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Research article

Direct connection between Navier and spherical harmonic kernels in elasticity

Dimitra Labropoulou, Panayiotis Vafeas* and George Dassios

Department of Chemical Engineering, University of Patras, 26504 Patras, Greece

* **Correspondence:** Email: vafeas@chemeng.upatras.gr; Tel: +302610996872.

Abstract: Linear isotropic elasticity is an interesting branch of continuum mechanics, described by the fundamental laws of Hooke and Newton, which are combined in order to construct the governing generalized Navier equation of the displacement within any material. Implying time-independence and in the absence of external body forces, the latter is reduced to the corresponding form of a homogeneous second-order partial differential equation, whose solution is given via the Papkovich differential representation, which expresses the displacement field in terms of harmonic functions. On the other hand, spherical geometry provides the most widely used framework in real-life applications, concerning interior and exterior problems in elasticity. The present work aims to provide a little progress, by producing ready-to-use basic functions for linear isotropic elasticity in spherical coordinates. Hence, we calculate the Papkovich eigensolutions, generated by the spherical harmonic eigenfunctions, obtaining connections between Navier and spherical harmonic kernels. A set of useful results are provided at the end of the paper in the form of examples, regarding the evaluation of displacement field inside and outside a sphere.

Keywords: solid mechanics; isotropic elasticity; Navier equation; analytical solutions; Papkovich representation; spherical harmonics
Mathematics Subject Classification: 74H10, 74B05, 35Q74, 35J05, 33C55

1. Introduction

Contemporary theoretical mechanics and engineering technology are mainly concerned with either the isotropic or the anisotropic behavior of elastic materials and structures [1,2]. Specifically, the classical theory of linear elasticity is considered to be an indivisible branch of the more general field of non-linear elasticity [3], wherein the mathematical description of the physical quantities, associated with such media [4,5], incorporates the fundamental characteristics of the corresponding physical problems. Therein, the relationship between the strain and the stress of any solid body, subjected to external forces, is considered to be linear. On the other hand, isotropic elasticity, being set in conjunction with problems of elasticity in the linear regime with [6] or without [7,8] body forces present, is an extensively developed area of continuum mechanics, embodying solid analysis that requires both analytical and numerical attention [9]. Even though the most attractive area for developing new methodologies appears to be the anisotropic linear elasticity [10,11], it is evident that there still exist open problems in the isotropic spectrum. In fact, the continuing interest in elaborating with such kind of aspects is still effective and the necessity in producing purely analytical solutions towards this direction, ready to accept proper numerical handling, stands in the frontline of the current research. Indeed, such a complete and comprehensive survey in linear isotropic elasticity brings insight to new elements of applied mathematics, which lead to the next step of studying the wave propagation [12,13] and, in general, the theory of scattering [14] in elastic materials.

Linear isotropic behavior of elastic media has an inherent mathematical interest due to the fact that even though the related theory is much simplified, many applications can accept the isotropic character without loss of robustness. In view of this aspect, this work is involved with the production of closed-type analytical formulae for the determination of the fundamental field in solid mechanics, i.e. the displacement. Doing so, we initially utilize the Hook's law, which provides the second order stress tensor in terms of the strain dyadic and the stiffness tetratic, while, in the sequel, we invoke the result into the Newton's law so as to obtain the generalized constitutive equation in elasticity. Thereafter, our purpose is twofold, that is we give special attention to elastostatics, neglecting the temporal derivatives and we exclude any external body forces, since any cause of disturbance can be entered into the boundary conditions of the particular physical problem. Moreover, we assume the specific expression for the stiffness tensor, which incorporates the appropriate components that inherit the isotropic character of the material. That way, we arrive at the known Navier partial differential equation for the displacement field, which accepts an analytical compact solution in the form of a partial differential operator, acting on harmonic functions with vector and scalar character, namely the Papkovich differential representation. This can be derived directly from the Naghdi-Hsu general solution [15] and since the spherical harmonics form a complete system, it also provides complete representations for the elastostatic fields. Hence, Papkovich representation of the solution of Navier equation is complete, that is any solution can be expressed in this form. The utility of such solutions is significant, considering the fact that they provide handy analytical expressions, for instance let us refer to a series of articles [16–18], which deal with elastic wave scattering at low frequencies around ellipsoidal solid bodies or cavities, using the Papkovich representation. Otherwise, the representation theory could be applied to inverse elastic scattering [19] or to more complicated mixed-type boundary value problems [20] in linear elasticity. Finally, let us not ignore a major advantage of Papkovich representation, which can offers us a certain degree of freedom, since the potentials needed to describe the field itself are redundant, thus the extra terms can be used conveniently, e.g. to compensate the force term in the case wherein it is included into the Navier equation.

It is the purpose of this research to provide progress in this interesting mathematical aspect, hence, using the spherical coordinate system [21], we obtain connection formulae, which relate the spherical harmonics that lead, through the Papkovich representation, to the displacement in linear isotropic elasticity. In other words, we calculate the displacement field, generated by the harmonic eigenfunctions [22], through the Papkovich representation, and then we provide the necessary analytical background so as to solve the inverse problem of identifying those harmonic eigenfunctions, which generate the same displacement fields. Henceforth, in the aim of solving sufficiently general interior and exterior problems, we use the internal and external solid spherical harmonic eigenfunctions in their complex form [22]. This way, we demonstrate the importance of attaining such kind of ready-to-use mathematical tools, by showing some interesting example problems inside and outside a sphere. The background of this mathematical procedure comes from the low-frequency scattering in elasticity [23] and shows how one can obtain a solution basis of a finite dimensional subspace. This reduces the calculation of the solution to the calculation of a finite number of scalar coefficients. However, our work leads to building blocks for constructing the solution of any relative problem.

Notwithstanding the existence of adequate and simultaneously convenient computational codes in solving problems in elasticity, we should not overlook the fact that pure analytical techniques are the backbone of numerical analysis. Hence, the important advantage of the performed mathematical analysis is based upon its ability to understand the physical background and to verify the credibility of numerical methods or other more sophisticated analytical models. On the other hand, the idea of building any solution from ready-to-use eigensolutions goes back to the classic references of Rayleigh [24], Kelvin [25], Maxwell [26], Sommerfeld [27] and Neuber [28]. In the present work, we tried to extend this idea to the theory of elasticity, wherein the ample literature survey of the fundamental classical references [29–40] verifies the necessity of such kind of analysis. Finally, by virtue of the representation theory, it is obvious that spherical geometry approximates sufficiently well most basic problems in linear isotropic elasticity. Nevertheless, the extension to spheroidal, ellipsoidal or even more complicated geometries [21,41] provides a challenging area for future investigation.

2. Physical and mathematical development

Let us introduce an arbitrarily defined smooth, either bounded or unbounded, three-dimensional elastic domain $\Omega\left(\mathbb{R}^3\right)$, which could be designated as interior or interior as the case may be. Then, each field within $\Omega\left(\mathbb{R}^3\right)$ is written in terms of its position vector $\mathbf{r} = x_1\hat{\boldsymbol{x}}_1 + x_2\hat{\boldsymbol{x}}_2 + x_3\hat{\boldsymbol{x}}_3$, expressed via the Cartesian basis $\hat{\boldsymbol{x}}_j$, $j = 1,2,3$ in Cartesian coordinates $\left(x_1, x_2, x_3\right)$. On the other hand, the current investigation excludes the dependence on time, since we operate according to the steady state status of the particular situation, while the necessary fundamental information, which are adequate for this work can be found collected in [11].

The physical interpretation of linear isotropic elasticity is involved with the displacement field $\mathbf{u}$, which comprises the measure of deformation of an elastic material. By means of the gradient ∇ and the Laplacian Δ operators, the displacement satisfies the well-known Navier equation in the presence of body forces, i.e.

$$\mu\Delta\mathbf{u}(\mathbf{r}) + (\lambda+\mu)\nabla\left[\nabla\cdot\mathbf{u}(\mathbf{r})\right] + \mathbf{f}(\mathbf{r}) = \mathbf{0} \quad \text{for} \quad \mathbf{r}\in\Omega\left(\mathbb{R}^3\right), \tag{1}$$

wherein $\mathbf{f}$ is an external applied force that renders expression (1) non-homogeneous, while $\lambda, \mu \in \mathbb{R}$ are the elastic parameters of the isotropic theory, being known as the Lamé constants. Papkovich proposed a differential representation of the solution for the homogeneous Navier equation [15] with no body forces, by considering $\mathbf{f}=\mathbf{0}$ within (1), which expresses the displacement field in differential form, in terms of two harmonic functions, one vector $\mathbf{A}$ and one scalar B, that is

$$\mathbf{u}(\mathbf{r})=\mathbf{A}(\mathbf{r})-\frac{\lambda+\mu}{2(\lambda+2\mu)}\nabla\left[\mathbf{r}\cdot\mathbf{A}(\mathbf{r})+\mathrm{B}(\mathbf{r})\right],\text{ where } \Delta\mathbf{A}(\mathbf{r})=\mathbf{0},\ \Delta\mathrm{B}(\mathbf{r})=0 \text{ for } \mathbf{r}\in\Omega\left(\mathbb{R}^3\right), \tag{2}$$

which is known to be complete. It is not hard to prove that (2) satisfies (1), bearing in mind the vector identity $\Delta(\mathbf{r}\cdot\mathbf{A})=\Delta\mathbf{r}\cdot\mathbf{A}+\mathbf{r}\cdot\Delta\mathbf{A}+2(\nabla\otimes\mathbf{r})^{\top}:(\nabla\otimes\mathbf{A})=2\tilde{\mathbf{I}}^{\top}:(\nabla\otimes\mathbf{A})=2\tilde{\mathbf{I}}:(\nabla\otimes\mathbf{A})=2\nabla\cdot\mathbf{A}$ and the interchange $\Delta\nabla=\nabla\Delta$, since it readily holds that $\Delta\mathbf{r}=\Delta\mathbf{A}=\mathbf{0}$ and $\Delta\mathrm{B}=0$, noting that $\tilde{\mathbf{I}}$ is the unit dyadic, "$\top$" denotes transposition, "$\otimes$" refers to the tensor product and "$:$" stands for the double inner product. The general differential solution (2) provides a powerful analytical tool for solving the homogeneous and time-independent linearized equation of classical dynamic elasticity.

