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Research article

Orthonormal Euler wavelets method for time-fractional Cattaneo equation with Caputo-Fabrizio derivative

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Abstract: In this paper, a new orthonormal wavelets based on the orthonormal Euler polynomials (OEPs) is constructed to approximate the numerical solution of time-fractional Cattaneo equation with Caputo-Fabrizio derivative. By applying the Gram-Schmidt orthonormalization process on sets of Euler polynomials of various degrees, an explicit representation of OEPs is obtained. The convergence analysis and error estimate of the orthonormal Euler wavelets expansion are studied. The exact formula of Caputo-Fabrizio fractional integral of orthonormal Euler wavelets are derived using Laplace transform. The applicability and validity of the proposed method are verified by some numerical examples.

Keywords: orthonormal Euler wavelets; Caputo-Fabrizio fractional integral; Cattaneo equation; Laplace transform; convergence analysis
Mathematics Subject Classification: 34A08, 34K28, 65T60

1. Introduction

Over the past few decades, fractional differential equations (FDEs) have gained importance in the modelling of various physical phenomena due to their ability to describe the memory and hereditary properties of different materials and processes. Various fractional models based on Caputo derivative have been successfully established, such as COVID-19 transmission [1], HIV-1 infection of $CD4^+$ T-cells [2], epidemiological MSEIR model [3], model of cancer chemotherapy effect [4], fractional model for Maxwell fluid [5]. Due to the singularity of the definition of Caputo derivatives, it causes some difficulties to the numerical and analytical methods of fractional calculus. Recently, Caputo and Fabrizio [6] proposed a new definition of fractional derivative by replacing the singular kernel in the Caputo derivative with exponential kernel, which was called Caputo-Fabrizio fractional derivative. This new operator has been successfully applied in human liver model [7], groundwater flow model [8, 9], electrical circuits model [10], COVID-19 model [11], Korteweg-de Vries-Burgers equation [12], fractional Maxwell fluid [13], pneumococcal pneumonia infection model [14] and transmission dynamics of brucellosis model [15].

The standard Cattaneo equation is normally obtained by using a generalized form of Fick's law, which describes a diffusion process with finite propagation velocity. Cattaneo equation is a hyperbolic partial differential equation as follows:

$$\frac{\partial u(x,t)}{\partial t}+\frac{\partial^2 u(x,t)}{\partial t^2}=\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{1.1}$$

where $u(x,t)$ denotes distribution function of diffusing-quantity/temperature [16]. However, the classical Cattaneo equation cannot describe the anomalous diffusion behavior observed in many natural systems. To address this issue, Compte and Metzler [17] extended the classical Cattaneo model to the time-fractional Cattaneo model, and studied the properties of the corresponding fractional Cattaneo equation. Following Compte and Metzler, Kosztolowicz and Lewandowska [18] proposed a theoretical basis for the study of subdiffusion impedance using a hyperbolic equation. Since then, many researchers have numerically considered the fractional Cattaneo equation in the sense of Caputo [19–25].

Caputo-Fabrizio derivative has many interesting properties. It can portray substance heterogeneities and configurations with different scales, which cannot be handled by the well-known local theory. In recent years, fractional Cattaneo equation based on Caputo-Fabrizio derivative has attracted much attention from many researchers. Since finding the solutions of these differential equations is a difficult task, it is necessary to find the solutions of partial differential equations(PDEs) by numerical methods. Liu [26] introduced a second order Crank-Nicolson scheme for fractional Cattaneo equation based on Caputo-Fabrizio derivative. Taghipour [27] considered a θ-finite difference scheme using cubic B-spline for the fractional Cattaneo equations. In [28], the spline-based collocation schemes are developed for the numerical solution of the time-fractional Cattaneo equations involving the Caputo-Fabrizio derivative. Li [29] proposed a fully discrete spectral method for fractional Cattaneo equation having Caputo-Fabrizio derivative, in which finite difference method is used in time and Legendre spectral approximation in space. Note that, the above-mentioned papers applied finite difference scheme in time direction, owing to the memory effect of fractional derivative, to compute the solution at the current time level, all previous solutions have to be saved, which would make the storage very expensive if low-order methods are employed for spatial discretization.

In this paper, we consider the following time-fractional Cattaneo equation with nonhomogeneous term [26]:

$$\frac{\partial u(x,t)}{\partial t}+\frac{\partial^\gamma u(x,t)}{\partial t^\gamma}=\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t), \tag{1.2}$$

where $(x,t)\in\Omega=[0,L]\times[0,T]$, $1<\gamma<2$, $f(x,t)\in C(\Omega)$ with initial conditions

$$u(x,0)=\phi(x),\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=\psi(x),\ 0\le x\le L, \tag{1.3}$$

and boundary conditions

$$u(0,t)=u(L,t)=0,\ t>0. \tag{1.4}$$

${}_{0}^{CF}D_t^{\gamma}u(x,t)$ denotes the Caputo-Fabrizio operator of the function $u(x,t)$ defined as

$$
{}_{0}^{CF}D_t^{\gamma}u(x,t) = \frac{M(\gamma)}{2-\gamma}\int_0^t u''(x,s)e^{\sigma(t-s)}ds, \tag{1.5}
$$

where $M(\gamma)$ is a normalization function such that $M(0) = M(1) = 1$ and $\sigma = \frac{1-\gamma}{2-\gamma}$.

As a global method, spectral method can achieve exponential rates of convergence for smooth problems and requires fewer grid points to produce highly accurate solution, which makes it widely used for solving the numerical solutions of different types of PDEs. Numerical methods based on some orthogonal or semi-orthogonal polynomials [30–33] are widely used to solve fractional differential equtions (FDEs). These methods utilize operational matrix of derivatives or integration to transform complicated FDEs into a system of linear or nonlinear algebraic equations. Recently, Genochi polynomial [34], Legendre polynomial [35] and Chebyshev polynomial [36] are applied to solve FDEs with Caputo-Fabrizio derivative. Orthogonal wavelet is a special orthogonal function with compact support, such as Legendre wavelet [37, 38], Chebyshev wavelet [39, 40] and Jacobi wavelet [41, 42], which can approximate discontinuous or rapidly changing functions at different resolutions. The main characteristic of wavelet-based technology is that the coefficient matrix of the discretized algebraic equation is a sparse matrix, which reduces the amount of calculation and accelerates the simulation speed. Since the Caputo-Fabrizio operator is a relatively new operator, as far as we know, there is no paper using orthogonal polynomial wavelets to solve fractional Cattaneo equations with Caputo-Fabrizio derivative. Euler wavelets have good accuracy for solving various differential equations [43–45], but do not have orthogonality. It is actually easier to use orthonormal Euler wavelets than Euler wavelets.

Motivated by the above discussion, our main target is to present an effective numerical method based on orthonormal Euler wavelets to solve time fractional Cattaneo equation with Caputo-Fabrizio derivative. Therefore, we construct orthonormal Euler wavelets by Gram-Schmidt orthogonalization of Euler polynomials. In addition, to obtain more accurate numerical solutions, Laplace transform is utilized to obtain the exact Caputo-Fabrizio fractional integral formula of Euler wavelets. The proposed method has the following advantages:

(i) Caputo-Fabrizio fractional integral operator for the orthonormal Euler wavelets has been derived directly with no approximations.

(ii) The present method is convenient for solving this problem, since the boundary and initial conditions are taken into account automatically in the algorithm.

(iii) This method is applicable to linear and nonlinear Cattaneo equations.

The paper is organized as follows. Section 2 introduces some fundamental definitions of fractional calculus and Laplace transform. Section 3 is devoted to the construction of orthonormal Euler wavelet. Convergence analysis and error estimate of the orthonormal Euler wavelets expansion are displayed in Section 4. An exact formula for the Caputo-Fabrizio fractional integral of orthonormal Euler wavelets is derived in Section 5. The description of the proposed method is presented in Section 6. Some illustrative examples are provided in Section 7. Finally, conclusion and discussion is given in Section 8.

2. Preliminaries and notations

In this section, some necessary definitions and mathematical preliminaries are given.

Definition 2.1. *(See [46]) The Laplace transform $F(s)$ of a locally integrable function $f(x)$ is defined by*

$$\mathfrak{L}[f(x)] = \int_0^\infty e^{-sx} f(x) dx, \tag{2.1}$$

where s is a complex number, and this operator has the following properties:

(1) $\mathfrak{L}[\lambda_1 f_1(x) + \lambda_2 f_2(x)] = \lambda_1 \mathfrak{L}[f_1(x)] + \lambda_2 \mathfrak{L}[f_2(x)]$,

(2) $\mathfrak{L}[f(x-a)\mu(x-a)] = e^{-as} F(s)$,

(3) $\mathfrak{L}[f * g] = \mathfrak{L}[f(x)]\mathfrak{L}[g(x)]$,

where λ_1, λ_2 and a are constants, $\mu(x)$ is the step function and $f * g$ is the convolution of two functions f and g.

