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Research article

Random uniform exponential attractors for non-autonomous stochastic Schrödinger lattice systems in weighted space

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Abstract: We mainly study the existence of random uniform exponential attractors for non-autonomous stochastic Schrödinger lattice system with multiplicative white noise and quasi-periodic forces in weighted spaces. Firstly, the stochastic Schrödinger system is transformed into a random system without white noise by the Ornstein-Uhlenbeck process, whose solution generates a jointly continuous non-autonomous random dynamical system Φ. Secondly, we prove the existence of a uniform absorbing random set for Φ in weighted spaces. Finally, we obtain the existence of a random uniform exponential attractor for the considered system Φ in weighted space.

Keywords: non-autonomous Schrödinger lattice system; random uniform exponential attractor; weighted space; quasi-periodic force; multiplicative white noise
Mathematics Subject Classification: 34F05, 37L60, 60H10

1. Introduction

Attractors are important concepts for characterizing the long-term behavior of dynamical systems. There are many types of attractors, including global attractors, uniform and pullback attractors for non-random dynamical systems (NDS), random attractors and random uniform attractors for random dynamical systems (RDS), see [1–3, 6–8, 11, 14, 18, 21, 23, 24, 26]. However, these attractors may attract orbits at very slow speeds and their dimensions may be infinite. In order to solve these problems, Eden et al. in [12] introduced the concept of exponential attractor for autonomous deterministic dynamical systems. Since then, the concept of exponential attractor has been greatly developed, including pullback and uniform exponential attractors for NDS, random exponential attractors and random uniform exponential attractors for RDS, see [4, 5, 9, 10, 13, 15–17, 20, 22, 25, 27–30]. In recent years, Han and Zhou in [16] defined the random uniform exponential attractor for non-autonomous random dynamical systems (NRDS), and presented an existence criterion and proved the existence of random uniform exponential attractors for the stochastic non-autonomous lattice systems.

Our goal is to prove the existence of random uniform exponential attractors for the following Schrödinger lattice system with quasi-periodic forces and multiplicative noise in the weighted space ℓ^2_ρ:

$$\begin{cases} i\dfrac{du_k}{dt} + (Au)_k + i\lambda_k u_k + f(|u_k|^2)u_k = g_k(\tilde{\sigma}(t)) + i(a+ib)u_k \circ \dot{W}, & t > 0, \\ u_k(0) = u_{k,0}, \qquad k \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where i is the imaginary unit; $\sigma \in \mathbb{T}^l$, $\tilde{\sigma}(t) = (\mathbf{x}t + \sigma)\mathrm{mod}(\mathbb{T}^l) \in \mathbb{T}^l$, $\mathbf{x} = (x_1, ..., x_l) \in \mathbb{R}^l$ is a fixed vector and $x_1, ..., x_l$ are rationally independent numbers; $a, b \in \mathbb{R}$; A is a linear coupled operator; for $k \in \mathbb{Z}$, $\lambda_k > 0$, u_k, $g_k(\tilde{\sigma}(t)) \in \mathbb{C}$, $f(|u_k|^2) \in \mathbb{R}$, $|u_k|$ is the absolute value of u_k; $W(t)$ is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$, $W(t, \omega) = \omega(t)$ for $\omega \in \Omega$; "$u_k \circ \dot{W}$" denotes the Stratonovich sense of the stochastic term; $u(0) = (u_k(0))_{k\in\mathbb{Z}} \in \ell^2_\rho$, the weighted space ℓ^2_ρ will be described in more detail below.

The nonlinear Schrödinger equation and its discrete form are mainly used to describe the motion law of the state of microscopic particles changing with time, it is one of the basic equations in quantum mechanics and is widely used in the fields of solid state physics and nuclear physics, see [19]. For the Schrödinger lattice system (1.1), Karachalios and Yannacopoulos [18], Chen et al. [8], Zhou and Tan [28], Cui and Zhou [7], Zhou et al. [29], Jiang et al. [17], Wang and Wang [24], Zhang and Zhou [30] proved the existence of global attractors, exponential attractors, random attractors, pullback and uniform exponential attractors, random exponential attractors, weak pullback random attractors, random uniform exponential attractors in ℓ^2 respectively.

Let $\rho(\cdot) : \mathbb{Z} \to (0, +\infty)$ be a positive-valued function and $\rho(k) = \rho_k$ for $k \in \mathbb{Z}$, $\rho = (\rho_k)_{k\in\mathbb{Z}}$. As we know, when $\rho_k \equiv 1$, $k \in \mathbb{Z}$, the weighted space $\ell^2_\rho = \{u = (u_k)_{k\in\mathbb{Z}} : \sum_{k\in\mathbb{Z}} \rho_k |u_k|^2 < \infty, u_k \in \mathbb{C}\}$ is reduced to the regular non-weighted standard space ℓ^2, thus $\ell^2 \subset \ell^2_\rho$. In addition, ℓ^2_ρ has other good properties: ℓ^2 is dense in ℓ^2_ρ; if $\sum_{k\in\mathbb{Z}} \rho_k < \infty$, then $\ell^2 \subset \ell^\infty \subset \ell^2_\rho$, thus ℓ^2_ρ contains infinite sequences with bounded components, see [15]. We will further study the existence of a random uniform exponential attractor for the system (1.1) in weighted space ℓ^2_ρ in this paper.

This paper is organized as follows. In Section 2, we present preliminaries and assumptions concerning the term of the system (1.1) and quote the existence criterion of random uniform exponential attractors for NRDS defined on the space of infinite sequences with complex-valued components. In Section 3, we study the existence of a random uniform exponential attractor for system (1.1) in ℓ^2_ρ.

2. Preliminaries and assumptions

In this section, we introduce some notations, concepts and a criterion concerning the existence of a random uniform exponential attractor for a jointly continuous NRDS, which is obtained directly from [6, 16, 30].

Let $\mathbb{T}^l$ be the l-dimensional torus

$$\mathbb{T}^l = \{\sigma = (\sigma_1, ..., \sigma_l) : \sigma_j \in [-\pi, \pi], \quad \forall j = 1, ..., l\}$$

with the identification

$$(\sigma_1, ..., \sigma_{j-1}, -\pi, \sigma_{j+1}, ..., \sigma_l) \sim (\sigma_1, ..., \sigma_{j-1}, \pi, \sigma_{j+1}, ..., \sigma_l), \quad \forall j = 1, ..., l,$$

and the norm in $\mathbb{T}^l$ is given by

$$\|\sigma\|_{\mathbb{T}^l} = \Big(\sum_{j=1}^{l} \sigma_j^2\Big)^{\frac{1}{2}}, \ \ \forall \sigma = (\sigma_1, ..., \sigma_l) \in \mathbb{T}^l.$$

Let $\mathbf{x} = (x_1, ..., x_l) \in \mathbb{R}^l$ be a fixed vector such that $x_1, ..., x_l$ are rationally independent. For $t \in \mathbb{R}$, define

$$\vartheta_t \sigma = (\mathbf{x}t + \sigma) mod(\mathbb{T}^l), \ \ \sigma \in \mathbb{T}^l,$$

then $\{\vartheta_t\}_{t\in\mathbb{R}}$ is a translation group on $\mathbb{T}^l$ with

$$\vartheta_t \mathbb{T}^l = \mathbb{T}^l, \ \ \forall t \in \mathbb{R},$$

and

$$(t, \sigma) \to \vartheta_t \sigma \ \ \textit{is continuous.}$$

$\mathcal{B}(\mathbb{T}^l)$ denotes the Borel σ-algebra of $\mathbb{T}^l$.

Denote by $\ell_\rho^2 = \{u = (u_k)_{k\in\mathbb{Z}} : \sum_{k\in\mathbb{Z}} \rho_k |u_k|^2 < \infty, u_k \in \mathbb{C}\}$ the Hilbert space with inner product $(u, v)_\rho = \sum_{k\in\mathbb{Z}} \rho_k u_k \overline{v_k}$ and induced norm $\|u\|_\rho^2 = (u, u)_\rho = \sum_{k\in\mathbb{Z}} \rho_k |u_k|^2$ for all $u = (u_k)_{k\in\mathbb{Z}}$, $v = (v_k)_{k\in\mathbb{Z}} \in \ell_\rho^2$. Particularly, $\rho_k \equiv 1$, for all $k \in \mathbb{Z}$, ℓ_ρ^2 becomes the standard space ℓ^2 with the inner product $(\cdot, \cdot)$ and induced norm $\|\cdot\|$. Thus, $(u, v)_\rho = \sum_{k\in\mathbb{Z}} (\rho_k u_k)\overline{v_k} = (\rho u, v) = (u, \rho v)$.

Let $\mathbb{X} \doteq \mathbb{T}^l \times \ell_\rho^2$ be the extended space with norm

$$\|\mathcal{X}\|_{\mathbb{X}} = (\|\sigma\|_{\mathbb{T}^l}^2 + \|x\|_\rho^2)^{\frac{1}{2}}, \quad \forall \mathcal{X} = \{\sigma\} \times \{x\} \in \mathbb{X}, \quad x \in \ell_\rho^2, \tag{2.1}$$

and Borel σ-algebra $\mathcal{B}(\mathbb{X})$. Norm $\|\cdot\|_{\mathbb{X}}$ induces a metric.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ be an ergodic metric dynamical system, see [1]. We call two groups $\{\vartheta_t\}_{t\in\mathbb{R}}$ and $\{\theta_t\}_{t\in\mathbb{R}}$ the base flows. Hereafter, for simplicity, we identify "a.e. $\omega \in \Omega$" as "$\omega \in \Omega$".

Given a NRDS ϕ, define a mapping $\pi : \mathbb{R}^+ \times \Omega \times \mathbb{X} \to \mathbb{X}$, and

$$\pi(t, \omega, \{\sigma\} \times \{x\}) = \{\vartheta_t \sigma\} \times \{\phi(t, \omega, \sigma, x)\}. \tag{2.2}$$

The π is called a skew-product cocycle generated by ϕ and ϑ. Note that π is continuous, that is, the mapping $\mathcal{X} \to \pi(\cdot, \cdot, \mathcal{X})$ is continuous in $\mathbb{X}$, if and only if ϕ is jointly continuous in $\mathbb{T}^l$ and ℓ_ρ^2. We often write $\pi(t, \omega, \mathcal{X})$ as $\pi(t, \omega)\mathcal{X}$ for convenience.

