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Research article

Investigation of the solvability of n- term fractional quadratic integral equation in a Banach algebra

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Abstract: In this paper, we consider a nonlinear n-term fractional quadratic integral equation. Our investigation is located in the space $C(J, \mathbb{R})$. We prove the existence and uniqueness of the solution for that problem by applying some fixed point theorems. Next, we establish the continuous dependence of the unique solution for that problem on some functions. Finally, we present some particular cases for n-term fractional quadratic integral equation and an example to illustrate our results.

Keywords: multi-term quadratic integral equation; existence and uniqueness results; continuous dependency
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1. Introduction and preliminaries

Several fixed point problems involving product of operators have been investigated in many literature and monographs, for example [1–5]. However, the problem considered in [6] is more general than those [5, 7–10].

Quadratic integral equations(QIEs) have been investigated from different points of view and using different techniques (see [11–18]). The QIEs can be widely applicable in more applications like the dynamic theory of gases, the theory of radiative exchange, the traffic theory, etc. see [4, 6, 16, 19]. For the case of Banach algebras, many recent references have been appeared, for example [20–22]. For solvability of some QIEs on unbounded interval see [19, 23, 24].

The multi-term fractional quadratic integral equation

$$x(t) = \sum_{i=1}^{n} f_i(t, x(t)) \int_0^t \frac{(t - s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x(s))\, ds, \quad \alpha_i > 0 \tag{1.1}$$

has been studied in a Banach algebra [25] by using some fixed point theorem [26].

The quadratic integral equation of fractional order

$$x(t) = k(t, x(\eta(t))) + f(t, x(\mu(t)))\Big(a(t) + \int_0^{\sigma(t)} v(t, s)\, g(s, x(v(s)))\, ds\Big)$$

has been investigated in [27] by applying the nonlinear alternative of Leray-Schauder type.

Motivated by these results and by the monographs that studied ϕ-fractional quadratic integral equations, in this article, we focus our attention on a nonlinear multi-term quadratic functional integral equation of fractional order

$$x(t) = \sum_{i=1}^{n} f_i(t, x(\eta_i(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x(\psi_i(s)))\phi'(s)\, ds\Big), \quad \alpha_i > 0, \tag{1.2}$$

for all $t \in J = [0, 1]$, where $a_i : J \to \mathbb{R}$, $f_i : J \times \mathbb{R} \to \mathbb{R}$, , $g_i : J \times \mathbb{R} \to \mathbb{R}$ and η_i, $\psi_i : J \to J$.

By a solution of the quadratic functional integral equation of fractional order (1.2), we mean a function $x \in C(J, \mathbb{R})$ that satisfies Eq (1.2), where $C(J, \mathbb{R})$ stands for the space of continuous real-valued functions on J.

For proving the existence results for the quadratic functional integral equation of fractional Eq (1.2). We recall the following fixed point theorem [26] which enables us to prove the existence theorem for solutions of the functional integral Eq (1.2).

Theorem 1.1. *Let n be a positive integer, and C be a nonempty, closed, convex and bounded subset of a Banach Algebra X. Assume that the operators $A_i : X \to X$ and $B_i : C \to X$, $i = 1, 2, ..., n$, satisfy*

(a) For each $i \in \{1, 2, ..., n\}$, A_i is $D-$Lipschitzian with a $D-$function ϕ_i;
(b) For each $i \in \{1, 2, ..., n\}$, B_i is continuous and $B_i(C)$ is precompact;
(c) For each $y \in C$, $x = \sum_{i=1}^{n} A_i x.B_i y$ implies that $x \in C$.

Then, the operator equation $x = \sum_{i=1}^{n} A_i x.B_i x$ has a solution provided that

$$\sum_{i=1}^{n} M_i \phi_i(r) < r, \quad \forall r > 0,$$

where $M_i = \sup_{x \in C} \|B_i x\|$, $i = 1, 2, ..., n$.

2. Existence results

Equation (1.2) is investigated under the assumptions:

(i^*) $g_i : J \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, ..., n$ satisfy Carathéodory condition (i.e., measurable in t for all $x \in \mathbb{R}$ and continuous in x for almost all $t \in J$) such that:

$$|g_i(t, x)| \leq m_i(t) + b_i|x|,\ b_i \geq 0,\ m_i \in L^1(J, \mathbb{R}),\ \ i = 1, 2, ..., n \quad \forall\ (t, x) \in J \times \mathbb{R}$$

and $k_i = \sup_{t \in J} I_\phi^{\beta_i} m_i(t)$ for any $\beta_i \leq \alpha_i$, $i = 1, 2, ..., n$ such that $k_i \neq 0\ \ \forall\ i$

(ii^*) $f_i : J \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, ..., n$ are continuous and bounded with $h_i = \sup_{(t,x)\in J\times\mathbb{R}} |f_i(t, x)|$, $i = 1, 2, ..., n$.

(iii^*) There exist constants L_i, $i = 1, 2, ..., n$ satisfying

$$|f_i(t, x) - f_i(t, y)| \leq L_i\, |x - y|, \quad i = 1, 2, ..., n$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

$(iv*)$ $\eta_i, \psi_i : J \to J$, $i = 1, 2, 3......, n$ are continuous functions.

$(v*)$ ϕ is increasing and absolutely continuous function.

$(vi*)$ $a_i : J \to \mathbb{R}, i = 1, ..., n$ are continuous and bound with $\mathfrak{k}_i = \sup_{t\in J} |a(t)|$.