Definition 2.2. *(See [47]) Let $n \geq 1$ and $\alpha \in [0, 1]$, the Caputo-Fabrizio fractional derivative with order α of a function $f(x)$ is defined by*

$$^{CF}D_a^{n+\alpha} f(x) = \frac{M(\alpha)}{1-\alpha} \int_a^x f^{(n+1)}(s) \exp\big(-\frac{\alpha(x-s)}{1-\alpha}\big) ds. \tag{2.2}$$

Definition 2.3. *(See [47]) Let $n \geq 1$, $\alpha \in [0, 1]$, and $f \in C^1[a, b]$. The Caputo-Fabrizio integral of order $\alpha + n$ of a function f is given by*

$$^{CF}J_a^{n+\alpha} f(x) = \frac{1}{M(\alpha) n!} \int_a^x (x-s)^{n-1} [\alpha(x-s) + n(1-\alpha)] f(s) ds, \tag{2.3}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

Lemma 2.1. *(See [47]) Let $\gamma \in (n, n+1)$, $\gamma = n+\alpha$, $n = [\gamma] \geq 0$, $\alpha \in [0, 1]$. Assume that $f(x) \in C^n[a, b]$, then the following statements hold:*

(1) $$^{CF}J_a^\gamma(^{CF}D_a^\gamma f(x)) = f(x) - \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i, \ i = 0, 1, 2, \cdots, n. \tag{2.4}$$

(2) $$^{CF}D_a^\gamma(^{CF}J_a^\gamma f(x)) = f(x) - f(a) \exp\big(-\alpha\frac{(x-a)}{1-\alpha}\big). \tag{2.5}$$

Definition 2.4. *(See [48]) The Riemann-Liouville fractional integral of order α ($\alpha > 0$) is defined as*

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) ds, \quad \alpha > 0. \tag{2.6}$$

3. Orthonormal Euler wavelets

The main goal of this section is to construct the orthonormal Euler wavelets (OEWs). To obtain orthonormal Euler polynomials (OEPs), we apply the Gram-Schmidt orthonormalization process on sets of Euler polynomials. Then, the orthonormal Euler wavelets will be constructed using the orthonormal Euler polynomials and their properties.

3.1. Orthonormal Euler polynomials

Definition 3.1. *(See [49]) The Euler polynomials $E_m(x)$ of degree m can be constructed from the following relation:*

$$\sum_{k=0}^{m}\binom{m}{k}E_k(x) + E_m(x) = 2x^m, \quad m = 0, 1, 2, \cdots . \tag{3.1}$$

From Eq (3.1), the first few terms of Euler polynomials are listed below:

$$\begin{aligned}&E_0(x) = 1,\ E_1(x) = x - \frac{1}{2},\ E_2(x) = x^2 - x,\ E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4},\ E_4(x) = x^4 - 2x^3 + x,\\ &E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2},\ \cdots .\end{aligned} \tag{3.2}$$

Euler polynomials has the following properties:

(1) $E_m^{'}(x) = mE_{m-1}(x),\ m = 1, 2, \cdots ,$

(2) $E_m(x + 1) + E_m(x) = 2x^m,$

(3) $\displaystyle\int_0^1 E_m(x)E_n(x)dx = (-1)^{n-1}\frac{m!(n + 1)!}{(m + n + 1)!}E_{m+n+1}(0),\ m, n \geq 1.$

Note that Euler polynomials are not mutually orthogonal. To overcome this difficulty, we try to apply the Gram-Schmidt orthogonal normalization process on $E_m(x)$ with respect to the weight function $w(x) = 1$. We denote the orthonormal Euler polynomials by $OE_m(x)$. For instance, for $m = 5$, we have

$$\begin{gathered}OE_0(x) = 1,\ OE_1(x) = \sqrt{3}(2x - 1),\ OE_2(x) = \sqrt{5}(6x^2 - 6x + 1),\\ OE_3(x) = \sqrt{7}(20x^3 - 30x^2 + 12x - 1),\\ OE_4(x) = \sqrt{9}(70x^4 - 140x^3 + 90x^2 - 20x + 1),\\ OE_5(x) = \sqrt{11}(252x^5 - 630x^4 + 560x^3 - 210x^2 + 30x - 1).\end{gathered}$$

By analyzing these coefficients, we can get an explicit representation of OEPs.

Definition 3.2. *The explicit form of the OEPs of degree m on the interval* $[0, 1]$ *is given by*

$$OE_m(x) = \sqrt{2m + 1}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{m + k}{k}x^k, \quad m = 0, 1, 2, \cdots . \tag{3.3}$$

The orthogonality property for these polynomials is as follows:

$$\int_0^1 OE_i(x)OE_j(x)dx = \delta_{i,j},\ i, j = 0, 1, 2, \cdots , \tag{3.4}$$

where $\delta_{i,j}$ is the Kronecker delta function. Furthermore, by analyzing the integrals of these polynomials from 0 to x, we obtain

$$\int_0^x OE_0(t)dt = \frac{1}{2}OE_0(x) + \frac{1}{2\sqrt{3}}OE_1(x),$$

and for $m \geq 1$,

$$\int_0^x OE_m(t)dt = \frac{1}{2\sqrt{(2m+1)(2m+3)}}OE_{m+1}(x) - \frac{1}{2\sqrt{(2m+1)(2m-1)}}OE_{m-1}(x). \tag{3.5}$$

Therefore, we also have

$$2\sqrt{2m+1}OE_m(x) = \frac{1}{\sqrt{2m+3}}OE'_{m+1}(x) - \frac{1}{\sqrt{2m-1}}OE'_{m-1}(x), \quad m \geq 1. \tag{3.6}$$

3.2. Construction of orthonormal Euler wavelets

Wavelets are a set of functions which can be defined from dilation and translation of a mother wavelet function $\psi(x)$. When the dilation and translation parameters vary continuously, we get a series of continuous wavelet functions [50]. The orthonormal Euler wavelets (OEWs) $\psi_{n,m}(x)$ have four arguments $n = 1, 2, \cdots, 2^{k-1}$ can assume any positive integer, m is the order of the orthonormal Euler polynomials and x is the normalized variable. They are defined on the interval $[0, 1)$ by

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k-1}{2}} OE_m(2^{k-1}x - n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.7}$$

where $m = 0, 1, \cdots$, and $OE_m(x)$ are the OEPs of degree m defined in Eq (3.3). It is easy to verify that the constructed OEWs $\psi_{n,m}(x)$, $n = 1, 2, \cdots, 2^{k-1}$, $m = 0, 1, \cdots$, constitutes an orthonormal set on the interval $[0, 1)$. Suppose $f(x) \in L^2[0, 1)$ can be expressed in terms of OEWs as

$$f(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) = C^T\Psi(x), \tag{3.8}$$

where C and $\Psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c_{1,0}, c_{1,1}, \cdots, c_{1,M-1}, c_{2,0}, \cdots, c_{2,M-1}, \cdots, c_{2^{k-1},0}, \cdots, c_{2^{k-1},M-1}]^T,$$

$$\Psi(x) = [\psi_{1,0}(x), \psi_{1,1}(x), \cdots, \psi_{1,M-1}(x), \psi_{2,0}(x), \cdots, \psi_{2,M-1}(x), \cdots, \psi_{2^{k-1},0}(x), \cdots, \psi_{2^{k-1},M-1}(x)]^T, \tag{3.9}$$

and

$$c_{n,m} = \langle f(x), \psi_{n,m}(x)\rangle_{L^2[0,1)} = \int_0^1 \psi_{n,m}(x)f(x)dx. \tag{3.10}$$

Two-dimensional orthonormal Euler wavelets can be expressed as product of 1D-OEWs as follows:

$$\psi_{n,m,n',m'}(x, y) = \begin{cases} \psi_{n,m}(x)\psi_{n',m'}(y), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}, \frac{n'-1}{2^{k'-1}} \leq y < \frac{n'}{2^{k'-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.11}$$

where $n' = 1, 2, \cdots, 2^{k'-1}$, $m' = 0, 1, \cdots$. It is easy to verify that $\{\psi_{n,m}(x)\psi_{n',m'}(y)\}$ is an orthonormal set over $[0, 1) \times [0, 1)$. Similarly, an arbitrary function of two variables $f(x, y)$ defined over $[0, 1) \times [0, 1)$ can be expanded into 2D-OEWs basis as

$$\begin{aligned} f(x,y) &= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} \sum_{n'=1}^{2^{k'-1}} \sum_{m'=0}^{\infty} c_{n,m,n',m'} \psi_{n,m}(x)\psi_{n',m'}(y) \\ &\simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \sum_{n'=1}^{2^{k'-1}} \sum_{m'=0}^{M'-1} c_{n,m,n',m'} \psi_{n,m}(x)\psi_{n',m'}(y) \\ &= \Psi^T(x) U \Psi(y), \end{aligned}$$

where $c_{n,m,n',m'} = \langle \psi_{n,m}(x), \langle f(x, y), \psi_{n',m'}(y)\rangle\rangle$ is the coefficient of 2D-OEWs, in which $\langle , \rangle$ denotes the inner product and U is a $2^{k-1}M \times 2^{k'-1}M'$ coefficient matrix.

4. Convergence analysis and error estimation

In this section, we prove the convergence of OEWs with respect to the L^2-norm. To this end, we first derive the upper error bound for the mentioned OEWs expansion.

Theorem 4.1. *Let* $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x)$ *be the orthonormal Euler wavelet expansion of* $f(x) \in L^2[0, 1)$ *and suppose* $f(x)$ *has a bounded second-order derivative, say* $|f''(x)| \leq B$*, then for* $m \geq 2$*, the following inequality holds:*

$$\left|c_{n,m}\right| \leq \frac{2\sqrt{3}B}{2^{\frac{5}{2}k}\sqrt{2m+1}\sqrt{(2m-3)^3}}, \tag{4.1}$$

where $c_{n,m} = \langle f(x), \psi_{n,m}(x)\rangle_{L^2[0,1)}$ *is given in (3.10).*

Proof. The orthonormal Euler wavelets coefficient is given as follows:

$$c_{n,m} = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(x) 2^{\frac{k-1}{2}} OE_m(2^{k-1}x - n + 1)dx.$$

By using the substitution $t = 2^{k-1}x - n + 1$ and the relation (3.6), we have

$$c_{n,m} = \frac{2^{-\frac{1+k}{2}}}{\sqrt{2m+1}} \int_0^1 f(\frac{t+n-1}{2^{k-1}}) d\Big[\frac{OE_{m+1}(t)}{\sqrt{2m+3}} - \frac{OE_{m-1}(t)}{\sqrt{2m-1}}\Big].$$