Let

$$\mathcal{D}_{\mathbb{X}} = \{\mathbb{B} : \mathbb{B} = \mathbb{T}^l \times B = \{\mathbb{T}^l \times B(\omega)\}_{\omega\in\Omega} \ \textit{and} \ B \in \mathcal{D}(\ell_\rho^2)\}. \tag{2.3}$$

Note that any element of $\mathcal{D}_{\mathbb{X}}$ is random set in $\mathbb{X}$. $\mathcal{D} = \mathcal{D}(\ell_\rho^2)$ be the collection of all tempered bounded random sets of ℓ_ρ^2, i.e.,

$$\mathcal{D} = \begin{Bmatrix} D : D \text{ is the bounded random sets in } \ell_\rho^2 \text{ satisfying} \\ e^{-\gamma t} \|D(\theta_{-t}\omega)\|_\rho^2 \xrightarrow{t\to+\infty} 0, \forall \gamma > 0, \omega \in \Omega \end{Bmatrix}.$$

We list the following assumptions on the continuous skew-product cocycle $\{\pi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ on the extended space $\mathbb{X}$:

(H1) There exist a tempered random variable R_ω and a tempered closed random set $\mathbb{D}_0 = \{\mathbb{D}_0(\omega)\}_{\omega\in\Omega}$ satisfying that $R_{\theta_t\omega}$ is continuous in $t \in \mathbb{R}$ and $sup_{\mathcal{X},\mathcal{Y}\in\mathbb{D}_0(\omega)}\|\mathcal{X} - \mathcal{Y}\|_{\mathbb{X}} \leq R_\omega < \infty$; Moreover, for any $\omega \in \Omega$ and $\mathbb{D}(\omega) \in \mathbb{D} \in \mathcal{D}_{\mathbb{X}}$, there exists a $T(\mathbb{D},\omega) \geq 0$ such that $\pi(t,\theta_{-t}\omega)\mathbb{D}(\theta_{-t}\omega) \subseteq \mathbb{D}_0(\omega)$ for all $t \geq T(\mathbb{D},\omega)$. For any $\omega \in \Omega$, $s \geq 0$, set

$$\chi(\theta_{-s}\omega) = \overline{\cup_{t\geq \max\{T(\mathbb{D}_0,\omega),T(\mathbb{D}_0,\theta_{-s}\omega)\}}\pi(t,\theta_{-t-s}\omega)\mathbb{D}_0(\theta_{-t-s}\omega)} \subseteq \mathbb{D}_0(\theta_{-s}\omega); \tag{2.4}$$

(H2) There exist positive numbers $\overline{\lambda}, \overline{\delta}, \overline{t}_0$, random variables $\overline{C}_0(\omega), \overline{C}_1(\omega) \geq 0$ and $k_0 \in \mathbb{N}$ such that for any $\omega \in \Omega$, $\mathcal{X}, \mathcal{Y} \in \chi(\omega)$, $t \geq 0$ and $\{\sigma_i\} \times \{x_0^i(\theta_{-t}\omega)\} \in \chi(\theta_{-t}\omega)$, $i = 1, 2$,

$$\|\pi(t,\omega)\mathcal{X} - \pi(t,\omega)\mathcal{Y}\|_{\mathbb{X}} \leq e^{\int_0^{\overline{t}_0}\overline{C}_0(\theta_s\omega)\,ds}\|\mathcal{X} - \mathcal{Y}\|_{\mathbb{X}}, \qquad \forall t \in [0,\overline{t}_0], \tag{2.5}$$

and

$$\begin{aligned}
&\sum_{|k|\geq 2k_0+1}\left|\phi_k(\overline{t}_0,\theta_{-\overline{t}_0}\omega,\sigma_1,x_0^1(\theta_{-t}\omega)) - \phi_k(\overline{t}_0,\theta_{-\overline{t}_0}\omega,\sigma_2,x_0^2(\theta_{-t}\omega))\right|^2\\
&\quad\leq\Big(e^{-\overline{\lambda}\overline{t}_0+\int_0^{\overline{t}_0}\overline{C}_1(\theta_s\omega)\,ds} + \frac{\overline{\delta}}{2}e^{\int_0^{\overline{t}_0}\overline{C}_0(\theta_s\omega)\,ds}\Big)^2\\
&\qquad\times\left(\|x_0^1(\theta_{-t}\omega) - x_0^2(\theta_{-t}\omega)\|_\rho^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^l}^2\right),
\end{aligned}\tag{2.6}$$

where $\overline{\lambda}, \overline{\delta}, \overline{t}_0, k_0$ are independent of ω;

(H3) $\overline{C}_0(\omega), \overline{C}_1(\omega), \overline{\lambda}, \overline{\delta}, \overline{t}_0$ satisfy:

$$\begin{cases}\overline{t}_0 = \frac{8\ln 2}{\overline{\lambda}}, \quad 0 \leq \mathbf{E}[\overline{C}_1(\omega)] \leq \frac{\overline{\lambda}}{16}, \quad 0 \leq \mathbf{E}[\overline{C}_0^2(\omega)] < \infty,\\ 0 < \overline{\delta} \leq \min\Big\{\frac{1}{16}, e^{-\frac{2}{\ln\frac{3}{2}}(2\overline{t}_0^2\mathbf{E}[\overline{C}_0^2(\omega)]+\overline{\lambda}\overline{t}_0^2\mathbf{E}[\overline{C}_0(\omega)])}\Big\}.\end{cases}\tag{2.7}$$

Theorem 2.1. *[16, 30] Assume that conditions (H1)–(H3) hold. Then the continuous skew-product cocycle π acting on $\mathbb{X}$ generated by jointly continuous NRDS $\{\phi(t,\omega,\sigma)\}_{t\geq 0,\omega\in\Omega,\sigma\in\mathbb{T}^l}$ with base flows $\{\theta_t\}_{t\in\mathbb{R}}$ and $\{\vartheta_t\}_{t\in\mathbb{R}}$ possesses a $\mathcal{D}_{\mathbb{X}}$-random uniform exponential attractor $\mathcal{O} = \{\mathcal{O}(\omega)\}_{\omega\in\Omega}$.*

To study the existence of a random uniform exponential attractor for system (1.1), we need to make assumptions about ρ_k, A, λ_k, f, $g_k(\sigma)$, a in the system (1.1). Denote by $c_i (i \in \mathbb{N})$ the positive constants.

(A0) $\forall k \in \mathbb{Z}$, $0 < \underline{\rho} \leq \rho_k = \rho(k) \leq \overline{\rho} < +\infty$, $\rho(k) \leq c_0\rho(k \pm 1)$, $|\rho(k \pm 1) - \rho(k)| \leq a_0\rho(k)$.

(A1) The linear coupled operator A is non-negative and self-adjoint with decomposition $A = D^*D = DD^*$, where D and D^* are defined by $(Du)_k = \sum_{j=-m_0}^{m_0} d_j u_{k+j}$, $(D^*u)_k = \sum_{j=-m_0}^{m_0} d_{-j}u_{k+j}$, for all $u = (u_k)_{k\in\mathbb{Z}}$, $|d_j| \leq c_1$, $k \in \mathbb{Z}$, $-m_0 \leq j \leq m_0$, $m_0 \in \mathbb{N}$ and D^* is the adjoint of D in ℓ^2.

(A2) There exist two constants $\overline{\lambda}, \underline{\lambda} > 0$ such that

$$0 < \underline{\lambda} \leqslant \lambda_k \leqslant \overline{\lambda} < +\infty, \qquad k \in \mathbb{Z}.$$

(A3) $f \in C^1(\mathbb{R},\mathbb{R})$, $f(0) = 0$, and there exist constants f_0, $p \geq 0$ such that $|f'(s)| \leq f_0(1 + |s|^p)$ for any $s \in \mathbb{R}$.

(A4) $g(\sigma)=(g_k(\sigma))_{k\in\mathbb{Z}} \in \mathrm{C}(\mathbb{T}^l, \ell^2_\rho)$, where $\mathrm{C}(\mathbb{T}^l, \ell^2_\rho)$ is the space composed of all continuous functions of the ℓ^2_ρ-value defined on $\mathbb{T}^l$, also there exist $d = (d_k)_{k\in\mathbb{Z}} \in \ell^2_\rho$, and $d_k > 0$, such that

$$|g_k(\sigma_1) - g_k(\sigma_2)| \le d_k\|\sigma_1 - \sigma_2\|_{\mathbb{T}^l}, \qquad k \in \mathbb{Z}.$$

(A5) a_0 in (A0) satisfies

$$a_0 \le \frac{\lambda}{2c_2c_0^m(2m_0+1)^2(c_1^3+c_1)},$$

where $c_0^m = \max\{1, c_0^{2m_0}\}$, $c_2 = c_0^{m_0-1} + c_0^{m_0-2} + \cdots + 1$.
(A6) $|a| < \frac{\sqrt{\pi}\lambda}{4}$.

Remark 2.1. *Note that ℓ^2_ρ-valued continuous functions defined on compact torus $\mathbb{T}^l$ are bounded. For each $\eta \in \mathrm{C}(\mathbb{T}^l, \ell^2_\rho)$, $\|\eta\|_c^2 = \max_{\sigma\in\mathbb{T}^l} \|\eta(\sigma)\|_\rho^2 < \infty$, and for every $\varepsilon > 0$, there exists $I(\varepsilon) \in \mathbb{N}$, such that $\max_{\sigma\in\mathbb{T}^l} \sum_{|k|>I(\varepsilon)} \rho_k\eta_k^2(\sigma) < \varepsilon$.*

For $\omega \in \Omega$ and $t \in \mathbb{R}$, denote by $z(\theta_t\omega)$ as Ornstein-Uhlenbeck process for the ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, and solves the equation $dz + zdt = dW$, and $z(\omega)$ has the following properties, see [22]:
(i) $\exists\tilde{\Omega} \subseteq \Omega$, such that $P(\tilde{\Omega}) = 1$, and $z(\omega)$ is tempered;
(ii) For $\omega \in \tilde{\Omega}$,

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{t} = \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_s\omega)\,ds = 0, \tag{2.8}$$

$$\lim_{t\to\pm\infty} \frac{\int_0^t |z(\theta_s\omega)|^r\,ds}{t} = \mathbf{E}[|z(\theta_s\omega)|^r] = \frac{\Gamma(\frac{1+r}{2})}{\sqrt{\pi}}, \ \ \forall r > 0, \ \ s \in \mathbb{R}, \tag{2.9}$$

$$\mathbf{E}[e^{\epsilon z(\theta_s\omega)}] \le \frac{4\sqrt{\pi} + 3e}{3\sqrt{\pi}}, \ \ \forall s \in \mathbb{R}, \ \ |\epsilon| \le 1, \tag{2.10}$$

$$\mathbf{E}[e^{\epsilon\int_\tau^{\tau+t}|z(\theta_s\omega)|\,ds}] \le e^{\epsilon t}. \ \ \forall\tau \in \mathbb{R}, \ \ t \ge 0, \tag{2.11}$$

where $\Gamma(\cdot)$ is the Gamma function, "$\mathbf{E}$" denotes the expectation.

