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Research article

On one coefficient inverse boundary value problem for a linear pseudoparabolic equation of the fourth order

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Abstract: In the present work, we consider an inverse boundary value problem for a fourth order pseudo parabolic equation with periodic and integral condition . Using analytical and operator-theoretic methods, as well as the Fourier method, the existence and uniqueness of the classical solution of this problem is proved. By the contraction mapping principle is formulated as an auxiliary inverse problem which, in turn, is reduced to the operator equation in a specified Banach space using the method of spectral analysis.

Keywords: inverse problems; nonlocal integral condition; classical solution
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1. Introduction

Modern problems of natural science lead to the need to generalize the classical problems of mathematical physics, as well as to the formulation of qualitatively new problems, which include non-local problems for differential equations. Among nonlocal problems, problems with integral conditions are of great interest. Integral conditions are encountered in the study of physical phenomena in the case when the boundary of the process flow region is inaccessible for direct measurements. Inverse problems arise in various fields of human activity, such as seismology, mineral exploration, biology, medical visualization, computed tomography, earth remote sensing, spectral analysis, nondestructive control, etc. Various inverse problems for certain types of partial differential equations have been studied in many works. A more detailed bibliography and a classification of problems are found in [1–5]. Inverse problems for one-dimensional pseudo-parabolic equations of third-order were studied in [6]. The existence and uniqueness of the solution of the inverse problem for the third order pseudoparabolic equation with integral over-determination condition is studied in [7]. Khompysh

[8] investigated the reconstruction of unknown coefficient in pseudo-parabolic inverse problem with the integral over determination condition and studied the uniqueness and existence of solution by means of method of successive approximations. Studies of wave propagation in cold plasma and magnetohydrodynamics also reduce to the partial differential equations of fourth-order. To the study of nonlocal boundary value problems (including integral conditions) for partial differential equations of the fourth-order are devoted large number of works, see, for example, [9,10]. It should be noted that boundary value problems with integral conditions are of particular interest. From physical considerations, the integral conditions are completely natural, and they arise in mathematical modelling in cases where it is impossible to obtain information about the process occurring at the boundary of the region of its flow using direct measurements or when it is possible to measure only some averaged (integral) characteristics of the desired quantity.

In this article, we study the an inverse boundary value problem for a fourth order pseudo parabolic equation with periodic and integral condition to identify the time-dependent coefficients along with the solution function theoretically, i.e. existence and uniqueness.

Statement of the problem and its reduction to an equivalent problem. In the domain $D_T = \{(x,t) : 0 \le x \le 1,\ 0 \le t \le T\}$, we consider an inverse boundary value problem of recovering the timewise dependent coefficients $p(t)$ in the pseudo-parabolic equation of the fourth-order

$$u_t(x,t) - bu_{txx}(x,t) + a(t)u_{xxxx}(x,t) = p(t)u(x,t) + f(x,t) \tag{1.1}$$

with the initial condition

$$u(x,0) + \delta u(x,T) = \varphi(x) \quad (0 \le x \le 1), \tag{1.2}$$

boundary conditions

$$u(0,t) = u(1,t), u_x(0,t) = u_x(1,t), u_{xx}(0,t) = u_{xx}(1,t) \quad (0 \le t \le T), \tag{1.3}$$

nonlocal integral condition

$$\int_0^1 u(x,t)dx = 0 (0 \le t \le T) \tag{1.4}$$

and with an additional condition

$$u(0,t) = \int_0^t \gamma(\tau)u(1,\tau)d\tau + h(t) \quad (0 \le t \le T), \tag{1.5}$$

where $b > 0$, $\delta \ge 0$-given numbers, $a(t) > 0,\ f(x,t),\ \varphi(x),\ \gamma(\tau), h(t)$ -given functions, $u(x,t)$ and $p(t)$ - required functions.

