	0.0666	0.0666
	$\lvert\widehat{\theta_3} - \theta_3\rvert$	0.0034	0.0034	0.0628	0.0628
Rayleigh	$\widehat{\theta_1}$	0.3972	0.3962	0.4336	0.4332
	$\widehat{\theta_2}$	0.3432	0.3442	0.3074	0.3076
	$\widehat{\theta_3}$	0.2597	0.2596	0.2591	0.2592
	$\lvert\widehat{\theta_1} - \theta_1\rvert$	0.1028	0.1038	0.0664	0.0668
	$\lvert\widehat{\theta_2} - \theta_2\rvert$	0.0432	0.0442	0.0074	0.0076
	$\lvert\widehat{\theta_3} - \theta_3\rvert$	0.0597	0.0596	0.0591	0.0592
Weibull	$\widehat{\theta_1}$	0.3756	0.3761	0.3992	0.3996
	$\widehat{\theta_2}$	0.3411	0.3417	0.3174	0.3176
	$\widehat{\theta_3}$	0.2834	0.2822	0.2834	0.2829
	$\lvert\widehat{\theta_1} - \theta_1\rvert$	0.1244	0.1239	0.1008	0.1004
	$\lvert\widehat{\theta_2} - \theta_2\rvert$	0.0411	0.0417	0.0174	0.0176
	$\lvert\widehat{\theta_3} - \theta_3\rvert$	0.0834	0.0822	0.0834	0.0829

3.3. Estimation of preference probabilities

The estimated preference probabilities highlighting the degree of prevailed utility of competing objects are compiled in Table 5. The estimates verify the preference norms established through the observed magnitude of the worth parameters. It is witnessed without exception that object 1 coined with the worth value of $\theta_1 = 0.5$, is overwhelmingly preferred over the objects characterized with $\theta_2 = 0.3$ and $\theta_3 = 0.2$. Similarly, the second item nominated with $\theta_2 = 0.3$ worth of the parameter value is preferred over the third available option. Also, the extent of preferences can be seen consistent with the magnitude of worth parameters. The larger the difference in the associated worth (utility) of the objects, the clearer will be the choices. Moreover, these realizations are seen regardless of the priors and sample sizes. It is to be noted, that all the findings verify that our proposed scheme successfully maintain the common rationale underlying the PC methods along with offering a general device capable of generating various PC models through the exponential family of distributions.

Table 5. The estimates of preference probabilities.

Models	Preferences	n_{ij} =15		n_{ij} =50	
		Jeffreys	Uniform	Jeffreys	Uniform
Beta	$\widehat{p_{12}}$	0.5636	0.5636	0.6603	0.6603
	$\widehat{p_{13}}$	0.6858	0.6858	0.7311	0.7311
	$\widehat{p_{23}}$	0.6281	0.6281	0.5829	0.5829
Exponential	$\widehat{p_{12}}$	0.5734	0.5626	0.6649	0.6595
	$\widehat{p_{13}}$	0.7028	0.6840	0.7374	0.7301
	$\widehat{p_{23}}$	0.6376	0.6273	0.5861	0.5828
Power	$\widehat{p_{12}}$	0.5734	0.5626	0.6649	0.6595
	$\widehat{p_{13}}$	0.7028	0.6840	0.7374	0.7301
	$\widehat{p_{23}}$	0.6376	0.6273	0.5861	0.5828
Gamma	$\widehat{p_{12}}$	0.5535	0.5535	0.6504	0.6504
	$\widehat{p_{13}}$	0.6635	0.6635	0.7185	0.7185
	$\widehat{p_{23}}$	0.6147	0.6147	0.5806	0.5806
Maxwell	$\widehat{p_{12}}$	0.5849	0.5849	0.9313	0.9313
	$\widehat{p_{13}}$	0.8837	0.8837	0.9838	0.9838
	$\widehat{p_{23}}$	0.8414	0.8414	0.7970	0.7970
Rayleigh	$\widehat{p_{12}}$	0.5725	0.5699	0.6655	0.6648
	$\widehat{p_{13}}$	0.7005	0.6996	0.7369	0.7364
	$\widehat{p_{23}}$	0.6359	0.6374	0.5846	0.5848
Weibull	$\widehat{p_{12}}$	0.5718	0.5714	0.6655	0.6657
	$\widehat{p_{13}}$	0.6995	0.7030	0.7365	0.7381
	$\widehat{p_{23}}$	0.6355	0.6397	0.5842	0.5859