The system (1.1) can be rewritten as the following vector form:

$$\begin{cases} i\dfrac{du}{dt} + Au + i\lambda u + f(|u|^2)u = g(\tilde{\sigma}(t)) + i(a+ib)u \circ \dot{W}, & t > 0, \\ u(0) = (u_{k,0})_{k\in\mathbb{Z}} = u_0, & k \in \mathbb{Z}, \end{cases} \tag{2.12}$$

where $u = (u_k)_{k\in\mathbb{Z}}$, $\lambda u = (\lambda_k u_k)_{k\in\mathbb{Z}}$, $f(|u|^2)u = (f(|u_k|^2)u_k)_{k\in\mathbb{Z}}$, $g(\tilde{\sigma}) = (g_k(\tilde{\sigma}))_{k\in\mathbb{Z}}$, $u \circ \dot{W} = (u_k \circ \dot{W})_{k\in\mathbb{Z}}$; $W(t)$ is a two-sided real-valued Wiener process on the probability space.

Introducing a variable transformation $\varphi(t,\omega) = e^{-(a+ib)z(\theta_t\omega)}u(t,\omega)$, where $u(t,\omega)$ is the solution of (2.12), then the system (2.12) is equivalent to the following random system without white noise term:

$$\begin{cases} \dfrac{d\varphi}{dt} = iA\varphi - \lambda\varphi + if(e^{2az(\theta_t\omega)}|\varphi|^2)\varphi - ie^{-(a+ib)z(\theta_t\omega)}g(\tilde{\sigma}(t)) + (a+ib)z(\theta_t\omega)\varphi, & t > 0, \\ \varphi(0,\omega) = \varphi_0(\omega) = \varphi_0 = e^{-(a+ib)z(\omega)}u_0. \end{cases} \tag{2.13}$$

3. Random uniform exponential attractor for (2.13) in ℓ^2_ρ

In this section, we will prove the existence of a random uniform exponential attractor for the system (2.13) in ℓ^2_ρ.

3.1. *The well-posedness of system* (2.13) *in* ℓ^2_ρ

We first consider the well-posedness of the system (2.13) in ℓ^2_ρ. For the system (2.13), we have the following lemma.

Lemma 3.1. *Assuming* (A0)–(A5) *holds, then for every* $\omega \in \Omega$*, the following properties hold:*
(i) $\forall \sigma \in \mathbb{T}^l$, $\varphi_0(\omega) \in \ell^2_\rho$*, problem* (2.13) *admits a unique solution* $\varphi(\cdot, \omega, \sigma, \varphi_0(\omega)) \in C([0, +\infty), \ell^2_\rho)$.
(ii) Let $\varphi_i(\cdot, \omega, \sigma_i, \varphi^i_0(\omega))$ *be the solution of problem* (2.13) *with* $\sigma_i \in \mathbb{T}^l$ *and* $\varphi^i_0(\omega) \in \ell^2_\rho$, $i = 1, 2$, $T > 0$ *is fixed, then there exists a constant* $c(T, \omega) > 0$ *such that for all* $t \in [0, T]$,

$$\|\varphi_1(t, \omega, \sigma_1, \varphi^1_0(\omega)) - \varphi_2(t, \omega, \sigma_2, \varphi^2_0(\omega))\|^2_\rho \leqslant e^{c(T,\omega)t}(\|\varphi^1_0 - \varphi^2_0\|^2_\rho + \|\sigma_1 - \sigma_2\|^2_{\mathbb{T}^l}).$$

Proof. (i) Let $T > 0$, $\omega \in \Omega$ be fixed, and $B(r) = \{\varphi \in \ell^2_\rho : \|\varphi\|_\rho \leqslant r\}$, $r > 0$. Let $L^2_1(r, \omega, T) = \frac{r^2}{\underline{\rho}} \sup_{t\in[0,T]} e^{2az(\theta_t\omega)}$, for all $r > 0$, $t \in [0, T]$, $\varphi_1 = (\varphi^1_k)_{k\in\mathbb{Z}}$, $\varphi_2 = (\varphi^2_k)_{k\in\mathbb{Z}} \in B(r)$, $e^{2az(\theta_t\omega)}|\varphi^1_k|^2$, $e^{2az(\theta_t\omega)}|\varphi^2_k|^2 \leqslant L^2_1(r, \omega, T)$, and

$$\begin{cases} \sum\limits_{k\in\mathbb{Z}} \rho_k f^2(e^{2az(\theta_t\omega)}|\varphi^1_k|^2)|\varphi^1_k - \varphi^2_k|^2 \leq \max\limits_{s\in[0,L^2_1(r,\omega,T)]} f^2(s)\|\varphi_1 - \varphi_2\|^2_\rho, \\ \sum\limits_{k\in\mathbb{Z}} \rho_k|\varphi^2_k|^2\left|f(e^{2az(\theta_t\omega)}|\varphi^1_k|^2) - f(e^{2az(\theta_t\omega)}|\varphi^2_k|^2)\right|^2 \leq 4f^2_0(1 + L^{2p}_1(r, \omega, T))^2 L^4_1(r, \omega, T)\|\varphi_1 - \varphi_2\|^2_\rho. \end{cases} \tag{3.1}$$

Denote

$$F(t, \omega, \varphi, \sigma) = iA\varphi - \lambda\varphi + if(e^{2az(\theta_t\omega)}|\varphi|^2)\varphi - ie^{-(a+ib)z(\theta_t\omega)}g(\tilde{\sigma}(t)) + (a + ib)z(\theta_t\omega)\varphi,$$

for all $r > 0$, $\varphi_1, \varphi_2 \in B(r)$, $\sigma \in \mathbb{T}^l$, $\omega \in \Omega$, we have

$$\begin{aligned} &\|F(t, \omega,\varphi_1, \sigma) - F(t, \omega, \varphi_2, \sigma)\|_\rho \\ &\quad\leq \Big[4(2m_0 + 1)^4 c^4_1 c^m_0 + 4\overline{\lambda}^2 + 8 \max_{s\in(0,L^2_1(r,\omega,T)]} f^2(s) \\ &\qquad + 32f^2_0(1 + L^{2p}_1(r, \omega, T))^2 L^4_1(r, \omega, T) + 4(a^2 + b^2) \max_{t\in[0,T]} |z(\theta_t\omega)|\Big]^{\frac{1}{2}} \|\varphi_1 - \varphi_2\|_\rho, \end{aligned}$$

where $c^m_0 = \max\{1, c^{2m_0}_0\}$, $t \in [0, T]$. Therefore, F satisfies local Lipschitz condition. By using a standard argument, we know that there exists a $T_{\max} \leq +\infty$, such that Eq (2.13) has a unique solution $\varphi(t) \in C([0, T_{\max}), \ell^2_\rho)$. Moreover, if $T_{\max} < +\infty$, then $\limsup_{t\to T_{\max}} \|\varphi(t)\|_\rho = +\infty$. We next prove that this local solution is a global one. Let $T \in [0, T_{\max})$, taking the real part of the inner product of the system (2.13) with φ in ℓ^2_ρ, we have

$$\frac{1}{2}\frac{d\|\varphi\|^2_\rho}{dt} = \mathrm{Re}(iA\varphi, \varphi)_\rho + \mathrm{Re}(-\lambda\varphi, \varphi)_\rho + \mathrm{Re}(-ie^{-(a+ib)z(\theta_t\omega)}g(\tilde{\sigma}(t)), \varphi)_\rho + az(\theta_t\omega)\|\varphi\|^2_\rho. \tag{3.2}$$

By [26] and (A5), we obtain

$$\mathrm{Re}(iA\varphi,\varphi)_\rho \le Q\|\varphi\|_\rho^2 \le \frac{\lambda}{4}\|\varphi\|_\rho^2, \tag{3.3}$$

where $Q = \frac{1}{2}a_0c_2c_0^mc_1^3(2m_0+1)^2 + \frac{1}{2}a_0c_1c_2c_0^m(2m_0+1)^2$.
According to (A2) and (A4), we get

$$\mathrm{Re}(-\lambda\varphi,\varphi)_\rho \le -\underline{\lambda}\|\varphi\|_\rho^2, \tag{3.4}$$

$$\mathrm{Re}(-ie^{-(a+ib)z(\theta_t\omega)}g(\tilde{\sigma}(t)),\varphi)_\rho \le \frac{1}{\underline{\lambda}}e^{-2az(\theta_t\omega)}\|g\|_c^2 + \frac{\lambda}{4}\|\varphi\|_\rho^2. \tag{3.5}$$

It follows from (3.2)–(3.5) that

$$\frac{d\|\varphi\|_\rho^2}{dt} \le (-\underline{\lambda} + 2az(\theta_t\omega))\|\varphi\|_\rho^2 + \frac{2}{\underline{\lambda}}e^{-2az(\theta_t\omega)}\|g\|_c^2. \tag{3.6}$$

Using Gronwall inequality to (3.6) on $[0,t]$ $(0 \le t < T_{\max})$, we get

$$\begin{aligned}&\|\varphi(t)\|_\rho^2\\ \le& e^{-\underline{\lambda}t+\int_0^t 2|a||z(\theta_s\omega)|\,ds}\|\varphi_0\|_\rho^2 + \frac{2}{\underline{\lambda}}\|g\|_c^2\int_0^t e^{\int_l^t(-\underline{\lambda}+2|a||z(\theta_s\omega)|)\,ds-2az(\theta_l\omega)}\,dl\\ \doteq& \theta^2(t,\omega,\|\varphi_0(\omega)\|_\rho).\end{aligned} \tag{3.7}$$