Denote

$$\tilde{C}^{4,1}(D_T) = \left\{u(x,t) : u(x,t) \in C^{2,1}(D_T), u_{txx}, u_{xxxx} \in C(D_T)\right\}.$$

Definition. By the classical solution of the inverse boundary value problem (1.1)-(1.5)we mean the pair $\{u(x,t), p(t)\}$ functions $u(x,t) \in \tilde{C}^{4,1}(D_T)$, $p(t) \in C[0,T]$ satisfying equation (1.1) in D_T, condition (1.2) in [0,1] and conditions (1.3)-(1.5) in [0,T].

Theorem 1. Let be $b > 0,\ \delta \ge 0,\ \ \varphi(x) \in C[0,1],\ f(x,t) \in C(D_T),\ \int_0^1 f(x,t)dx = 0,\ 0 < a(t) \in C[0,T],\ h(t) \in C^1[0,T], h(t) \ne 0\ (0 \le t \le T), \gamma(t) \in C[0,T], \delta\gamma(t) = 0\ (0 \le t \le T)$ and

$$\int_0^1 \varphi(x)dx = 0, \varphi(0) = h(0) + \delta h(T).$$

Then the problem of finding a solution to problem (1.1)-(1.5) is equivalent to the problem of determining the functions $u(x,t) \in \tilde{C}^{4,1}(D_T)$ and $p(t) \in C[0,T]$, from (1.1)-(1.3) and

$$u_{xxx}(0,t) = u_{xxx}(1,t) \quad (0 \le t \le T), \tag{1.6}$$

$$\gamma(t)u(1,t) + h'(t) - bu_{txx}(0,t) + a(t)u_{xxxx}(0,t) =$$
$$= p(t)\left(\int_0^t \gamma(\tau)u(1,\tau)d\tau + h(t)\right) + f(0,t) \ \ (0 \le t \le T). \tag{1.7}$$

Proof. Let be $\{u(x,t),\ p(t)\}$ is a classical solution to problem (1.1)-(1.5). Integrating equation (1.1) with respect to x from 0 to 1, we get:

$$\frac{d}{dt}\int_0^1 u(x,t)dx - b(u_{tx}(1,t) - u_{tx}(0,t)) + a(t)(u_{xxx}(1,t) - u_{xxx}(0,t)) =$$

$$= p(t)\int_0^1 u(x,t)dx + \int_0^1 f(x,t)dx \ \ (0 \le t \le T). \tag{1.8}$$

Assuming that $\int_0^1 f(x,t)dx = 0$, taking into account (1.3) and (1.4), we arrive at the fulfillment of (1.6).

Further, considering $h(t) \in C^1[0,T]$ and differentiating with respect to t (1.5), we get:

$$u_t(0,t) = \gamma(t)u(1,t) + h'(t) \quad (0 \le t \le T) \tag{1.9}$$

Substituting $x = 0$ into equation (1.1), we have:

$$u_t(0,t) - bu_{txx}(0,t) + a(t)u_{xxxx}(0,t) = p(t)u(0,t) + f(0,t) \ \ (0 \le t \le T). \tag{1.10}$$

Now, suppose that $\{u(x,t),\ p(t)\}$ is a solution to problem (1.1)-(1.3), (1.6), (1.7). Then from (1.8), taking into account (1.3) and (1.6), we find:

$$\frac{d}{dt}\int_0^1 u(x,t)dx - p(t)\int_0^1 u(x,t)dx = 0 \ \ (0 \le t \le T). \tag{1.11}$$

Due to (1.2) and $\int_0^1 \varphi(x)dx = 0$, it's obvious that

$$\int_0^1 u(x,0)dx + \delta\int_0^1 u(x,T)dx = \int_0^1 \varphi(x)dx = 0. \tag{1.12}$$

Obviously, the general solution(1.11) has the form:

$$\int_0^1 u(x,t)dx = ce^{-\int_0^t p(\tau)d\tau} \quad (0 \le t \le T). \tag{1.13}$$

From here, taking into account (1.12), we obtain:

$$\int_0^1 u(x,0)dx + \delta\int_0^1 u(x,T)dx = c(1 + \delta e^{-\int_0^T p(\tau)d\tau}) = 0. \tag{1.14}$$