At this point we have to mention that even though representation (2) is complete, it is not unique, since we have four potentials (three components of the vector $\mathbf{A}$ and one component, which is the scalar B) to determine the three-dimensional vector displacement field. Actually, this is more than an advantage rather than a disadvantage, considering the fact that we are equipped with certain degrees of freedom to deploy conveniently the proper manner, which depends on the boundary value problem at hand. On the other end, this flexibility of the representation theory could be used to cancel the difficulty when the source term $\mathbf{f}$ is not absent (see equation (1) for instance). More precisely, in the same sense, we could use a similar differential solution like (2), in order to obtain a general solution for the non-homogeneous Navier equation (1). In fact, we can keep the same form for the displacement field, potential $\mathbf{A}$ could be harmonic, but then potential B should satisfy the mixed-type equation $\Delta(\nabla\mathrm{B})\equiv\nabla(\Delta\mathrm{B})=2\mathbf{f}/(\lambda+\mu)$, in order for (1) to be satisfied. However, obtaining B is quite a difficult task.

3. Interrelation between Navier and spherical harmonics

This section includes the main results of our work, which are focused on the direct connection between the homogeneous Navier equation and the harmonic kernels of the implicated potentials $\mathbf{A}$ and B, using the spherical geometry. Before we proceed to the analysis in spherical coordinates, we take advantage of the already discussed flexibility of the differential general solution (2), thus and without loss of generality, we may suppose that the vector harmonic potential $\mathbf{A}$ is adequate enough to provide us with a complete solution for the displacement field $\mathbf{u}$ via (2). Consequently, assuming that $\mathrm{B}\equiv 0$, the representation (2) becomes

$$\mathbf{u}(\mathbf{r})=\mathbf{A}(\mathbf{r})-\frac{\lambda+\mu}{2(\lambda+2\mu)}\nabla\left[\mathbf{r}\cdot\mathbf{A}(\mathbf{r})\right],\text{ where } \Delta\mathbf{A}(\mathbf{r})=\mathbf{0} \text{ for } \mathbf{r}\in\Omega\left(\mathbb{R}^3\right), \tag{3}$$

which constitutes a complete solution for the Navier equation (1) if $\mathbf{f}=\mathbf{0}$. Taking the *curl* on both sides of (3) and defining a new arbitrary function Φ, we have

$$\nabla\times\mathbf{u}(\mathbf{r})=\nabla\times\mathbf{A}(\mathbf{r}) \Rightarrow \mathbf{u}(\mathbf{r})-\mathbf{A}(\mathbf{r})=\nabla\Phi(\mathbf{r}) \Rightarrow \mathbf{A}(\mathbf{r})=\mathbf{u}(\mathbf{r})-\nabla\Phi(\mathbf{r}) \text{ for } \mathbf{r}\in\Omega\left(\mathbb{R}^3\right). \tag{4}$$

Next, we apply the *div* on (3), which leads us to

$$\nabla\cdot\mathbf{u}(\mathbf{r})=\nabla\cdot\mathbf{A}(\mathbf{r})-\frac{\lambda+\mu}{2(\lambda+2\mu)}\Delta\left[\mathbf{r}\cdot\mathbf{A}(\mathbf{r})\right]$$

$$=\nabla\cdot\mathbf{A}(\mathbf{r})-\frac{\lambda+\mu}{\lambda+2\mu}\nabla\cdot\mathbf{A}(\mathbf{r})$$

$$=\frac{\mu}{\lambda+2\mu}\nabla\cdot\mathbf{A}(\mathbf{r})\;\Rightarrow\;\nabla\cdot\mathbf{A}(\mathbf{r})=\frac{\lambda+2\mu}{\mu}\nabla\cdot\mathbf{u}(\mathbf{r})\;\text{ for }\;\mathbf{r}\in\Omega\left(\mathbb{R}^3\right). \quad (5)$$

Next, we take the divergence of relation (4) and use simultaneously the outcome (5), to arrive at

$$\nabla\cdot\mathbf{A}(\mathbf{r})=\nabla\cdot\mathbf{u}(\mathbf{r})-\Delta\Phi(\mathbf{r})\;\Rightarrow\;\Delta\Phi(\mathbf{r})=-\frac{\lambda+\mu}{\mu}\nabla\cdot\mathbf{u}(\mathbf{r})\;\text{ for }\;\mathbf{r}\in\Omega\left(\mathbb{R}^3\right), \quad (6)$$

which is a Poisson equation with respect to the unknown function Φ, whose solution is the result of implying the fundamental solution of the Laplace's operator [20], obtaining

$$\Phi(\mathbf{r})=\frac{\lambda+\mu}{4\pi\mu}\iiint\limits_{\Omega\left(\mathbb{R}^3\right)}\frac{\nabla_{\mathbf{r}'}\cdot\mathbf{u}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V(\mathbf{r}')\;\text{ for }\;\mathbf{r}\in\Omega\left(\mathbb{R}^3\right). \quad (7)$$

So, if we insert Φ from (7) into (4), we recover potential $\mathbf{A}$ as

$$\mathbf{A}(\mathbf{r})=\mathbf{u}(\mathbf{r})-\frac{\lambda+\mu}{4\pi\mu}\nabla\iiint\limits_{\Omega\left(\mathbb{R}^3\right)}\frac{\nabla_{\mathbf{r}'}\cdot\mathbf{u}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V(\mathbf{r}')\;\text{ for }\;\mathbf{r}\in\Omega\left(\mathbb{R}^3\right). \quad (8)$$

Therefore, given the displacement $\mathbf{u}$, the harmonic potential $\mathbf{A}$ is given by (8). Recapitulating the above reasoning, we calculate the displacement field $\mathbf{u}$, generated by the harmonic function $\mathbf{A}$ through the Papkovich representation (3) and then we face the inverse problem of determining this harmonic function $\mathbf{A}$, which generates the displacement field $\mathbf{u}$, via the integro-differential formula (8). This procedure is also invertible, in the sense that we can start with solutions of the form (8) and recover the displacement field via the Papkovich form (3). Both ways lead to the same result, independently of the coordinate system.

Aiming to approach applications that require an easily amenable expression for the displacement, we focus our attention to the spherical geometry [19], so in terms of the spherical coordinate system

$$x_1=r\sin\theta\cos\varphi,\;\; x_2=r\sin\theta\sin\varphi\;\text{ and }\; x_3=r\cos\theta \quad (9)$$

for $0\le r<+\infty$, $0\le\theta\le\pi$ and $0\le\varphi<2\pi$, the differential operators appearing into the Papkovich representation assume the forms

$$\nabla=\hat{\mathbf{r}}\frac{\partial}{\partial r}+\frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial\theta}+\frac{\hat{\boldsymbol{\varphi}}}{r\sin\theta}\frac{\partial}{\partial\varphi}\;\text{ and }\;\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}, \quad (10)$$

where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ (see [19]) denote the coordinate vectors of the spherical system with position vector $\mathbf{r}=r\hat{\mathbf{r}}$ and unit dyadic $\tilde{\mathbf{I}}=\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}+\hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}+\hat{\boldsymbol{\varphi}}\otimes\hat{\boldsymbol{\varphi}}$. For every value of the nature $n\ge 0$, there exist $(2n+1)$ linearly independent complex spherical surface harmonics Y_n^m of degree n and of order

$|m| \le n$ [20], given in the orthonormalized form by

$$Y_n^m(\hat{\mathbf{r}}) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i}m\varphi} \quad \text{for} \quad \hat{\mathbf{r}} = (\theta, \varphi) \in [0, \pi] \times [0, 2\pi) \tag{11}$$

and with inner product

$$\oiint_{S^2} Y_n^m(\hat{\mathbf{r}}) \bar{Y}_n^m(\hat{\mathbf{r}}) \mathrm{d}s(\hat{\mathbf{r}}) = 1 \quad \text{for} \quad \hat{\mathbf{r}} = (\theta, \varphi) \in [0, \pi] \times [0, 2\pi), \tag{12}$$

wherein $P_n^{|m|}$ are the associated Legendre functions of the first kind [20], $\bar{Y}_n^m$ denote the complex conjugate surface spherical harmonics and S^2 is the unit sphere in $\mathbb{R}^3$. Thereafter, we introduce the interior $u_{n,\mathrm{in}}^m$ (regular as $r \to 0^+$) and the exterior $u_{n,\mathrm{ex}}^m$ (regular as $r \to +\infty$), solid spherical harmonic eigenfunctions for every $n = 0,1,2,...$ and $m = -n,...,-1,0,1,...,n$, which are given by the expression

$$u_{n,\mathrm{in}}^m(\mathbf{r}) = r^n Y_n^m(\hat{\mathbf{r}}) \quad \text{and} \quad u_{n,\mathrm{ex}}^m(\mathbf{r}) = r^{-(n+1)} Y_n^m(\hat{\mathbf{r}}) \quad \text{for} \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{13}$$

respectively. Relationships (13) comprise a complete set of eigenfunctions for harmonic functions and belong to the kernel space of the Laplace's operator from (10), i.e. $\Delta u_{n,\mathrm{in}}^m = 0$ and $\Delta u_{n,\mathrm{ex}}^m = 0$ for $n \ge 0$ and $|m| \le n$, while they are obtained once the classical method of separation of variables [21,22] is applied.