Taking integration by parts and using the relation (3.6) yield that

$$\begin{aligned} c_{n,m} = &-\frac{2^{\frac{1-3k}{2}}}{\sqrt{2m+1}} \int_0^1 f'(\frac{t+n-1}{2^{k-1}}) d\Big[\frac{OE_{m+2}(t)}{2(2m+3)\sqrt{2m+5}} - \frac{OE_m(t)}{2(2m+3)\sqrt{2m+1}} \\ &- \frac{OE_m(t)}{2(2m-1)\sqrt{2m+1}} + \frac{OE_{m-2}(t)}{2(2m-1)\sqrt{2m-3}}\Big]. \end{aligned}$$

By using the integration by parts again, we have

$$
\begin{aligned}
c_{n,m} =& -\frac{2^{\frac{3-5k}{2}}}{\sqrt{2m+1}}\int_0^1 f''(\frac{t+n-1}{2^{k-1}})\Big[\frac{OE_{m+2}(t)}{2(2m+3)\sqrt{2m+5}}-\frac{OE_m(t)}{2(2m+3)\sqrt{2m+1}}\\
&-\frac{OE_m(t)}{2(2m-1)\sqrt{2m+1}}+\frac{OE_{m-2}(t)}{2(2m-1)\sqrt{2m-3}}\Big]dt\\
=&\frac{2^{\frac{3-5k}{2}}}{\sqrt{2m+1}}\int_0^1 f''(\frac{t+n-1}{2^{k-1}})\Big[\frac{(2m+1)\sqrt{2m+5}OE_{m+2}(t)-(2m+5)\sqrt{2m+1}OE_m(t)}{2(2m+1)(2m+3)(2m+5)}\\
&-\frac{(2m-3)\sqrt{2m+1}OE_m(t)-(2m+1)\sqrt{2m-3}OE_{m-2}(t)}{2(2m-3)(2m-1)(2m+1)}\Big]dt.
\end{aligned}
$$

By applying the Cauchy-Schwarz inequality and using the assumption that $|f''(x)| \le B$, we obtain

$$
\begin{aligned}
&\Big|\int_0^1 f''(\frac{t+n-1}{2^{k-1}})\Big[\frac{(2m+1)\sqrt{2m+5}OE_{m+2}(t)-(2m+5)\sqrt{2m+1}OE_m(t)}{2(2m+1)(2m+3)(2m+5)}\\
&-\frac{(2m-3)\sqrt{2m+1}OE_m(t)-(2m+1)\sqrt{2m-3}OE_{m-2}(t)}{2(2m-3)(2m-1)(2m+1)}\Big]dt\Big|^2\\
\le&\int_0^1\Big|f''(\frac{t+n-1}{2^{k-1}})\Big|^2\Big|\frac{(2m+1)\sqrt{2m+5}OE_{m+2}(t)-(2m+5)\sqrt{2m+1}OE_m(t)}{2(2m+1)(2m+3)(2m+5)}\\
&-\frac{(2m-3)\sqrt{2m+1}OE_m(t)-(2m+1)\sqrt{2m-3}OE_{m-2}(t)}{2(2m-3)(2m-1)(2m+1)}\Big|^2dt\\
\le&B^2\int_0^1\Big[\frac{(2m-3)^2(2m-1)^2(2m+1)^2(2m+5)OE_{m+2}^2(t)}{4(2m-3)^2(2m-1)^2(2m+1)^2(2m+3)^2(2m+5)^2}\\
&+\frac{4(2m-3)^2(2m+1)^2(2m+5)^2(2m+1)OE_m^2(t)}{4(2m-3)^2(2m-1)^2(2m+1)^2(2m+3)^2(2m+5)^2}\\
&+\frac{(2m+1)^2(2m+3)^2(2m+5)^2(2m-3)OE_{m-2}^2(t)}{4(2m-3)^2(2m-1)^2(2m+1)^2(2m+3)^2(2m+5)^2}\Big]dt\\
\le&B^2\Big[\frac{1}{4(2m+3)^2(2m+5)}+\frac{4}{4(2m-1)^2(2m+1)}+\frac{1}{4(2m-1)^2(2m-3)}\Big]\\
\le&\frac{3B^2}{2(2m-1)^2(2m-3)}.
\end{aligned}
$$

Therefore, for $m \ge 2$,

$$
\left|c_{n,m}\right| \le \frac{2\sqrt{3}B}{2^{\frac{5}{2}k}\sqrt{2m+1}\sqrt{(2m-3)^3}}. \tag{4.2}
$$

Theorem 4.2. *Let $f(x) \in L^2[0,1)$ be a bounded function with second derivative say $|f''(x)| \le B$. Then the following inequality holds:*

$$
\left\|f(x)-C^T\Psi(x)\right\|_2 \le \frac{B}{2^{2k}\sqrt{(2M-4)^3}}. \tag{4.3}
$$

Proof. By the definition of the L^2-norm, we have

$$\begin{aligned}\left\|f(x)-C^T\Psi(x)\right\|_2^2 &= \int_0^1 (f(x)-C^T\Psi(x))^2dx\\ &= \int_0^1 (\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x))^2dx\\ &= \int_0^1 (\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} c_{n,m}^2\psi_{n,m}^2(x))dx\\ &= \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} c_{n,m}^2.\end{aligned}$$

The above equation holds due to the fact that the constructed Euler wavelets are orthonormal. By the Theorem 4.1, we have

$$\left\|f(x)-C^T\Psi(x)\right\|_2^2 = \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} c_{n,m}^2 \le \frac{3B^2}{2^{4k}2^3}\sum_{m=M}^{\infty}\frac{1}{(m-2)^4}. \tag{4.4}$$

Using the integral inequality yields that

$$\sum_{m=M}^{\infty}\frac{1}{(m-2)^4} = \sum_{m=M-2}^{\infty}\frac{1}{m^4} \le \int_{M-2}^{\infty}\frac{1}{m^4}dx = \frac{1}{3(M-2)^3}. \tag{4.5}$$

Therefore,

$$\left\|f(x)-C^T\Psi(x)\right\|_2^2 \le \frac{B^2}{2^{4k}2^3}\frac{1}{(M-2)^3}. \tag{4.6}$$

Here, take the square root for both sides of (4.6), then

$$\left\|f(x)-C^T\Psi(x)\right\|_2 \le \frac{B}{2^{2k}}\frac{1}{\sqrt{(2M-4)^3}}. \tag{4.7}$$

The proof is completed. □

Corollary 4.1. The upper bound of error given in Theorem 4.2 is related to k and M. Obviously, for a fixed k, $\frac{B}{2^{2k}}\frac{1}{\sqrt{(2M-4)^3}}$ tends to zero as M tends to ∞. Therefore, we conclude that $C^T\Psi(x)$ converges to $f(x)$ with respect to the L^2-norm.

Theorem 4.3. *Let $f \in L^2[0,1)$ be a bounded function with second derivative say $|f''(x)| \le B$. Then the following inequality holds:*

$$\left\|{}^{CF}J_a^{\gamma}f(x) - {}^{CF}J_a^{\gamma}f_{k,M}(x)\right\|_2 \le \frac{1}{\Gamma(\eta)}\frac{1}{\sqrt{2\eta(2\eta-1)}}\frac{B}{2^{2k}\sqrt{(2M-3)^3}}, \tag{4.8}$$

where $\gamma = \eta+\alpha$, $\eta = [\gamma]$, $f_{k,M}(x) = C^T\Psi(x)$.

Proof. The definition of Caputo-Fabrizio integral can be further simplified as follows:

$$ {}^{CF}J_a^{\eta+\alpha} f(x) = \frac{1-\alpha}{M(\alpha)} I^{\eta} f(x) + \frac{\alpha}{M(\alpha)} I^{\eta+1} f(x), \tag{4.9} $$

using the Cauchy formula [51], where

$$ I^{\eta} f(x) = \frac{1}{\Gamma(\eta)} \int_a^x (x-s)^{\eta-1} f(s) ds. \tag{4.10} $$

By the Hölder's inequality, we can estimate the difference between ${}^{CF}J_a^{\gamma} f(x)$ and ${}^{CF}J_a^{\gamma} f_{k,M}(x)$ at $x \in [0,1]$, as follows:

$$ \begin{aligned} & \left| {}^{CF}J_a^{\gamma} f(x) - {}^{CF}J_a^{\gamma} f_{k,M}(x) \right| \\ \leq & \frac{1-\alpha}{M(\alpha)} \frac{1}{\Gamma(\eta)} \int_a^x \frac{|f(s) - f_{k,M}(s)|}{(x-s)^{1-\eta}} ds + \frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\eta+1)} \int_a^x \frac{|f(s) - f_{k,M}(s)|}{(x-s)^{-\eta}} ds \\ \leq & \frac{1-\alpha}{M(\alpha)} \frac{1}{\Gamma(\eta)} \Big(\int_a^x \frac{ds}{(x-s)^{2-2\eta}} \Big)^{1/2} \Big(\int_a^x (f(s) - f_{k,M}(s)) ds \Big)^{1/2} \\ & + \frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\eta+1)} \Big(\int_a^x \frac{ds}{(x-s)^{2\eta}} \Big)^{1/2} \Big(\int_a^x (f(s) - f_{k,M}(s)) ds \Big)^{1/2} \\ \leq & \Big(\frac{1-\alpha}{M(\alpha)} \frac{1}{\Gamma(\eta)} \frac{1}{\sqrt{(2\eta-1)}} (x-a)^{\eta-\frac{1}{2}} + \frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\eta+1)} \frac{1}{\sqrt{(2\eta+1)}} (x-a)^{\eta+\frac{1}{2}} \Big) \left\| f(x) - C^T \Psi(x) \right\|_2. \end{aligned} \tag{4.11} $$