By virtue of $\delta \geq 0$, from (1.14) we get that $c = 0$,and substituting into (1.13) we conclude, that $\int_0^1 u(x,t)dx = 0 \ \ (0 \leq t \leq T)$. Therefore, condition (1.4) is also satisfied.
Further, from (1.7) and (1.10), we obtain:

$$\frac{d}{dt}\left[u(0,t) - \left(\int_0^t \gamma(\tau)u(1,\tau)d\tau + h(t)\right)\right] =$$

$$= p(t)\left[u(0,t) - \left(\int_0^t \gamma(\tau)u(1,\tau)d\tau + h(t)\right)\right] \ (0 \leq t \leq T). \tag{1.15}$$

Let introduce the notation:

$$y(t) \equiv u(0,t) - \left(\int_0^t \gamma(\tau)u(1,\tau)d\tau + h(t)\right) \ (0 \leq t \leq T) \tag{1.16}$$

and rewrite the last relation in the form:

$$y'(t) + p(t)y(t) = 0 \ \ (0 \leq t \leq T). \tag{1.17}$$

From (1.16), taking into account (1.2), $\delta\gamma(t) = 0 \ (0 \leq t \leq T)$ and $\varphi(0) = h(0) + \delta h(T)$, it is easy to see that

$$y(0) + \delta y(T) = \ u(0,0) - h(0) + \delta\left[u(0,T) - \left(\int_0^T \gamma(\tau)u(1,\tau)d\tau + h(T)\right)\right] = u(0,0)+$$

$$+ \delta u(0,T) - (h(0) + \delta h(T)\,) - \delta\int_0^T \gamma(\tau)u(1,\tau)d\tau = \varphi(0) - (h(0) + \delta h(T)) = 0. \tag{1.18}$$

Obviously, the general solution (1.17) has the form:

$$y(t) = ce^{-\int_0^t p(\tau)d\tau} \quad (0 \leq t \leq T). \tag{1.19}$$

From here, taking into account (1.18), we obtain:

$$y(0) + \delta y(T) = c(1 + \delta e^{-\int_0^T \frac{a_0(\tau)}{a_1(\tau)}d\tau}) = 0. \tag{1.20}$$

By virtue of $\delta \geq 0$, from (1.20) we get that $c = 0$, and substituting into (1.19) we conclude that $y(t) = 0 \ \ (0 \leq t \leq T)$. Therefore, from (1.16) it is clear that the condition (1.5). The theorem has been proven.

2. The existence and uniqueness of the classical solution of the inverse boundary value problem.

It is known [5] that the system

$$1, \cos\lambda_1 x, \sin\lambda_1 x, ..., \cos\lambda_k x, \sin\lambda_k x, ... \tag{2.1}$$

forms the basis of $L_2(0,1)$, where $\lambda_k = 2k\pi \ \ (k = 0, 1, ...)$.

Since system (2.1) forms a basis in $L_2(0,1)$, it is obvious that for each solution $\{u(x,t), a(t)\}$ problem (1.1)–(1.3), (1.6),(1.7):

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t)\sin\lambda_k x \ \ (\lambda_k = 2\pi k), \tag{2.2}$$

where

$$u_{10}(t) = \int_0^1 u(x,t)dx \ \ , u_{1k}(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx \ \ (k = 1,2,...)\,,$$

$$u_{2k}(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx \ \ (k = 1,2,...).$$

Applying the formal scheme of the Fourier method, to determine the desired coefficients $u_{1k}(t)$ $(k = 0,1,...)$ and $u_{2k}(t)$ $(k = 1,2,...)$ functions $u(x,t)$ from (1.1) and (1.2) we get:

$$u''_{10}(t) = F_{10}(t;u,p) \ \ (0 \le t \le T), \tag{2.3}$$

$$(1 + b\lambda_k^2)u'_{ik}(t) + a(t)\lambda_k^4 u_{ik}(t) = F_{ik}(t;u,p) \ (i = 1,2;\ 0 \le t \le T;\ \ k = 1,2,...), \tag{2.4}$$

$$u_{10}(0) + \delta u_{10}(T) = \varphi_{10} \ \ , \tag{2.5}$$

$$u_{ik}(0) + \delta u_{ik}(T) = \varphi_{ik} \ \ (\ i = 1,2;\ k = 1,2,...), \tag{2.6}$$