3.4. Inferential aspects

This sub-section is dedicated to delineate the attainment of rational decision making through inferentially workable scheme by the launch of sound utility theory. We proceed by drawing conjoint posterior samples of worth parameters, that is $\theta_i's$ and $\theta_j's$, using Eqs (5) and (7). The exercise is conducted under both priors, both samples and with respect to all sub-cases. Table 6 reports the respective hypothesis and their posterior probabilities. Furthermore, the extent of the significance of the dis-agreement between the worth parameters is quantified through the Bayes factor (BF). We decide between the hypothesis, H_{ij}: $\theta_i \geq \theta_j$ *vs* H_{ji}: $\theta_i < \theta_j$ by calculating the posterior probabilities, such as:

$$\mathrm{p_{ij}} = \int_{\zeta=0}^{1} \int_{\eta=\zeta}^{(1+\zeta)/2} \mathrm{p}(\zeta, \eta|\omega)\, \mathrm{d}\eta \mathrm{d}\zeta,$$

where, posterior probability of H_{ji} will be $p_{ji} = 1 - p_{ij}$. Moreover, η represents worth parameter, such as θ_i and ζ denotes the difference of utility associated with comparative strategies, such as, $\zeta = \theta_i - \theta_j$. The BF is then remains quantifiable as a ratio of the above given posterior probabilities that is, $BF = p_{ij}/p_{ji}$ through well known criteria highlighting the degree of dis-agreement between hypotheses as:

$BF \geq 1$, support H_{ij}

$10^{-0.5} \leq BF \leq 1$, minimal evidence against H_{ij}

$$10^{-1} \leq BF \leq 10^{-0.5}, \quad \text{substantial evidence against } H_{ij}$$

$$10^{-2} \leq BF \leq 10^{-1}, \quad \text{strong evidence against } H_{ij}$$

$$BF \leq 10^{-2}, \quad \text{decisive evidence against } H_{ij}.$$

Through Table 6, we observe that, the pre-fix preference ordering, that is $\theta_1 > \theta_2 > \theta_3$, remains maintained with overwhelming statistical evidences. This outcome is vividly observable through the table, regardless of priors and for both sample sizes with respect to all sub-cases. Moreover, as the mutual difference of the worth parameters increases, the evidence of choice ordering moves from being substantial to decisive. These findings are consistent with usual PC theory and thus validate the legitimacy of the proposed strategy.

Table 6. Posterior probabilities of hypotheses and associated Bayes factor.

