Optical soliton solutions for Lakshmanan-Porsezian-Daniel equation with parabolic law nonlinearity by trial function method

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Abstract: In this paper, the trial function method is used to address the Lakshmanan-Porsezian-Daniel (LPD) equation with parabolic law nonlinearity. Implementing the traveling wave hypothesis reduces the LPD equation to an ordinary differential equation (ODE). From this ODE, many exact solutions, such as kink solitary wave solutions, bell shaped solitary wave solutions, triangular function solutions, periodic function solutions, singular solutions and Jacobian elliptic function solutions, are retrieved. Among them, some solutions are new. By suitable choice of parameters, we also draw 3D surface and 2D graphs of density, contour and level curves of some precise solutions for intuitive investigation.

Keywords: solitons; trial function method; Lakshmanan-Porsezian-Daniel equation
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1. Introduction

Optical soliton dynamics is mainly described by non-linear Schrödinger equations (NLSEs) [1–5]. The primary purposes of research on optical solitons are to develop the telecommunications industry and optimize communications network technology. NLSEs consist of many well-known ones, like the Schrödinger-Hirota equation, Gerdjikov-Ivanov equation, Radhakrishnan-Kundu-Lakshmanan equation, Kundu-Eckhaus equation, Maxwell-Bloch equation, Biswas-Milovic equation, Chen-Lee-Liu equation and so on. Among NLSEs, the Lakshmanan-Porsezian-Daniel (LPD) equation, proposed in the category of Heisenberg spin chains [6–9], governs the transmission of solitons through a variety of waveguides and has been widely studied by many authors [10–12]. It is worth mentioning that the LPD equation, which follows from the so-called AKNS hierarchy and has non-autonomous generalized reductions and integrated models, has gained huge interest recently [13–15]. The studies on optical solitons with LPD have a great effect on modern electronic communication systems and reflect the propagation rules for nonlinear optics.
The trial function method was proposed by Liu [16] for finding certain types of solutions for nonlinear evolution equations (NLEEs). This method has been reliably and effectively used to seek the solitary traveling wave solutions from various NLEE models [17–21]. Generally speaking, although many advanced techniques, such as the undetermined coefficients method [22], the modified simple equation method [23], the improved \( \tan \frac{\phi(n)}{2} \)-expansion method [24], the modified auxiliary equation method [25] and the modified extended direct algebraic method [26], have been adopted to study the solutions of the governing LPD model exhibited in the following, some of its soliton solutions have not been explored. Our purpose is to retrieve new optical soliton solutions for the model hereunder by the trial function method.

In this paper, we consider the dimensionless form of the LPD model in the presence of spatio-temporal dispersion (STD) with full nonlinearity and higher order dispersion [22]:

\[
\begin{align*}
    iu_t + au_{xx} + bu_{xt} + F(|u|^2)u &= \rho u_{xxxx} + \alpha (u_x)^2 u + \beta |u|^2 u_{xx} + \lambda u^2 u_{xx}^* + \delta |u|^4 u. \\
    \text{(1.1)}
\end{align*}
\]

In (1.1), the independent variables \( x \) and \( t \) denote the space and time, respectively, while the independent variable \( u(x,t) \) represents the complex-valued wave profile. The coefficients of group-velocity dispersion and STD are represented by parameters \( a \) and \( b \), respectively. The real-valued functional \( F \), which is decided by the refractive index of the propagation medium, such as optical fiber, provides the general form of nonlinearity, including Kerr law, parabolic law, power law, dual-power law and log law nonlinearities. In particular, \( F(|u|^2)u \) is a \( k \)-times continuously differentiable function [1].

\[
F(|u|^2)u \in \bigcup_{n,m=1}^{\infty} \mathbb{C}^k((-n,n) \times (-m,m) : \mathbb{R}^2).
\]

Then, for the terms on the right hand side of (1.1), \( \sigma \) is the coefficient of fourth-order dispersion; \( \alpha, \beta, \gamma \) and \( \lambda \) indicate the other perturbation terms with nonlinear effects. The two-photon absorption is represented by parameter \( \delta \). For \( b = 0 \) and removing \( F(|u|^2)u \), the model (1.1) becomes the original LPD model [3,4] that is extensively studied. Therefore, we consider the LPD model with functional \( F \) and \( b \neq 0 \).