Theorem 2.1. *Let the assumptions (i*)-(v*) be satisfied. Furthermore, if*

$$\sum_{i=1}^{n} \frac{b_i h_i}{\Gamma(\alpha_i + 1)} < 1$$

then the quadratic integral Eq (1.2) has at least one solution in $C(J, \mathbb{R})$.

Proof. Set $X = C(J, \mathbb{R})$. Consider the closed ball $\overline{\mathbb{B}}_r(0)$ in X centered at origin 0 and of radius r, where $r = \sum_{i=1}^{n} \left[\frac{h_i k_i}{\Gamma(\alpha_i - \beta_i + 1)} + \mathfrak{k}_i\right] \left[1 - \sum_{i=1}^{n} \frac{b_i h_i}{\Gamma(\alpha_i + 1)}\right]^{-1} > 0.$

Consider the mapping $A_i : X \to X$ and $B_i : \overline{\mathbb{B}}_r(0) \to X$ on $C(J, \mathbb{R})$ defined by:

$$\begin{aligned}
(A_i x)(t) &= f_i(t, x(\eta(t))) \\
(B_i x)(t) &= a_i(t) + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x(\psi_i(s)))\phi'(s)\, ds.
\end{aligned}$$

Then the quadratic integral Eq (1.2) can be written in the form:

$$\mathcal{T} x(t) = \sum_{i=1}^{n} A_i x(t).B_i x(t). \tag{2.1}$$

We shall show that A_i and B_i satisfy all the conditions of Theorem 1.1.
Let us define a subset C of $C(J, \mathbb{R})$ by

$$C := \{x \in C(J, \mathbb{R}),\ \|x\| \leq r\}.$$

Obviously, C is nonempty, bounded, convex and closed subset of $C(J, \mathbb{R})$.

For all $t \in J$, since the assumptions (ii*) and (iii*) are satisfied, the mapping f_i is well defined and the function $A_i x$ is continuous and bounded on J. Again, since each g_i is continuous in x and each ψ_i is continuous function, then the function $B_i x$ is also, continuous and bounded by $m_i'(t) = m_i(t) + b_i r$, $m_i' \in L^1(J, \mathbb{R})$ (in view of assumptions (i*) and (vi*)). Firstly, we show that A_i is Lipschitz on X. Let $x, y \in X$ be arbitrary. Then by assumptions (ii*) and (iii*)

$$\begin{aligned}
|A_i x(t) - A_i y(t)| &= |f_i(t, x(\eta_i(t))) - f_i(t, y(\eta_i(t)))| \\
&\leq L_i |x(\eta_i(t)) - y(\eta_i(t))| \\
&\leq L_i\, \|x - y\|,\ i = 1, 2, .., n.
\end{aligned}$$

For all $t \in J$. Taking supremum over t

$$\|A_i x - A_i y\| \leq L_i \, \|x - y\|, \; i = 1, 2, .., n$$

for all $x, \; y \in X$. This shows that A_i is a Lipschitz mapping on X with the Lipschitz constants L_i. Secondly, we show that B_i is continuous and compact operator on $\overline{\mathbb{B}}_r(0)$. First we show that each B_i is continuous on $\overline{\mathbb{B}}_r(0)$. To do this, let us fix arbitrary $\epsilon > 0$ and let $\{x_n\}$ be a sequence of functions in $\overline{\mathbb{B}}_r(0)$ converging to $x \in \overline{\mathbb{B}}_r(0)$. Then we get

$$\begin{aligned}
|(B_i x_n)(t) - (B_i x)(t)| &\leq \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_)} \left| g_i(s, x_n(\psi_i(s))) - g_i(s, x(\psi_i(s))) \right| \phi'(s) \, ds \\
&\leq \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} \Big[|g_i(s, x_n(\psi_i(s)))| + |g_i(s, x(\psi_i(s)))| \Big] \phi'(s) \, ds \\
&\leq 2 \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} m_i(s) \phi'(s) ds \\
&+ 2b_i \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} |x(\psi_i(s))| \phi'(s) ds \\
&\leq 2 I_\phi^{\alpha_i - \beta_i} I_\phi^{\beta_i} m_i(t) + 2 b_i r \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} \phi'(s) ds \\
&\leq 2k_i \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - \beta_i - 1}}{\Gamma(\alpha_i - \beta_i)} \phi'(s) ds + 2 b_i r \frac{(\phi(t) - \phi(s))^{\alpha_i}}{\Gamma(\alpha_i + 1)} \\
&\leq 2k_i \frac{(\phi(t) - \phi(s))^{\alpha_i - \beta_i}}{\Gamma(\alpha_i - \beta_i + 1)} + 2 b_i r \frac{(\phi(t) - \phi(s))^{\alpha_i}}{\Gamma(\alpha_i + 1)} \\
&\leq \frac{2k_i}{\Gamma(\alpha_i - \beta_i + 1)} + \frac{2 b_i r}{\Gamma(\alpha_i + 1)} \\
&\leq \epsilon \quad \text{for } \; t \in J.
\end{aligned}$$