Models	Hypotheses	n_{ij} =15				n_{ij} =50			
		Jeffreys		Uniform		Jeffreys		Uniform	
		P_{ij}	BF	P_{ij}	BF	P_{ij}	BF	P_{ij}	BF
Beta	$H_{12}: \theta_1 \geq \theta_2$	0.7331	2.7467	0.7234	2.6153	0.9975	399.0000	0.9971	343.8276
	$H_{13}: \theta_1 \geq \theta_3$	0.9696	31.8947	0.9671	29.3951	0.9999	9999.0000	0.9999	9999.0000
	$H_{23}: \theta_2 \geq \theta_3$	0.8958	8.5969	0.8882	7.9445	0.9209	11.6422	0.9082	9.8932
Exponential	$H_{12}: \theta_1 \geq \theta_2$	0.7320	2.7313	0.7228	2.6075	0.9975	399.0000	0.9971	343.8276
	$H_{13}: \theta_1 \geq \theta_3$	0.9691	31.3625	0.9666	28.9401	0.9999	9999.0000	0.9999	9999.0000
	$H_{23}: \theta_2 \geq \theta_3$	0.8954	8.5602	0.8879	7.9206	0.9207	11.6103	0.9081	9.8932
Power	$H_{12}: \theta_1 \geq \theta_2$	0.7320	2.7313	0.7228	2.6075	0.9975	399.0000	0.9971	343.8276
	$H_{13}: \theta_1 \geq \theta_3$	0.9691	31.3625	0.9666	28.9401	0.9999	9999.0000	0.9999	9999.0000
	$H_{23}: \theta_2 \geq \theta_3$	0.8954	8.5602	0.8879	7.9206	0.9207	11.6103	0.9081	9.8932
Gamma	$H_{12}: \theta_1 \geq \theta_2$	0.7218	2.5945	0.7156	2.5162	0.9968	311.5000	0.9965	284.7143
	$H_{13}: \theta_1 \geq \theta_3$	0.9626	25.7380	0.9606	24.3807	0.9999	9999.0000	0.9999	9999.0000
	$H_{23}: \theta_2 \geq \theta_3$	0.8876	7.8968	0.8816	7.4459	0.9175	11.1212	0.9066	9.7066
Maxwell	$H_{12}: \theta_1 \geq \theta_2$	0.7379	2.8153	0.7135	2.4904	0.9977	433.7826	0.9967	302.0303
	$H_{13}: \theta_1 \geq \theta_3$	0.9730	36.0370	0.9676	29.8642	0.9999	9999.0000	0.9999	9999.0000
	$H_{23}: \theta_2 \geq \theta_3$	0.9008	9.0806	0.8852	7.7108	0.9217	11.7714	0.8961	8.6246
Rayleigh	$H_{12}: \theta_1 \geq \theta_2$	0.7398	2.8432	0.7212	2.5868	0.9977	433.7826	0.9969	321.5806
	$H_{13}: \theta_1 \geq \theta_3$	0.9730	36.0370	0.9688	31.0513	0.9999	9999.0000	0.9999	9999.0000
	$H_{23}: \theta_2 \geq \theta_3$	0.9014	9.1420	0.8890	8.0090	0.9216	11.7551	0.9020	9.2041
Weibull	$H_{12}: \theta_1 \geq \theta_2$	0.7405	2.8536	0.7118	2.4698	0.9977	433.7826	0.9964	276.7778
	$H_{13}: \theta_1 \geq \theta_3$	0.9736	36.8788	0.9674	29.6748	0.9999	9999.0000	0.9999	9999.0000
	$H_{23}: \theta_2 \geq \theta_3$	0.9023	9.2354	0.8843	7.6430	0.9211	11.6743	0.8912	8.1912

4. Empirical Evaluation: An application to water brand preference data

The applicability of the proposed generalization is demonstrated by using the water brand preference data gathered through a balance paired comparison (PC) experiment. Thirty-five households of Islamabad were requested to report their preferences for drinking water while the pairwise comparing three of leading brands of Pakistan that is, Aquafina (AQ), Nestle (NT) and Kinley (KL). Table 7 comprehends the reported choice data of the respondents. An initial analysis reveals that, when comparing AQ and NT, 40% of the respondents reported AQ as their preferred brand, whereas 60% chose NT over AQ. Further, around 63% of the participants preferred AQ brand over KL brand, where remaining 37% were of the favor of KL. Lastly, in comparison of NT and KL, almost 57% contestants favored NT, while 43% participants stayed with KL. Figure 2 depicts the choice data.

Table 7. Drinking water brand preference data.