The current paper consists of the following parts. In Section 2, we will demonstrate the algorithm of the trial function method. In Section 3, Eq (1.1) with parabolic law nonlinearity will be reduced to an ODE via the traveling wave hypothesis. In order to retrieve the exact traveling wave solutions from this ODE, we will employ the trial function methodology united with the complete discrimination system method [27]. In Section 4, kink, periodic, bell shaped and triangle solitons are extracted for specific parameter values and depicted through graphical demonstration. Section 5 is the conclusion of this paper.

2. The trial function method

The basic steps of the trial function method [16] are itemized as follows.

Step 1: Assume equation

\[
N(u, u_t, u_x, u_{xt}, u_{xx}, u_{xxx}, \cdots) = 0
\]

(2.1)
denotes the nonlinear evolution equation (NLEE) for unknown function \( u = u(t, x) \). Implementing complex traveling wave transformation \( u(x, t) = U(\xi)e^{i\phi(x, t)} \), \( \xi = x - ct \), converts (2.1) to an ODE,

\[
M(U, U', U'', \cdots) = 0,
\]

(2.2)

where \( U' = \frac{du}{d\xi} \) and \( M \) denotes the polynomial of \( U \) and its derivatives with respect to \( \xi \).

**Step 2:** In this paper, the simple trial function is considered as

\[
(U')^2 = F(U) = \sum_{i=0}^{K} a_i U^i,
\]

(2.3)

where integer \( K \) is determined by the balance principle, and the coefficients \( a_i(i = 1, \cdots K) \) will be retrieved later. A polynomial equation \( G(U) = 0 \) is obtained by substituting (2.3) into (2.2). By making all the coefficients of \( G(U) \) equal to zero, one can get the algebraic equation system for \( a_i(i = 1, \cdots K) \) and solve it. Then, \( F(U) \) becomes fixed.

**Step 3:** By separating variables and integrating, (2.2) can be reduced to

\[
\pm (\xi - \xi_0) = \int \frac{dU}{\sqrt{F(U)}},
\]

(2.4)

where \( \xi_0 \) is an integral constant. The classification of the roots for polynomial \( F(U) \), which are extracted via the polynomial complete discriminant system method [27], eventually addresses exact traveling wave solutions for Eq (2.1).

It is worth mentioning that the trial equation can be chosen in several forms besides (2.3) to deal with different NLEEs.

### 3. Applications

The assumption is that the exact solutions for Eq (1.1) satisfy the traveling wave hypothesis [1]

\[
u(x, t) = U(\xi)e^{i\zeta(x, t)}, \quad \xi = x - ct,
\]

(3.1)

where \( U \) is the shape of the wave profile with \( c \) representing the wave velocity, and \( \zeta(x, t) = -kx + \omega t + \varphi_0 \) is the phase component. The parameter \( k \) is frequency, \( \omega \) is the wave number, and \( \varphi_0 \) represents the phase constant. By plugging the hypothesis (3.1) into (1.1) and decomposing the real and imaginary parts, we obtain the real part as

\[
\sigma U''' - (a - bc + 6\sigma k^2)U'' + (\omega - \omega bk + ak^2 + \sigma k^4)U - F(U^2)U \\
- (\alpha - \beta + \gamma + \lambda)k^2 U^3 + \delta U^5 + (\alpha + \beta)U(U')^2 + (\gamma + \lambda)U^2 U'' = 0,
\]

(3.2)

while the imaginary portion reads as

\[
(bck - c + b\omega - 2ak - 4\sigma k^3)U' + 2(\alpha + \gamma - \lambda)k U^2 U' + 4\sigma k U''' = 0.
\]

(3.3)

Equating the linearly independent coefficients in (3.3) to zero imposes the constraint conditions

\[
\sigma = 0,
\]

(3.4)
\[\alpha + \gamma - \lambda = 0, \quad (3.5)\]

and the velocity of the soliton is attained as

\[c = \frac{2ak - b\omega}{bk - 1}, \text{ for } bk \neq 1. \quad (3.6)\]