Thus

$$|(B_i x_n)(t) - (B_i x)(t)| \leq \epsilon \quad \text{as } n \to \infty.$$

Furthermore, let us assume that $t \in J$. Then, by Lebesgue dominated convergence theorem, we obtain the estimate:

$$\begin{aligned}
\lim_{n \to \infty} (B_i x_n)(t) &= a_i(t) + \lim_{n \to \infty} \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} \; g_i(s, x_n(\psi_i(s))) \phi'(s) \, ds \\
&= a_i(t) + \int_0^t \frac{\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} \; g_i(s, x(\psi_i(s))) \phi'(s) \, ds \\
&= (B_i x)(t)
\end{aligned}$$

for all $t \in J$. Thus, $B_i x_n \to B_i x$ as $n \to \infty$ uniformly and hence each B_i is a continuous operator on $\overline{\mathbb{B}}_r(0)$ into $\overline{\mathbb{B}}_r(0)$ has a Cauchy subsequence. Now by (i*) and (vi*)

$$\begin{aligned}
|B_i x_n(t)| &\leq |a_i(t)| + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} \; |g_i(s, x_n(\psi_i(s)))| \phi'(s) \, ds \\
&\leq \mathfrak{k}_i + \frac{k_i}{\Gamma(\alpha_i - \beta_i + 1)} + \frac{b_i r}{\Gamma(\alpha_i + 1)}
\end{aligned}$$

for all $t \in J.$ Then $\|B_i x_n(t)\| \leq ŧ_i + \frac{k_i}{\Gamma(\alpha_i-\beta_i+1)} + \frac{b_i r}{\Gamma(\alpha_i+1)}$ for all $n \in N$. This shows that $\{B_i x_n\}$ is a uniformly bounded sequence in $B_i(\overline{\mathbb{B}}_r(0))$.

Now, we proceed to show that it is also equicontinuous. Let $t_1,\ t_2 \in J$ (without loss of generality assume that $t_1 < t_2$), then we have

$$\begin{aligned}
|B_i x_n(t_2) - B_i x_n(t_1)| &\leq |a_i(t_2) - a_i(t_1)| + \bigg| \int_0^{t_2} \frac{(\phi(t_2) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_n(\psi_i(s)))\phi'(s)\, ds \\
&- \int_0^{t_1} \frac{(\phi(t_1) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_n(\psi_i(s)))\phi'(s)\, ds \bigg| \\
&\leq |a_i(t_2) - a_i(t_1)| + \bigg| \int_0^{t_1} \frac{(\phi(t_2) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_n(\psi_i(s)))\phi'(s)\, ds \\
&+ \int_{t_1}^{t_2} \frac{(\phi(t_2) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_n(\psi_i(s)))\phi'(s)\, ds \\
&- \int_0^{t_1} \frac{(\phi(t_1) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_n(\psi_i(s)))\phi'(s)\, ds \bigg| \\
&\leq |a_i(t_2) - a_i(t_1)| + \bigg| \int_0^{t_1} \frac{(\phi(t_2) - \phi(s))^{\alpha_i - 1} - (\phi(t_1) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_n(\psi_i(s)))\phi'(s)\, ds \\
&+ \int_{t_1}^{t_2} \frac{(\phi(t_2) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_n(\psi_i(s)))\phi'(s)\, ds \bigg| \\
&\leq |a_i(t_2) - a_i(t_1)| + \int_{t_1}^{t_2} \frac{(\phi(t_2) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} |g_i(s, x_n(\psi_i(s)))| \phi'(s)\, ds \\
&+ \int_0^{t_1} \frac{(\phi(t_2) - \phi(s))^{\alpha_i - 1} - (\phi(t_1) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} |g_i(s, x_n(\psi_i(s)))| \phi'(s)\, ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|B_i x_n(t_2) - B_i x_n(t_1)| &\leq |a_i(t_2) - a_i(t_1)| + \int_{t_1}^{t_2} \frac{(\phi(t_2) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} [m_i(t) + b_i r]\phi'(s)\, ds \\
&+ \int_0^{t_1} \frac{(\phi(t_1) - \phi(s))^{\alpha_i - 1} - (\phi(t_1) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} [m_i(t) + b_i r]\phi'(s)\, ds \\
&\leq |a_i(t_2) - a_i(t_1)| + k_i \int_{t_1}^{t_2} \frac{(\phi(t_2) - \phi(s))^{\alpha_i - \beta_i - 1}}{\Gamma(\alpha_i - \beta_i)} \phi'(s)\, ds \\
&+ b_i r \int_{t_1}^{t_2} \frac{(\phi(t_2) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} \phi'(s)\, ds \\
&\leq |a_i(t_2) - a_i(t_1)| + \frac{k_i(\phi(t_2) - \phi(t_1))^{\alpha_i - \beta_i}}{\Gamma(\alpha_i - \beta_i + 1)} + \frac{b_i r(\phi(t_2) - \phi(t_1))^{\alpha_i}}{\Gamma(\alpha_i + 1)}.
\end{aligned}$$

Using the uniform continuity of the functions a_i and ϕ on $J,$ we obtain

$$|B_i x_n(t_2) - B_i x_n(t_1)| \to 0 \quad \text{as } t_2 \to t_1.$$

As a consequence, $|B_i x_n(t_2) - B_i x_n(t_1)| \to 0$ as $t_2 \to t_1$. This shows that $\{B_i x_n\}$ is an equicontinuous sequence in X. Now an application of Arzela-Ascoli [28] theorem yields that $\{B_i x_n\}$ has a uniformly convergent subsequence on the the compact subset J of $\mathbb{R}$. without loss of generality, call the subsequence it self. We show that $\{B_i x_n\}$ is Cauchy sequence in X. Now $|B_i x_n(t) - B_i x(t)| \to 0$ as $n \to \infty$ for all $t \in J$. Then for given $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$. Therefore (or $m, n \geq n_0$), we have

$$\begin{aligned} |B_i x_m(t) - B_i x_n(t)| &\leq \sup_{t\in J} \left| \int_0^t \frac{(\phi(t)-\phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} \Big[g_i(s, x_m(\psi_i(s))) - g_i(s, x_n(\psi_i(s))) \Big] \phi'(s)\, ds \right| \\ &\leq \sup_{t\in J} \int_0^t \frac{(\phi(t)-\phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} |g_i(s, x_m(\psi_i(s))) - g_i(s, x_n(\psi_i(s)))| \phi'(s)\, ds < \epsilon. \end{aligned}$$

This shows that $\{B_i x_n\} \subset F(\overline{\mathbb{B}}_r(0)) \subset X$ is Cauchy sequence. Since X is complete, $\{B_i x_n\}$ converges to a point in X. Hence $B_i(\overline{\mathbb{B}}_r(0))$ is relatively compact and consequently B_i is a continuous and compact operator on $\overline{\mathbb{B}}_r(0)$.