Adopting the above mathematical analysis, the harmonic function $\mathbf{A}$ in differential representation (3) admits series expansion in terms of functions (13), i.e.

$$\mathbf{A}(\mathbf{r}) = \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \left[\mathbf{c}_{n,\mathrm{in}}^m u_{n,\mathrm{in}}^m(\mathbf{r}) + \mathbf{c}_{n,\mathrm{ex}}^m u_{n,\mathrm{ex}}^m(\mathbf{r}) \right] \quad \text{for} \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{14}$$

where

$$\mathbf{c}_{n,\mathrm{in}}^m = c_{n,\mathrm{in}}^{m,1} \hat{\boldsymbol{x}}_1 + c_{n,\mathrm{in}}^{m,2} \hat{\boldsymbol{x}}_2 + c_{n,\mathrm{in}}^{m,3} \hat{\boldsymbol{x}}_3 \quad \text{and} \quad \mathbf{c}_{n,\mathrm{ex}}^m = c_{n,\mathrm{ex}}^{m,1} \hat{\boldsymbol{x}}_1 + c_{n,\mathrm{ex}}^{m,2} \hat{\boldsymbol{x}}_2 + c_{n,\mathrm{ex}}^{m,3} \hat{\boldsymbol{x}}_3 \quad \text{with} \quad n \ge 0 \quad \text{and} \quad |m| \le n \tag{15}$$

are arbitrary constant coefficients. Expansion (14) expresses the completeness of the interior and the exterior solid spherical harmonics. Consequently, substituting the potential (14) into the general solution (3), we obtain

$$\mathbf{u}(\mathbf{r}) = \frac{1}{2(\lambda+2\mu)} \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \Big[(\lambda+3\mu) \mathbf{c}_{n,\mathrm{in}}^m u_{n,\mathrm{in}}^m(\mathbf{r}) - (\lambda+\mu) (\mathbf{c}_{n,\mathrm{in}}^m \cdot \mathbf{r}) \nabla u_{n,\mathrm{in}}^m(\mathbf{r})$$
$$+ (\lambda+3\mu) \mathbf{c}_{n,\mathrm{ex}}^m u_{n,\mathrm{ex}}^m(\mathbf{r}) - (\lambda+\mu) (\mathbf{c}_{n,\mathrm{ex}}^m \cdot \mathbf{r}) \nabla u_{n,\mathrm{ex}}^m(\mathbf{r}) \Big] \quad \text{for} \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{16}$$

where we used the trivial differential identities $\nabla(\mathbf{r} \cdot \mathbf{A}) = \mathbf{A} + (\nabla \otimes \mathbf{A}) \cdot \mathbf{r}$ (note that $\nabla \otimes \mathbf{r} = \tilde{\mathbf{I}}$) and $\nabla(\mathbf{c}_{n,\mathrm{y}}^m u_{n,\mathrm{y}}^m) = \nabla u_{n,\mathrm{y}}^m \otimes \mathbf{c}_{n,\mathrm{y}}^m$ for $\mathrm{y} = \mathrm{in}, \mathrm{ex}$, where $n \ge 0$ and $|m| \le n$. Nevertheless, expression (16) needs further elaboration in order for its processing to be feasible. To this direction, we analytically work as follows. Since the vector character of the harmonic function $\mathbf{A}$ is reflected upon the constant coefficients, which are written in Cartesian coordinates, we are obliged to work in the Cartesian system. This is attainable but requires the expression of the displacement field $\mathbf{u}$ in Cartesian coordinates involving constants and spherical surface harmonics. Therefore, we are able to transfer the connection between $\mathbf{A}$ and $\mathbf{u}$ to the corresponding connection via the constant coefficients. In order to do that, it is necessary to express the terms $\nabla u_{n,\mathrm{in}}^{m}$ and $\nabla u_{n,\mathrm{ex}}^{m}$ for $n \geq 0$ and $|m| \leq n$ as a function of spherical surface harmonics in Cartesian coordinates. This is possible, since these terms belong to the subspace that is produced by the spherical surface harmonics, while this task requires certain steps.

Hence, in the interest of making this work complete and independent, we provide recurrence relations for the associated Legendre functions [20], which, by definition of the conveniently chosen variable $x = \cos\theta \in [-1,1]$, since $\theta \in [0,\pi]$, they are furnished by the Rodrigues formula

$$\mathrm{P}_n^{|m|}(x) = \left(1-x^2\right)^{|m|/2} \frac{1}{2^n n!} \frac{\mathrm{d}^{|m|+n}}{\mathrm{d}x^{|m|+n}} \left(x^2-1\right)^n \quad \text{for} \quad n \geq 0 \quad \text{and} \quad |m| \leq n, \tag{17}$$

where obviously $\mathrm{P}_n^{|m|}(x) = 0$ if $|m| > n$. Thus, the associated Legendre functions of the first kind satisfy

$$(2n+1)x\mathrm{P}_n^{|m|}(x) = (n+|m|)\mathrm{P}_{n-1}^{|m|}(x) + (n-|m|+1)\mathrm{P}_{n+1}^{|m|}(x), \tag{18}$$

$$(2n+1)\sqrt{1-x^2}\,\mathrm{P}_n^{|m|}(x) = \mathrm{P}_{n+1}^{|m|+1}(x) - \mathrm{P}_{n-1}^{|m|+1}(x)$$

$$= (n+|m|)(n+|m|-1)\mathrm{P}_{n-1}^{|m|-1}(x) - (n-|m|+1)(n-|m|+2)\mathrm{P}_{n+1}^{|m|-1}(x), \tag{19}$$

$$\frac{2mx}{\sqrt{1-x^2}}\mathrm{P}_n^{|m|}(x) = \mathrm{P}_n^{|m|+1}(x) + (n+|m|)(n-|m|+1)\mathrm{P}_n^{|m|-1}(x) \tag{20}$$

and the first derivative relation

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}_n^{|m|}(x) = \left(1-x^2\right)^{-1/2}\left[-|m|x\left(1-x^2\right)^{-1/2}\mathrm{P}_n^{|m|}(x) + \mathrm{P}_n^{|m|+1}(x)\right], \tag{21}$$

all relationships (18)–(21) being provided for every value of $n \geq 0$ and $|m| \leq n$. On the other hand, the classical trigonometric functions imply the trivial identities

$$\sin\varphi \sin|m|\varphi = \frac{1}{2}\left[\cos(|m|-1)\varphi - \cos(|m|+1)\varphi\right], \tag{22}$$

$$\cos\varphi \cos|m|\varphi = \frac{1}{2}\left[\cos(|m|-1)\varphi + \cos(|m|+1)\varphi\right], \tag{23}$$

$$\cos\varphi \sin|m|\varphi = \frac{1}{2}\left[\sin(|m|+1)\varphi + \sin(|m|-1)\varphi\right], \tag{24}$$

$$\sin\varphi\cos|m|\varphi=\frac{1}{2}\left[\sin\left(|m|+1\right)\varphi-\sin\left(|m|-1\right)\varphi\right] \tag{25}$$

for any $|m|\le n$ ($n\ge 0$). The derivation process to obtain handy expressions for $\nabla u_{n,\mathrm{in}}^{m}$ and $\nabla u_{n,\mathrm{ex}}^{m}$ with $n\ge 0$ and $|m|\le n$ involves long and tedious calculations, however it is useful to provide the basic steps that gives us the necessary tools to accomplish this task. Under this aim, the procedure we follow is based on the utilization of the gradient operator within (10) in terms of the decomposition of the orthonormal spherical basis to the Cartesian one via the formulae

$$\hat{\mathbf{r}}=\sin\theta\cos\varphi\hat{\boldsymbol{x}}_1+\sin\theta\sin\varphi\hat{\boldsymbol{x}}_2+\cos\theta\hat{\boldsymbol{x}}_3, \tag{26}$$

$$\hat{\boldsymbol{\theta}}=\cos\theta\cos\varphi\hat{\boldsymbol{x}}_1+\cos\theta\sin\varphi\hat{\boldsymbol{x}}_2-\sin\theta\hat{\boldsymbol{x}}_3, \tag{27}$$

$$\hat{\boldsymbol{\varphi}}=-\sin\varphi\hat{\boldsymbol{x}}_1+\cos\varphi\hat{\boldsymbol{x}}_2. \tag{28}$$