where

$$F_{1k}(t;u,a,b) = p(t)u_{1k}(t) + f_{1k}(t) \ \ (k = 0,1,...)\ ,$$

$$f_{10}(t) = \int_0^1 f(x,t)dx \ , f_{1k}(t) = 2\int_0^1 f(x,t)\cos\lambda_k x dx \ (k = 1,2,...)\ ,$$

$$\varphi_{10} = \int_0^1 \varphi(x)dx, \qquad \varphi_{1k} = 2\int_0^1 \varphi(x)\cos\lambda_k x dx \qquad (k = 1,2,...)\,,$$

$$F_{2k}(t;u,a,b) = p(t)u_{2k}(t) + f_{2k}(t),$$

$$f_{2k}(t) = 2\int_0^1 f(x,t)\sin\lambda_k x dx \quad (k = 1,2,...)\ ,\varphi_{2k} = 2\int_0^1 \varphi(x)\sin\lambda_k x dx \quad (k = 1,2,...)\,.$$

Solving problem (2.3)-(2.6), we find:

$$u_{10}(t) = (1+\delta)^{-1}\left(\varphi_{10} - \delta\int_0^T F_0(\tau;u,p)d\tau\right) + \int_0^t F_{10}(\tau;u,p)d\tau \ (0 \le t \le T), \tag{2.7}$$

$$u_{ik}(t) = \frac{e^{-\int_0^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}{1+\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}dss}}\varphi_{ik} + \frac{1}{1+b\lambda_k^2}\int_0^t F_{ik}(\tau;u,p)e^{-\int_\tau^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}d\tau -$$

$$-\frac{\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}{1+\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}\frac{1}{1+b\lambda_k^2}\int_0^T F_{ik}(\tau;u,p)e^{-\int_\tau^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}d\tau \ \ (i = 1,2; 0 \le t \le T; k = 1,2,...). \tag{2.8}$$

After substituting the expression $u_{1k}(t)$ $(k = 0, 1, ...)$, $u_{2k}(t)$ $(k = 1, 2, ...)$ in (2.2), to define a component $u(x,t)$ solution of problem (1.1)-(1.3), (1.6), (1.7), we obtain:

$$u(x,t) = (1+\delta)^{-1}\left(\varphi_0 - \delta\int_0^T F_0(\tau;u,p)d\tau\right) + \int_0^t F_0(\tau;u,p)d\tau +$$

$$+\sum_{k=1}^{\infty}\left\{\frac{e^{-\int_0^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}{1+\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}dss}}\varphi_{1k}\;{}_k + \frac{1}{1+b\lambda_k^2}\int_0^t F_{1k}(\tau;u,p)e^{-\int_\tau^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}d\tau - \right.$$

$$\left. - \frac{\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}{1+\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}\frac{1}{1+b\lambda_k^2}\int_0^T F_{1k}(\tau;u,p)e^{-\int_\tau^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}d\tau\right\}\cos\lambda_k x +$$

$$+\sum_{k=1}^{\infty}\left\{\frac{e^{-\int_0^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}{1+\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}dss}}\varphi_{2k}\;{}_k + \frac{1}{1+b\lambda_k^2}\int_0^t F_{2k}(\tau;u,p)e^{-\int_\tau^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}d\tau - \right.$$

$$\left. - \frac{\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}{1+\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}\frac{1}{1+b\lambda_k^2}\int_0^T F_{2k}(\tau;u,p)e^{-\int_\tau^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}d\tau\right\}\sin\lambda_k x. \tag{2.9}$$

Now from (1.7), taking into account (2.2), we have:

$$p(t) = [h(t)]^{-1}\left\{h'(t) - f(0,t) + \gamma(t)u_{10}(t) - p(t)\int_0^t \gamma(\tau)u_{10}(\tau)d\tau + \right.$$

$$\left. + \sum_{k=1}^{\infty}\left(b\lambda_k^2 u'_{1k}(t) + a(t)\lambda_k^4 u_{1k}(t) + \gamma(t)u_{1k}(t) - p(t)\int_0^t \gamma(\tau)u_{1k}(\tau)d\tau\right).\right. \tag{2.10}$$