Next, we show that $A_i x B_i y = x \Rightarrow x \in \overline{\mathbb{B}}_r(0)$ for all $y \in \overline{\mathbb{B}}_r(0)$. Then,

$$\begin{aligned} |x(t)| &\leq |A_i x(t)||B_i y(t)| \\ &\leq |f_i(t, x(\eta_i(t)))|[I^{\alpha_i}|g_i(t, x(\psi_i(t)))|] \\ &\leq h_i\Big(\mathfrak{k}_i + \frac{k_i}{\Gamma(\alpha_i - \beta_i + 1)} + \frac{b_i r}{\Gamma(\alpha_i + 1)}\Big) \\ &\leq h_i\Big(\mathfrak{k}_i + \frac{k_i}{\Gamma(\alpha_i - \beta_i + 1)} + \frac{b_i r}{\Gamma(\alpha_i + 1)}\Big) \leq r \quad \text{for all} \quad t \in J. \end{aligned}$$

Taking the supremum over t. we obtain $\|x\| \leq r$ for all $y \in \overline{\mathbb{B}}_r(0)$, $r = \sum_{i=1}^{n} \left[\frac{h_i k_i}{\Gamma(\alpha_i-\beta_i+1)} + \mathfrak{k}_i\right] \left[1 - \sum_{i=1}^{n} \frac{b_i h_i}{\Gamma(\alpha_i+1)}\right]^{-1}$. Hence hypothesis (c) of Theorem 1.1 holds. Here,

$$M_i = \|B_i x\| \leq \mathfrak{k}_i + I^{\alpha_i}|g_i(t, x(\psi_i(t)))| \leq \mathfrak{k}_i + \frac{k_i}{\Gamma(\alpha_i - \beta_i + 1)} + \frac{b_i r}{\Gamma(\alpha_i + 1)}.$$

Therefore we can get, for every $x \in C$ we have

$$|(\mathcal{T}x)(t)| \leq \sum_{i=1}^{n} \left[\mathfrak{k}_i h_i + \frac{h_i k_i}{\Gamma(\alpha_i - \beta_i + 1)} + \frac{b_i h_i r}{\Gamma(\alpha_i + 1)}\right] = r.$$

Then, $\mathcal{T}x \in C$ and hence $TC \subset C$.

Since all conditions of Theorem 1.1 are satisfied, then the operator $\mathcal{T} = \sum_{i=1}^{n} A_i.B_i$ has a fixed point in C.

3. Features of the solution

In this section, we shall demonstrate some characteristics for the solutions of the n-term quadratic integral Eq (1.2).

3.1. Uniqueness of the solution

In aim of proving the uniqueness of the solution of (1.2), we replace assumption (i^*) by the following assumption:

> $g_i : J \times \mathbb{R} \rightarrow \mathbb{R}, \ i = 1, 2, ..., n$ satisfy Carathéodory condition (i.e., measurable in t for all $x \in \mathbb{R}$ and continuous in x for almost all $t \in J$) and
>
> $$|g_i(t, x) - g_i(t, y)| \leq c_i|x - y|, \quad i = 1, 2, ..., n \quad \forall \ (t, x) \in J \times \mathbb{R}$$
>
> and $\rho_i = \sup_{t \in J} |g_i(t, 0)|$ for any $i = 1, 2, ..., n$ such that $k_i \neq 0 \ \forall \ i$.

Theorem 3.1. *Let the assumptions (i) and (ii∗)–(v∗) be satisfied. Moreover, if*

$$\sum_{i=1}^{n} \Big(L_i(\text{ŧ}_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}) + \frac{h_i c_i}{\Gamma(\alpha_i + 1)} \Big) < 1$$

then the quadratic integral Eq (1.2) has a unique solution in $C(J, \mathbb{R})$.

Proof. From the assumption (i^*) we have

$$\begin{aligned} |g_i(s, x(s))| - |g_i(s, 0)| &\leq |g_i(s, x(s)) - g_i(s, 0)| \leq c_i|x| \\ |g_i(s, x(s))| &\leq c_i|x| + |g_i(s, 0)| \\ |g_i(s, x(s))| &\leq c_i|x| + \rho_i. \end{aligned}$$