Hence,

$$ \begin{aligned} & \left\| {}^{CF}J_a^{\gamma} f - {}^{CF}J_a^{\gamma} f_{k,M} \right\|_2 \\ \leq & \Big(\frac{1-\alpha}{M(\alpha)} \frac{1}{\Gamma(\eta)} \frac{1}{\sqrt{(2\eta-1)}} \left\| (x-a)^{\eta-\frac{1}{2}} \right\|_2 \\ & + \frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\eta+1)} \frac{1}{\sqrt{(2\eta+1)}} \left\| (x-a)^{\eta+\frac{1}{2}} \right\|_2 \Big) \left\| f(x) - C^T \Psi(x) \right\|_2 \\ \leq & \Big(\frac{1-\alpha}{M(\alpha)} \frac{1}{\Gamma(\eta)} \frac{1}{\sqrt{2\eta(2\eta-1)}} + \frac{\alpha}{M(\alpha)} \frac{1}{\Gamma(\eta)} \frac{1}{\sqrt{(2\eta+1)(2\eta+2)}} \Big) \frac{B}{2^{2k}} \frac{1}{\sqrt{(2M-4)^3}} \\ \leq & \frac{1}{\Gamma(\eta)} \frac{1}{\sqrt{2\eta(2\eta-1)}} \frac{B}{2^{2k}} \frac{1}{\sqrt{(2M-4)^3}}. \end{aligned} \tag{4.12} $$

The proof is completed. □

Theorem 4.4. *If a continuous function* $f(x,y) \in L^2([0,1) \times [0,1))$ *has bounded mixed fourth partial derivative* $\left| \frac{\partial^4 f(x,y)}{\partial x^2 \partial y^2} \right| \leq \tilde{B}$, *then 2D-OEWs expansion of* $f(x,y)$ *converges uniformly to the function* $f(x,y)$, *that is*

$$ f(x,y) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} \sum_{n'=1}^{2^{k'-1}} \sum_{m'=0}^{\infty} c_{n,m,n',m'} \psi_{n,m}(x) \psi_{n',m'}(y), \tag{4.13} $$

where $c_{n,m,n',m'} = \langle \psi_{n,m}(x), \langle f(x,y), \psi_{n',m'}(y) \rangle \rangle$.

Proof.

$$\begin{aligned} c_{n,m,n',m'} &= \int_0^1\int_0^1 f(x,y)\psi_{n,m}(x)\psi_{n',m'}(y)dxdy \\ &= \int_0^1 \psi_{n',m'}(y)(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(x,y)2^{\frac{k-1}{2}}OE_m(2^{k-1}x-n+1)dx)dy \\ &= \int_{\frac{n-1}{2^{k'-1}}}^{\frac{n}{2^{k'-1}}} 2^{\frac{k'-1}{2}}OE_{m'}(2^{k'-1}y-n'+1)(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(x,y)2^{\frac{k-1}{2}}OE_m(2^{k-1}x-n+1)dx)dy. \end{aligned}$$

By change of variable $s = 2^{k-1}x - n + 1$, $t = 2^{k'-1}y - n' + 1$, $dx = \frac{1}{2^{k-1}}ds$ and $dy = \frac{1}{2^{k'-1}}dt$, we obtain

$$c_{n,m,n',m'} = 2^{\frac{1-k'}{2}}\int_0^1 OE_{m'}(t)(\int_0^1 f(\frac{s+n-1}{2^{k-1}}, y)2^{\frac{1-k}{2}}OE_m(s)ds)dt. \tag{4.14}$$

Similar to the previous steps used in the Theorem 4.1, we have

$$\begin{aligned} \left|c_{n,m,n',m'}\right| &\leq \frac{2\sqrt{3}\tilde{B}}{2^{\frac{5}{2}k}\sqrt{2m+1}\sqrt{(2m-3)^3}}\frac{2\sqrt{3}}{2^{\frac{5}{2}k'}\sqrt{2m'+1}\sqrt{(2m'-3)^3}} \\ &\leq \frac{12\tilde{B}}{2^{\frac{5}{2}(k+k')}(2m-3)^2(2m'-3)^2}. \end{aligned}$$

Since $n \leq 2^{k-1}$, $n' \leq 2^{k'-1}$, we get

$$\left|c_{n,m,n',m'}\right| \leq \frac{12\tilde{B}}{(2n)^{\frac{5}{2}}(2m-3)^2(2n')^{\frac{5}{2}}(2m'-3)^2}. \tag{4.15}$$

Therefore, $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{\infty} c_{n,m,n',m'}\psi_{n,m}(x)\psi_{n',m'}(y)$ is absolutely convergent. If $n = n'$, $m = m'$, then we have

$$\left|c_{n,m,n',m'}\right| \leq \frac{12\tilde{B}}{(2n)^5(2m-3)^4}.$$

5. Caputo-Fabrizio fractional integral for orthonormal Euler wavelets

In this section, we will derive the exact formula for the fractional integral of orthonormal Euler wavelets $\psi_{n,m}(x)$ in the sense of Caputo-Fabrizio. In fact, the $\alpha - th$ repeated integral of $f(x)$ is the special case of Riemann-Liouville fractional integral, when α is a positive integer. To derive the Caputo-Fabrizio fractional integral of orthonormal Euler wavelets, first we need to derive the Riemann-Liouville fractional integral of order α of the orthonormal Euler wavelets.

Theorem 5.1. *The Caputo-Fabrizio fractional integral of order $\gamma > 0$ of orthonormal Euler wavelets $\psi_{n,m}(x)$ is given by*

$${}^{CF}J_a^{\gamma}\psi_{n,m}(x) = \frac{1-\beta}{M(\beta)}I^n\psi_{n,m}(x) + \frac{\beta}{M(\beta)}I^{n+1}\psi_{n,m}(x), \tag{5.1}$$

where $\gamma = n + \beta$, $n = [\gamma] \geq 0$, $\beta \in [0, 1]$ and $I^n\psi_{n,m}(x)$ is given by the following theorem. To make the theorem more general, we derive the Riemann-Liouville fractional integral of order $\alpha > 0$ of the orthonormal Euler wavelets.

Theorem 5.2. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of the orthonormal Euler wavelets $\psi_{n,m}(x)$ is given by*

$$I^{\alpha}\psi_{n,m}(x) = \begin{cases} 0, & 0 \leq x \leq \frac{n-1}{2^{k-1}}, \\ U(x), & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}, \\ U(x) - V(x), & \frac{n}{2^{k-1}} \leq x \leq 1, \end{cases} \tag{5.2}$$

where

$$U(x) = 2^{\frac{k-1}{2}}\sqrt{2m+1}\sum_{i=0}^{m}(-1)^{m-i}C_m^iC_{m+i}^i2^{i(k-1)}\frac{\Gamma(i+1)}{\Gamma(i+\alpha+1)}(x-\frac{n-1}{2^{k-1}})^{i+\alpha},$$

and

$$V(x) = 2^{\frac{k-1}{2}}\sqrt{2m+1}\sum_{i=0}^{m}(-1)^{m-i}C_m^iC_{m+i}^i\sum_{r=0}^{i}C_i^r2^{r(k-1)}\frac{\Gamma(r+1)}{\Gamma(r+\alpha+1)}(x-\frac{n}{2^{k-1}})^{r+\alpha}.$$

Proof. See Appendix.

For instance, in the case of $k = 2$, $M = 3$, $\gamma = 5/2$, $x = 0.65$, we obtain

$${}^{CF}J_0^{5/2}\Psi_{6\times1}(0.65) = \begin{pmatrix} 0.17338847530345 \\ -0.07144345083118 \\ 0.00329403922934 \\ 0.00835269885277 \\ -0.01160828498416 \\ 0.00925321971836 \end{pmatrix},$$

where $\Psi_{6\times1}(x) = (\psi_{1,0}(x), \psi_{1,1}(x), \psi_{1,2}(x), \psi_{2,0}(x), \psi_{2,1}(x), \psi_{2,2}(x))^T$.

6. Description of the proposed method

For simplicity, we consider the following time-fractional Cattaneo equation with $T = 1$, $L = 1$:

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial^{\gamma}u(x,t)}{\partial t^{\gamma}} = \frac{\partial^2u(x,t)}{\partial x^2} + f(x,t), \tag{6.1}$$

$$u(x,0) = \phi(x),\ \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x),\ 0 \leq x \leq 1, \tag{6.2}$$

$$u(0,t) = u(1,t) = 0,\ 0 \leq t \leq 1. \tag{6.3}$$

To solve the time-fractional Cattaneo eqution (6.1), we assume

$$\frac{\partial^{2+\gamma}u(x,t)}{\partial x^2\partial t^{\gamma}} = \Psi^T(x)U\Psi(t), \tag{6.4}$$

where $U = (u_{ij})_{2^{k-1}M\times 2^{k-1}M}$ is an unknown wavelet coefficient matrix and $\Psi(\cdot)$ is the vector defined in (3.9). By applying the Caputo-Fabrizio fractional integral of order γ with respect to t on both sides of Eq (6.4) and considering the property of Caputo-Fabrizio fractional integral, we obtain

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \Psi^T(x)U^{CF}J_0^{\gamma}\Psi(t) + \frac{\partial^2 u(x,t)}{\partial x^2}\Big|_{t=0} + t\frac{\partial}{\partial t}\Big(\frac{\partial^2 u(x,t)}{\partial x^2}\Big)\Big|_{t=0}. \tag{6.5}$$

Using the conditions (6.2), we yield

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \phi''(x) + t\psi''(x) + \Psi^T(x)U^{CF}J_0^{\gamma}\Psi(t). \tag{6.6}$$

By integrating Eq (6.6) two times with respect to x and combining the conditions (6.2), we have

$$u(x,t) = x\frac{\partial u}{\partial x}\Big|_{x=0} + \phi(x) - \phi(0) - x\phi'(0) + t(\psi(x) - \psi(0) - x\psi'(0)) + (I^2\Psi(x))^T U^{CF}J_0^{\gamma}\Psi(t). \tag{6.7}$$

Putting $x = 1$ in Eq (6.7), we get

$$-\frac{\partial u}{\partial x}\Big|_{x=0} = \phi(1) - \phi(0) - \phi'(0) + t(\psi(1) - \psi(0) - \psi'(0)) + (I^2\Psi(1))^T U^{CF}J_0^{\gamma}\Psi(t). \tag{6.8}$$