Further, from (2.4), taking into account (2.8), we obtain:

$$b\lambda_k^2 u'_{1k}(t) + a(t)\lambda_k^4 u_{1k}(t) + \gamma(t)u_{1k}(t) = F_{1k}(t;u,p) - u'_{1k}(t) + \gamma(t)u_{1k}(t) =$$

$$= \frac{b\lambda_k^2}{1+b\lambda_k^2}F_{1k}(t;u,p) + \left(\frac{a(t)\lambda_k^4}{1+b\lambda_k^2} + \gamma(t)\right)u_{1k}(t) =$$

$$= \frac{b\lambda_k^2}{1+b\lambda_k^2}F_k(t;u,p) + \left(\frac{a(t)\lambda_k^4}{1+b\lambda_k^2} + \gamma(t)\right)\left[\frac{e^{-\int_0^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}{1+\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}dss}}\varphi_{1k} + \right.$$

$$+\frac{1}{1+b\lambda_k^2}\int_0^t F_{1k}(\tau;u,p)e^{-\int_\tau^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}d\tau -$$

$$\left. - \frac{\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}{1+\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}\frac{1}{1+b\lambda_k^2}\int_0^T F_{1k}(\tau;u,p)e^{-\int_\tau^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}d\tau\right] \quad (0 \le t \le T; k = 1,2,...). \tag{2.11}$$

$$p(t) = [h(t)]^{-1}\left\{h'(t) - f(0,t) + \right.$$

$$+\gamma(t))\left[(1+\delta)^{-1}\left(\varphi_{10} - \delta\int_0^T F_0(\tau;u,p)d\tau\right) + \int_0^t F_{10}(\tau;u,p)d\tau\right] -$$

$$-p(t)\int_0^t \gamma(\tau)u_{10}(\tau)d\tau + \sum_{k=1}^{\infty}\left[\frac{b\lambda_k^2}{1+b\lambda_k^2}F_{1k}(t;u,p) + \right.$$

$$+\left(\frac{a(t)\lambda_k^4}{1+b\lambda_k^2} + \gamma(t)\right)\left[\frac{e^{-\int_0^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}{1+\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}dss}}\varphi_{1k} + \frac{1}{1+b\lambda_k^2}\int_0^t F_{1k}(\tau;u,p)e^{-\int_\tau^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}d\tau - \right.$$

$$+\frac{1}{1+b\lambda_k^2}\int_0^t F_{1k}(\tau;u,p)e^{-\int_\tau^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}d\tau -$$

$$\left. - \frac{\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}{1+\delta e^{-\int_0^T \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}}\frac{1}{1+b\lambda_k^2}\int_0^T F_{1k}(\tau;u,p)e^{-\int_\tau^t \frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}d\tau\right] +$$

$$\left.\left. +p(t)\int_0^t \gamma(\tau)u_{1k}(\tau)d\tau\right]\right\}. \tag{2.12}$$

Thus, the solution of problem (1.1)–(1.3), (1.6), (1.7)is reduced to the solution of system (2.9), (2.12) with respect to unknown functions $u(x,t)$ and $p(t)$.

To study the question of the uniqueness of the solution of problem (1.1)–(1.3), (1.6), (1.7) the following plays an important role.

Lemma 1. If $\{u(x,t),\ p(t)\}$-any solution of problem (1.1)–(1.3), (1.6), (1.7), then the functions

$$u_{10}(t) = \int_0^1 u(x,t)dx\ \ , u_{1k}(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx\ \ (k = 1,2,...),$$

$$u_{2k}(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx\ \ (k = 1,2,...)$$

satisfy the system consisting of equations (27), (28) on $[0,T]$.

It is obvious that if $u_{10}(t) = \int_0^1 u(x,t)dx$, $u_{1k}(t) = 2\int_0^1 u(x,t)\cos\lambda_k x dx\ \ (k = 1,2,...)$, $u_{2k}(t) = 2\int_0^1 u(x,t)\sin\lambda_k x dx\ \ (k = 1,2,...)$ is a solution to system (2.7), (2.8), then the pair $\{u(x,t), p(t)\}$ functions $u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t)\cos\lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t)\sin\lambda_k x\ \ (\lambda_k = 2\pi k)$ and $p(t)$ is a solution to system (2.9), (2.12).