Let x_1, x_2 be two solutions of the integral Eq (1.2), then

$$\begin{aligned} |x_1(t) - x_2(t)| &= \Big| \sum_{i=1}^{n} f_i(t, x_1(\eta_i(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_1(\psi_i(s)))\phi'(s)\, ds\Big) \\ &- \sum_{i=1}^{n} f_i(t, x_2(\eta_i(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_2(\psi_i(s)))\phi'(s)\, ds\Big) \\ &+ \sum_{i=1}^{n} f_i(t, x_2(\eta_i(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_1(\psi_i(s)))\phi'(s)\, ds\Big) \\ &- \sum_{i=1}^{n} f_i(t, x_2(\eta_i(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_1(\psi_i(s)))\phi'(s)\, ds\Big)\Big| \\ &\leq \sum_{i=1}^{n} \Big| f_i(t, x_1(\eta_i(t))) - f_i(t, x_2(\eta_i(t)))\Big| \Big| a_i(t) + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x_1(\psi_i(s)))\phi'(s)\, ds\Big| \\ &+ \sum_{i=1}^{n} |f_i(t, x_2(\eta_i(t)))| \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} \Big| g_i(s, x_1(\psi_i(s))) - g_i(s, x_2(\psi_i(s)))\Big| \phi'(s)\, ds \\ &\leq \sum_{i=1}^{n} L_i|x_1(\eta_i(t)) - x_2(\eta_i(t))|\Big(\text{ŧ}_i + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} |g_i(s, x_1(\psi_i(s)))|\phi'(s)\, ds\Big) \\ &+ \sum_{i=1}^{n} h_i \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} c_i|x_1(\psi_i(s)) - x_2(\psi_i(s))|\phi'(s)\, ds \\ &\leq \sum_{i=1}^{n} L_i|x_1(\eta_i(t)) - x_2(\eta_i(t))|\Big(\text{ŧ}_i + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} (\rho_i + c_i|x_1(\psi_i(s))|)\phi'(s)\, ds\Big) \end{aligned}$$

$$
\begin{aligned}
&+ \quad \sum_{i=1}^{n} h_i \int_0^t \frac{(\phi(t) \ - \ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} \, c_i \sup_{s\in J} |x_1(s) - x_2(s)| \phi'(s) \, ds \\
&\leq \quad \sum_{i=1}^{n} L_i |x_1(t) - x_2(t)| \Big(ŧ_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)} \Big) + \sum_{i=1}^{n} \frac{h_i c_i \|x_1 - x_2\|}{\Gamma(\alpha_i + 1)} \\
&\leq \quad \sum_{i=1}^{n} L_i \|x_1 - x_2\| \Big(ŧ_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)} \Big) + \sum_{i=1}^{n} \frac{h_i c_i \|x_1 - x_2\|}{\Gamma(\alpha_i + 1)} \\
&\leq \quad \sum_{i=1}^{n} \Big(L_i (ŧ_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}) + \frac{h_i c_i}{\Gamma(\alpha_i + 1)} \Big) \|x_1 - x_2\|.
\end{aligned}
$$

Then

$$
\Big(1 - \sum_{i=1}^{n} \Big(L_i (ŧ_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}) + \frac{h_i c_i}{\Gamma(\alpha_i + 1)} \Big) \Big) \|x_1 - x_2\| \leq 0.
$$

Since $\sum_{i=1}^{n} \Big(L_i (ŧ_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i+1)}) + \frac{h_i c_i}{\Gamma(\alpha_i+1)} \Big) < 1,$ then $x_1(t) = x_2(t).$ □

3.2. Continuous dependency

Firstly, we discuss the continuous dependence of the unique solution of the Eq (1.2) on the delays functions η_i and ψ_i.

Definition 1. *The solutions of the quadratic functional integral Eq (1.2) depends continuously on the delay functions* η_i *and* ψ_i *if* $\forall \ \epsilon > 0, \ \exists \delta, \ \sigma > 0$*, such that*

$$
|\eta_i(t) - \eta_i^*(t)| \leq \delta \text{ and } |\psi_i(t) - \psi_i^*(t)| \leq \sigma \ \Rightarrow \ \|x - x^*\| \leq \epsilon, \ i = 1, 2, ..., n.
$$

Theorem 3.2. *Let the assumptions of Theorem 3.1 be satisfied, then the solution of the functional integral Eq (1.2) depends continuously on the delay functions* η_i *and* ψ_i.

Proof. Let $\delta > 0$ and $\sigma > 0$ be given such that $|\eta_i(t) - \eta_i^*(t)| \leq \delta$ and $|\psi_i(t) - \psi_i^*(t)| \leq \sigma \ \forall t \ \in J$, then for

$$
x^*(t) = \sum_{i=1}^{n} f_i(t, x^*(\eta_i^*(t))) \Big(a_i(t) + \int_0^t \frac{(\phi(t) \ - \ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} \, g_i(s, x^*(\psi_i^*(s))) \phi'(s) \, ds \Big), \quad t \ \in \ J, \quad \alpha_i > 0,
$$

$$
\begin{aligned}
|x(t) - x^*(t)| \quad = \quad & \Big| \sum_{i=1}^{n} f_i(t, x(\eta_i(t))) \Big(a_i(t) + \int_0^t \frac{(\phi(t) \ - \ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} \, g_i(s, x(\psi_i(s))) \phi'(s) \, ds \Big) \\
- \quad & \sum_{i=1}^{n} f_i(t, x^*(\eta_i^*(t))) \Big(a_i(t) + \int_0^t \frac{(\phi(t) \ - \ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} \, g_i(s, x^*(\psi_i^*(s))) \phi'(s) \, ds \Big) \\
+ \quad & \sum_{i=1}^{n} f_i(t, x(\eta_i^*(t))) \Big(a_i(t) + \int_0^t \frac{(\phi(t) \ - \ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} \, g_i(s, x(\psi_i(s))) \phi'(s) \, ds \Big) \\
- \quad & \sum_{i=1}^{n} f_i(t, x(\eta_i^*(t))) \Big(a_i(t) + \int_0^t \frac{(\phi(t) \ - \ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} \, g_i(s, x(\psi_i(s))) \phi'(s) \, ds \Big) \\
+ \quad & \sum_{i=1}^{n} f_i(t, x(\eta_i^*(t))) \Big(a_i(t) + \int_0^t \frac{(\phi(t) \ - \ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} \, g_i(s, x^*(\psi_i^*(s))) \phi'(s) \, ds \Big)
\end{aligned}
$$