The parabolic law nonlinearity is presented by

\[F(u) = c_1u + c_2u^2, \quad (3.7)\]

where \(c_1\) and \(c_2\) are real parameters. Inserting (3.4) and (3.7) into (3.2) yields

\[(bc - a)U'' + (\omega - \omega bk + ak^2)U - (c_1 U^2 + c_2 U^4)U - (2\alpha - \beta + 2\gamma)k^2 U^3 + \delta U^5 + (\alpha + \beta)U(U')^2 + (\alpha + 2\gamma)U^2 U'' = 0. \quad (3.8)\]

In order to apply the extended trial function algorithm to tackle (3.8), we shall assume that the structure of solutions is in accord with the form

\[(U')^2 = \sum_{i=0}^{n} a_i U^i. \quad (3.9)\]

Balancing the orders between \(U(U')^2\) and \(U^5\) yields \(n = 4\). Then, it is easy to compute

\[U'' = 2a_4 U^3 + \frac{3}{2}a_3 U^2 + a_2 U + \frac{a_1}{2}. \quad (3.10)\]

By substituting (3.9) and (3.10) in (3.8) and solving the obtained algebraic equation system about the coefficients of each power of \(U\), the following results are received:

\[a_4 = \frac{c_2 - \delta}{3\alpha + \beta + 4\gamma}, \quad a_3 = 0, \quad a_2 = \frac{(3\alpha + \beta + 4\gamma)[c_1 + (2\alpha - \beta + 2\gamma)k^2 - 2(bc - a)(c_2 - \delta)]}{(3\alpha + \beta + 4\gamma)(2\alpha + \beta + 2\gamma)}, \quad a_1 = 0, \quad a_0 = \frac{2(bc - a)^2(c_2 - \delta) - (bc - a)(3\alpha + \beta + 4\gamma)[c_1 + (2\alpha - \beta + 2\gamma)k^2]}{(\alpha + \beta)(3\alpha + \beta + 4\gamma)(2\alpha + \beta + 2\gamma)} - \frac{\omega - \omega bk + ak^2}{\alpha + \beta}, \quad (3.11)\]

and (3.9) consequently becomes \((U')^2 = a_4 U^4 + a_2 U^2 + a_0\). Implementing the transformation

\[U = \pm \sqrt{(4a_4)^{-\frac{1}{2}} \phi}, \quad \xi_1 = (4a_4)^{\frac{1}{4}} \xi, \quad q = 4a_2(4a_4)^{-\frac{1}{4}}, \quad r = 4a_0(4a_4)^{-\frac{1}{4}}, \quad (3.12)\]

reduces the Eq (3.9) to

\[(\phi \xi_1)^2 = \phi(\phi^2 + q\phi + r). \quad (3.13)\]

Then, after separating variables and integrating, (3.13) becomes

\[\pm (\xi_1 - \xi_0) = \int \frac{d\phi}{\sqrt{\phi F(\phi)}} = \int \frac{d\phi}{\sqrt{\phi(\phi^2 + q\phi + r)}}, \quad (3.14)\]
where \( \xi_0 \), an arbitrary constant, is the integration constant.

Now, the second-order complete discrimination system

\[
F(\phi) = \phi^2 + q\phi + r, \quad \Delta = q^2 - 4r,
\]

is used to classify the roots of \( F(U) \). Four cases of the solutions for Eq (3.14) are studied, as follows:

**Case 1:** \( \Delta = 0 \), so that \( a_2^2 = 4a_0a_1 \). In this case, \( \phi > 0 \) is valid, and (3.14) is reduced to

\[
\pm (\xi_1 - \xi_0) = \int \frac{d\phi}{\sqrt{\phi + \frac{q}{2}}},
\]