Thus, $\theta(t,\omega,\|\varphi_0(\omega)\|_\rho)$ is continuous in t, which implies that $T_{\max} = +\infty$ and property (i) holds.
(ii) Let $y(t,\omega) = \varphi_1(t,\omega,\sigma_1,\varphi_0^1(\omega)) - \varphi_2(t,\omega,\sigma_2,\varphi_0^2(\omega))$, then

$$\begin{cases}\dfrac{dy}{dt} = iAy - \lambda y + if(e^{2az(\theta_t\omega)}|\varphi_1|^2)\varphi_1 - if(e^{2az(\theta_t\omega)}|\varphi_2|^2)\varphi_2\\ \qquad -ie^{-(a+ib)z(\theta_t\omega)}(g(\tilde{\sigma}_1(t)) - g(\tilde{\sigma}_2(t))) + (a+ib)z(\theta_t\omega)y, \qquad t>0,\\ y(0,\omega) = y_0(\omega) = y_0 = \varphi_0^1(\omega) - \varphi_0^2(\omega).\end{cases} \tag{3.8}$$

Taking the real part of the inner product of the system (3.8) with y in ℓ_ρ^2, we get

$$\begin{aligned}\frac{d\|y\|_\rho^2}{dt} =& 2\mathrm{Re}(iAy,y)_\rho + 2\mathrm{Re}(-\lambda y,y)_\rho + 2\mathrm{Re}(if(e^{2az(\theta_t\omega)}|\varphi_1|^2)\varphi_1 - if(e^{2az(\theta_t\omega)}|\varphi_2|^2)\varphi_2, y)_\rho\\ &+ 2\mathrm{Re}(-ie^{-(a+ib)z(\theta_t\omega)}(g(\tilde{\sigma}_1(t)) - g(\tilde{\sigma}_2(t))), y)_\rho + 2az(\theta_t\omega)\|y\|_\rho^2.\end{aligned} \tag{3.9}$$

Similar to (3.3) and (3.4), we have

$$2\mathrm{Re}(iAy,y)_\rho + 2\mathrm{Re}(-\lambda y,y)_\rho \le -\frac{3}{2}\underline{\lambda}\|y\|_\rho^2. \tag{3.10}$$

By (3.7)

$$\|\varphi_i(t,\omega)\|_\rho^2 \le \theta^2(t,\omega,\|\varphi_0^i(\omega)\|_\rho), \qquad i = 1,2, \tag{3.11}$$

where $\theta(t,\omega,\|\varphi_0^i(\omega)\|_\rho)$ is continuous in $t\in[0,+\infty)$. Then, by (3.11), we get

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\rho_k\Big|f(e^{2az(\theta_t\omega)}|\varphi_k^1|^2)y_k\Big|^2\\ &\le f_0^2(1+\frac{1}{\underline{\rho}^p}e^{2apz(\theta_t\omega)}\theta^{2p}(t,\omega,\|\varphi_0^1(\omega)\|_\rho))^2e^{4az(\theta_t\omega)}\frac{1}{\underline{\rho}^2}\theta^4(t,\omega,\|\varphi_0^1(\omega)\|_\rho)\|y\|_\rho^2\\ &\doteq\theta_1(t,\omega)\|y\|_\rho^2,\end{aligned}\tag{3.12}$$

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\rho_k|\varphi_k^2|^2\Big|f(e^{2az(\theta_t\omega)}|\varphi_k^1|^2)-f(e^{2az(\theta_t\omega)}|\varphi_k^2|^2)\Big|^2\\ &\le\Big[f_0^2\Big(1+\frac{1}{\underline{\rho}^p}e^{2apz(\theta_t\omega)}(\theta^2(t,\omega,\|\varphi_0^1(\omega)\|_\rho)+\theta^2(t,\omega,\|\varphi_0^2(\omega)\|_\rho))^p\Big)^2\\ &\times e^{4az(\theta_t\omega)}\frac{1}{\underline{\rho}^2}(3\theta^4(t,\omega,\|\varphi_0^2(\omega)\|_\rho)+\theta^4(t,\omega,\|\varphi_0^1(\omega)\|_\rho))\Big]\|y\|_\rho^2\\ &\doteq\theta_2(t,\omega)\|y\|_\rho^2.\end{aligned}\tag{3.13}$$

It follows from (3.9), (3.12) and (3.13) that

$$\begin{aligned}&2\mathrm{Re}(if(e^{2az(\theta_t\omega)}|\varphi_1|^2)\varphi_1-if(e^{2az(\theta_t\omega)}|\varphi_2|^2)\varphi_2,y)_\rho\\ &\le\frac{\underline{\lambda}}{2}\|y\|_\rho^2+\frac{4}{\underline{\lambda}}(\theta_1(t,\omega)+\theta_2(t,\omega))\|y\|_\rho^2\\ &=(\frac{\underline{\lambda}}{2}+\frac{4}{\underline{\lambda}}\theta_1(t,\omega)+\frac{4}{\underline{\lambda}}\theta_2(t,\omega))\|y\|_\rho^2.\end{aligned}\tag{3.14}$$

By (A4),

$$\begin{aligned}&2\mathrm{Re}(-ie^{-(a+ib)z(\theta_t\omega)}(g(\tilde\sigma_1(t))-g(\tilde\sigma_2(t))),y)_\rho\\ &\le e^{-az(\theta_t\omega)}\|d\|_\rho^2\cdot\|\sigma_1-\sigma_2\|_{\mathbb{T}^l}^2+e^{-az(\theta_t\omega)}\|y\|_\rho^2.\end{aligned}\tag{3.15}$$

It follows from (3.9), (3.10), (3.14) and (3.15), that

$$\begin{aligned}\frac{d}{dt}\|y\|_\rho^2\le&(-\underline{\lambda}+\frac{4}{\underline{\lambda}}\theta_1(t,\omega)+\frac{4}{\underline{\lambda}}\theta_2(t,\omega)+e^{-az(\theta_t\omega)}+2az(\theta_t\omega))\|y\|_\rho^2\\ &+e^{-az(\theta_t\omega)}\|d\|_\rho^2\cdot\|\sigma_1-\sigma_2\|_{\mathbb{T}^l}^2,\end{aligned}$$

which implies

$$\begin{aligned}&\frac{d}{dt}(\|y\|_\rho^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^l}^2)\\ &\le(\underline{\lambda}+\frac{4}{\underline{\lambda}}\theta_1(t,\omega)+\frac{4}{\underline{\lambda}}\theta_2(t,\omega)+e^{-az(\theta_t\omega)}+2|a||z(\theta_t\omega)|+e^{-az(\theta_t\omega)}\|d\|_\rho^2)(\|y\|_\rho^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^l}^2).\end{aligned}\tag{3.16}$$

Applying Gronwall inequality to (3.16) on $[0,t]$ $(0\le t\le T)$, we have

$$\begin{aligned}\|y\|_\rho^2\le&\|y\|_\rho^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^l}^2\\ &\le e^{\int_0^t\left(\underline{\lambda}+\frac{4}{\underline{\lambda}}\theta_1(s,\omega)+\frac{4}{\underline{\lambda}}\theta_2(s,\omega)+e^{-az(\theta_s\omega)}+2|a||z(\theta_s\omega)|+e^{-az(\theta_s\omega)}\|d\|_\rho^2\right)ds}\\ &\quad\times(\|y_0\|_\rho^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^l}^2).\end{aligned}$$

Let $c(T,\omega) = \underline{\lambda} + \max\limits_{0\le s\le T}(\frac{4}{\underline{\lambda}}\theta_1(s,\omega) + \frac{4}{\underline{\lambda}}\theta_2(s,\omega) + e^{-az(\theta_s\omega)} + 2|a||z(\theta_s\omega)| + e^{-az(\theta_s\omega)}\|d\|_\rho^2)$, then property (ii) holds. The proof is completed. □

According to Lemma 3.1, we have the solution of the system (2.13), which generates a jointly continuous NRDS $\Phi(t,\omega,\sigma): \ell_\rho^2 \to \ell_\rho^2$, and

$$\Phi(t,\omega,\sigma): \varphi_0(\omega) \to \Phi(t,\omega,\sigma,\varphi_0(\omega)) = \varphi(t,\omega,\sigma,\varphi_0(\omega)), \quad \forall t \ge 0, \quad \omega\in\Omega, \quad \sigma\in\mathbb{T}^l.$$

In the following, we will prove the existence of a $\mathcal{D}$-random uniform exponential attractor for Φ.

3.2. Uniformly absorbing set of system (2.13) *in* ℓ_ρ^2

For any $\omega\in\Omega$, $\sigma\in\mathbb{T}^l$, $t\ge 0$, let $\varphi(r) = \varphi(r,\theta_{-t}\omega,\sigma,\varphi_0(\theta_{-t}\omega))(r\ge 0)$ be the solution of (2.13) with $\sigma\in\mathbb{T}^l$ and initial value $\varphi_0(\theta_{-t}\omega)\in\ell_\rho^2$, then

$$\begin{cases}\dfrac{d\varphi}{dr} = iA\varphi - \lambda\varphi + if(e^{2az(\theta_{r-t}\omega)}|\varphi|^2)\varphi - ie^{-(a+ib)z(\theta_{r-t}\omega)}g(\tilde{\sigma}(r)) + (a+ib)z(\theta_{r-t}\omega)\varphi, \quad r\ge 0,\\ \varphi(0,\theta_{-t}\omega) = \varphi_0(\theta_{-t}\omega) = e^{-(a+ib)z(\theta_{-t}\omega)}u_0, \quad t\ge 0.\end{cases} \tag{3.17}$$

Lemma 3.2. *Let* (A1)–(A6) *hold. Then for every* $\omega\in\Omega$ *and* $D\in\mathcal{D}$, *there exist a* $T = T(\omega,D)\ge 0$ *and a tempered random variable* $M_0(\omega) > 0$ *such that for* $\varphi_0(\theta_{-t}\omega)\in D(\theta_{-t}\omega)$,

$$\begin{aligned}\|\varphi(t,\theta_{-t}\omega,\sigma,\varphi_0(\theta_{-t}\omega))\|_\rho^2 + \frac{\lambda}{2}\int_0^t & e^{\int_l^t(-\frac{\lambda}{2}+2|a||z(\theta_{s-t}\omega)|)\,ds}\\ &\times \|\varphi(l,\theta_{-t}\omega,\sigma,\varphi_0(\theta_{-t}\omega))\|_\rho^2\,dl\\ &\le M_0^2(\omega), \ t\ge T\end{aligned}$$

holds uniformly for $\sigma\in\mathbb{T}^l$.