$$
\begin{aligned}
& - \sum_{i=1}^{n} f_i(t, x(\eta_i^*(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} g_i(s, x^*(\psi_i^*(s)))\phi'(s)\, ds\Big)\Big| \\
\leq & \sum_{i=1}^{n} |f_i(t, x(\eta_i(t))) - f_i(t, x(\eta_i^*(t)))|\Big(a_i(t) + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} |g_i(s, x(\psi_i(s)))|\phi'(s)\, ds\Big) \\
+ & \sum_{i=1}^{n} |f_i(t, x(\eta_i^*(t)))| \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} |g_i(s, x(\psi_i(s))) - g_i(s, x^*(\psi_i^*(s)))|\phi'(s)\, ds \\
+ & \sum_{i=1}^{n} |f_i(t, x(\eta_i^*(t))) - f_i(t, x^*(\eta_i^*(t)))| \\
. & \Big(|a_i(t)| + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} |g_i(s, x^*(\psi_i^*(s)))|\phi'(s)\, ds\Big)
\end{aligned}
$$

$$
\begin{aligned}
\leq & \sum_{i=1}^{n} L_i|x(\eta_i(t)) - x(\eta_i^*(t))|\Big(|a_i(t)| + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} (\rho_i + c_i|x(\psi_i(s))|)\phi'(s)\, ds\Big) \\
+ & \sum_{i=1}^{n} |f_i(t, x(\eta_i^*(t)))| \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} |g_i(s, x(\psi_i(s))) - g_i(s, x(\psi_i^*(s)))|\phi'(s)\, ds \\
+ & \sum_{i=1}^{n} |f_i(t, x(\eta_i^*(t)))| \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} |g_i(s, x(\psi_i^*(s))) - g_i(s, x^*(\psi_i^*(s)))|\phi'(s)\, ds \\
+ & \sum_{i=1}^{n} L_i|x(\eta_i^*(t)) - x^*(\eta_i^*(t))|\Big(\text{ŧ}_i + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)} (\rho_i + c_i|x^*(\psi_i^*(s))|)\phi'(s)\, ds\Big) \\
\leq & \sum_{i=1}^{n} L_i|\eta_i(t) - \eta_i^*(t)|\Big(\text{ŧ}_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}\Big) + \sum_{i=1}^{n} h_i \frac{c_i}{\Gamma(\alpha_i + 1)} \sup_{t \in J} |\psi_i(t) - \psi_i^*(t)| \\
+ & \sum_{i=1}^{n} h_i \frac{c_i}{\Gamma(\alpha_i + 1)} \sup_{t \in J} |x(t) - x^*(t)| + \sum_{i=1}^{n} L_i \sup_{t \in J} |x(t) - x^*(t)|\Big(\text{ŧ}_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}\Big) \\
\leq & \sum_{i=1}^{n} L_i\delta\Big(\text{ŧ}_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}\Big) + \sum_{i=1}^{n} h_i \frac{c_i}{\Gamma(\alpha_i + 1)}\sigma \\
+ & \sum_{i=1}^{n} \Big(L_i\text{ŧ}_i + \frac{h_i c_i + L_i(\rho_i + c_i r)}{\Gamma(\alpha_i + 1)}\Big)\|x - x^*\|.
\end{aligned}
$$

Then

$$
\begin{aligned}
\Big[1 - \sum_{i=1}^{n} \Big(\text{ŧ}_i L_i + \frac{h_i c_i + L_i(\rho_i + c_i r)}{\Gamma(\alpha_i + 1)}\Big)\Big]\|x - x^*\| \leq & \sum_{i=1}^{n} \Big(L_i\delta\Big(\text{ŧ}_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}\Big) + h_i \frac{c_i}{\Gamma(\alpha_i + 1)}\sigma\Big) \\
\|x - x^*\| \leq & \sum_{i=1}^{n} \Big(L_i\delta\Big(\text{ŧ}_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}\Big) + h_i \frac{c_i}{\Gamma(\alpha_i + 1)}\sigma\Big) \\
. & \Big[1 - \sum_{i=1}^{n} \Big(L_i\text{ŧ}_i + \frac{h_i c_i + L_i(\rho_i + c_i r)}{\Gamma(\alpha_i + 1)}\Big)\Big]^{-1}
\end{aligned}
$$

□

Next, we investigate the continuous dependence of the unique solution of the Eq (1.2) on the functions f_i and g_i.

Definition 2. *The solutions of the quadratic functional integral Eq (1.2) depends continuously on the functions f_i and g_i if* $\forall\ \epsilon > 0,\ \exists\delta,\ \sigma > 0$*, such that*

$$|f_i(t,x) - f_i^*(t,x)| \leq \delta \text{ and } |g_i(t,x) - g_i^*(t,x)| \leq \sigma \ \Rightarrow\ \|x - x^*\| \leq \epsilon,\ i = 1, 2, ..., n.$$

Theorem 3.3. *Let the assumptions of Theorem 3.1 be satisfied, then the solution of the functional integral equation (1.2) depends continuously on the functions f_i and g_i.*