When \( q < 0 \), the solutions of Eq (1.1) are obtained as

\[
u_1(x,t) = \pm \sqrt{\frac{2(bc-a)(c_2-\delta)-(3\alpha+\beta+4\gamma)[c_1+(2\alpha-\beta+2\gamma)k^2]}{2(2\alpha+\beta+2\gamma)(c_2-\delta)}} \times \tanh \left[ \sqrt{\frac{2(bc-a)(c_2-\delta)-(3\alpha+\beta+4\gamma)[c_1+(2\alpha-\beta+2\gamma)k^2]}{2(2\alpha+\beta+2\gamma)(c_2-\delta)}} \left( x - \frac{2ak - b\omega}{bk-1} t - \xi_0 \right) \right].
\]

\[
u_2(x,t) = \pm \sqrt{\frac{2(bc-a)(c_2-\delta)-(3\alpha+\beta+4\gamma)[c_1+(2\alpha-\beta+2\gamma)k^2]}{2(2\alpha+\beta+2\gamma)(c_2-\delta)}} \times \coth \left[ \sqrt{\frac{2(bc-a)(c_2-\delta)-(3\alpha+\beta+4\gamma)[c_1+(2\alpha-\beta+2\gamma)k^2]}{2(2\alpha+\beta+2\gamma)(c_2-\delta)}} \left( x - \frac{2ak - b\omega}{bk-1} t - \xi_0 \right) \right].
\]

When \( q > 0 \), the solutions of Eq (1.1) are obtained as

\[
u_3(x,t) = \pm \sqrt{\frac{(3\alpha+\beta+4\gamma)[c_1+(2\alpha-\beta+2\gamma)k^2]-2(bc-a)(c_2-\delta)}{2(2\alpha+\beta+2\gamma)(c_2-\delta)}} \times \tan \left[ \sqrt{\frac{(3\alpha+\beta+4\gamma)[c_1+(2\alpha-\beta+2\gamma)k^2]-2(bc-a)(c_2-\delta)}{2(2\alpha+\beta+2\gamma)(c_2-\delta)}} \left( x - \frac{2ak - b\omega}{bk-1} t - \xi_0 \right) \right].
\]

When \( p = 0 \), so that solutions will appear for \( a_2 = a_0 = 0 \), the solutions of Eq (1.1) are

\[
u_4(x,t) = \pm \sqrt{\frac{3\alpha+\beta+4\gamma}{4(c_2-\delta)}} \left( x - \frac{2ak - b\omega}{bk-1} t - \xi_0 \right)^{-1} \times e^{i(-kx + ut + \varphi_0)}.
\]

**Case 2:** For \( \Delta > 0 \) and \( r = 0 \) \( (a_0 = 0) \), Eq (3.14) is reduced to

\[
\pm (\xi_1 - \xi_0) = \int \frac{d\phi}{\phi \sqrt{\phi + q}}.
\]

If \( q > 0 \), provided \( -q < \phi < 0 \) is valid, the solutions of Eq (1.1) are procured as
\[
    u_5(x, t) = \pm \sqrt{\frac{2(bc - a)(c_2 - \delta) - (3\alpha + \beta + 4\gamma)(c_1 + (2\alpha - \beta + 2\gamma)k^2)}{(2\alpha + \beta + 2\gamma)(c_2 - \delta)}} e^{i(-kx + \omega t + \phi_0)} \tag{3.22}
\]
\[
    \times \text{sech}\left[ \frac{(3\alpha + \beta + 4\gamma)[c_1 + (2\alpha - \beta + 2\gamma)k^2] - 2(bc - a)(c_2 - \delta)}{(3\alpha + \beta + 4\gamma)(2\alpha + \beta + 2\gamma)} \left( x - \frac{2ak - b\omega}{bk - 1} t - \xi_0 \right) \right]
\]

and when \( \phi > 0 \), the solutions of Eq (1.1) are

\[
    u_6(x, t) = \pm \sqrt{\frac{2(bc - a)(c_2 - \delta) - (3\alpha + \beta + 4\gamma)(c_1 + (2\alpha - \beta + 2\gamma)k^2)}{(2\alpha + \beta + 2\gamma)(c_2 - \delta)}} e^{i(-kx + \omega t + \phi_0)} \tag{3.23}
\]
\[
    \times \text{csch}\left[ \frac{(3\alpha + \beta + 4\gamma)[c_1 + (2\alpha - \beta + 2\gamma)k^2] - 2(bc - a)(c_2 - \delta)}{(3\alpha + \beta + 4\gamma)(2\alpha + \beta + 2\gamma)} \left( x - \frac{2ak - b\omega}{bk - 1} t - \xi_0 \right) \right].
\]