Pairs	r_{ij}	r_{ji}	n_{ij}
$(AQ, NT = 1,2)$	14 (40%)	21 (60%)	35
$(AQ, KL = 1,3)$	22 (62.8%)	13 (37.1%)	35
$(NT, KL = 2,3)$	20 (57.1%)	15 (42.8%)	35

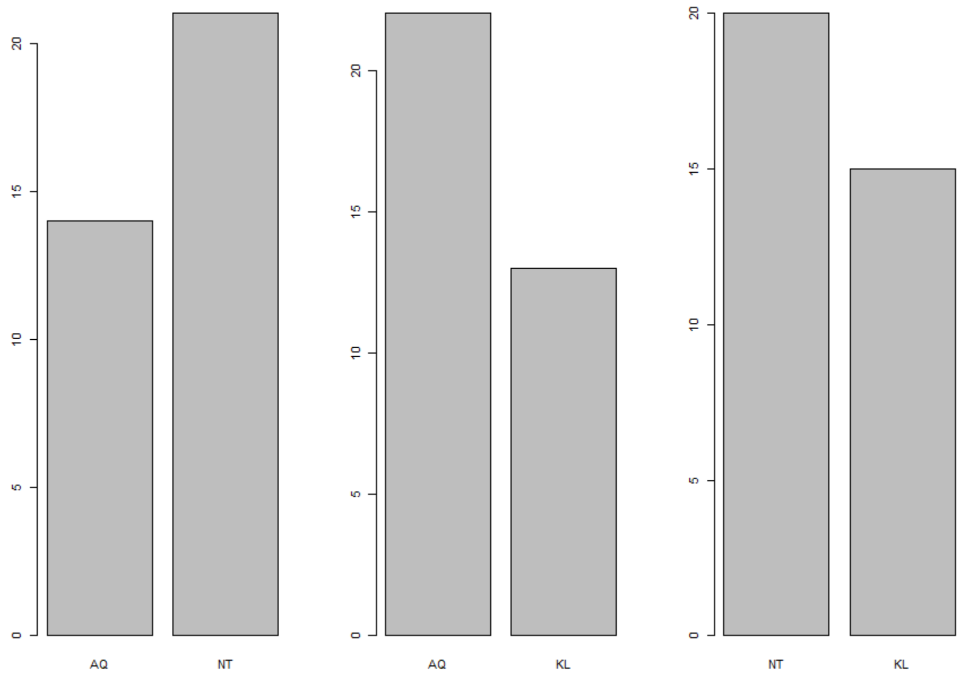


Figure 2. Graphical display of the choices of drinking water brands data.

Next, the estimation of worth parameters deriving the utility of competing brands as latent phenomena is persuaded while considering all sub-cases of the devised generalization and both prior distributions. The results related to estimated worth parameters and resultant preference probabilities are comprehended in the Table 8. Most obviously, the empirical estimation of the proposed procedure stays consistent with the simulation evaluation. Firstly, regardless of the prior distributions, all members of the suggested scheme capably retain the preferences exhibited through the comparative data. The uncovered choice hierarchy is estimated such that, $\hat{\theta}_{NT} > \hat{\theta}_{AQ} > \hat{\theta}_{KL}$ indicating Nestle as the most preferred brand followed by Aquafina which is then stayed preferable in comparison to Kinley. Secondly, all of the members showed an equally tendency of using prior information fetched from the considered prior distributions, however, intra-model variations exist. These delicacies are projected in the resulting estimated preference probabilities compiled in the Table 9. One may notice the vivid functional dependency of the preference ordering over the associated worth parameter. As the utility of the object enhances so is the extent of preference increase. The Bayes factors provided in table 10 also reveal the same trends. As the utility associated with water brands vary, so does the evidence of likely preference. For example, there is substantial evidence in the favor of NT as compared to AQ but this evidence turns to be decisive when NT is compared with KL. These variations are in fact the projection of varying degrees of comparative utility prevalent in the choices of the respondents.

Table 8. Estimated values of worth parameters.