Therefore, we have

$$\begin{aligned} u(x,t) =&(\phi(x) - \phi(0) - x\phi'(0)) + t(\psi(x) - \psi(0) - x\psi'(0)) + (I^2\Psi(x))^T U^{CF}J_0^{\gamma}\Psi(t) \\ &- x[\phi(1) - \phi(0) - \phi'(0) + t(\psi(1) - \psi(0) - \psi'(0)) + (I^2\Psi(1))^T U^{CF}J_0^{\gamma}\Psi(t)]. \end{aligned} \tag{6.9}$$

Taking the first derivative of Eq (6.9) with respect to t, we get

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} =&\psi(x) - \psi(0) - x\psi'(0) + (I^2\Psi(x))^T U^{CF}J_0^{\gamma-1}\Psi(t) - x[\psi(1) - \psi(0) - \psi'(0) \\ &+ (I^2\Psi(1))^T U^{CF}J_0^{\gamma-1}\Psi(t)]. \end{aligned} \tag{6.10}$$

Again, taking the Caputo-Fabrizio fractional derivative of order γ of Eq (6.9) with respect to t and considering the Lemma 2.1, we obtain

$$\frac{\partial^{\gamma} u(x,t)}{\partial t^{\gamma}} = (I^2\Psi(x))^T U(\Psi(t) - \Psi(0)e^{\frac{\alpha t}{1-\alpha}}) - x(I^2\Psi(1))^T U(\Psi(t) - \Psi(0)e^{\frac{\alpha t}{1-\alpha}}). \tag{6.11}$$

Now, by substituting Eqs (6.6), (6.10) and (6.11) into Eq (6.1) and considering the collocation points $x_i = \frac{2i-1}{2^k M}$, $t_j = \frac{2j-1}{2^k M}$, $i, j = 1, 2, \cdots, 2^{k-1}M$, we obtain the following linear system of algebraic equation:

$$\begin{aligned} &(I^2\Psi(x_i) - x_i I^2\Psi(1))^T U\big({}^{CF}J_0^{\gamma-1}\Psi(t_j) + \Psi(t_j) - \Psi(0)e^{\frac{\alpha t_j}{1-\alpha}}\big) - \Psi^T(x_i)U^{CF}J_0^{\gamma}\Psi(t_j) \\ =&f(x_i,t_j) - (\psi(x_i) - \psi(0) - x_i\psi'(0)) + x_i(\psi(1) - \psi(0) - \psi'(0)) + (\phi''(x_i) + t_j(\psi''(x_i))). \end{aligned} \tag{6.12}$$

Solving this system, we obtain the unknown Euler wavelet coefficient vector U and thereafter substituting U into Eq (6.9), we obtain the approximate solutions of the given time-fractional Cattaneo equation (6.1).

7. Numerical examples

In this section, we provide several numerical examples with exact solution to demonstrate the accuracy of the orthonormal Euler wavelets collocation method. All the numerical experiments are performed by Matlab 7.0. We report errors by using
(i) Mean root square norm (L_2):

$$L_2 - error = \sqrt{\frac{\sum_{i=0}^{m}\sum_{j=0}^{m}(u_{k,M}(x_i,t_j) - u(x_i,t_j))^2}{m^2}},$$

(ii) Maximum error norm (L_∞):

$$L_\infty - error = \max_{1\le i\le m, 1\le j\le m} |u_{k,M}(x_i,t_j) - u(x_i,t_j)|,$$

where $u_{k,M}(x_i,t_j)$ is the approximated solution and $u(x_i,t_j)$ is the exact solution.

Example 1. Consider the following time-fractional Cattaneo equation:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), & 0 < x < 1,\ 0 \le t \le 1,\ 1 < \gamma < 2, \\ u(0,t) = 0, & u(1,t) = 0, \\ u(x,0) = 0, & \frac{\partial u}{\partial t}\Big|_{t=0} = 0, \end{cases} \tag{7.1}$$

where $f(x,t) = 2(1-x^2)x^{\frac{16}{3}}(t + \frac{1-e^{\sigma t}}{\gamma-1}) + t^2(\frac{418}{9}x^{\frac{16}{3}} - \frac{208}{9}x^{\frac{10}{3}})$. The exact solution is $u(x,t) = t^2(1-x^2)x^{\frac{16}{3}}$. Figure 1 shows the numerical solution (left) and absolute error (right) for some different points of $[0,1]\times[0,1]$ with $\gamma = 5/4$, $k = 5$ and $M = 5$. For various γ, Table 1 gives the L_∞, L_2 errors with different k and M.

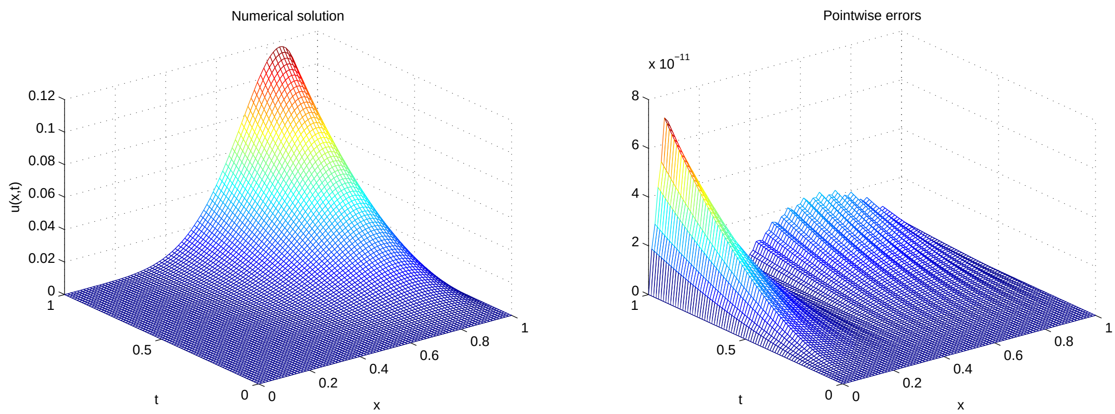

Figure 1. The plot of numerical solution and pointwise errors for Example 1 with $\gamma = 5/4$, $k = 5$ and $M = 5$.

Table 1. Error estimate for Example 1 with different k and M.

γ	norm	$k=3, M=3$	$k=4, M=4$	$k=5, M=5$	$k=6, M=6$
1.2	L_∞	4.4291e-4	4.8747e-6	6.1905e-11	5.2767e-13
	L_2	1.2791e-4	1.4450e-6	1.1791e-11	1.0202e-13
1.4	L_∞	4.3108e-4	4.7223e-6	6.1689e-11	5.2645e-13
	L_2	1.2334e-4	1.3840e-6	1.1756e-11	1.0039e-13
1.6	L_∞	4.2223e-4	4.5005e-6	5.8791e-11	5.2395e-13
	L_2	1.2122e-4	1.2886e-6	1.6430e-11	9.8481e-14
1.8	L_∞	4.6023e-4	3.7309e-6	1.9333e-9	5.5668e-13
	L_2	1.7954e-4	9.8998e-7	9.6276e-10	2.8539e-13
1.9	L_∞	9.6641e-4	2.9868e-5	9.7490e-8	5.9616e-11
	L_2	4.9498e-4	1.3332e-5	4.4653e-8	2.7277e-11

Example 2. Consider the following time-fractional Cattaneo equation with the exact solution $u(x,t) = e^t\sin(\pi x)$:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), & 0<x<1,\ 0\le t\le 1,\ 1<\gamma<2, \\ u(0,t)=0, & u(1,t)=0, \\ u(x,0)=\sin(\pi x), & \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin(\pi x), \end{cases} \tag{7.2}$$

where $f(x,t) = e^t\sin(\pi x) + \frac{1}{2-\gamma}\frac{1}{1-\sigma}(e^t - e^{\sigma t})\sin(\pi x) + e^t\pi^2\sin(\pi x)$.

Figure 2 shows the numerical solution and pointwise errors on $[0,1]\times[0,1]$ with $\gamma = 1.9$, $k=5$ and $M=5$. For various γ, Table 2 gives the L_∞, L_2 errors with different k and M. In Figure 3, we depict the logarithmic plot for the maximal absolute errors L_∞ and mean root square error L_2 at selected points with $k=4$, $\gamma=1.9$ and $M=3$ through 9.

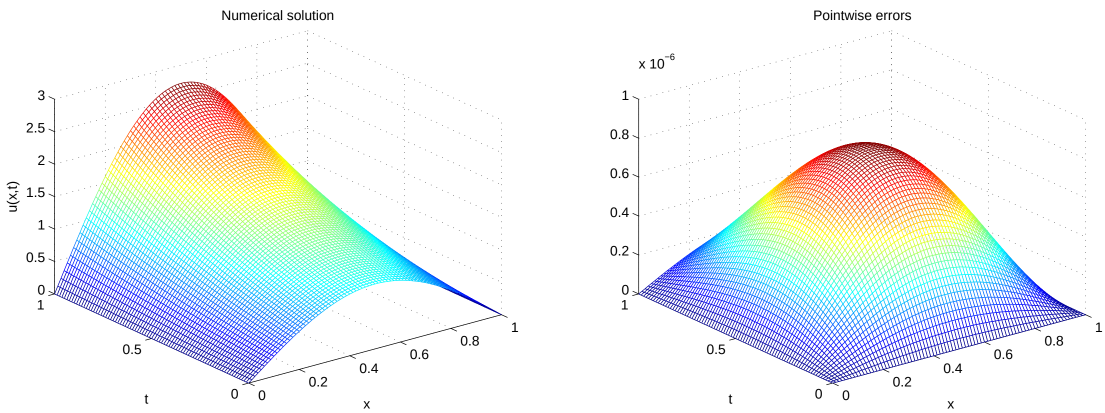

Figure 2. The plot of numerical solution and pointwise errors for Example 2 with $\gamma = 1.9$, $k=5$ and $M=5$.