Thus, substituting above (26)–(28) into the spherical gradient operator from (10), we initially invoke the surface spherical harmonics from (11) into the corresponding solid spherical harmonics (13), next we write $e^{\mathrm{i}m\varphi}$ in the form $e^{\pm\mathrm{i}|m|\varphi}=\cos|m|\varphi\pm\mathrm{i}\sin|m|\varphi$ for every $|m|\le n$ ($n\ge 0$) and, finally, by virtue of $\cos\theta=x$ and $\sin\theta=\sqrt{1-x^2}$ with $x\in[-1,1]$, we use recurrence relations (17)–(21) for the associated Legendre functions of the first kind, as well as relationships (22)–(25) for the trigonometric functions, so as to perform our analysis. Consequently, combining properly the relative terms to reproduce surface spherical harmonics, we derive for the interior solid spherical harmonic eigenfunctions $u_{n,\mathrm{in}}^{m}$ in $\Omega\left(\mathbb{R}^3\right)$ the expressions

$$\begin{aligned}\nabla u_{n,\mathrm{in}}^{m}(\mathbf{r})={}&\frac{1}{2}\kappa_n^m\left[\left(n+|m|\right)\left(n+|m|-1\right)\frac{Y_{n-1}^{m-1}(\hat{\mathbf{r}})}{\kappa_{n-1}^{m-1}}-\frac{Y_{n-1}^{m+1}(\hat{\mathbf{r}})}{\kappa_{n-1}^{m+1}}\right]r^{n-1}\hat{\mathbf{x}}_1\\&+\frac{\mathrm{i}}{2}\kappa_n^m\left[\left(n+|m|\right)\left(n+|m|-1\right)\frac{Y_{n-1}^{m-1}(\hat{\mathbf{r}})}{\kappa_{n-1}^{m-1}}+\frac{Y_{n-1}^{m+1}(\hat{\mathbf{r}})}{\kappa_{n-1}^{m+1}}\right]r^{n-1}\hat{\mathbf{x}}_2\\&+\kappa_n^m\left[\left(n+|m|\right)\frac{Y_{n-1}^{m}(\hat{\mathbf{r}})}{\kappa_{n-1}^{m}}+\frac{Y_{n-1}^{m+1}(\hat{\mathbf{r}})}{\kappa_{n-1}^{m+1}}\right]r^{n-1}\hat{\mathbf{x}}_3\quad\text{for } n\ge 0 \text{ and } m=-n,\ldots,-1,1,\ldots,n\end{aligned} \tag{29}$$

and for the case $m=0$

$$\begin{aligned}\nabla u_{n,\mathrm{in}}^{0}(\mathbf{r})={}&-\frac{1}{2}\kappa_n^0\left[\frac{Y_{n-1}^{1}(\hat{\mathbf{r}})}{\kappa_{n-1}^{1}}-\frac{Y_{n-1}^{-1}(\hat{\mathbf{r}})}{\kappa_{n-1}^{-1}}\right]r^{n-1}\hat{\mathbf{x}}_1\\&+\frac{\mathrm{i}}{2}\kappa_n^0\left[\frac{Y_{n-1}^{1}(\hat{\mathbf{r}})}{\kappa_{n-1}^{1}}+\frac{Y_{n-1}^{-1}(\hat{\mathbf{r}})}{\kappa_{n-1}^{-1}}\right]r^{n-1}\hat{\mathbf{x}}_2\\&+\kappa_n^0\left[n\frac{Y_{n-1}^{0}(\hat{\mathbf{r}})}{\kappa_{n-1}^{0}}-\frac{Y_{n-1}^{-1}(\hat{\mathbf{r}})}{\kappa_{n-1}^{-1}}\right]r^{n-1}\hat{\mathbf{x}}_3\quad\text{for } n\ge 0,\end{aligned} \tag{30}$$

while for the exterior solid spherical harmonic eigenfunctions $u_{n,\mathrm{ex}}^m$ in $\Omega\left(\mathbb{R}^3\right)$, we similarly receive the expressions

$$\nabla u_{n,\mathrm{ex}}^m\left(\mathbf{r}\right)=\frac{1}{2}\kappa_n^m\left[\left(n-|m|+1\right)\left(n-|m|+2\right)\frac{Y_{n+1}^{m-1}\left(\hat{\mathbf{r}}\right)}{\kappa_{n+1}^{m-1}}-\frac{Y_{n+1}^{m+1}\left(\hat{\mathbf{r}}\right)}{\kappa_{n+1}^{m+1}}\right]r^{-(n+2)}\hat{\mathbf{x}}_1$$

$$+\frac{\mathrm{i}}{2}\kappa_n^m\left[\left(n-|m|+1\right)\left(n-|m|+2\right)\frac{Y_{n+1}^{m-1}\left(\hat{\mathbf{r}}\right)}{\kappa_{n+1}^{m-1}}+\frac{Y_{n+1}^{m+1}\left(\hat{\mathbf{r}}\right)}{\kappa_{n+1}^{m+1}}\right]r^{-(n+2)}\hat{\mathbf{x}}_2$$

$$-\kappa_n^m\left[\left(n-|m|+1\right)\frac{Y_{n+1}^{m}\left(\hat{\mathbf{r}}\right)}{\kappa_{n+1}^{m}}+\frac{Y_{n+1}^{m+1}\left(\hat{\mathbf{r}}\right)}{\kappa_{n+1}^{m+1}}\right]r^{-(n+2)}\hat{\mathbf{x}}_3 \quad \text{for} \quad n\geq 0 \quad \text{and} \quad m=-n,\ldots,-1,1,\ldots,n \tag{31}$$

and for the case $m=0$

$$\nabla u_{n,\mathrm{ex}}^0\left(\mathbf{r}\right)=-\frac{1}{2}\kappa_n^0\left[\frac{Y_{n+1}^{1}\left(\hat{\mathbf{r}}\right)}{\kappa_{n+1}^{1}}+\frac{Y_{n+1}^{-1}\left(\hat{\mathbf{r}}\right)}{\kappa_{n+1}^{-1}}\right]r^{-(n+2)}\hat{\mathbf{x}}_1$$

$$+\frac{\mathrm{i}}{2}\kappa_n^0\left[\frac{Y_{n+1}^{1}\left(\hat{\mathbf{r}}\right)}{\kappa_{n+1}^{1}}-\frac{Y_{n+1}^{-1}\left(\hat{\mathbf{r}}\right)}{\kappa_{n+1}^{-1}}\right]r^{-(n+2)}\hat{\mathbf{x}}_2$$

$$-\kappa_n^0\left[\left(n+1\right)\frac{Y_{n+1}^{0}\left(\hat{\mathbf{r}}\right)}{\kappa_{n+1}^{0}}\right]r^{-(n+2)}\hat{\mathbf{x}}_3 \quad \text{for} \quad n\geq 0\,, \tag{32}$$

where

$$\kappa_n^m=\sqrt{\frac{2n+1}{4\pi}\frac{\left(n-|m|\right)!}{\left(n+|m|\right)!}} \quad \text{for} \quad n\geq 0 \quad \text{and} \quad |m|\leq n \tag{33}$$

are the normalizing constants of the spherical surface harmonics. It is obvious, from the definition of the associated Legendre functions, that

$$Y_{-n}^m\left(\hat{\mathbf{r}}\right)\equiv 0 \quad \text{for} \quad n\geq 0 \quad \text{and} \quad |m|\leq n \quad \text{with} \quad \hat{\mathbf{r}}=\left(\theta,\varphi\right)\in\left[0,\pi\right]\times\left[0,2\pi\right), \tag{34}$$

while

$$Y_{n}^m\left(\hat{\mathbf{r}}\right)\equiv 0 \quad \text{for} \quad n\geq 0 \quad \text{and} \quad |m|> n \quad \text{with} \quad \hat{\mathbf{r}}=\left(\theta,\varphi\right)\in\left[0,\pi\right]\times\left[0,2\pi\right), \tag{35}$$

rendering formulae (29)–(32) with (33) applicable for every value of n and m.