Proof. Taking the real part of the inner product of the system (3.17) with $\varphi(r)$ in ℓ_ρ^2, we obtain

$$\frac{d\|\varphi\|_\rho^2}{dr} \le (-\frac{\lambda}{2} + 2|a||z(\theta_{r-t}\omega)|)\|\varphi\|_\rho^2 + \frac{2}{\underline{\lambda}}e^{-2az(\theta_{r-t}\omega)}\|g\|_c^2 - \frac{\lambda}{2}\|\varphi\|_\rho^2. \tag{3.18}$$

Using Gronwall inequality to (3.18) on $[0,t]$, we get

$$\begin{aligned}\|\varphi(t,&\theta_{-t}\omega,\sigma,\varphi_0(\theta_{-t}\omega))\|_\rho^2 + \frac{\lambda}{2}\int_0^t e^{\int_l^t(-\frac{\lambda}{2}+2|a||z(\theta_{s-t}\omega)|)\,ds}\\ &\times\|\varphi(l,\theta_{-t}\omega,\sigma,\varphi_0(\theta_{-t}\omega))\|_\rho^2\,dl\\ \le& e^{-\int_{-t}^0(\frac{\lambda}{2}-2|a||z(\theta_s\omega)|)\,ds}\|\varphi_0(\theta_{-t}\omega)\|_\rho^2 + \frac{1}{2}M_0^2(\omega),\end{aligned} \tag{3.19}$$

where

$$M_0^2(\omega) = \frac{4}{\underline{\lambda}}\|g\|_c^2\int_{-\infty}^0 e^{-\int_l^0(\frac{\lambda}{2}-2|a||z(\theta_s\omega)|)\,ds - 2az(\theta_l\omega)}\,dl. \tag{3.20}$$

Since $z(\theta_t\omega)$ is tempered and by (A6), for $\varphi_0(\theta_{-t}\omega) \in D(\theta_{-t}\omega)$, we have

$$\begin{aligned} & e^{-\int_{-t}^{0}(\frac{\lambda}{2}-2|a||z(\theta_s\omega)|)\,ds}\|\varphi_0(\theta_{-t}\omega)\|_\rho^2 \\ \leq & \sup_{\varphi\in D(\theta_{-t}\omega)} e^{-\int_{-t}^{0}(\frac{\lambda}{2}-2|a||z(\theta_s\omega)|)\,ds}\|\varphi\|_\rho^2 \xrightarrow{t\to+\infty} 0, \end{aligned}$$

and for any $\kappa > 0$,

$$\begin{aligned} e^{-\kappa t}M_0^2(\theta_{-t}\omega) = e^{-\kappa t}\frac{4}{\underline{\lambda}}\|g\|_c^2 \int_{-\infty}^{-t} e^{-\int_s^{-t}(\frac{\lambda}{2}-2|a||z(\theta_l\omega)|)\,dl-2az(\theta_s\omega)}\,ds \\ \longrightarrow 0(t \to +\infty). \end{aligned}$$

Thus, $M_0^2(\omega)$ is tempered. This completes the proof. □

Remark 3.1. *Let*

$$B_0(\omega) = \{\varphi \in \ell_\rho^2 : \|\varphi\|_\rho^2 \leq M_0^2(\omega)\}, \qquad \forall\omega \in \Omega. \tag{3.21}$$

By Lemma 3.2, $B_0 \in \mathcal{D}$ is a uniformly(with respect to $\sigma \in \mathbb{T}^l$) absorbing set for Φ, which is a bounded closed random set.

3.3. Estimation of the tail of solution for system (2.13) *in* ℓ_ρ^2

For every $\omega \in \Omega$, let $T_*(\omega) = T(\omega, B_0)$ and

$$\tilde{\mathbb{B}}(\theta_{-s}\omega) = \overline{\cup_{t\geq\max\{T_*(\theta_{-s}\omega),T_*(\omega)\}}\pi(t,\theta_{-t-s}\omega)(\mathbb{T}^l \times B_0(\theta_{-t-s}\omega))}, \quad s \geq 0, \tag{3.22}$$

where π is the skew-product cocycle generated by Φ and ϑ;

$$\pi(t,\theta_{-t-s}\omega)(\mathbb{T}^l \times B_0(\theta_{-t-s}\omega)) = \cup_{\sigma\in\mathbb{T}^l}\{\vartheta_t\sigma\} \times \Phi(t,\theta_{-t-s}\omega,\sigma)B_0(\theta_{-t-s}\omega).$$

Evidently, $P_{\ell_\rho^2}\tilde{\mathbb{B}} \subseteq B_0$, $P_{\mathbb{T}^l}\tilde{\mathbb{B}} = \mathbb{T}^l$, where $P_{\ell_\rho^2}$ is the projection from $\mathbb{T}^l\times\ell_\rho^2$ to ℓ_ρ^2, $P_{\mathbb{T}^l}$ is the projection from $\mathbb{T}^l\times\ell_\rho^2$ to $\mathbb{T}^l$. The definition of projection P is given in [16]. Given an increasing function $\xi \in C^1(\mathbb{R}^+,\mathbb{R})$ satisfying

$$\begin{cases} \xi(s) = 0, & 0 \leqslant s \leqslant 1; \\ 0 \leqslant \xi(s) \leqslant 1, & 1 \leqslant s \leqslant 2; \\ \xi(s) = 1, & 2 \leq s < +\infty; \\ |\xi'(s)| \leqslant H_1, & \forall s \in \mathbb{R}^+ \ \textit{and some constant}\ H_1 > 0. \end{cases}$$

Lemma 3.3. *Let $\varphi(r) = \varphi(r,\theta_{-t}\omega,\sigma,\varphi_0(\theta_{-t}\omega))$ be the solution of* (3.17) *with $\{\sigma\}\times\{\varphi_0(\theta_{-t}\omega)\} \in \tilde{\mathbb{B}}(\theta_{-t}\omega)$, then for every $\omega \in \Omega$, $\varepsilon > 0$ and $I(\geq 1) \in \mathbb{N}$, there exists a $T_\varepsilon(\omega) > 0$ such that*

$$\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{I})\rho_k|\varphi_k(t,\theta_{-t}\omega,\sigma,\varphi_0(\theta_{-t}\omega))|^2 \leq \varepsilon + c_4(\frac{1}{I} + \gamma_{1,I})M_0^2(\omega), \quad t \geq T_\varepsilon(\omega),$$

where $\gamma_{1,I}$ is defined in (3.28).

Proof. Taking the real part of the inner product of the system (3.17) with $x = (\xi(\frac{|k|}{I})\varphi_k(r))_{k\in\mathbb{Z}}$ in ℓ^2_ρ, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dr}\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{I})\rho_k|\varphi_k(r)|^2 \leq& \operatorname{Re}(iA\varphi, x)_\rho - \underline{\lambda}\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{I})\rho_k|\varphi_k(r)|^2\\ &+\operatorname{Re}\big(-ie^{-(a+ib)z(\theta_{r-t}\omega)}g(\tilde{\sigma}(r)), x\big)_\rho + az(\theta_{r-t}\omega)\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{I})\rho_k|\varphi_k(r)|^2.\end{aligned} \tag{3.23}$$

From (A0) and (A1), we get

$$\operatorname{Re}(iA\varphi, x)_\rho = -\operatorname{Im}(D\varphi, D(\rho x)) = -\operatorname{Im}\sum_{k\in\mathbb{Z}}(D\varphi)_k\overline{(D(\rho x))_k} \leq \frac{c_3}{I}\|\varphi\|^2_\rho, \tag{3.24}$$

where $c_3 = c_1^2(c_0^m)^2H_1m_0(2m_0+1)^2$, and

$$\begin{aligned}&\operatorname{Re}\big(-ie^{-(a+ib)z(\theta_{r-t}\omega)}g(\tilde{\sigma}(r)), x\big)_\rho\\ \leq&\frac{1}{3\underline{\lambda}}e^{-2az(\theta_{r-t}\omega)}\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{I})\rho_kg_k^2(\tilde{\sigma}(r)) + \frac{3\underline{\lambda}}{4}\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{I})\rho_k|\varphi_k|^2.\end{aligned} \tag{3.25}$$

It follows from (3.23)–(3.25) that

$$\begin{aligned}\frac{d}{dr}\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{I})\rho_k|\varphi_k|^2 \leq& \big(-\frac{\underline{\lambda}}{2} + 2|a||z(\theta_{r-t}\omega)|\big)\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{I})\rho_k|\varphi_k|^2\\ &+\frac{2c_3}{I}\|\varphi\|^2_\rho + \frac{2}{3\underline{\lambda}}e^{-2az(\theta_{r-t}\omega)}\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{I})\rho_k\big|g_k(\tilde{\sigma}(r))\big|^2.\end{aligned} \tag{3.26}$$

Applying Gronwall inequality to (3.26) on $[0,t]$ and by (3.19), we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{I})\rho_k|\varphi_k(t,\theta_{-t}\omega,\sigma,\varphi_0(\theta_{-t}\omega))|^2\\ \leq&(1+\frac{4c_3}{I\underline{\lambda}})e^{\int_{-t}^0(-\frac{\underline{\lambda}}{2}+2|a|\cdot|z(\theta_s\omega)|)\,ds}\|\varphi_0(\theta_{-t}\omega)\|^2_\rho + c_4(\frac{1}{I}+\gamma_{1,I})M_0^2(\omega),\end{aligned} \tag{3.27}$$

where

$$c_4 = \frac{2c_3}{\underline{\lambda}} + \frac{1}{6\|g\|^2_c}, \quad \gamma_{1,I} = \max_{\sigma\in\mathbb{T}^l}\sum_{|k|\geq I}\rho_k|g_k(\sigma)|^2. \tag{3.28}$$