Table 2. Error estimate for Example 2 with different k and M.

γ	norm	$k=3, M=3$	$k=4, M=4$	$k=5, M=5$	$k=6, M=6$
1.2	L_∞	1.0143e-4	8.0837e-7	3.3529e-11	6.0396e-14
	L_2	3.6737e-5	2.3647e-7	8.8511e-12	1.9451e-14
1.4	L_∞	1.0483e-4	7.6592e-7	3.1084e-11	9.9476e-14
	L_2	4.0664e-5	2.2115e-7	8.7714e-12	2.7726e-14
1.6	L_∞	1.6201e-4	5.4781e-7	1.2204e-10	1.6875e-13
	L_2	8.8981e-5	1.3852e-7	6.4412e-11	3.6683e-14
1.8	L_∞	0.0016	1.1935e-5	1.5296e-8	3.9275e-12
	L_2	8.7472e-4	6.2400e-6	8.0685e-9	2.0768e-12
1.9	L_∞	0.0083	2.5393e-4	8.2131e-7	5.0065e-10
	L_2	0.0042	1.2783e-4	4.1361e-7	2.5214e-10

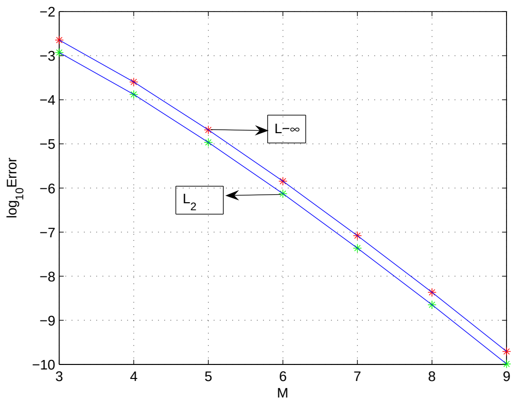

Figure 3. The logarithmic plot for L_∞ and L_2 error for Example 2.

Example 3. Consider the following time-fractional Cattaneo equation with the exact solution $u(x,t) = e^t x^2(1-x)^2$:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), & 0 < x < 1,\ 0 \le t \le 1,\ 1 < \gamma < 2, \\ u(0,t) = 0, & u(1,t) = 0, \\ u(x,0) = x^2(1-x)^2, & \frac{\partial u}{\partial t}\Big|_{t=0} = x^2(1-x)^2, \end{cases} \tag{7.3}$$

where $f(x,t) = e^t x^2(1-x)^2 + \frac{1}{2-\gamma}\frac{1}{1-\sigma}(e^t - e^{\sigma t})x^2(1-x)^2 - e^t(12x^2 - 12x + 2)$.

Figure 4 shows the numerical solution and pointwise errors on $[0,1]\times[0,1]$ with $\gamma = 1.7$, $k = 5$ and $M = 5$. For various γ, Table 3 gives the L_∞, L_2 errors with different k and M. In Figure 5, we depict the logarithmic plot for the maximal absolute errors L_∞ and mean root square error L_2 at selected points with $k = 3$, $\gamma = 1.7$ and $M = 3$ through 9.

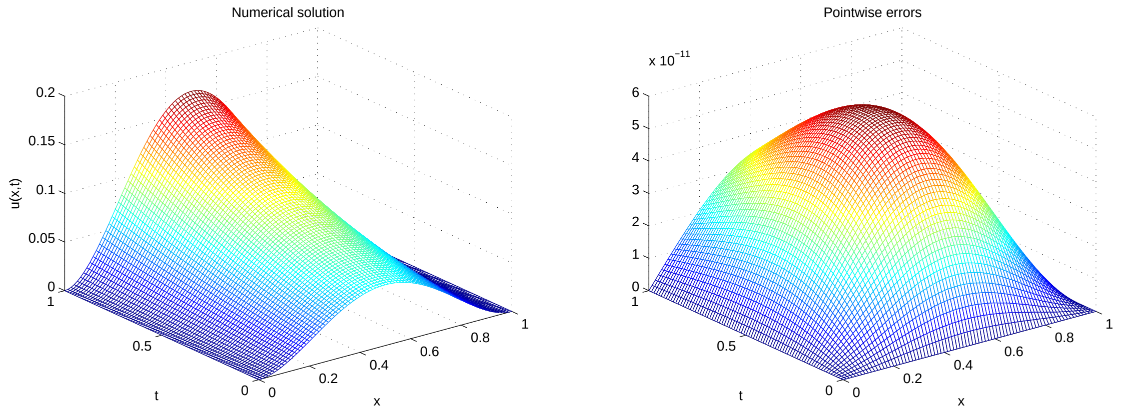

Figure 4. The plot of numerical solution and pointwise errors for Example 3 with $\gamma = 1.7$, $k = 5$ and $M = 5$.

Table 3. Error estimate for Example 3 with different k and M.

γ	norm	$k = 3, M = 3$	$k = 4, M = 4$	$k = 5, M = 5$	$k = 6, M = 6$
1.2	L_∞	2.0143e-6	3.2792e-9	8.3658e-13	5.7454e-15
	L_2	1.1090e-6	1.8354e-9	4.8755e-13	8.2946e-16
1.4	L_∞	2.4907e-6	2.6650e-9	9.5716e-13	1.2768e-15
	L_2	1.4089e-6	1.5146e-9	5.5222e-13	2.1428e-16
1.6	L_∞	7.6440e-6	1.0909e-8	7.1870e-12	5.4678e-15
	L_2	4.2624e-6	6.0637e-9	4.0095e-12	9.1325e-16
1.8	L_∞	9.1918e-5	6.7269e-7	8.5751e-10	2.1959e-13
	L_2	4.8255e-5	3.5286e-7	4.4978e-10	1.1499e-13
1.9	L_∞	4.6337e-4	1.4125e-5	4.5675e-8	2.7842e-11
	L_2	2.3351e-4	7.1249e-6	2.3041e-8	1.4045e-11

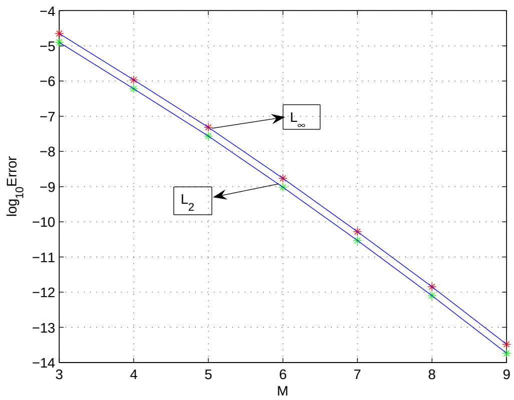

Figure 5. The logarithmic plot for L_∞ and L_2 error for Example 3.

Example 4. Consider the following time-fractional Cattaneo equation with the exact solution $u(x,t) = e^t(1-x)x^{\frac{16}{3}}$:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), & 0 < x < 1, 0 \le t \le 1, 1 < \gamma < 2, \\ u(0,t) = 0, & u(1,t) = 0, \\ u(x,0) = (1-x)x^{\frac{16}{3}}, & \frac{\partial u}{\partial t}\Big|_{t=0} = (1-x)x^{\frac{16}{3}}, \end{cases} \tag{7.4}$$

where $f(x,t) = e^t(1-x)x^{\frac{16}{3}} + \frac{1}{2-\gamma}\frac{1}{1-\sigma}(e^t - e^{\sigma t})(1-x)x^{\frac{16}{3}} + e^t(\frac{304}{9}x^{\frac{13}{3}} - \frac{208}{9}x^{\frac{10}{3}})$.

Figure 6 displays the numerical solution and pointwise errors on $[0,1]\times[0,1]$ with $\gamma = 1.5$, $k = 5$ and $M = 5$. In Table 4, we list the L_∞, L_2 errors with different k, M and γ.

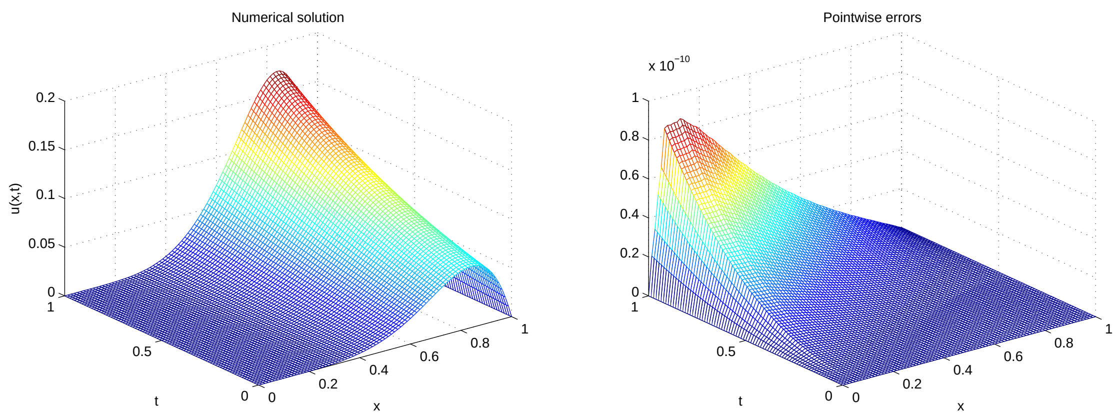

Figure 6. The plot of numerical solution and pointwise errors for Example 4 with $\gamma = 1.5$, $k = 5$ and $M = 5$.

Table 4. Error estimate for Example 4 with different k and M.