In order to validate the correctness and demonstrate the effectiveness of the produced formulae, we provide a simple example of evaluating $\nabla u_{1,\mathrm{in}}^1$, utilizing two different approaches, one with direct calculation and the other with the aid of expression (29) for $n=1$ and $m=1$, showing that both the two results coincide. Towards this direction, from (11) and (33) we obtain

$$Y_1^1(\hat{\mathbf{r}}) = \kappa_1^1 P_1^1(\cos\theta)e^{\mathrm{i}\varphi} = \kappa_1^1 \sin\theta e^{\mathrm{i}\varphi} \quad \text{with} \quad \kappa_1^1 = \sqrt{\frac{3}{8\pi}} \text{ for } \hat{\mathbf{r}} = (\theta,\varphi) \in [0,\pi]\times[0,2\pi), \tag{36}$$

since $P_1^1(\cos\theta) = \sin\theta$, therefore (13) yields

$$u_{1,\mathrm{in}}^1(\mathbf{r}) = r^1 Y_1^1(\hat{\mathbf{r}}) = \kappa_1^1 r P_1^1(\cos\theta)e^{\mathrm{i}\varphi} = \kappa_1^1 r \sin\theta e^{\mathrm{i}\varphi} \quad \text{for} \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{37}$$

Thus, by direct action of the gradient operator from (10) on the interior harmonic (37) and in view of equations (26)–(28), it holds

$$\begin{aligned}
\nabla u_{1,\mathrm{in}}^1(\mathbf{r}) &= \left[\hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\boldsymbol{\varphi}}}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]\left(\kappa_1^1 r\sin\theta e^{\mathrm{i}\varphi}\right) \\
&= \kappa_1^1 \hat{\mathbf{r}} \sin\theta e^{\mathrm{i}\varphi} + \kappa_1^1 \hat{\boldsymbol{\theta}} \cos\theta e^{\mathrm{i}\varphi} + \mathrm{i}\kappa_1^1 \hat{\boldsymbol{\varphi}}\, e^{\mathrm{i}\varphi} \\
&= \kappa_1^1\left(\sin\theta\cos\varphi\hat{\mathbf{x}}_1 + \sin\theta\sin\varphi\hat{\mathbf{x}}_2 + \cos\theta\hat{\mathbf{x}}_3\right)\sin\theta e^{\mathrm{i}\varphi} \\
&+ \kappa_1^1\left(\cos\theta\cos\varphi\hat{\mathbf{x}}_1 + \cos\theta\sin\varphi\hat{\mathbf{x}}_2 - \sin\theta\hat{\mathbf{x}}_3\right)\cos\theta e^{\mathrm{i}\varphi} \\
&+ \mathrm{i}\kappa_1^1\left(-\sin\varphi\hat{\mathbf{x}}_1 + \cos\varphi\hat{\mathbf{x}}_2\right)e^{\mathrm{i}\varphi} \\
&= \kappa_1^1(\cos\varphi - \mathrm{i}\sin\varphi)\,\hat{\mathbf{x}}_1 e^{\mathrm{i}\varphi} + \kappa_1^1(\sin\varphi + \mathrm{i}\cos\varphi)\,\hat{\mathbf{x}}_2 e^{\mathrm{i}\varphi} \quad \text{for} \quad \mathbf{r} \in \Omega(\mathbb{R}^3)
\end{aligned} \tag{38}$$

or

$$\nabla u_{1,\mathrm{in}}^1(\mathbf{r}) = \kappa_1^1 \hat{\mathbf{x}}_1 + \mathrm{i}\kappa_1^1 \hat{\mathbf{x}}_2 = \kappa_1^1\left(\hat{\mathbf{x}}_1 + \mathrm{i}\,\hat{\mathbf{x}}_2\right) \quad \text{for} \quad \mathbf{r} \in \Omega(\mathbb{R}^3) \tag{39}$$

because $e^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi$. Otherwise, if we utilize the derived formula (29) for $n = m = 1$ and we take profit from the property (35), yielding $Y_0^1 \equiv 0$ and $Y_0^2 \equiv 0$, we have

$$\nabla u_{1,\mathrm{in}}^1(\mathbf{r}) = \frac{1}{2}\kappa_1^1\left[2\frac{Y_0^0(\hat{\mathbf{r}})}{\kappa_0^0}\right]\hat{\mathbf{x}}_1 + \frac{\mathrm{i}}{2}\kappa_1^1\left[2\frac{Y_0^0(\hat{\mathbf{r}})}{\kappa_0^0}\right]\hat{\mathbf{x}}_2 \quad \text{for} \quad \mathbf{r} \in \Omega(\mathbb{R}^3) \tag{40}$$

or

$$\nabla u_{1,\mathrm{in}}^1(\mathbf{r}) = \kappa_1^1 \hat{\mathbf{x}}_1 + \mathrm{i}\kappa_1^1 \hat{\mathbf{x}}_2 = \kappa_1^1\left(\hat{\mathbf{x}}_1 + \mathrm{i}\,\hat{\mathbf{x}}_2\right) \quad \text{for} \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{41}$$

since $\kappa_0^0 = 1$ and $Y_0^0 = 1$. Obviously, (39) and (41) are identical, not only validating the obtained expression (29), but also confirming that our approach is much more efficient and faster than the direct calculation of the gradient on solid spherical harmonics. It is evident that following a similar way, we can reproduce the subspace of any space of harmonic functions in the Cartesian basis, while a general procedure can be readily established.

Concluding, the result (16) for the displacement field, via the Papkovich differential representation, can be now rewritten in a handy form, by substituting the outcomes (29)–(32) with (33)–(35) into (16), using the Cartesian expressions for the implicated constant coefficients (15) and the interior, as well as the exterior harmonic eigenfunctions (13). Even though the current theory is applied on spherical boundaries, it can be extended to any surface. For instance, we indicate as an example that in a scattering problem by an arbitrary compact body we can always consider a sphere surrounding the scatterer. In this case, utilizing Green's second identity [22] we can transfer any information from the boundary of the scatterer to the sphere including the scatterer. This way, we reduce the exterior problem to the spherical geometry, where the boundary conditions are globally introduced. After this, it is obvious that the displacement field is now easily accessible to be applied to any kind of boundary value problem that concerns the wide area of linear isotropic elasticity.

4. Application: Displacement field inside and outside a spherical boundary

The purpose of this section is to demonstrate the usefulness and the efficiency of the proposed analytical methodology, by invoking some special case problems in spherical geometry, which can be found in simple but quite important physical problems in linear isotropic elasticity. The specific choices of the examples come from specific boundary value problems appearing in the theory of low-frequency scattering [23]. In fact, the assumed elastic fields describe the leading low-frequency approximations of the incident excitation field. Consequently, the forthcoming examples come from real physical problems and they are not artificial.

We consider a spherical body of radius a, whose center coincides with the center of the Cartesian coordinate system. Therefore, the surrounding boundary $\partial\Omega \equiv S$ corresponds to the spherical variable at $r = a$ for any $\theta \in [0,\pi]$ and $\varphi \in [0,2\pi)$. This spherical surface separates the domain of interest $\Omega(\mathbb{R}^3) \equiv \Omega$ into two subdomains, one interior Ω^- for every $r < a$ and one exterior Ω^+ for every $r > a$, such as $\Omega = \Omega^- \cup \Omega^+ \cup S$. In order to facilitate our calculations, we define the ratio of the main phase velocities in an elastic medium [14] via the formula

$$\tau = \sqrt{\frac{\mu}{\lambda + 2\mu}}, \tag{42}$$

hence, the differential representation of the displacement field (3) becomes

$$\mathbf{u}^-(\mathbf{r}) = \frac{\tau^2+1}{2}\mathbf{A}^-(\mathbf{r}) + \frac{\tau^2-1}{2}\left[\nabla \otimes \mathbf{A}^-(\mathbf{r})\right]\cdot \mathbf{r}, \text{ where } \Delta\mathbf{A}^-(\mathbf{r}) = \mathbf{0} \text{ for } \mathbf{r} \in \Omega^- \tag{43}$$

inside the sphere and

$$\mathbf{u}^+(\mathbf{r}) = \frac{\tau^2+1}{2}\mathbf{A}^+(\mathbf{r}) + \frac{\tau^2-1}{2}\left[\nabla \otimes \mathbf{A}^+(\mathbf{r})\right]\cdot \mathbf{r}, \text{ where } \Delta\mathbf{A}^+(\mathbf{r}) = \mathbf{0} \text{ for } \mathbf{r} \in \Omega^+ \tag{44}$$

outside the sphere, where we have once more utilized the identity $\nabla(\mathbf{r}\cdot\mathbf{A}^{\mp}) = \mathbf{A}^{\mp} + (\nabla \otimes \mathbf{A}^{\mp})\cdot\mathbf{r}$. The harmonic potentials $\mathbf{A}^-$ and $\mathbf{A}^+$ admit expansions similar to (14) with (15), taking profit of their interior and exterior character, respectively, thus, using (13), we have

$$\mathbf{A}^-(\mathbf{r}) = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n} \mathbf{c}_{n,\text{in}}^m u_{n,\text{in}}^m(\mathbf{r}) = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n} \mathbf{c}_{n,\text{in}}^m r^n Y_n^m(\hat{\mathbf{r}}) \text{ for } \mathbf{r} \in \Omega^-, \tag{45}$$

in order for the potential $\mathbf{A}^-$ to be regular at the origin and

$$\mathbf{A}^+(\mathbf{r}) = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n} \mathbf{c}_{n,\text{ex}}^m u_{n,\text{ex}}^m(\mathbf{r}) = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n} \mathbf{c}_{n,\text{ex}}^m r^{-(n+1)} Y_n^m(\hat{\mathbf{r}}) \quad \text{for } \mathbf{r}\in\Omega^+, \tag{46}$$

in order for the potential $\mathbf{A}^+$ to remain bounded as we move towards infinity. The unknown vector constant coefficients

$$\mathbf{c}_{n,\text{y}}^m = c_{n,\text{y}}^{m,1}\hat{\boldsymbol{x}}_1 + c_{n,\text{y}}^{m,2}\hat{\boldsymbol{x}}_2 + c_{n,\text{y}}^{m,3}\hat{\boldsymbol{x}}_3 \quad \text{for } \text{y} = \text{in, ex}, \text{ where } n \geq 0 \text{ and } |m| \leq n \tag{47}$$

within (45) and (46) are calculated, when a particular set of boundary conditions is applied on the surface boundary S at $r = a$. In the sequel, we focus ourselves in providing general solutions of particular interest in Ω^- and Ω^+ for specific values of the degree $n \geq 0$, without getting involved with the boundary value problem itself.