Since $\lim_{t\to+\infty}(1+\frac{4c_3}{I\underline{\lambda}})e^{\int_{-t}^0(-\frac{\underline{\lambda}}{2}+2|a|\cdot|z(\theta_s\omega)|)\,ds}\|\varphi_0(\theta_{-t}\omega)\|^2_\rho = 0$, we have that for any $\varepsilon > 0$, $\omega\in\Omega$, there exists $T_\varepsilon(\omega) > 0$ such that $(1+\frac{4c_3}{I\underline{\lambda}})e^{\int_{-t}^0(-\frac{\underline{\lambda}}{2}+2|a|\cdot|z(\theta_s\omega)|)\,ds}\|\varphi_0(\theta_{-t}\omega)\|^2_\rho \leq \varepsilon$ for $t \geq T_\varepsilon(\omega)$. The proof is completed. □

3.4. Existence of a random uniform exponential attractor

In this subsection, we assume that conditions (A0)–(A6) hold. For every $\omega\in\Omega$, $s\geq 0$ and $\varepsilon > 0$, set

$$\mathbb{B}(\theta_{-s}\omega) = \overline{\cup_{t\geq\max\{T_*(\theta_{-s}\omega),T_*(\omega),T_*(\theta_{-T_\varepsilon(\omega)}\omega)\}+T_\varepsilon(\omega)}\pi(t,\theta_{-t-s}\omega)\big(\mathbb{T}^l\times B_0(\theta_{-t-s}\omega)\big)}. \tag{3.29}$$

Then it is easy to check that $\mathbb{B}$ possesses the following properties:

(a1) For all $r \geq 0$, $t \geq 0$, $\omega \in \Omega$, we have

$$\pi(r, \theta_{-t}\omega)\mathbb{B}(\theta_{-t}\omega) \subseteq \mathbb{B}(\theta_{r-t}\omega), \tag{3.30}$$

which along with $P_{\ell^2_\rho}\mathbb{B} \subseteq B_0$ imply that for all $r \geq 0$, $t \geq 0$, $\omega \in \Omega$,

$$\Phi(r, \theta_{-t}\omega, \sigma, \varphi) \in B_0(\theta_{r-t}\omega), \quad \forall\{\sigma\} \times \{\varphi\} \in \mathbb{B}(\theta_{-t}\omega). \tag{3.31}$$

(a2) For any $\{\sigma\} \times \{\varphi\} \in \mathbb{B}(\omega)$, the following holds:

$$\sum_{k\in\mathbb{Z}} \xi(\frac{|k|}{I})\rho_k|\varphi_k|^2 \leq \varepsilon + c_4(\frac{1}{I} + \gamma_{1,I})M_0^2(\omega). \tag{3.32}$$

To obtain the existence of a random uniform exponential attractor for Φ, we need to check that π satisfies conditions (H2) and (H3) of Theorem 2.1. on $\mathbb{B}$.

For any $r \geq 0$, $t \geq 0$, $\omega \in \Omega$, $\{\sigma_i\} \times \{\varphi_0^i(\theta_{-t}\omega)\} \in \mathbb{B}(\theta_{-t}\omega)$, $i = 1, 2$, let

$$\tilde{y}(r) = \varphi_1(r) - \varphi_2(r), \qquad \varphi_i(r) = \varphi(r, \theta_{-t}\omega, \sigma_i, \varphi_0^i(\theta_{-t}\omega)), \qquad i = 1, 2. \tag{3.33}$$

By (3.31), we have

$$\varphi_i(r) \in B_0(\theta_{r-t}\omega), \qquad \|\varphi_i(r)\|_\rho \leq M_0(\theta_{r-t}\omega), \qquad i = 1, 2. \tag{3.34}$$

It follows from (3.17) and (3.33) that

$$\begin{cases} \dfrac{d\tilde{y}}{dr} = iA\tilde{y} - \lambda\tilde{y} + if(e^{2az(\theta_{r-t}\omega)}|\varphi_1|^2)\varphi_1 - if(e^{2az(\theta_{r-t}\omega)}|\varphi_2|^2)\varphi_2 \\ \qquad -ie^{-(a+ib)z(\theta_{r-t}\omega)}(g(\tilde{\sigma}_1(r)) - g(\tilde{\sigma}_2(r))) + (a + ib)z(\theta_{r-t}\omega)\tilde{y}, \qquad r \geq 0, \\ \tilde{y}(0, \theta_{-t}\omega) = \tilde{y}_0(\theta_{-t}\omega) = \varphi_0^1(\theta_{-t}\omega) - \varphi_0^2(\theta_{-t}\omega), \qquad t \geq 0. \end{cases} \tag{3.35}$$

Lemma 3.4. *For any $r \geq 0$, $t \geq 0$, $\omega \in \Omega$ and $\{\sigma_i\} \times \{\varphi_0^i(\theta_{-t}\omega)\} \in \mathbb{B}(\theta_{-t}\omega)$, $i = 1, 2$, there exists a random variable $C_1(\omega) \geq 0$ such that the following holds:*

$$\begin{aligned} &\|\pi(t, \theta_{-t}\omega)\{\sigma_1\} \times \{\varphi_0^1(\theta_{-t}\omega)\} - \pi(t, \theta_{-t}\omega)\{\sigma_2\} \times \{\varphi_0^2(\theta_{-t}\omega)\}\|_{\mathbb{X}} \\ &\quad \leq e^{\int_{-t}^0 C_1(\theta_s\omega)\,ds}(\|\varphi_0^1(\theta_{-t}\omega) - \varphi_0^2(\theta_{-t}\omega)\|_\rho^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^l}^2)^{\frac{1}{2}}. \end{aligned} \tag{3.36}$$

Proof. Taking the real part of the inner product of the system (3.35) with $\tilde{y}(r)$ in ℓ^2_ρ, we get

$$\begin{aligned} \frac{d}{dr}\|\tilde{y}\|_\rho^2 = &- 2\mathrm{Im}(A\tilde{y}, \tilde{y})_\rho + 2\mathrm{Re}(-\lambda\tilde{y}, \tilde{y})_\rho \\ &- 2\mathrm{Im}(f(e^{2az(\theta_{r-t}\omega)}|\varphi_1|^2)\varphi_1 - f(e^{2az(\theta_{r-t}\omega)}|\varphi_2|^2)\varphi_2, \tilde{y})_\rho \\ &+ 2\mathrm{Im}(e^{-(a+ib)z(\theta_{r-t}\omega)}(g(\tilde{\sigma}_1(r)) - g(\tilde{\sigma}_2(r))), \tilde{y})_\rho \\ &+ 2\mathrm{Re}((a + ib)z(\theta_{r-t}\omega)\tilde{y}, \tilde{y})_\rho. \end{aligned} \tag{3.37}$$

By (A0)–(A2),

$$-2\mathrm{Im}(A\tilde{y}, \tilde{y})_\rho \leq 2Q\|\tilde{y}\|_\rho^2, \quad 2\mathrm{Re}(-\lambda\tilde{y}, \tilde{y})_\rho \leq -2\underline{\lambda}\|\tilde{y}\|_\rho^2. \tag{3.38}$$

By (A0), (A3) and (3.34),

$$\begin{aligned}&-2\mathrm{Im}\big(f(e^{2az(\theta_{r-t}\omega)}|\varphi_1|^2)\varphi_1-f(e^{2az(\theta_{r-t}\omega)}|\varphi_2|^2)\varphi_2,\tilde{y}\big)_\rho\\ \le&\Big[2(f_0(1+e^{2apz(\theta_{r-t}\omega)}\cdot\frac{M_0^{2p}(\theta_{r-t}\omega)}{\underline{\rho}^p})e^{2az(\theta_{r-t}\omega)}\frac{M_0^2(\theta_{r-t}\omega)}{\underline{\rho}})\\ &+2(f_0(1+e^{2apz(\theta_{r-t}\omega)}\cdot\frac{2^p\times M_0^{2p}(\theta_{r-t}\omega)}{\underline{\rho}^p})e^{2az(\theta_{r-t}\omega)}\frac{2M_0^2(\theta_{r-t}\omega)}{\underline{\rho}})\Big]\|\tilde{y}\|_\rho^2.\end{aligned}\tag{3.39}$$

By (A4),

$$\begin{aligned}&2\mathrm{Im}\big(e^{-(a+ib)z(\theta_{r-t}\omega)}(g(\tilde{\sigma}_1(r))-g(\tilde{\sigma}_2(r))),\tilde{y}\big)_\rho\\ \le&\frac{1}{\underline{\lambda}}e^{-2az(\theta_{r-t}\omega)}\|d\|_\rho^2\cdot\|\sigma_1-\sigma_2\|_{\mathbb{T}^l}^2+\underline{\lambda}\|\tilde{y}\|_\rho^2,\end{aligned}\tag{3.40}$$

and

$$2\mathrm{Re}\big((a+ib)z(\theta_{r-t}\omega)\tilde{y},\tilde{y}\big)_\rho=2az(\theta_{r-t}\omega)\|\tilde{y}\|_\rho^2.\tag{3.41}$$

It follows from (3.37)–(3.41) that

$$\frac{d}{dr}(\|\tilde{y}\|_\rho^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^l}^2)\le 2C_1(\theta_{r-t}\omega)(\|\tilde{y}\|_\rho^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^l}^2),\tag{3.42}$$

where

$$\begin{aligned}2C_1(\omega)=&2Q+\underline{\lambda}+2f_0e^{2az(\omega)}\frac{M_0^2(\omega)}{\underline{\rho}}(3+\frac{e^{2apz(\omega)}}{\underline{\rho}^p}M_0^{2p}(\omega)(2^{p+1}+1))\\ &+\frac{1}{\underline{\lambda}}e^{-2az(\omega)}\|d\|_\rho^2+2|a|\cdot|z(\omega)|.\end{aligned}\tag{3.43}$$

Using Gronwall inequality to (3.42) on $[0,r]$, we obtain

$$\|\tilde{y}(r)\|_\rho^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^l}^2\le e^{\int_0^r 2C_1(\theta_{s-t}\omega)\,ds}(\|\tilde{y}_0(\theta_{-t}\omega)\|_\rho^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^l}^2).\tag{3.44}$$

The proof is completed. □

Lemma 3.5. *For any $t\ge 0$, $\omega\in\Omega$ and $I(\ge 1)\in\mathbb{N}$, there exist random variables $C_2(\omega)$, $C_3(\omega)\ge 0$ such that for any $\{\sigma_i\}\times\{\varphi_0^i(\theta_{-t}\omega)\}\in\mathbb{B}(\theta_{-t}\omega)$, $i=1,2$, the following holds:*

$$\begin{aligned}&\Big(\sum_{|k|\ge 4I+1}\rho_k|\tilde{y}_k(t)|^2\Big)^{\frac{1}{2}}\\ \le&\Big(e^{-\underline{\lambda}t+\int_{-t}^0 C_2(\theta_s\omega)\,ds}+\frac{\delta_I}{2}e^{\int_{-t}^0 C_3(\theta_s\omega)\,ds}\Big)(\|\tilde{y}_0(\theta_{-t}\omega)\|_\rho^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^l}^2)^{\frac{1}{2}},\end{aligned}\tag{3.45}$$

where δ_I is defined in (3.56).