Consequence. Let system (29), (32) have a unique solution. Then problem (1.1)–(1.3), (1.6), (1.7) cannot have more than one solution, i.e. if problem (1.1)-(1.3), (1.6), (1.7) has a solution, then it is unique.

In order to study the problem (1.1)–(1.3), (1.6), (1.7) consider the following spaces.

Denote by $B^{\alpha}_{2,T}$ [6] the set of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_{1k}(t)\cos\lambda_k x+\sum_{k=1}^{\infty}u_{2k}(t)\sin\lambda_k x\ \ (\lambda_k=2\pi k),$$

considered in D_T, where each of the functions $u_{1k}(t)\ \ (k=0,\ 1,...)$, $u_{2k}(t)\ \ (k=1,2,...)$ continuous on $[0,T]$ and

$$J(u)=\|u_{10}(t)\|_{C[0,T]}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^{\alpha}\|u_{1k}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^{\alpha}\|u_{2k}(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}<+\infty,$$

$\alpha\geq 0$. We define the norm in this set as follows:

$$\|u(x,t)\|_{B^{\alpha}_{2,T}}=J(u).$$

Through E^{α}_{T} denote the space $B^{\alpha}_{2,T}\times C[0,T]$ vector - functions $z(x,t)=\{u(x,t),\ p(t)\}$ with norm

$$\|z(x,t)\|_{E^{\alpha}_{T}}=\|u(x,t)\|_{B^{\alpha}_{2,T}}+\|p(t)\|_{C[0,T]}\,.$$

It is known that $B^{\alpha}_{2,T}$ and E^{α}_{T} are Banach spaces.
Now consider in space E^{5}_{T} operator

$$\Phi(u,p)=\{\Phi_1(u,p),\Phi_2(u,p)\}\ ,$$

operator

$$\Phi_1(u,p))=\tilde{u}(x,t)\equiv\sum_{k=0}^{\infty}\tilde{u}_{1k}(t)\cos\lambda_k x+\sum_{k=1}^{\infty}\tilde{u}_{2k}(t)\sin\lambda_k x\ ,\Phi_2(u,p)=\tilde{p}(t),$$

$\tilde{u}_{10}(t),\tilde{u}_{ik}(t)\ \ (i=1,2;\ k=1,2,...),\ \tilde{p}(t)$ are equal to the right-hand sides of (2.7), (2.8) and (2.12), respectively.

It is easy to see that

$$1+b\lambda_k^2>b\lambda_k^2,\quad 1+\delta\geq 1,.1+\delta e^{-\int_0^T\frac{a(s)\lambda_k^4}{1+b\lambda_k^2}ds}\geq 1.$$

Then, we have:

$$\|\tilde{u}_0(t)\|_{C[0,T]}\leq|\varphi_{10}|+(1+\delta)\sqrt{T}\left(\int_0^T|f_{10}(\tau)|^2\,d\tau\right)^{\frac{1}{2}}+(1+\delta)T\,\|p(t)\|_{C[0,T]}\,\|u_{10}(t)\|_{C[0,T]}\ ,\tag{2.13}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\leq\sqrt{3}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_{ik}|)^2\right)^{\frac{1}{2}}+\frac{\sqrt{3}(1+\delta)}{b}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3\,|f_{ik}(\tau)|)^2d\tau\right)^{\frac{1}{2}}+$$

$$+\frac{\sqrt{3}(1+\delta)}{b}T\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{ik}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\ \ (i=1,2),\tag{2.14}$$

$$\|\tilde{p}(t)\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\|h'(t)-f(0,t)\|_{C[0,T]}+\right.$$

$$+\|\gamma(t)\|_{C[0,T]}\left[|\varphi_0|+(1+\delta)\sqrt{T}\left(\int_0^T|f_0(\tau)|^2\,d\tau\right)^{\frac{1}{2}}+(1+\delta)T\,\|p(t)\|_{C[0,T]}\,\|u_0(t)\|_{C[0,T]}\right]+$$