– 1st *Example*: If $\mathbf{c}$ is a constant vector, then it is not difficult to prove, performing direct substitution, that the Papkovich potentials

$$\mathbf{A}^-(\mathbf{r}) = \mathbf{c}, \; r < a \;\text{ and }\; \mathbf{A}^+(\mathbf{r}) = \frac{\mathbf{c}}{r}, \; r > a \tag{48}$$

generate the displacement fields

$$\mathbf{u}^-(\mathbf{r}) = \frac{\tau^2+1}{2}\mathbf{c} \quad \text{for } \mathbf{r}\in\Omega^- \tag{49}$$

and

$$\mathbf{u}^+(\mathbf{r}) = \left[\frac{\tau^2+1}{2}\tilde{\mathbf{I}} - \frac{\tau^2-1}{2}\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right]\cdot\frac{\mathbf{c}}{r} \quad \text{for } \mathbf{r}\in\Omega^+, \tag{50}$$

respectively.

– 2nd *Example*: If $\tilde{\mathbf{C}}$ is a constant tensor, let us introduce the Papkovich potentials

$$\mathbf{A}^-(\mathbf{r}) = \tilde{\mathbf{C}}\cdot\mathbf{r}, \; r < a \;\text{ and }\; \mathbf{A}^+(\mathbf{r}) = \tilde{\mathbf{C}}\cdot\frac{\mathbf{r}}{r^3}, \; r > a, \tag{51}$$

which, by virtue of the identities

$$\nabla\otimes\mathbf{A}^-(\mathbf{r}) = \left(\nabla\otimes\tilde{\mathbf{C}}\right)\cdot\mathbf{r} + \left(\nabla\otimes\mathbf{r}\right)\cdot\tilde{\mathbf{C}}^{\mathrm{T}} = \tilde{\mathbf{C}}^{\mathrm{T}} \tag{52}$$

and

$$\nabla\otimes\mathbf{A}^+(\mathbf{r}) = \left(\nabla\otimes\tilde{\mathbf{C}}\right)\cdot\frac{\mathbf{r}}{r^3} + \left(\nabla\otimes\frac{\mathbf{r}}{r^3}\right)\cdot\tilde{\mathbf{C}}^{\mathrm{T}} = \frac{\tilde{\mathbf{I}} - 3\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}}{r^3}\cdot\tilde{\mathbf{C}}^{\mathrm{T}} \tag{53}$$

generate the displacement fields

$$\mathbf{u}^{-}(\mathbf{r})=\left(\frac{\tau^{2}+1}{2}\tilde{\mathbf{C}}+\frac{\tau^{2}-1}{2}\tilde{\mathbf{C}}^{\mathrm{T}}\right)\cdot\mathbf{r}\quad\text{for }\ \mathbf{r}\in\Omega^{-} \tag{54}$$

and

$$\mathbf{u}^{+}(\mathbf{r})=\left[\frac{\tau^{2}+1}{2}\tilde{\mathbf{C}}+\frac{\tau^{2}-1}{2}\left(\tilde{\mathbf{I}}-3\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)\cdot\tilde{\mathbf{C}}^{\mathrm{T}}\right]\cdot\frac{\mathbf{r}}{r^{3}}\quad\text{for }\ \mathbf{r}\in\Omega^{+}, \tag{55}$$

respectively. Here, we must remark that, if we consider the potentials $\mathbf{A}^{-}=\mathbf{r}\cdot\tilde{\mathbf{C}}=\tilde{\mathbf{C}}^{\mathrm{T}}\cdot\mathbf{r}$ and $\mathbf{A}^{+}=\left(\mathbf{r}/r^{3}\right)\cdot\tilde{\mathbf{C}}=\tilde{\mathbf{C}}^{\mathrm{T}}\cdot\left(\mathbf{r}/r^{3}\right)$, we obtain the displacements generated by $\tilde{\mathbf{C}}^{\mathrm{T}}$ instead of $\tilde{\mathbf{C}}$. On the other hand, if $\tilde{\mathbf{C}}$ is symmetric, i.e. if $\tilde{\mathbf{C}}=\tilde{\mathbf{C}}^{\mathrm{T}}$, then the interior and exterior displacements (54) and (55) are rewritten as

$$\mathbf{u}^{-}(\mathbf{r})=\tau^{2}\left(\tilde{\mathbf{C}}\cdot\mathbf{r}\right)\ \text{ for }\ \mathbf{r}\in\Omega^{-}\ \text{ and }\ \mathbf{u}^{+}(\mathbf{r})=\left[\tau^{2}\tilde{\mathbf{C}}-\frac{3}{2}\left(\tau^{2}-1\right)\left(\tilde{\mathbf{C}}:\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)\tilde{\mathbf{I}}\right]\cdot\frac{\mathbf{r}}{r^{3}}\ \text{ for }\ \mathbf{r}\in\Omega^{+}, \tag{56}$$

respectively. In particular, if $\tilde{\mathbf{C}}=\tilde{\mathbf{I}}$, then (56) simplify to

$$\mathbf{u}^{-}(\mathbf{r})=\tau^{2}\mathbf{r}\ \text{ for }\ \mathbf{r}\in\Omega^{-}\ \text{ and }\ \mathbf{u}^{+}(\mathbf{r})=\frac{3-\tau^{2}}{2}\frac{\mathbf{r}}{r^{3}}\ \text{ for }\ \mathbf{r}\in\Omega^{+}, \tag{57}$$

respectively. Finally, if $\tilde{\mathbf{C}}$ is antisymmetric, i.e. if $\tilde{\mathbf{C}}^{\mathrm{T}}=-\tilde{\mathbf{C}}$, then obviously

$$\tilde{\mathbf{C}}^{\mathrm{T}}=\frac{1}{2}\left(\tilde{\mathbf{C}}^{\mathrm{T}}+\tilde{\mathbf{C}}^{\mathrm{T}}\right)=\frac{1}{2}\left(\tilde{\mathbf{C}}^{\mathrm{T}}-\tilde{\mathbf{C}}\right) \tag{58}$$

and since

$$\tilde{\mathbf{C}}^{\mathrm{T}}:\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}=\tilde{\mathbf{C}}:\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}, \tag{59}$$

we obtain

$$\tilde{\mathbf{C}}^{\mathrm{T}}:\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}=\frac{1}{2}\tilde{\mathbf{C}}^{\mathrm{T}}:\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}-\frac{1}{2}\tilde{\mathbf{C}}:\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}=0. \tag{60}$$

Taking into account (58)–(60), the interior and exterior fields (54) and (55) are transformed to

$$\mathbf{u}^{-}(\mathbf{r})=\tilde{\mathbf{C}}\cdot\mathbf{r}\ \text{ for }\ \mathbf{r}\in\Omega^{-}\ \text{ and }\ \mathbf{u}^{+}(\mathbf{r})=\tilde{\mathbf{C}}\cdot\frac{\mathbf{r}}{r^{3}}\ \text{ for }\ \mathbf{r}\in\Omega^{+}, \tag{61}$$

respectively, which coincide with the harmonic potentials (51).

– ***3rd Example*:** If $\tilde{\mathbf{C}}$ is a constant dyadic, then we want to find the displacement field that the exterior harmonic potential

$$\mathbf{A}^{+}(\mathbf{r})=\nabla\left[\tilde{\mathbf{C}}:\nabla\otimes\nabla\frac{1}{r}\right],\quad r>a\quad\text{with }\ \tilde{\mathbf{C}}=\sum_{i=1}^{3}\boldsymbol{a}_{i}\otimes\boldsymbol{b}_{i}, \tag{62}$$

produces. To prove it we first show that

$$\nabla \frac{1}{r} = -\frac{\mathbf{r}}{r^3} \quad \text{and} \quad \nabla \otimes \nabla \frac{1}{r} = \nabla \otimes \left(-\frac{\mathbf{r}}{r^3} \right) = \frac{3\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} - \tilde{\mathbf{I}}}{r^3}. \tag{63}$$