Proof. Let $\delta > 0$ and $\sigma > 0$ be given such that $|f_i(t,x) - f_i^*(t,x)| \leq \delta$ and $|g_i(t,x) - g_i^*(t,x)| \leq \sigma\ \forall t\ \in J$, then for

$$x^*(t) = \sum_{i=1}^{n} f_i^*(t, x^*(\eta_i(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i - 1}}{\Gamma(\alpha_i)}\ g_i^*(s, x^*(\psi_i(s)))\phi'(s)\, ds\Big), \qquad t\ \in\ J, \quad \alpha_i > 0,$$

$$\begin{aligned}
|x(t) - x^*(t)| \ &= \ \Big|\sum_{i=1}^{n} f_i(t, x(\eta_i(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ g_i(s, x(\psi_i(s)))\phi'(s)\, ds\Big) \\
&- \ \sum_{i=1}^{n} f_i^*(t, x^*(\eta_i(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ g_i^*(s, x^*(\psi_i(s)))\phi'(s)\, ds\Big) \\
&+ \ \sum_{i=1}^{n} f_i^*(t, x(\eta_i(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ g_i(s, x(\psi_i(s)))\phi'(s)\, ds\Big) \\
&- \ \sum_{i=1}^{n} f_i^*(t, x(\eta_i(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ g_i(s, x(\psi_i(s)))\phi'(s)\, ds\Big) \\
&+ \ \sum_{i=1}^{n} f_i^*(t, x(\eta_i(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ g_i^*(s, x^*(\psi_i(s)))\phi'(s)\, ds\Big) \\
&- \ \sum_{i=1}^{n} f_i^*(t, x(\eta_i(t)))\Big(a_i(t) + \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ g_i^*(s, x^*(\psi_i(s)))\phi'(s)\, ds\Big)\Big| \\
&\leq \ \sum_{i=1}^{n} |f_i(t, x(\eta_i(t))) - f_i^*(t, x(\eta_i(t)))|\Big(a_i(t) + \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ |g_i(s, x(\psi_i(s)))|\phi'(s)\, ds\Big) \\
&+ \ \sum_{i=1}^{n} |f_i^*(t, x(\eta_i(t)))| \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ |g_i(s, x(\psi_i(s))) - g_i^*(s, x^*(\psi_i(s)))|\phi'(s)\, ds \\
&+ \ \sum_{i=1}^{n} |f_i^*(t, x(\eta_i(t))) - f_i^*(t, x^*(\eta_i(t)))|\Big(|a_i(t)| + \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ |g_i^*(s, x^*(\psi_i(s)))|\phi'(s)\, ds\Big) \\
&\leq \ \sum_{i=1}^{n} \delta\Big(|a_i(t)| + \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ (\rho_i + c_i|x(\psi_i(s))|)\phi'(s)\, ds\Big) \\
&+ \ \sum_{i=1}^{n} |f_i^*(t, x(\eta_i(t)))| \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ |g_i(s, x(\psi_i(s))) - g_i^*(s, x(\psi_i(s)))|\phi'(s)\, ds \\
&+ \ \sum_{i=1}^{n} |f_i^*(t, x(\eta_i(t)))| \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ |g_i^*(s, x(\psi_i(s))) - g_i^*(s, x^*(\psi_i(s)))|\phi'(s)\, ds \\
&+ \ \sum_{i=1}^{n} L_i|x(\eta_i(t)) - x^*(\eta_i(t))|\Big(|a_i(t)| + \int_0^t \frac{(\phi(t)\ -\ \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)}\ (\rho_i + c_i|x^*(\psi_i(s))|)\phi'(s)\, ds\Big) \\
&\leq \ \sum_{i=1}^{n} \delta\Big(\text{ŧ}_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}\Big) + \sum_{i=1}^{n} h_i \frac{\sigma}{\Gamma(\alpha_i + 1)}
\end{aligned}$$

$$+ \sum_{i=1}^{n} h_i \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} c_i|x(\psi_i(s)) - x^*(\psi_i(s))|\phi'(s)\, ds$$
$$+ \sum_{i=1}^{n} L_i|x(\eta_i(t)) - x^*(\eta_i(t))|\Big(ŧ_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}\Big)$$

$$\leq \sum_{i=1}^{n} \delta\Big(ŧ_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}\Big) + \sum_{i=1}^{n} h_i \frac{\sigma}{\Gamma(\alpha_i + 1)}$$
$$+ \sum_{i=1}^{n} \frac{h_i c_i}{\Gamma(\alpha_i + 1)} \|x - x^*\| + \sum_{i=1}^{n} L_i \|x - x^*\| \Big(ŧ_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}\Big).$$

Then

$$\Big[1 - \sum_{i=1}^{n} \Big(L_i ŧ_i + \frac{h_i c_i + L_i(\rho_i + c_i r)}{\Gamma(\alpha_i + 1)}\Big)\Big]\|x - x^*\| \leq \sum_{i=1}^{n} \Big[\delta\Big(ŧ_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}\Big) + \frac{h_i\sigma}{\Gamma(\alpha_i + 1)}\Big]$$
$$\|x - x^*\| \leq \sum_{i=1}^{n} \Big[\delta\Big(ŧ_i + \frac{\rho_i + c_i r}{\Gamma(\alpha_i + 1)}\Big) + \frac{h_i\sigma}{\Gamma(\alpha_i + 1)}\Big]$$
$$. \Big[1 - \sum_{i=1}^{n} \Big(L_i ŧ_i + \frac{h_i c_i + L_i(\rho_i + c_i r)}{\Gamma(\alpha_i + 1)}\Big)\Big]^{-1}$$

□

4. Discussion and remarks

- In case $a_i = 0,$ the operators B_i have the following form

$$(B_i x)(t) = \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} g_i(s, x(\psi_i(s)))\phi'(s)\, ds.$$

  Then we get the n-term quadratic integral equation of fractional order

$$x(t) = \sum_{i=1}^{n} f_i(t, x(\eta_i(t))). \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_i-1}}{\Gamma(\alpha_i)} g_i(s, x(\psi_i(s)))\phi'(s)\, ds, \ t \in J. \tag{4.1}$$

  By a simple calculation we can verify that the operators B_i satisfy the assumptions of Theorem 1.1 and hence the quadratic integral Eq (4.1) has a solution in C. For $\phi(t) = t$ in Eq (4.1), we obtain the quadratic equation which is studied in [25].
- Taking $n = 1$ we obtain the quadratic integral equation

$$x(t) = f(t, x(\psi_1(t))) \Big[a(t) + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha-1}}{\Gamma(\alpha)} g(s, x(\psi_2(s)))\, \phi'(s)\, ds\Big] \tag{4.2}$$

  and putting $\phi(t) = t,$ then we have the same result studied in [27].