γ	norm	$k=3, M=3$	$k=4, M=4$	$k=5, M=5$	$k=6, M=6$
1.2	L_∞	9.4908e-5	9.8307e-7	8.7095e-11	4.4699e-13
	L_2	2.6285e-5	2.8646e-7	2.0943e-11	9.9711e-14
1.4	L_∞	9.1462e-5	9.4324e-7	8.6046e-11	4.4436e-13
	L_2	2.5169e-5	2.7294e-7	2.0245e-11	9.7045e-14
1.6	L_∞	8.8202e-5	8.8560e-7	8.4203e-11	4.4144e-13
	L_2	2.4450e-5	2.5215e-7	1.8668e-11	9.3449e-14
1.8	L_∞	9.8768e-5	6.7160e-7	4.4743e-10	4.4437e-13
	L_2	3.8992e-5	1.9644e-7	2.1858e-10	1.2035e-13
1.9	L_∞	2.5128e-4	7.7518e-6	2.5265e-8	1.5449e-11
	L_2	1.2672e-4	3.5255e-6	1.1706e-8	7.1740e-12

Example 5. Consider the following time-fractional Cattaneo equation with the exact solution $u(x,t) = e^t(\sin(\pi x))^2$:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), & 0 < x < 1, 0 \le t \le 1, 1 < \gamma < 2, \\ u(0,t) = 0, & u(1,t) = 0, \\ u(x,0) = (\sin(\pi x))^2, & \frac{\partial u}{\partial t}\Big|_{t=0} = (\sin(\pi x))^2, \end{cases} \tag{7.5}$$

where $f(x,t) = e^t((\sin(\pi x))^2 + \frac{1}{2-\gamma}\frac{1}{1-\sigma}(e^t - e^{\sigma t})(\sin(\pi x))^2 - e^t 2\pi^2(\cos(\pi x))^2$.

In Figure 7, we plot the approximate solution and pointwise errors on $[0,1]\times[0,1]$ when $\gamma = 1.9$, $k = 5$ and $M = 5$. For various γ, Table 5 gives the L_∞, L_2 errors with different k and M. In Figure 8, we depict the logarithmic plot for the maximal absolute errors L_∞ and mean root square error L_2 at selected points with $k = 4$, $\gamma = 1.9$ and $M = 3$ through 9.

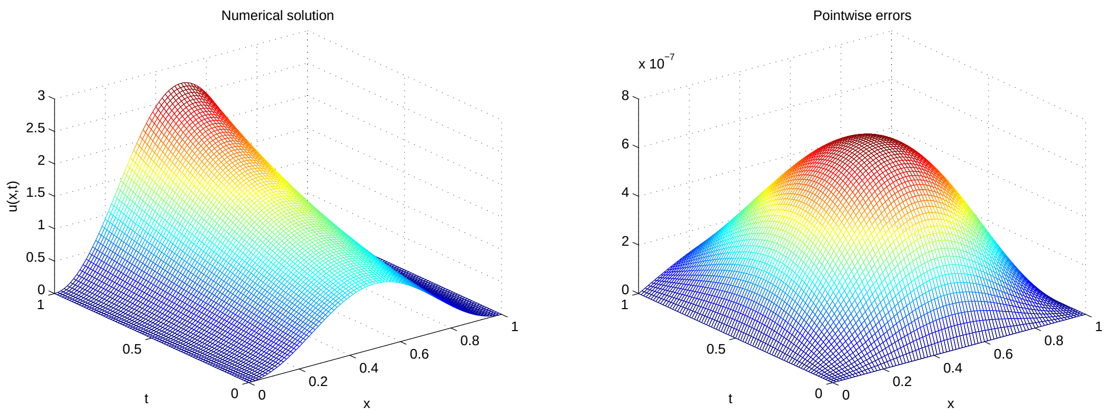

Figure 7. The plot of numerical solution and pointwise errors for Example 5 with $\gamma = 1.9$, $k = 5$ and $M = 5$.

Table 5. Error estimate for Example 5 with different k and M.

γ	norm	$k=3, M=3$	$k=4, M=4$	$k=5, M=5$	$k=6, M=6$
1.2	L_∞	0.0014	1.2982e-5	3.2843e-9	6.0574e-12
	L_2	3.0838e-4	2.9801e-6	7.6865e-10	1.4442e-12
1.4	L_∞	0.0014	1.2562e-5	3.1775e-9	5.9392e-12
	L_2	2.9784e-4	2.8451e-6	7.3457e-10	1.3902e-12
1.6	L_∞	0.0014	1.1862e-5	2.9698e-9	5.7323e-12
	L_2	3.1058e-4	2.5919e-6	6.5691e-10	1.3124e-12
1.8	L_∞	0.0018	9.3156e-6	1.2832e-8	6.6569e-12
	L_2	8.8981e-4	4.5346e-6	6.5375e-9	2.5842e-12
1.9	L_∞	0.0072	2.1421e-4	6.9554e-7	4.2455e-10
	L_2	0.0037	1.0762e-4	3.5099e-7	2.1461e-10

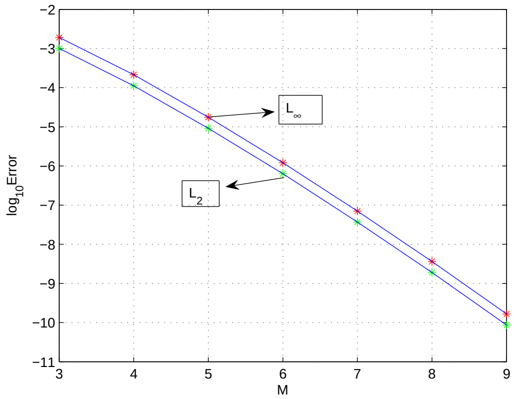

Figure 8. The logarithmic plot for L_∞ and L_2 error for Example 5.

Example 6. Consider the following nonlinear time-fractional Cattaneo equation with the exact solution $u(x,t) = e^t x^2 \sin(\pi x)$:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial^{1.5} u(x,t)}{\partial t^{1.5}} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(1-u) + f(x,t), & 0 < x < 1, 0 \le t \le 1, \\ u(0,t) = 0, & u(1,t) = 0, \\ u(x,0) = x^2 \sin(\pi x), & \left.\frac{\partial u}{\partial t}\right|_{t=0} = x^2 \sin(\pi x), \end{cases} \tag{7.6}$$

where $f(x,t) = \frac{x^2 \sin(\pi x)}{2-\gamma} \frac{1}{1-\sigma}(e^t - e^{\sigma t}) - (2\sin(\pi x) + 4\pi x \cos(\pi x) - \pi^2 x^2 \sin(\pi x))e^t + (e^t x^2 \sin(\pi x))^2$. In [36], the authors solved the problem using shifted Chebyshev polynomial collocation method and Caputo-Fabrizio operational matrix of derivative. To compare our numerical findings with those results in [36], we list absolute errors at different spatial points at time $t = 0.1$, and for different value of M with $k = 1$ in Table 6. It can be seen that we obtain more accurate numerical solutions with the same parameters. For $k = 2$ and $M = 6$, the numerical solution and pointwise errors on $[0,1]\times[0,1]$ are given in Figure 9. In Figure 10, we depict the logarithmic plot for the maximal absolute errors L_∞ and root mean square error L_2 at selected points with $k = 2$ and $M = 3$ through 9.

Table 6. Absolute errors for $M = 4$ and $M = 6$ in Example 6.

	Method in [36]		Present method	
x	$M = 4$	$M = 6$	$M = 4$	$M = 6$
0.2	6.7e-3	3.7e-4	1.7e-5	2.4e-7
0.4	5.2e-3	8.2e-4	1.0e-5	1.0e-7
0.6	1.6e-3	3.6e-4	1.2e-5	1.1e-7
0.8	9.1e-3	4.2e-4	1.9e-5	1.7e-7

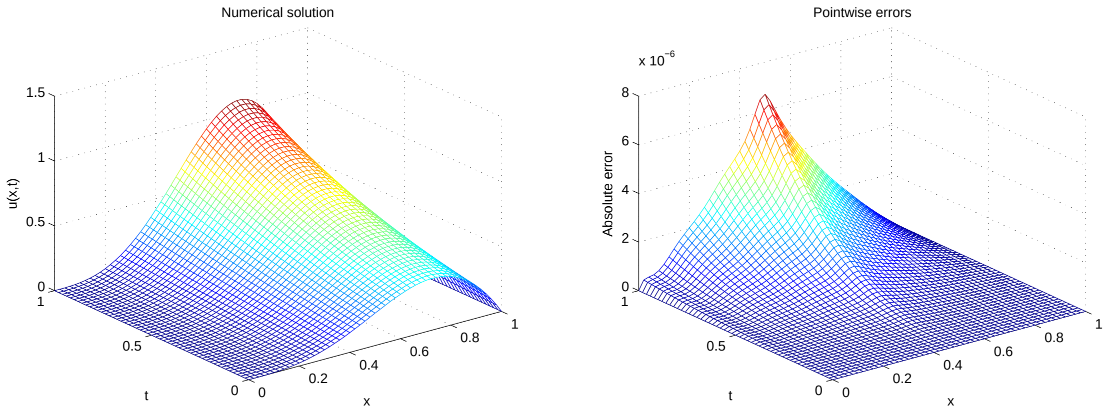

Figure 9. The plot of numerical solution and pointwise errors for Example 6 with $k = 2$ and $M = 6$.

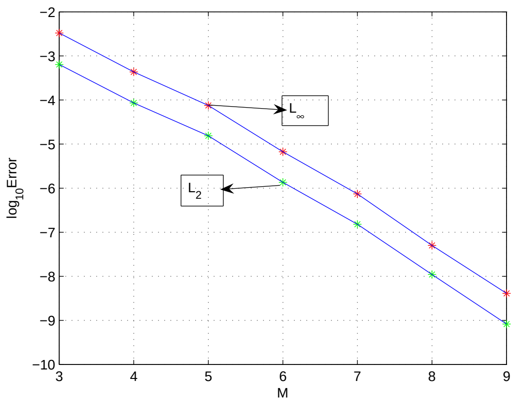

Figure 10. The logarithmic plot for L_∞ and L_2 error for Example 6.