Models	Estimated Parameters	Jeffreys	Uniform
Beta	$\hat{\theta}_{AQ}$	0.3398	0.3398
	$\hat{\theta}_{NT}$	0.4057	0.4057
	$\hat{\theta}_{KL}$	0.2544	0.2544
Exponential	$\hat{\theta}_{AQ}$	0.3414	0.3396
	$\hat{\theta}_{NT}$	0.4108	0.4091
	$\hat{\theta}_{KL}$	0.2478	0.2514
Power	$\hat{\theta}_{AQ}$	0.3414	0.3396
	$\hat{\theta}_{NT}$	0.4108	0.4091
	$\hat{\theta}_{KL}$	0.2478	0.2514
Gamma	$\hat{\theta}_{AQ}$	0.3388	0.3388
	$\hat{\theta}_{NT}$	0.4455	0.4455
	$\hat{\theta}_{KL}$	0.2157	0.2157
Maxwell	$\hat{\theta}_{AQ}$	0.3520	0.3520
	$\hat{\theta}_{NT}$	0.4137	0.4137
	$\hat{\theta}_{KL}$	0.2343	0.2343
Rayleigh	$\hat{\theta}_{AQ}$	0.3384	0.3381
	$\hat{\theta}_{NT}$	0.3723	0.3726
	$\hat{\theta}_{KL}$	0.2892	0.2893
Weibull	$\hat{\theta}_{AQ}$	0.3370	0.3369
	$\hat{\theta}_{NT}$	0.3593	0.3597
	$\hat{\theta}_{KL}$	0.3037	0.3034

Table 9. Estimated preference probabilities.

Models	Preference probabilities	Jeffreys	Uniform
Beta	$\hat{p}_{AQ,NT}$	0.4533	0.4533
	$\hat{p}_{AQ,KL}$	0.5749	0.5749
	$\hat{p}_{NT,KL}$	0.6200	0.6200
Exponential	$\hat{p}_{AQ,NT}$	0.4539	0.4536
	$\hat{p}_{AQ,KL}$	0.5794	0.5746
	$\hat{p}_{NT,KL}$	0.6237	0.6194
Power	$\hat{p}_{AQ,NT}$	0.4539	0.4536
	$\hat{p}_{AQ,KL}$	0.5794	0.5746
	$\hat{p}_{NT,KL}$	0.6237	0.6194

Continued on next page

Models	Preference probabilities	Jeffreys	Uniform
	$\hat{p}_{AQ,NT}$	0.4564	0.4564
Gamma	$\hat{p}_{AQ,KL}$	0.5710	0.5710
	$\hat{p}_{NT,KL}$	0.6127	0.6127
	$\hat{p}_{AQ,NT}$	0.3987	0.3987
Maxwell	$\hat{p}_{AQ,KL}$	0.7395	0.7395
	$\hat{p}_{NT,KL}$	0.8124	0.8124
	$\hat{p}_{AQ,NT}$	0.4524	0.4516
Rayleigh	$\hat{p}_{AQ,KL}$	0.5779	0.5773
	$\hat{p}_{NT,KL}$	0.6237	0.6239
	$\hat{p}_{AQ,NT}$	0.4521	0.4510
Weibull	$\hat{p}_{AQ,KL}$	0.5774	0.5779
	$\hat{p}_{NT,KL}$	0.6235	0.6250

Table 10. Posterior probabilities of hypotheses and associated Bayes factor.