If \( q < 0 \), the solutions of Eq (1.1) are obtained as

\[
    u_7(x, t) = \pm \sqrt{\frac{2(bc - a)(c_2 - \delta) - (3\alpha + \beta + 4\gamma)(c_1 + (2\alpha - \beta + 2\gamma)k^2)}{(2\alpha + \beta + 2\gamma)(c_2 - \delta)}} e^{i(-kx + \omega t + \phi_0)} \tag{3.24}
\]
\[
    \times \sec\left[ \frac{2(bc - a)(c_2 - \delta) - (3\alpha + \beta + 4\gamma)(c_1 + (2\alpha - \beta + 2\gamma)k^2)}{(3\alpha + \beta + 4\gamma)(2\alpha + \beta + 2\gamma)} \left( x - \frac{2ak - b\omega}{bk - 1} t - \xi_0 \right) \right].
\]

**Case 3:** \( \Delta > 0 \) and \( r \neq 0 \), which leads to \( F(U) \) having two distinct roots. For convenience, by assuming \( \lambda_1 < \lambda_2 < \lambda_3 \), satisfying that one of these equals 0, Eq (3.14) can be denoted by

\[
    \pm (\xi_1 - \xi_0) = \int \frac{d\phi}{\sqrt{(\phi - \lambda_1)(\phi - \lambda_2)(\phi - \lambda_3)}}. \tag{3.25}
\]

When \( \phi > \lambda_3 \), by using the transformation \( \phi = \frac{-\lambda_1 \sin^2 \psi + \lambda_1}{\cos \psi} \), we can attain the solutions of Eq (1.1), as

\[
    u_8(x, t) = \pm \sqrt{\frac{4c_2 - 4\delta}{3\alpha + \beta + 4\gamma}} \frac{1}{\lambda_3 - \lambda_2} \frac{1}{\lambda_2} \frac{1}{m} \frac{1}{\lambda_3} \frac{1}{m} \times e^{i(-kx + \omega t + \phi_0)}. \tag{3.26}
\]

When \( \lambda_1 < \phi < \lambda_2 \), implementing the transformation \( \phi = \lambda_1 + (\lambda_2 - \lambda_1) \sin^2 \psi \), the Jacobian elliptic function solutions of Eq (1.1) are given by

\[
    u_9(x, t) = \pm e^{i(-kx + \omega t + \phi_0)} \tag{3.27}
\]
\[
\times \left[ \frac{4c_2 - 4\delta}{3\alpha + \beta + 4\gamma} \right] ^{1/4} \left[ \lambda_1 + (\lambda_2 - \lambda_1) \sin^2 \left( \frac{4c_2 - 4\delta}{3\alpha + \beta + 4\gamma} \right) \left( x - \frac{2ak - b\omega}{bk - 1} t - \xi_0 \right) \right],
\]

where \( m^2 = \frac{d_2 - \lambda_1}{d_3 - \lambda_1} \).
**Case 4:** $\Delta < 0$. Implementing the transformation $\phi = \sqrt{r} \tan^2 \frac{\psi}{2}$ converts Eq (3.14) to

$$\pm (\xi_1 - \xi_0) = r^{-\frac{1}{2}} \int \frac{d\psi}{\sqrt{1 - m^2 \sin^2 \psi}},$$