- Letting $\alpha_i \to 1, i = 1, 2$. Then we have the n- term quadratic integral equation

$$x(t) = \sum_{i=1}^{n} f_i(t, x(\eta_i(t)))\Big(a_i(t) + \int_0^t g_i(s, x(\psi_i(s)))\, ds\Big), \; t \in J$$

and for $n = 2$, we have

$$x(t) = f_1(t, x(\eta_1(t)))\Big(a_1(t) + \int_0^t g_1(s, x(\psi_1(s)))\, ds\Big) + f_2(t, x(\eta_2(t)))\Big(a_2(t) + \int_0^t g_2(s, x(\psi_2(s)))\, ds\Big).$$

Therefore we obtain the same result obtained in [29] when $a_i = 0, i = 1, 2$ and $\eta_i(t) = \psi_i(t) = t$.

- Letting $n = 2$ we obtain the two term quadratic integral equation

$$\begin{aligned} x(t) &= f_1(t, x(\eta_1(t))) \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_1 - 1}}{\Gamma(\alpha_1)} g_1(s, x(\psi_1(s)))\phi'(s)\, ds \\ &+ f_2(t, x(\eta_2(t))) \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha_2 - 1}}{\Gamma(\alpha_2)} g_2(s, x(\psi_2(s)))\phi'(s)\, ds, \; t \in J. \end{aligned} \tag{4.3}$$

- Taking $\phi(t) = t$, $\eta_1(t) = \eta_2(t) = t$, $\psi_1(t) = \psi_2(t) = t$ and $f_1(t, x) = 1$ in (4.3), then we get the hybrid differential equation of fractional order

$${}^R D^{\alpha_2}\Big(\frac{x(t) - I^{\alpha_1} g_1(t, x(t))}{f_2(t, x(t))}\Big) = g_2(t, x(t)), \; t \in J,$$

where ${}^R D^{\alpha_2}$ is the Riemann-Liouville fractional derivative of order $\alpha_2 \in (0, 1)$.

- Taking $\alpha_2 \to 1$, $g_1(t, x) = 0$, $\phi(t) = t$, $\eta_1(t) = \eta_2(t) = t$, $\psi_1(t) = \psi_2(t) = t$ and $f_1(t, x) = 1$ in (4.3) we have the initial value problem

$$\Big(\frac{x(t)}{f_2(t, x(t))}\Big)' = g_2(t, x(t)), \; t \in J, \quad x(0) = x_0 \in \mathbb{R}.$$

Example 1. *For $n = 2$, consider the nonlinear quadratic integral equation*

$$\begin{aligned} x(t) &= f_1(t, x(t))\Big(\frac{2t + sin(t)}{10} + \int_0^t \frac{(t - s)^{\frac{1}{2}} - 1}{\Gamma(\frac{1}{2})} g_1(s, x(s))\, ds\Big) \\ &+ f_2(t, x(t))\Big(1 + \int_0^t \frac{(t - s)^{\frac{1}{4}} - 1}{\Gamma(\frac{1}{4})} g_2(s, x(s))\, ds\Big), \; t \in [0, 1], \end{aligned}$$

where

$$\begin{aligned} f_1(t, x(t)) &= 1 + \frac{cos(t)x(t)}{1 + t}, \qquad f_2(t, x(t)) = \sqrt{t + 3} + |x(t)| \\ g_1(t, x(t)) &= sin(t) + \frac{3|x(t)|}{10(0.1 + |x(t)|)}, \qquad g_2(t, x(t)) = \frac{1 + 2t}{5} + \frac{1}{2}.\frac{|x(t)|}{1 + |x(t)|} \\ a_1(t) &= \frac{2t + sin(t)}{10}, \quad a_2(t) = 1. \end{aligned}$$

We can easily verify that the functions f_1, f_2, g_1 and g_2 satisfy all assumptions of Theorem 2.1. Then $h_1 = h_2 = 1$, $b_1 = \frac{1}{2}$, $b_2 = \frac{3}{10}$, which implies that $\frac{b_1 h_1}{\Gamma(\alpha_1+1)} + \frac{b_2 h_2}{\Gamma(\alpha_2+1)} = \frac{0.3}{\Gamma(\frac{1}{2}+1)} + \frac{0.5}{\Gamma(\frac{1}{4}+1)} = 0.898437 < 1$. Moreover, we get $r > 0$.

5. Conclusions

Many previous papers have discussed quadratic integral equations of fractional order by using different techniques in several classes of functions [3, 18, 30–34].

We have investigated existence and uniqueness of the solutions for a multi-term ϕ-quadratic integral equation of fractional order in $C(J, \mathbb{R})$. The giving problem is converted into an analogous fixed point problem which is solved using typical functional analysis tools to prove our results. As a pursuit of this, sufficient conditions are given for the existence of solutions to that singular quadratic integral equation. Next, the continuous dependence of the solution on some functions has been proved. Finally, some particular cases, remarks and example to validate our results.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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