8. Conclusions

In this work, a numerical algorithmn based on orthonormal Euler wavelets together with collocation method is proposed for the time-fractional Cattaneo equation with Caputo-Fabrizio derivative. The orthonormal Euler wavelets are constructed by applying the Gram-Schmidt orthonormalization process on sets of Euler polynomials. The convergence analysis and error estimate of the orthonormal Euler wavelets expansion are investigated. The exact formula of the Caputo-Fabrizio fractional integral of orthogonal Euler wavelets is obtained for the first time. The main characteristic behind the method is that the problem under consideration is converted into a system of algebraic equations, which greatly simplifies the problem. The effectiveness of the proposed method for time-fractional Cattaneo equation with Caputo-Fabrizio derivative is verified by the graphical and tabular demonstrations. It can be seen from the Tables 1–5 that maximum absolute error L_∞ and root mean square error L_2 become smaller and smaller with increasing k and M. One can see from the Figures 3, 5, 8 and 10 that L_∞ and L_2 errors show exponential decay in the coordinate, the error variations are essentially linear versus the polynomial degrees. In addition, we also compare our results in Example 6 with the results of shifted

Chebyshev polynomial collocation method. We find that Euler wavelets collocation method in this work provides much better results than the aforementioned numerical method. The outcomes show that the proposed method is suitable for solving time-fractional Cattaneo equations having Caputo-Fabrizio derivative.

In our future work, we will consider 2D and 3D linear and nonlinear fractional partial differential equations involving Caputo-Fabrizio derivative, such as fractional KdV, KdV-Burgers equations, time distributed-order differential equations.

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Conflict of interest

The authors declare they have no competing interests.

Appendix

In this section, we prove Theorem 5.2.

Proof. According to the definition of orthonormal Euler wavelets $\psi_{n,m}(x)$, they are given as follows:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k-1}{2}} OE_m(2^{k-1}x - n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}. \end{cases} \tag{A.1}$$

Using the unit step function, orthonormal Euler wavelets can be expressed as

$$\psi_{n,m}(x) = \mu_{\frac{n-1}{2^{k-1}}}(x) 2^{\frac{k-1}{2}} OE_m(2^{k-1}x - n + 1) - \mu_{\frac{n}{2^{k-1}}}(x) 2^{\frac{k-1}{2}} OE_m(2^{k-1}x - n + 1),$$

where $\mu_c(x)$ is the unit step function defined as

$$\mu_c(x) = \begin{cases} 1, & x \geq c, \\ 0, & x < c. \end{cases} \tag{A.2}$$

Recalling the definition of Caputo-Fabrizio integral of order $\eta = n + \beta$:

$$^{CF}J_a^{n+\beta} f(x) = \frac{1}{M(\beta)n!} \int_a^x (x-s)^{n-1} [\beta(x-s) + n(1-\beta)] f(s) ds, \tag{A.3}$$

it can be further simplified as follows using the Cauchy formula $I^n f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-s)^{n-1} f(s) ds$ and the relation $\Gamma(n) = (n-1)!$,

$$^{CF}J_a^{n+\beta} f(x) = \frac{1-\beta}{M(\beta)} I^n f(x) + \frac{\beta}{M(\beta)} I^{n+1} f(x). \tag{A.4}$$

Note that $I^n f(x)$ is the special case of Riemann-Liuville integral $I^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s)ds$. Furthermore, by the aid of convolution operation, $I^\alpha f(x)$ can be express as

$$I^\alpha \psi_{n,m}(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} * \psi_{n,m}(x). \tag{A.5}$$

By employing the Laplace transform, we obtain

$$\mathfrak{L}\{I^\alpha \psi_{n,m}(x)\} = \mathfrak{L}\{\frac{1}{\Gamma(\alpha)} x^{\alpha-1}\}\mathfrak{L}\{\psi_{n,m}(x)\} = \frac{1}{s^\alpha}\mathfrak{L}\{\psi_{n,m}(x)\}. \tag{A.6}$$

Next, we focus on the derivation of $\mathfrak{L}\{\psi_{n,m}(x)\}$. By using the delay property of Laplace transform $\mathfrak{L}\{\mu_c(x) f(x)\} = e^{-cs}\mathfrak{L}\{f(x+c)\}$, we have

$$\begin{aligned}\mathfrak{L}\{\psi_{n,m}(x)\} =& 2^{\frac{k-1}{2}}\mathfrak{L}\{\mu_{\frac{n-1}{2^{k-1}}}(x) OE_m(2^{k-1}x - n + 1)\} \\ &- 2^{\frac{k-1}{2}}\mathfrak{L}\{\mu_{\frac{n}{2^{k-1}}}(x) OE_m(2^{k-1}x - n + 1)\} \\ =& e^{-\frac{n-1}{2^{k-1}}s} 2^{\frac{k-1}{2}}\mathfrak{L}\{OE_m(2^{k-1}(x + \frac{n-1}{2^{k-1}}) - n + 1)\} \\ &- e^{-\frac{n}{2^{k-1}}s} 2^{\frac{k-1}{2}}\mathfrak{L}\{OE_m(2^{k-1}(x + \frac{n}{2^{k-1}}) - n + 1)\} \\ =& e^{-\frac{n-1}{2^{k-1}}s} 2^{\frac{k-1}{2}}\mathfrak{L}\{OE_m(2^{k-1}x)\} - e^{-\frac{n}{2^{k-1}}s} 2^{\frac{k-1}{2}}\mathfrak{L}\{OE_m(2^{k-1}x + 1)\}.\end{aligned} \tag{A.7}$$

From Eq (3.3), we have

$$OE_m(2^{k-1}x) = \sqrt{2m+1}\sum_{i=0}^{m}(-1)^{m-i} C_m^i C_{m+i}^i 2^{i(k-1)} x^i, \tag{A.8}$$

and

$$OE_m(2^{k-1}x + 1) = \sqrt{2m+1}\sum_{i=0}^{m}(-1)^{m-i} C_m^i C_{m+i}^i \sum_{r=0}^{i} C_i^r 2^{r(k-1)} x^r. \tag{A.9}$$

By substituting Eqs (A.8) and (A.9) into Eq (A.7), we obtain

$$\begin{aligned}\mathfrak{L}\{\psi_{n,m}(x)\} =& e^{-\frac{n-1}{2^{k-1}}s} 2^{\frac{k-1}{2}}\sqrt{2m+1}\sum_{i=0}^{m}(-1)^{m-i} C_m^i C_{m+i}^i 2^{i(k-1)}\mathfrak{L}\{x^i\} \\ &- e^{-\frac{n}{2^{k-1}}s} 2^{\frac{k-1}{2}}\sqrt{2m+1}\sum_{i=0}^{m}(-1)^{m-i} C_m^i C_{m+i}^i \sum_{r=0}^{i} C_i^r 2^{r(k-1)}\mathfrak{L}\{x^r\}.\end{aligned}$$

Since $\mathfrak{L}\{x^r\} = \frac{\Gamma(r+1)}{s^{r+1}}$, we have

$$\begin{aligned}\mathfrak{L}\{\psi_{n,m}(x)\} =& 2^{\frac{k-1}{2}}\sqrt{2m+1}\sum_{i=0}^{m}(-1)^{m-i} C_m^i C_{m+i}^i 2^{i(k-1)}\Gamma(i+1)\frac{e^{-\frac{n-1}{2^{k-1}}s}}{s^{i+1}} \\ &- 2^{\frac{k-1}{2}}\sqrt{2m+1}\sum_{i=0}^{m}(-1)^{m-i} C_m^i C_{m+i}^i \sum_{r=0}^{i} C_i^r 2^{r(k-1)}\Gamma(r+1)\frac{e^{-\frac{n}{2^{k-1}}s}}{s^{r+1}}.\end{aligned}$$

From the definition of $I^{\alpha}\psi_{n,m}(x)$, we get

$$\begin{aligned}\mathfrak{L}\{I^{\alpha}\psi_{n,m}(x)\} =&\mathfrak{L}\{\frac{1}{\Gamma(\alpha)}x^{\alpha-1}\}\mathfrak{L}\{\psi_{n,m}(x)\} = \frac{1}{s^{\alpha}}\mathfrak{L}\{\psi_{n,m}(x)\}\\ =&2^{\frac{k-1}{2}}\sqrt{2m+1}\sum_{i=0}^{m}(-1)^{m-i}C_m^iC_{m+i}^i2^{i(k-1)}\Gamma(i+1)\frac{e^{-\frac{n-1}{2^{k-1}}s}}{s^{i+\alpha+1}}\\ &-2^{\frac{k-1}{2}}\sqrt{2m+1}\sum_{i=0}^{m}(-1)^{m-i}C_m^iC_{m+i}^i\sum_{r=0}^{i}C_i^r2^{r(k-1)}\Gamma(r+1)\frac{e^{-\frac{n}{2^{k-1}}s}}{s^{r+\alpha+1}}.\end{aligned}$$

By taking the inverse Laplace transform and using the delay property of Laplace transform power function $\mathfrak{L}\{(x-a)^n\mu(x-a)\} = e^{-as}\frac{\Gamma(n+1)}{s^{a+1}}$, we obtain

$$\begin{aligned}I^{\alpha}\psi_{n,m}(x) =&2^{\frac{k-1}{2}}\sqrt{2m+1}\sum_{i=0}^{m}(-1)^{m-i}C_m^iC_{m+i}^i2^{i(k-1)}\frac{\Gamma(i+1)}{\Gamma(i+\alpha+1)}(x-\frac{n-1}{2^{k-1}})^{i+\alpha}\mu_{\frac{n-1}{2^{k-1}}}(x)\\ &-2^{\frac{k-1}{2}}\sqrt{2m+1}\sum_{i=0}^{m}(-1)^{m-i}C_m^iC_{m+i}^i\sum_{r=0}^{i}C_i^r2^{r(k-1)}\frac{\Gamma(r+1)}{\Gamma(r+\alpha+1)}(x-\frac{n}{2^{k-1}})^{r+\alpha}\mu_{\frac{n}{2^{k-1}}}(x).\end{aligned}$$

The theorem then follows.□

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