Then, combining (62) and (63), we have

$$\mathbf{A}^+(\mathbf{r}) = \nabla \left[\tilde{\mathbf{C}} : \frac{3\hat{\mathbf{r}} \otimes \hat{\mathbf{r}} - \tilde{\mathbf{I}}}{r^3} \right] = \sum_{i=1}^{3} \nabla \left[\boldsymbol{a}_i \otimes \boldsymbol{b}_i : \left(\frac{3}{r^5} \mathbf{r} \otimes \mathbf{r} - \frac{1}{r^3} \tilde{\mathbf{I}} \right) \right]$$

$$= \sum_{i=1}^{3} \nabla \left[\frac{3}{r^5} (\boldsymbol{a}_i \cdot \mathbf{r})(\boldsymbol{b}_i \cdot \mathbf{r}) - \frac{1}{r^3} (\boldsymbol{a}_i \cdot \boldsymbol{b}_i) \right]$$

$$= \sum_{i=1}^{3} \left[-\frac{15}{r^7} \boldsymbol{a}_i \otimes \boldsymbol{b}_i : \mathbf{r} \otimes \mathbf{r} \otimes \mathbf{r} + \frac{3}{r^5} \boldsymbol{a}_i \otimes \boldsymbol{b}_i \cdot \mathbf{r} + \frac{3}{r^5} \boldsymbol{b}_i \otimes \boldsymbol{a}_i \cdot \mathbf{r} + \frac{3}{r^5} (\boldsymbol{a}_i \cdot \boldsymbol{b}_i) \mathbf{r} \right]$$

$$= \frac{3}{r^5} \sum_{i=1}^{3} \left[-5\boldsymbol{a}_i \otimes \boldsymbol{b}_i : \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} \otimes \mathbf{r} + \boldsymbol{a}_i \otimes \boldsymbol{b}_i \cdot \mathbf{r} + \boldsymbol{b}_i \otimes \boldsymbol{a}_i \cdot \mathbf{r} + (\boldsymbol{a}_i \cdot \boldsymbol{b}_i) \mathbf{r} \right] \tag{64}$$

or

$$\mathbf{A}^+(\mathbf{r}) = \frac{3}{r^5} \left[(\tilde{\mathbf{C}} : \tilde{\mathbf{I}}) \tilde{\mathbf{I}} + (\tilde{\mathbf{C}} + \tilde{\mathbf{C}}^\top) - 5 (\tilde{\mathbf{C}} : \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}) \tilde{\mathbf{I}} \right] \cdot \mathbf{r}, \quad r > a. \tag{65}$$

Thereafter, the gradient of the Papkovich potential (65), i.e. $\nabla \otimes \mathbf{A}^+$, is then calculated, using classical identities and algebra as follows,

$$\nabla \otimes \mathbf{A}^+(\mathbf{r}) = 3\nabla \left[\left((\tilde{\mathbf{C}} : \tilde{\mathbf{I}}) \tilde{\mathbf{I}} + (\tilde{\mathbf{C}} + \tilde{\mathbf{C}}^\top) \right) \cdot \frac{\mathbf{r}}{r^5} \right] - 15 \sum_{i=1}^{3} \nabla \left[\frac{1}{r^7} (\boldsymbol{a}_i \cdot \mathbf{r})(\boldsymbol{b}_i \cdot \mathbf{r}) \mathbf{r} \right]$$

$$= 3 \left(\nabla \otimes \frac{\mathbf{r}}{r^5} \right) \cdot \left[(\tilde{\mathbf{C}} : \tilde{\mathbf{I}}) \tilde{\mathbf{I}} + (\tilde{\mathbf{C}} + \tilde{\mathbf{C}}^\top) \right]^\top$$

$$-15 \sum_{i=1}^{3} \left[-\frac{7}{r^5} \boldsymbol{a}_i \otimes \boldsymbol{b}_i : \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \frac{1}{r^5} \boldsymbol{a}_i \otimes \boldsymbol{b}_i \cdot \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \frac{1}{r^5} \boldsymbol{b}_i \otimes \boldsymbol{a}_i \cdot \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \frac{1}{r^5} (\boldsymbol{a}_i \otimes \boldsymbol{b}_i : \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}) \tilde{\mathbf{I}} \right]$$

$$= 3 \left[-\frac{5}{r^5} \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} + \frac{1}{r^5} \tilde{\mathbf{I}} \right] \cdot \left[(\tilde{\mathbf{C}} : \tilde{\mathbf{I}}) \tilde{\mathbf{I}} + (\tilde{\mathbf{C}} + \tilde{\mathbf{C}}^\top) \right]$$

$$+15 \left[\frac{7}{r^5} (\tilde{\mathbf{C}} : \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}) \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} - \frac{1}{r^5} (\tilde{\mathbf{C}} + \tilde{\mathbf{C}}^\top) \cdot \hat{\mathbf{r}} \otimes \hat{\mathbf{r}} - \frac{1}{r^5} (\tilde{\mathbf{C}} : \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}) \tilde{\mathbf{I}} \right], \quad r > a. \tag{66}$$

Contracting $\nabla\otimes\mathbf{A}^+$ of (66) with $\mathbf{r}$ from the right, we obtain

$$\left[\nabla\mathbf{A}^+(\mathbf{r})\right]\cdot\mathbf{r}=\left[-\frac{15}{r^5}\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}+\frac{3}{r^5}\tilde{\mathbf{I}}\right]\cdot\left[\left(\tilde{\mathbf{C}}:\tilde{\mathbf{I}}\right)\mathbf{r}+\left(\tilde{\mathbf{C}}+\tilde{\mathbf{C}}^\top\right)\cdot\mathbf{r}\right]$$

$$+15\left[\frac{7}{r^5}\left(\tilde{\mathbf{C}}:\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)\mathbf{r}-\frac{1}{r^5}\left(\tilde{\mathbf{C}}+\tilde{\mathbf{C}}^\top\right)\cdot\mathbf{r}-\frac{1}{r^5}\left(\tilde{\mathbf{C}}:\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)\mathbf{r}\right]$$

$$=-\frac{15}{r^5}\left(\tilde{\mathbf{C}}:\tilde{\mathbf{I}}\right)\mathbf{r}+\frac{3}{r^5}\left(\tilde{\mathbf{C}}:\tilde{\mathbf{I}}\right)\mathbf{r}-\frac{15}{r^5}\left(\tilde{\mathbf{C}}+\tilde{\mathbf{C}}^\top\right):\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\otimes\mathbf{r}$$

$$+\frac{3}{r^5}\left(\tilde{\mathbf{C}}+\tilde{\mathbf{C}}^\top\right)\cdot\mathbf{r}+\frac{90}{r^5}\left(\tilde{\mathbf{C}}:\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)\mathbf{r}-\frac{15}{r^5}\left(\tilde{\mathbf{C}}+\tilde{\mathbf{C}}^\top\right)\cdot\mathbf{r}$$

$$=-\frac{12}{r^5}\left(\tilde{\mathbf{C}}:\tilde{\mathbf{I}}\right)\mathbf{r}+\frac{60}{r^5}\left(\tilde{\mathbf{C}}:\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)\mathbf{r}-\frac{12}{r^5}\left(\tilde{\mathbf{C}}+\tilde{\mathbf{C}}^\top\right)\cdot\mathbf{r}$$

$$=-4\frac{3}{r^5}\left[\left(\tilde{\mathbf{C}}:\tilde{\mathbf{I}}\right)\tilde{\mathbf{I}}+\left(\tilde{\mathbf{C}}+\tilde{\mathbf{C}}^\top\right)-5\left(\tilde{\mathbf{C}}:\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)\tilde{\mathbf{I}}\right]\cdot\mathbf{r}$$

$$=-4\mathbf{A}^+(\mathbf{r}),\quad r>a\,. \tag{67}$$

Substituting (67) into the Papkovich representation (44) for the exterior field, we obtain

$$\mathbf{u}^+(\mathbf{r})=\frac{\tau^2+1}{2}\mathbf{A}^+(\mathbf{r})+\frac{\tau^2-1}{2}\left[-4\mathbf{A}^+(\mathbf{r})\right]=\frac{5-3\tau^2}{2}\mathbf{A}^+(\mathbf{r})\ \text{ for }\ \mathbf{r}\in\Omega^+, \tag{68}$$

which is the requested. Herein, we remark that, since $\mathbf{A}^+$ is harmonic and $\mathbf{u}^+$ solves the equation of elastostatics (3), we obtain that $\nabla\nabla\cdot\mathbf{A}^+=\mathbf{0}$. Indeed,

$$\nabla\nabla\cdot\mathbf{A}^+(\mathbf{r})=\nabla\nabla\cdot\left[\nabla\left(\tilde{\mathbf{C}}:\nabla\otimes\nabla\frac{1}{r}\right)\right]=\nabla\Delta\left[\tilde{\mathbf{C}}:\nabla\otimes\nabla\frac{1}{r}\right]=\nabla\left[\tilde{\mathbf{C}}:\nabla\otimes\nabla\left(\Delta\frac{1}{r}\right)\right]=\mathbf{0},\quad r>a\,. \tag{69}$$

Another interesting remark is that if $\tilde{\mathbf{C}}$ is an antisymmetric dyadic, i.e. if $\tilde{\mathbf{C}}^\top=-\tilde{\mathbf{C}}$, then it is not hard to prove

$$\mathbf{A}^+(\mathbf{r})=\nabla\left[\tilde{\mathbf{C}}:\nabla\nabla\frac{1}{r}\right]=\mathbf{0},\quad r>a\ \text{ and therefore }\ \mathbf{u}^+(\mathbf{r})=\mathbf{0}\ \text{ for }\ \mathbf{r}\in\Omega^+. \tag{70}$$

This means that the potential $\mathbf{A}^+$ is symmetric and it generates a symmetric displacement. In particular, any $\tilde{\mathbf{C}}$ generates a symmetric $\mathbf{A}^+$ and thus, a symmetric $\mathbf{u}^+$.