(3.28)

where $m^2 = \frac{1}{2}(1 - \frac{q}{2\sqrt{r}})$. In this case, the Jacobian elliptic function solutions of Eq (1.1) are given by

$$u_{10}(x, y) = \pm \left\{ \frac{2(bc - a)^2(c_2 - \delta)^2 - (bc - a)(3\alpha + \beta + 4\gamma)(c_1 + (2\alpha - \beta + 2\gamma)k^2)}{(\alpha + \beta)(2\alpha + \beta + 2\gamma)} \right\}^{\frac{1}{4}} e^{-kx + \omega t + \theta} \times \left[ 1 + \sum_{2((bc-a)(3\alpha + \beta + 4\gamma)k^2)}^{2} \frac{2}{(\alpha + \beta)(3\alpha + \beta + 4\gamma)(2\alpha + \beta + 2\gamma)} \right]^{\frac{1}{4}} (x - \frac{2ak - bk - 1}{2k}t - \xi_0), m \right\}^{\frac{1}{4}}. \tag{3.29}
$$

4. Graphical illustrations

This section includes the graphical presentations for some obtained solutions. We select suitable values of parameters to simulate exact solutions of Eq (1.1) and choose $+$ instead of $\pm$ to plot all the graphs using Maple software. The envelope of kink solitary wave solution $u_1$, periodic function solution $u_3$, bell shaped solitary wave solution $u_5$ and triangle analytical solution $u_7$ are represented in Figures 1(a)–4(a), respectively. Contour plots and density plots of these solutions are depicted in Figures 1(b), 1(c)–4(b), 4(c). 2D plots in Figures 1(d)–4(d) capture the level curves at different times $t = 10, 20, 30$. In recently published works, researchers mainly describe the structures for the absolute values of the obtained solutions and illustrated some interesting results. In this paper, we present the envelope of the obtained solutions to help in understanding properties of the LPD model, which describes many phenomena in nonlinear science.

(a) 3D surface plot of (3.17)  (b) Contour plot of (3.17)  (c) Density plot of (3.17)  (d) 2D plot at $t = 10, 20, 30$

**Figure 1.** The profile of kink soliton solution $U_1$ with $k = 4$, $\omega = 3$ and $c = \frac{1}{3}$. The parameters are: $a = \frac{1}{2}$, $b = 1$, $c_1 = -51 \frac{5}{12}$, $c_2 = 3$, $\alpha = \beta = \gamma = \delta = 1$ and $\xi_0 = 0$. 

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Figure 2. The profile of periodic soliton solution $U_3$ with $k = 4$, $\omega = 3$ and $c = \frac{1}{3}$. The parameters are: $a = \frac{1}{2}$, $b = 1$, $c_1 = -44\frac{1}{3}$, $c_2 = 3$, $\alpha = \beta = \gamma = \delta = 1$ and $\xi_0 = 0$.

Figure 3. The profile of bell shaped soliton solution $U_5$ with $k = 2$, $\omega = \frac{7}{4}$ and $c = \frac{1}{4}$. The parameters are: $a = \frac{1}{2}$, $b = 1$, $c_1 = -6\frac{31}{32}$, $c_2 = -1$, $\alpha = \beta = \gamma = \delta = 1$ and $\xi_0 = 0$.

Figure 4. The profile of triangle function solution $U_7$ with $k = 2$, $\omega = 1$ and $c = 1$. The parameters are: $a = \frac{1}{2}$, $b = 1$, $c_1 = -21\frac{3}{4}$, $c_2 = 3$, $\alpha = \beta = \gamma = \delta = 1$ and $\xi_0 = 0$.

5. Conclusions

In this work, the Lakshmanan-Porsezian-Daniel equation with parabolic law nonlinearity is studied by applying the trial function methodology and complete discrimination system for polynomials method. Many new complex exact solutions of the LPD model are obtained, such as kink soliton
solutions, bell shaped soliton solutions, triangular function solutions, periodic function solutions, singular solutions and Jacobian function solutions, which are not obtained by other techniques. In [23–25], the solutions are obtained by setting the constrain conditions $\alpha + \beta = 0$ and $\gamma + \lambda = 0$ for the coefficients in Eq (1.1). Comparing our results with the previous works, we address the model without the constrain conditions above and attain some new results. Also, Figures 1–4 depict the solutions in (3.17), (3.19), (3.22) and (3.24) with suitable values of parameters, which can be expected to reveal more dynamic behaviors and physical properties of the LPD model. We think the methodology used in this work is able to solve more complicated and various nonlinear partial differential equations.

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Conflict of interest

The authors declare no conflicts of interest.

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