Models	Hypotheses	$n_{ij} = 35$			
		Jeffreys		Uniform	
		P_{ij}	BF	P_{ij}	BF
	$H_{12}: \theta_{AQ} \geq \theta_{NT}$	0.2297	0.2982	0.2297	0.2982
Beta	$H_{13}: \theta_{AQ} \geq \theta_{KL}$	0.8505	5.6890	0.8505	5.6890
	$H_{23}: \theta_{NT} \geq \theta_{KL}$	0.9580	22.8095	0.9580	22.8095
	$H_{12}: \theta_{AQ} \geq \theta_{NT}$	0.2311	0.3006	0.2236	0.2880
Exponential	$H_{13}: \theta_{AQ} \geq \theta_{KL}$	0.8506	5.6934	0.8434	5.3857
	$H_{23}: \theta_{NT} \geq \theta_{KL}$	0.9579	22.7530	0.9554	21.4215
	$H_{12}: \theta_{AQ} \geq \theta_{NT}$	0.2311	0.3006	0.2236	0.2880
Power	$H_{13}: \theta_{AQ} \geq \theta_{KL}$	0.8506	5.6934	0.8434	5.3857
	$H_{23}: \theta_{NT} \geq \theta_{KL}$	0.9579	22.7530	0.9554	21.4215
	$H_{12}: \theta_{AQ} \geq \theta_{NT}$	0.2417	0.3187	0.2417	0.3187
Gamma	$H_{13}: \theta_{AQ} \geq \theta_{KL}$	0.8508	5.7024	0.8508	5.7024
	$H_{23}: \theta_{NT} \geq \theta_{KL}$	0.9561	21.7790	0.9561	21.7790
	$H_{12}: \theta_{AQ} \geq \theta_{NT}$	0.2035	0.2555	0.2035	0.2555
Maxwell	$H_{13}: \theta_{AQ} \geq \theta_{KL}$	0.8349	5.0569	0.8349	5.0569
	$H_{23}: \theta_{NT} \geq \theta_{KL}$	0.9530	20.2766	0.9530	20.2766
	$H_{12}: \theta_{AQ} \geq \theta_{NT}$	0.2088	0.2639	0.2118	0.2687
Rayleigh	$H_{13}: \theta_{AQ} \geq \theta_{KL}$	0.8393	5.2228	0.8420	5.3291
	$H_{23}: \theta_{NT} \geq \theta_{KL}$	0.9546	21.0264	0.9556	21.5225
	$H_{12}: \theta_{AQ} \geq \theta_{NT}$	0.1912	0.2364	0.1956	0.2432
Weibull	$H_{13}: \theta_{AQ} \geq \theta_{KL}$	0.8260	4.7471	0.8301	4.8858
	$H_{23}: \theta_{NT} \geq \theta_{KL}$	0.9497	18.8807	0.9512	14.9.4918

5. Conclusions

This research elucidates the proposition of a generalized framework to assist rational decision-making while dealing with binary choices. The objectives are achieved by devising a general paired comparison modeling scheme by employing an exponential family of distributions. The methodological environment is enriched by illuminating various parametric settings such as sample size and prior distributions. Through tiresome evaluation operations, it is delineated that the suggested model is not capable of retaining the preference hierarchy exhibited through the observed data but also treats seven mainstream paired comparison models as sub-cases. It is estimated that the members of the proposed family robustly use the prior information when offered non-informative priors such as Jeffery's prior and Uniform prior. However, the deducted sub-cases reveal a varying degree of estimating accuracy. Also, the suggested generalized scheme capably elaborates the choice hierarchy among the competing objects as a function of associated utility. This realization is in fact consistent with the theoretical understanding of rational decision-making. Moreover, the inferential aspects of the model are explored in depth through the launch of the Bayesian approach. The outcomes of the investigation demonstrate with clarity that as the utility of rival objects distinguishes the statistical evidence establishing the choice ranking varies accordingly. These mentioned realizations are in support of the fondness for professional and commercial research circles exploring capable mechanisms assisting optimal decision-making by defining sound interlinks between utility theory and its inferential dynamics. Thus, it remains deducible that the study encapsulating various choice behaviors and associated utility functional in accordance with choice axioms, is worth pursuing. The estimating hierarchy of the contemporary sub-cases of the newly devised family is observed such that, $Gamma > Exponntial = Power > Beta > Maxwell > Rayleigh > Weibull.$

At this stage, it is appropriate to mention that this article demonstrates the utility of non-informative priors for the estimation of the worth parameters and directed preferences. In the future, it will be interesting to compare the dynamic behaviors of the devised family while using informative priors as well. Also, it is well known that the self-reported choice data remains vulnerable to the contaminations such as desirability bias, order effects and time-varying subtleties. A more comprehensive framework capable of handling these complexities is an attractive research pursuit.

Conflict of interest

The authors declare no conflict of interest.

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