$$+T\,\|\gamma(t)\|_{C[0,T]}\,\|p(t)\|_{C[0,T]}\,\left\|u_{10}(t)\right\|_{C[0,T]}+$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left(\sum_{k=1}^{\infty}(\lambda_k\,\|f_{1k}(t)\|)^2_{C[0,T]}\right)^{\frac{1}{2}}+\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,\|u_{1k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\right.$$

$$+\left(\|\gamma(t)\|_{C[0,T]}+\frac{1}{b}\,\|a(t)\|_{C[0,T]}\right)\left[\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\varphi_{1k}|)^2\right)^{\frac{1}{2}}+\frac{\sqrt{T}(1+\delta)}{b}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k\,|f_{1k}(\tau)|)^2d\tau\right)^{\frac{1}{2}}+\right.$$

$$\left.\left.+\frac{T(1+\delta)}{b}\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{1k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]++T\,\|\gamma(t)\|_{C[0,T]}\,\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|u_{1k}(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\Bigg\}. \tag{2.15}$$

Let us assume that the data of problem (1.1)–(1.3), (1.6), (1.7) satisfy the following conditions:

$$1.\varphi(x)\in W_2{}^{(5)}(0,1),\quad \varphi(0)=\varphi(1),\varphi'(0)=\varphi'(1),$$

$$\varphi''(0)=\varphi''(1),\ \ \varphi'''(0)=\varphi'''(1),\ \ \varphi^{(4)}(0)=\varphi^{(4)}(1);$$

$$2.f(x,t),\ \ f_x(x,t),\ \ f_{xx}(x,t)\in C(D_T),\ \ f_{xxx}(x,t)\in L_2(D_T)\,,$$

$$f(0,t)=\ \ f(1,t)\ ,f_x(0,t)=\ \ f_x(1,t)\,,\ f_{xx}(0,t)=\ \ f_{xx}(1,t)\quad(0\le t\le T);$$

$$3.b>0,\ \delta\ge 0,\ \gamma(t),\ \ a(t)\in C[0,T],\ \ \ h(t)\in C^1[0,T],\ \ \ h(t)\neq 0\quad(0\le t\le T).$$

Then from (2.10)–(2.12), we have:

$$\|\tilde{u}(x,t)\|_{B^5_{2,T}}\le A_1(T)+B_1(T)\,\|p(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{2.16}$$

$$\|\tilde{p}(t)\|_{C[0,T]}\le A_2(T)+B_2(T)\,\|p(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}}\,, \tag{2.17}$$

where

$$A_1(T)=\|\varphi(x)\|_{L_2(0,1)}+(1+\delta)\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)}+2\sqrt{3}\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}+$$

$$+\frac{2\sqrt{3}}{b}(1+\delta)\sqrt{T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\,,\qquad B_1(T)=(1+\delta)\left(1+\frac{\sqrt{3}}{b}\right)T,$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\Big\{\|h'(t)-f(0,t)\|_{C[0,T]}+$$

$$+\|\gamma(t)\|_{C[0,T]}\left(\|\varphi(x)\|_{L_2(0,1)}+(1+\delta)\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)}\right)+$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}+\right.$$

$$\left.+\left(\|\gamma(t)\|_{C[0,T]}+\frac{1}{b}\,\|a(t)\|_{C[0,T]}\right)\left(\left\|\varphi^{(3)}(x)\right\|_{L_2(0,1)}+\frac{\sqrt{T}(1+\delta)}{b}\,\|f_x(x,t)\|_{L_2(D_T)}\right)\right]\Bigg\},$$

$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2}\right)^{\frac{1}{2}} \left[\left(\|\gamma(t)\|_{C[0,T]} + \frac{1}{b}\|a(t)\|_{C[0,T]}\right)\frac{T(2+\delta)}{b} + \right.$$

$$\left. +T\|\gamma(t)\|_{C[0,T]} + 1\right].$$

From inequalities (2.16), (2.17) we conclude:

$$\|u(x,t)\|_{B^5_{2,T}} + \|\tilde{p}(t)\|_{C[0,T]} \leq A(T) + B(T)\|p(t)\|_{C[0,T]}\|u(x,t)\|_{B^5_{2,T}}, \tag{2.18}$$

$$A(T) = A_1(T) + A_2(T),\ B(T) = B_1(T) + B_2(T).$$

We can prove the following theorem.