Proof. Let $M(\geq 1) \in \mathbb{N}$, set $q_k = \xi(\frac{|k|}{M})\tilde{y}_k$, $q = (q_k)_{k\in\mathbb{Z}}$. Taking the real part of the inner product of the system (3.35) with q in ℓ^2_ρ, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dr}\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{M})\rho_k|\tilde{y}_k|^2 =&\mathrm{Re}(iA\tilde{y},q)_\rho + \mathrm{Re}(-\lambda\tilde{y},q)_\rho\\ &+\mathrm{Re}\big(if(e^{2az(\theta_{r-t}\omega)}|\varphi_1|^2)\varphi_1 - if(e^{2az(\theta_{r-t}\omega)}|\varphi_2|^2)\varphi_2,q\big)_\rho\\ &+\mathrm{Re}\big(-ie^{-(a+ib)z(\theta_{r-t}\omega)}(g(\tilde{\sigma}_1(r))-g(\tilde{\sigma}_2(r))),q\big)_\rho\\ &+\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{M})\rho_k az(\theta_{r-t}\omega)|\tilde{y}_k|^2.\end{aligned}\tag{3.46}$$

Similar to (3.24), we arrive

$$\mathrm{Re}(iA\tilde{y},q)_\rho + \mathrm{Re}(-\lambda\tilde{y},q)_\rho \leq \frac{c_3}{M}\|\tilde{y}\|^2_\rho - \underline{\lambda}\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{M})\rho_k|\tilde{y}_k|^2.\tag{3.47}$$

$$\begin{aligned}&\mathrm{Re}\big(if(e^{2az(\theta_{r-t}\omega)}|\varphi_1|^2)\varphi_1 - if(e^{2az(\theta_{r-t}\omega)}|\varphi_2|^2)\varphi_2,q\big)_\rho\\ \leq&2\sum_{k\in\mathbb{Z}}\rho_k f_0 e^{2az(\theta_{r-t}\omega)}(|\varphi^1_k|^2+|\varphi^2_k|^2)\xi(\frac{|k|}{M})|\overline{\tilde{y}_k}|^2\\ &+2^{p+2}\sum_{k\in\mathbb{Z}}\rho_k f_0 e^{2a(p+1)z(\theta_{r-t}\omega)}(|\varphi^1_k|^{2(p+1)}+|\varphi^2_k|^{2(p+1)})\xi(\frac{|k|}{M})|\overline{\tilde{y}_k}|^2.\end{aligned}$$

By (3.30) and (3.32), for $|k| \geq 2I$, $I(\geq 1) \in \mathbb{N}$,

$$\begin{aligned}&|\varphi^1_k|^2+|\varphi^2_k|^2 \leq \frac{2}{\underline{\rho}}[\varepsilon + c_4(\frac{1}{I}+\gamma_{1,I})M_0^2(\theta_{r-t}\omega)],\\ &|\varphi^1_k|^{2(p+1)}+|\varphi^2_k|^{2(p+1)} \leq 2^{p+2}\Big(\frac{\varepsilon^{p+1}}{\underline{\rho}^{p+1}}+\frac{c_4^{p+1}(\frac{1}{I}+\gamma_{1,I})^{p+1}M_0^{2(p+1)}(\theta_{r-t}\omega)}{\underline{\rho}^{p+1}}\Big).\end{aligned}$$

Thus, for $M \geq 2I$, we get

$$\begin{aligned}&\mathrm{Re}\big(if(e^{2az(\theta_{r-t}\omega)}|\varphi_1|^2)\varphi_1 - if(e^{2az(\theta_{r-t}\omega)}|\varphi_2|^2)\varphi_2,q\big)_\rho\\ \leq&\Big(\frac{4\varepsilon f_0 e^{2az(\theta_{r-t}\omega)}}{\underline{\rho}}+\frac{2^{2(p+2)}\varepsilon^{p+1}f_0e^{2a(p+1)z(\theta_{r-t}\omega)}}{\underline{\rho}^{p+1}}\Big)\sum_{k\in\mathbb{Z}}\rho_k\xi(\frac{|k|}{M})|\overline{\tilde{y}_k}|^2\\ &+\frac{4c_4f_0e^{2az(\theta_{r-t}\omega)}M_0^2(\theta_{r-t}\omega)}{\underline{\rho}}(\frac{1}{I}+\gamma_{1,I})\sum_{k\in\mathbb{Z}}\rho_k\xi(\frac{|k|}{M})|\overline{\tilde{y}_k}|^2\\ &+\frac{2^{3p+5}c_4^{p+1}f_0e^{2a(p+1)z(\theta_{r-t}\omega)}M_0^{2(p+1)}(\theta_{r-t}\omega)}{\underline{\rho}^{p+1}}(\frac{1}{I^{p+1}}+\gamma_{1,I}^{p+1})\sum_{k\in\mathbb{Z}}\rho_k\xi(\frac{|k|}{M})|\overline{\tilde{y}_k}|^2.\end{aligned}\tag{3.48}$$

$$\begin{aligned}&\mathrm{Re}\big(-ie^{-(a+ib)z(\theta_{r-t}\omega)}(g(\tilde{\sigma}_1(r))-g(\tilde{\sigma}_2(r))),q\big)_\rho\\ \leq&\frac{1}{2\varepsilon}\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^l}\sum_{k\in\mathbb{Z}}\rho_k\xi(\frac{|k|}{M})d_k^2+\frac{\varepsilon}{2}e^{-2az(\theta_{r-t}\omega)}\sum_{k\in\mathbb{Z}}\rho_k\xi(\frac{|k|}{M})|\overline{\tilde{y}_k}|^2,\ \ (0<\varepsilon\leq 1).\end{aligned}\tag{3.49}$$

From (3.46)–(3.49), we have that for $M \geq 2I$,

$$\begin{aligned}
&\frac{d}{dr}\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{M})\rho_k|\tilde{y}_k|^2 + (2\underline{\lambda} - 2C_2(\theta_{r-t}\omega))\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{M})\rho_k|\tilde{y}_k|^2\\
\leq&(2c_3 + \frac{8c_4f_0}{\underline{\rho}})(\frac{1}{I} + \gamma_{1,I})(1 + e^{2az(\theta_{r-t}\omega)}M_0^2(\theta_{r-t}\omega))\|\tilde{y}\|_\rho^2\\
&+ \frac{2^{3p+6}c_4^{p+1}f_0}{\underline{\rho}^{p+1}}(\frac{1}{I^{p+1}} + \gamma_{1,I}^{p+1})e^{2a(p+1)z(\theta_{r-t}\omega)}M_0^{2(p+1)}(\theta_{r-t}\omega)\|\tilde{y}\|_\rho^2\\
&+ \frac{1}{\varepsilon}\gamma_{2,I}\|\sigma_1 - \sigma_2\|_{\mathbb{T}^l}^2\\
\leq&c_5(\varepsilon)\tilde{\delta}_I(1 + e^{2a(p+1)z(\theta_{r-t}\omega)}M_0^{2(p+1)}(\theta_{r-t}\omega))\\
&\times e^{\int_0^r 2C_1(\theta_{s-t}\omega)\,ds}(\|\tilde{y}_0(\theta_{-t}\omega)\|_\rho^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^l}^2),
\end{aligned} \tag{3.50}$$

where

$$2C_2(\omega) = \frac{8\varepsilon f_0 e^{2az(\omega)}}{\underline{\rho}} + \frac{2^{2p+5}\varepsilon^{p+1}f_0e^{2a(p+1)z(\omega)}}{\underline{\rho}^{p+1}} + \varepsilon e^{-2az(\omega)} + 2|az(\omega)|, \tag{3.51}$$

$$\gamma_{2,I} = \sum_{|k|\geq I}\rho_k d_k^2, \quad c_5(\varepsilon) = \frac{1}{\varepsilon} + (2c_3 + \frac{8c_4f_0}{\underline{\rho}})\frac{2p+1}{p+1} + \frac{2^{3p+6}c_4^{p+1}f_0}{\underline{\rho}^{p+1}}, \tag{3.52}$$

$$\tilde{\delta}_I = \frac{1}{I} + \gamma_{1,I} + \frac{1}{I^{p+1}} + \gamma_{1,I}^{p+1} + \gamma_{2,I}. \tag{3.53}$$

Applying Gronwall inequality to (3.50) on $[0, t]$, we have that for $M \geq 2I$,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}}\xi(\frac{|k|}{M})\rho_k|\tilde{y}_k(t)|^2 \leq& e^{\int_{-t}^0(-2\underline{\lambda}+2C_2(\theta_s\omega))\,ds}(\|\tilde{y}_0\|_\rho^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^l}^2)\\
&+ \tilde{\delta}_I(\|\tilde{y}_0\|_\rho^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^l}^2)e^{\int_{-t}^0(2C_1(\theta_s\omega)+2C_2(\theta_s\omega))\,ds}\\
&\times\int_{-t}^0 c_5(\varepsilon)e^{2\underline{\lambda}s}(1 + e^{2(p+1)az(\theta_s\omega)}M_0^{2(p+1)}(\theta_s\omega))\,ds.
\end{aligned} \tag{3.54}$$

Note that

$$\begin{aligned}
&\int_{-t}^0 c_5(\varepsilon)e^{2\underline{\lambda}s}(1 + e^{2(p+1)az(\theta_s\omega)}M_0^{2(p+1)}(\theta_s\omega))\,ds\\
\leq&(\int_{-t}^0 e^{4\underline{\lambda}s}\,ds)^{\frac{1}{2}}(\int_{-t}^0 c_5^2(\varepsilon)(1 + e^{2(p+1)az(\theta_s\omega)}M_0^{2(p+1)}(\theta_s\omega))^2\,dl)^{\frac{1}{2}}\\
\leq&\frac{1}{2\sqrt{\underline{\lambda}}}e^{\int_{-t}^0 2c_5^2(\varepsilon)(1+e^{4a(p+1)z(\theta_s\omega)}M_0^{4(p+1)}(\theta_s\omega))\,ds}, \qquad (by \quad \sqrt{x} \leq e^x, \quad x \geq 0).
\end{aligned} \tag{3.55}$$