– ***4th Example*:** In this case, we consider the equation of elastostatics (1) for $\mathbf{f}=\mathbf{0}$, which in terms of (42) is rewritten as

$$\tau^2\,\Delta\mathbf{u}^-(\mathbf{r})+\left(1-\tau^2\right)\nabla\left[\nabla\cdot\mathbf{u}^-(\mathbf{r})\right]=\mathbf{0}\ \text{ for }\ \mathbf{r}\in\Omega^- \tag{71}$$

and we wish to find the coefficients α,β,γ for which the linear combination of the displacement field

$$\mathbf{u}^{-}(\mathbf{r}) = \alpha\tilde{\mathbf{S}}:\mathbf{r}\otimes\mathbf{r}\otimes\mathbf{r} + \beta r^2\tilde{\mathbf{S}}\cdot\mathbf{r} + \gamma S r^2\mathbf{r} \quad \text{for} \quad \mathbf{r}\in\Omega^{-} \tag{72}$$

belongs to the $\ker\left[\tau^2\,\Delta + \left(1-\tau^2\right)\nabla\nabla\cdot\right]$, where $\tilde{\mathbf{S}} = \mathbf{a}\otimes\mathbf{b} = \mathbf{b}\otimes\mathbf{a} = \tilde{\mathbf{S}}^{\mathsf{T}}$ is a constant symmetric dyadic and $S = \mathbf{a}\cdot\mathbf{b}$ is its trace. This example makes use of the previous three examples, as well as the linearity of the Navier equation, to expand the physical important interior displacement field in a three-dimensional solution subspace and to calculate the actual solution by calculating the three scalar coefficients. We primarily have to evaluate the following expressions,

$$\Delta\left(\tilde{\mathbf{S}}:\mathbf{r}\otimes\mathbf{r}\otimes\mathbf{r}\right) = \Delta\left[(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})\mathbf{r}\right] = \Delta\left[(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})\right]\mathbf{r} + (\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})\Delta\mathbf{r} + 2\nabla\left[(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})\right]\mathbf{r}\cdot\nabla\otimes\mathbf{r}$$

$$= 2(\mathbf{a}\cdot\mathbf{b})\mathbf{r} + 2(\mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a})\cdot\mathbf{r} = 2S\mathbf{r} + 4\tilde{\mathbf{S}}\cdot\mathbf{r}, \tag{73}$$

$$\Delta\left(r^2\tilde{\mathbf{S}}\cdot\mathbf{r}\right) = \left(\Delta r^2\right)\tilde{\mathbf{S}}\cdot\mathbf{r} + r^2\Delta\left(\tilde{\mathbf{S}}\cdot\mathbf{r}\right) + 2\left(\nabla r^2\right)\cdot\nabla\left(\tilde{\mathbf{S}}\cdot\mathbf{r}\right) = 6\tilde{\mathbf{S}}\cdot\mathbf{r} + 4\mathbf{r}\cdot\tilde{\mathbf{S}} = 10\tilde{\mathbf{S}}\cdot\mathbf{r}, \tag{74}$$

$$\Delta\left(r^2\mathbf{r}\right) = 6\mathbf{r} + 2\cdot 2\mathbf{r}\cdot\tilde{\mathbf{I}} = 10\mathbf{r}, \tag{75}$$

$$\nabla\nabla\cdot\left(\tilde{\mathbf{S}}:\mathbf{r}\otimes\mathbf{r}\otimes\mathbf{r}\right) = \nabla\nabla\cdot\left[(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})\mathbf{r}\right] = \nabla\left[(\mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a}):\mathbf{r}\otimes\mathbf{r} + (\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})3\right]$$

$$= 5\nabla\left(\tilde{\mathbf{S}}:\mathbf{r}\otimes\mathbf{r}\right) = 5(\mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a})\cdot\mathbf{r} = 10\tilde{\mathbf{S}}\cdot\mathbf{r}, \tag{76}$$

$$\nabla\nabla\cdot\left(r^2\tilde{\mathbf{S}}\cdot\mathbf{r}\right) = \nabla\left[2\mathbf{r}\cdot\tilde{\mathbf{S}}\cdot\mathbf{r} + r^2\nabla\cdot\left(\tilde{\mathbf{S}}\cdot\mathbf{r}\right)\right] = \nabla\left[2(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r}) + r^2\tilde{\mathbf{S}}:\tilde{\mathbf{I}}\right]$$

$$= 2(\mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a})\cdot\mathbf{r} + 2S\mathbf{r} = 4\tilde{\mathbf{S}}\cdot\mathbf{r} + 2S\mathbf{r} \tag{77}$$

and

$$\nabla\nabla\cdot\left(r^2\mathbf{r}\right) = \nabla\left[2\mathbf{r}\cdot\mathbf{r} + 3r^2\right] = 5\nabla r^2 = 10\mathbf{r}, \tag{78}$$

wherein we have assumed that $\tilde{\mathbf{S}} = (\mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a})/2$, so that $\tilde{\mathbf{S}}:\mathbf{r}\otimes\mathbf{r} = (\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})$ and $S = \mathbf{a}\cdot\mathbf{b}$. In view of the above formulae (73)–(78) and with respect to (71) and (72), we now obtain

$$\left[\tau^2\Delta + \left(1-\tau^2\right)\nabla\nabla\cdot\right]\mathbf{u}^{-}(\mathbf{r}) = \alpha\tau^2\left[2S\mathbf{r} + 4\tilde{\mathbf{S}}\cdot\mathbf{r}\right] + \beta\tau^2\left[10\tilde{\mathbf{S}}\cdot\mathbf{r}\right] + \gamma\tau^2\left[10\mathbf{r}\right]S$$

$$+\alpha\left(1-\tau^2\right)\left[10\tilde{\mathbf{S}}\cdot\mathbf{r}\right] + \beta\left(1-\tau^2\right)\left[2S\mathbf{r} + 4\tilde{\mathbf{S}}\cdot\mathbf{r}\right] + \gamma\left(1-\tau^2\right)\left[10\mathbf{r}\right]S$$

$$= \left[2\alpha\tau^2 S + 10\gamma\tau^2 S + 2\beta\left(1-\tau^2\right)S + 10\gamma\left(1-\tau^2\right)S\right]\mathbf{r}$$

$$+\left[4\alpha\tau^2 S + 10\beta\tau^2 + 10\alpha\left(1-\tau^2\right) + 4\beta\left(1-\tau^2\right)\right]\tilde{\mathbf{S}}\cdot\mathbf{r}$$

$$= \left[2\alpha\tau^2 S + 2\beta\left(1-\tau^2\right)S + 10\gamma S\right]\mathbf{r}$$

$$+\left[\alpha\left(10-6\tau^2\right) + \beta\left(4+6\tau^2\right)\right]\tilde{\mathbf{S}}\cdot\mathbf{r} = \mathbf{0} \quad \text{for} \quad \mathbf{r}\in\Omega^{-}, \tag{79}$$

which implies

$$\tau^2 S\alpha + \left(1-\tau^2\right) S\beta = -5\gamma S \quad \text{and} \quad \left(5-3\tau^2\right)\alpha + \left(2+3\tau^2\right)\beta = 0\,, \tag{80}$$

leading to the constants

$$\alpha = 3\tau^2 + 2\,, \quad \beta = 3\tau^2 - 5 \quad \text{and} \quad \gamma = 1 - 2\tau^2\,. \tag{81}$$

Hence, the solution is provided via (72) by

$$\mathbf{u}^-(\mathbf{r}) = \left(3\tau^2 + 2\right)\tilde{\mathbf{S}} : \mathbf{r}\otimes\mathbf{r}\otimes\mathbf{r} + \left(3\tau^2 - 5\right) r^2 \tilde{\mathbf{S}}\cdot\mathbf{r} + \left(1-2\tau^2\right) S r^2 \mathbf{r} \quad \text{for} \quad \mathbf{r}\in\Omega^-\,, \tag{82}$$

which is the interior displacement field of the specific static elasticity problem.

Recapitulating, we offered the analytical solution for the displacement fields of four interesting problems in linear isotropic elastostatics in the absence of body forces, inside and outside a spherical boundary in the absence of boundary conditions. These solutions are solid and can be directly derived from the theory of differential representation, which we developed and analyzed in the previous section.

5. Conclusions and discussions

In this study, we presented a mathematical method for recovering the main spherical components of the well-known Navier equation in linear isotropic elastostatics, under the circumstance of no external forces present. To this end, we primarily combined the Hooke's and Newton's law, via the correlation of the displacement field with the strain, the stress and the stiffness tensors, reproducing the linearized equation of dynamic isotropic elasticity. In the sequel, we introduced the Papkovich differential representation, which offered solutions in terms of scalar and vector harmonic functions. Then, connection formulae were obtained, by which we transformed any solution of the Navier system from the Papkovich to the potential-type eigenform and vice versa. In order to enhance this procedure, we implied the commonly used spherical geometry and we calculated the time-independent displacement field, generated by the well-known spherical harmonic eigenfunctions. In order to demonstrate the effectiveness of our method, we presented some important degenerate cases for the evaluation of the interior and the exterior displacement field on either side of a spherical boundary.

Work under progress involves research directed towards the extension of the current analysis to more complicated geometries, e.g. spheroidal and ellipsoidal, producing ready-to-use functions and their Navier counterparts.

Conflict of interest

The authors have declared no conflict of interest.

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