Theorem 2. Let conditions 1-3 be satisfied and

$$(A(T)+2)^2 B(T) < 1. \tag{2.19}$$

Then problem (1.1)–(1.3), (1.6), (1.7) has in $K = K_R(\|z\|_{E^5_T} \leq R = A(T)+2)$ in the space E^5_T only one solution.

Proof. In space E^5_T consider the equation

$$z = \Phi z, \tag{2.20}$$

where $z = \{u,p\}$, components $P\ \Phi_1(u,p), \Phi_2(u,p)$ of operators $\Phi(u,p)$ are defined by the right-hand sides of equations (2.9) and (2.12).

Consider the operator $\Phi(u,p)$ in a ball $K = K_R$ from E^5_T. Similarly to (2.18) we obtain that for any $z = \{u,p\}, z_1 = \{u_1,p_1\}, z_2 = \{u_2,p_2\} \in K_R$:

$$\|\Phi z\|_{E^5_T} \leq A(T) + B(T)\|p(t)\|_{C[0,T]}\|u(x,t)\|_{B^5_{2,T}}, \tag{2.21}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^5_T} \leq B(T)R(\|p_1(t)-p_2(t)\|_{C[0,T]} + \|u_1(x,t)-u_2(x,t)\|_{B^5_{2,T}}). \tag{2.22}$$

Then from estimates (2.21), (2.22), taking into account (2.19), it follows that the operator Φ acts in a ball $K = K_R$ and is contractive. Therefore, in the ball $K = K_R$ operator Φ has a single fixed point $\{u,p\}$, which is the only one in the ball $K = K_R$ solution of equation (2.20), i.e. is the only one solution in the ball $K = K_R$ of system (2.9), (2.12) in the ball.

Functions $u(x,t)$, as an element of space $B^5_{2,T}$ is continuous and has continuous derivatives $u_x(x,t),\ u_{xx}(x,t), u_{xxx}(x,t),\ u_{xxxx}(x,t)$ in D_T.

From (2.4), it is easy to see that

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k\left\|u'_{ik}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \leq \frac{\sqrt{2}}{b}\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^5\|u_{ik}(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} +$$

$$+\frac{\sqrt{2}}{b}\left\|\|f_x(x,t)+p(t)u_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\ (i=1,2).$$

Hence it follows that $u_t(x,t)$ and u_{txx} continuous in D_T.

It is easy to check that equation (1.1) and conditions (1.2), (1.3), (1.6), (1.7) are satisfied in the usual sense. Consequently, $\{u(x,t), p(t)\}$ is a solution to problem (1.1)–(1.3), (1.6), (1.7). By the corollary of Lemma 1, it is unique in the ball $K = K_R$. The theorem has been proven.

With the help of Theorem 1, the unique solvability of the original problem (1.1)–(1.5) immediately follows from the last theorem.

Theorem 3. Let all the conditions of Theorem 1 be satisfied, $\int_0^1 f(x,t)dx = 0$ $(0 \le t \le T)$, $\delta\gamma(t) = 0$ $(0 \le t \le T)$ and the matching condition is met:

$$\int_0^1 \varphi(x)dx = 0, \varphi(0) = h(0) + \delta h(T).$$

Then problem (1.1)–(1.5) has in the ball $K = K_R(\|z\|_{E_T^5} \le R = A(T) + 2)$ from E_T^5 the only classical solution.

3. Conclusions

The article considered an inverse boundary value problem with a periodic and integral condition, when the unknown coefficient depends on time for a linear pseudoparabolic equation of the fourth order. An existence and uniqueness theorem for the classical solution of the problem is proved.

Conflict of interest

The authors have declared no conflict of interest.

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