It follows from (3.54) and (3.55) that for $M \geq 2I$,

$$\sum_{|k|\geq 4I+1} \rho_k |\tilde{y}_k(t)|^2 \leq \sum_{k\in\mathbb{Z}} \xi(\frac{|k|}{M})\rho_k|\tilde{y}_k(t)|^2$$
$$\leq (e^{\int_{-t}^{0}(-2\underline{\lambda}+2C_2(\theta_s\omega))\,ds} + \frac{(\delta_I)^2}{4} e^{\int_{-t}^{0} 2C_3(\theta_s\omega)\,ds})(\|\tilde{y}_0\|_\rho^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^l}^2),$$

where

$$C_3(\omega) = C_1(\omega) + C_2(\omega) + c_5^2(\varepsilon)(1 + e^{4a(p+1)z(\omega)} M_0^{4(p+1)}(\omega)), \qquad \frac{(\delta_I)^2}{4} = \frac{1}{2\sqrt{\underline{\lambda}}}\tilde{\delta}_I. \tag{3.56}$$

The proof is completed. □

Lemma 3.6. *Assume that* $|a| > 0$ *and* $\varepsilon = \varepsilon_0 > 0$ *are small enough such that*

$$|a| \leq \min\{\frac{\sqrt{\pi}\underline{\lambda}}{32}, \frac{1}{32p+32}, \frac{1}{8}\}, \tag{3.57}$$

$$\frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}}(\frac{4f_0\varepsilon_0}{\underline{\rho}} + \frac{2^{2p+4}f_0\varepsilon_0^{p+1}}{\underline{\rho}^{p+1}} + \frac{\varepsilon_0}{2}) \leq \frac{\underline{\lambda}}{32}, \tag{3.58}$$

then

$$0 \leq \boldsymbol{E}(C_2(\omega)) \leq \frac{\underline{\lambda}}{16}, \qquad 0 \leq \boldsymbol{E}(C_3^2(\omega)) < \infty. \tag{3.59}$$

Proof. By (2.9), (2.10), (3.51), (3.57) and (3.58),

$$\begin{aligned}\mathbf{E}(C_2(\omega)) =& \frac{4f_0\varepsilon_0}{\underline{\rho}}\mathbf{E}(e^{2az(\omega)}) + \frac{2^{2p+4}f_0\varepsilon_0^{p+1}}{\underline{\rho}^{p+1}}\mathbf{E}(e^{2a(p+1)z(\omega)}) + \frac{\varepsilon_0}{2}\mathbf{E}(e^{-2az(\omega)}) + \mathbf{E}(|az(\omega)|)\\ \leq& (\frac{4f_0\varepsilon_0}{\underline{\rho}} + \frac{2^{2p+4}f_0\varepsilon_0^{p+1}}{\underline{\rho}^{p+1}} + \frac{\varepsilon_0}{2})\frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}} + \frac{|a|}{\sqrt{\pi}} \leq \frac{\underline{\lambda}}{16}.\end{aligned} \tag{3.60}$$

By (3.56),

$$\begin{aligned}\mathbf{E}(C_3^2(\omega)) =& 3\mathbf{E}(C_1^2(\omega)) + 3\mathbf{E}(C_2^2(\omega)) + 6\mathbf{E}(c_5^4(\varepsilon_0)) + 3\mathbf{E}\Big(c_5^4(\varepsilon_0)e^{16(p+1)az(\omega)}\Big)\\ &+ 3\mathbf{E}\Big(c_5^4(\varepsilon_0)M_0^{16(p+1)}(\omega)\Big).\end{aligned} \tag{3.61}$$

By (2.9), (2.10), (3.43) and (3.57),

$$\begin{aligned}\mathbf{E}(C_1^2(\omega)) =& 8Q^2 + 2\underline{\lambda}^2 + \mathbf{E}(18f_0^2e^{8az(\omega)}) + \mathbf{E}\Big(18f_0^2\frac{M_0^8(\omega)}{\underline{\rho}^4}\Big) + \mathbf{E}\Big(\frac{2f_0^2(2^{p+1}+1)^2}{\underline{\rho}^{2(p+1)}}e^{8a(p+1)z(\omega)}\Big)\\ &+ \mathbf{E}\Big(2f_0^2(2^{p+1}+1)^2\frac{M_0^{8(p+1)}(\omega)}{\underline{\rho}^{2(p+1)}}\Big) + \mathbf{E}(\frac{2}{\underline{\lambda}^2}e^{-4az(\omega)}\|d\|_\rho^4) + \mathbf{E}(8|az(\omega)|^2)\\ =& c_6 + 18f_0^2\frac{1}{\underline{\rho}^4}\mathbf{E}(M_0^8(\omega)) + 2(2^{p+1}+1)^2f_0^2\frac{1}{\underline{\rho}^{2(p+1)}}\mathbf{E}(M_0^{8(p+1)}(\omega)),\end{aligned} \tag{3.62}$$

where

$$c_6 = 8Q^2 + 2\underline{\lambda}^2 + \mathbf{E}(18f_0^2 e^{8az(\omega)}) + \mathbf{E}\Big(\frac{2f_0^2(2^{p+1}+1)^2}{\underline{\rho}^{2(p+1)}} e^{8a(p+1)z(\omega)}\Big) + \mathbf{E}(\frac{2}{\underline{\lambda}^2} e^{-4az(\omega)} \|d\|_\rho^4) + \mathbf{E}(8|az(\omega)|^2) < +\infty. \tag{3.63}$$

By (3.51), (3.52) and (3.57),

$$\mathbf{E}(C_2^2(\omega)) < +\infty, \quad \mathbf{E}(c_5^4(\varepsilon_0)) < +\infty, \quad \mathbf{E}(c_5^4(\varepsilon_0) e^{16(p+1)az(\omega)}) < +\infty. \tag{3.64}$$

By (3.20), (3.57) and Hölder inequality,

$$\mathbf{E}(M_0^{16(p+1)}(\omega)) \le \frac{1}{2}(\frac{4}{\underline{\lambda}}\|g\|_c^2)^{8p+8}(\frac{8p+7}{2p+2})^{8p+7}\int_{-\infty}^{0} e^{2(p+1)\underline{\lambda}l}(e^{-(32p+32)|a|l} + \frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}})\,dl < +\infty. \tag{3.65}$$

In the same way

$$\mathbf{E}(M_0^8(\omega)) < \infty, \quad \mathbf{E}(M_0^{8(p+1)}(\omega)) < \infty. \tag{3.66}$$

It follows from (3.61)–(3.66) that $0 \le \mathbf{E}(C_3^2(\omega)) < \infty$. The proof is completed. □

Theorem 3.1. *Assume that (A0)–(A6),* (3.57) *and* (3.58) *hold, then* $\{\Phi(t,\omega,\sigma)\}_{t\ge 0,\omega\in\Omega,\sigma\in\mathbb{T}^l}$ *has a* $\mathcal{D}(\ell_\rho^2)$*-random uniform exponential attractor* $\{\mathcal{O}(\omega)\}_{\omega\in\Omega}$ *in* ℓ_ρ^2 *with properties:*

(i) $\mathcal{O}$ *is a random compact set;*

(ii) *There exists an* $I_0 \in \mathbb{N}$ *such that* $dim_f \mathcal{O}(\omega) \le \frac{2(l+8I_0+1)\ln(\frac{2\sqrt{(l+8I_0+1)}}{\delta_{I_0}}+1)}{\ln\frac{4}{3}}$, $\forall \omega \in \Omega$*;*

(iii) *For every* $\omega \in \Omega$*,* $D \in \mathcal{D}(\ell_\rho^2)$*, there exist random variables* $\tilde{T}(\omega,\mathbb{D})$ *and* $\tilde{Q}(\omega,\|D(\omega)\|_\rho) > 0$ *such that*

$$\sup_{\sigma\in\mathbb{T}^l} dist_{\ell_\rho^2}(\varphi(t,\theta_{-t}\omega,\vartheta_{-t}\sigma)D(\theta_{-t}\omega),\mathcal{O}(\omega)) \le \tilde{Q}(\omega,\|D(\omega)\|_\rho)e^{-\frac{\underline{\lambda}\ln\frac{4}{3}}{32\ln 2}t}, \quad t \ge \tilde{T}(\omega,\mathbb{D}),$$

where $\mathbb{D} = \mathbb{T}^l \times D$*.*

Proof. From Lemma 3.2 and (3.29), Φ satisfies condition (H1). Taking $t = t_0 = \frac{8\ln 2}{\underline{\lambda}}$ in (3.36) and (3.45), Φ satisfies condition (H2) on $\mathbb{B}$. From Lemma 3.6, it follows that the number $\nu = 2t_0^2\mathbf{E}[C_3^2(\omega)] + \underline{\lambda}t_0^2\mathbf{E}[C_3(\omega)] < \infty$. Let

$$\eta = \min\{\frac{1}{16}, e^{-\frac{2}{\ln\frac{3}{2}}\nu}\}.$$

Since $\delta_I \to 0$, as $I \to +\infty$, thus, we can choose $I = I_0$ big enough such that $\delta_I \le \eta$. Thus, Φ satisfies condition (H3). Finally, by Theorem 2.1, we complete the proof. □

Remark 3.2. *(i) The positive integer I in Lemma 3.5 is related to weights* ρ_k *of* ℓ_ρ^2 *according to the results of the proof. So the upper bound of* $dim_f\mathcal{O}(\omega)$ *in Theorem 3.2 is related to weights* ρ_k *of* ℓ_ρ^2*.*
(ii) The attracting speed of the random uniform exponential attractor $\{\mathcal{O}(\omega)\}_{\omega\in\Omega}$ *is also related to weights* ρ_k *of* ℓ_ρ^2 *based on* $\tilde{Q}(\omega,\|D(\omega)\|_\rho)$ *in (iii) Theorem 3.1.*

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