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Research article

Vanishing viscosity limit of incompressible flow around a small obstacle: A special case

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Abstract: In this paper, we consider two dimensional viscous flow around a small obstacle. In [4], the authors proved that the solutions of the Navier-Stokes system around a small obstacle of size ε converge to solutions of the Euler system in the whole space under the condition that the size of the obstacle ε is smaller than a suitable constant K times the kinematic viscosity ν. We show that, if the Euler flow is antisymmetric, then this smallness condition can be removed.

Keywords: Navier-Stokes equations; Euler equations; vanishing viscosity limit; exterior domain; boundary layer
Mathematics Subject Classification: 35Q30,76D05,76D10

1. Introduction

Let $\mathcal{O}$ be a smooth, simply connected and bounded domain in $\mathbb{R}^2$. Let $\varepsilon > 0$, and we set $\mathcal{O}_\varepsilon = \varepsilon\mathcal{O}$, $\Omega_\varepsilon = \mathbb{R}^2 \setminus \varepsilon\overline{\mathcal{O}}$. Let Γ_ε be the boundary of $\mathcal{O}_\varepsilon$. The Navier-Stokes equations

$$\begin{cases} \partial_t \boldsymbol{u}^{\nu,\varepsilon} + \boldsymbol{u}^{\nu,\varepsilon} \cdot \nabla \boldsymbol{u}^{\nu,\varepsilon} + \nabla p^{\nu,\varepsilon} = \nu \Delta \boldsymbol{u}^{\nu,\varepsilon} & \text{in } \Omega_\varepsilon \times (0,\infty), \quad (1.1) \\ \operatorname{div} \boldsymbol{u}^{\nu,\varepsilon} = 0 & \text{in } \Omega_\varepsilon \times [0,\infty) \quad (1.2) \end{cases}$$

are assumed to describe the motion of viscous fluid substances in the exterior domain Ω_ε. Here, $\boldsymbol{u}^{\nu,\varepsilon}(x,t)$ is the velocity field, the scalar function p represents the pressure, and $\nu > 0$ is the kinematic viscosity. We assume that the velocity field vanishes at infinity and satisfies the non-slip boundary conditions, that is,

$$\begin{cases} \boldsymbol{u}^{\nu,\varepsilon} = 0 & \text{on } \Gamma_\varepsilon \times [0,\infty], \quad (1.3) \\ \boldsymbol{u}^{\nu,\varepsilon}(x,t) \to 0 & as\ |x| \to \infty, t \in [0,\infty)\,. \quad (1.4) \end{cases}$$

Formally, if we set $\nu = 0$ and $\varepsilon = 0$, we obtain the Euler flow in the whole plane:

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = 0 & \text{in } \mathbb{R}^2 \times [0, \infty), \quad (1.5) \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \mathbb{R}^2 \times [0, \infty), \quad (1.6) \\ \boldsymbol{u}(x, t) \to 0 & as\ |x| \to \infty, t \in [0, \infty). \quad (1.7) \end{cases}$$

As we know, the vanishing viscosity limit problem is largely open in fluid mechanics. Let us mention some well known results. For the case of the whole space $\mathbb{R}^2$, the problem is much more tractable, and the convergence was verified in several studies (see [1, 11]). For the three dimensional case, we refer to [5, 12]. For a compact manifold without boundary of any dimension, we refer to [3]. For Navier type boundary condition, the convergence was established in [2]; see also [7, 8, 10, 15]. For the non-characteristic boundary case, the vanishing viscosity limit was established in [14]. For the case in a bounded domain with Dirichlet boundary conditions, whether the vanishing viscosity limit holds even for a short time is largely an open problem. Kato [13] proposed the criterion for the vanishing viscosity limit in bounded domains, which shows that the vanishing of energy dissipation in a small layer near the boundary is equivalent to the validity of the zero-viscosity limit in the energy space.

In this article, we consider the vanishing viscosity limit problem by assuming additionally that the size ε of the obstacle also tends to zero. To some degree, we are making the flow more viscous at its scale when making our obstacle smaller. In [4], the authors showed that the solution of Eqs (1.1)–(1.4) converges to the solution of Eqs (1.5)–(1.7) in $L^\infty([0, T]; L^2(\mathbb{R}^2))$ for arbitrary $T > 0$ provided that there exists a constant K such that

$$\varepsilon \leq K\nu. \tag{C}$$

Moreover, the convergence rate was established:

$$\|\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}\|_{L^2(\mathbb{R}^2)} \leq K \sqrt{\nu}. \tag{1.8}$$

The purpose of the present work is to weaken the smallness condition on the size of the obstacle. We find that, when the initial data $\boldsymbol{u}_0$ of the Euler flow is antisymmetric, the condition (C) could be left out. More precisely, suppose the initial data $\boldsymbol{u}_0$ of the Euler flow is antisymmetric and belongs to $H^3(\mathbb{R}^2)$. Then, we can construct a family of approximations $\boldsymbol{u}_0^\varepsilon$ of $\boldsymbol{u}_0$ such that the solutions of Eqs (1.1)−(1.4) with initial data $\boldsymbol{u}_0^\varepsilon$ converge to the solution of Eqs (1.5)−(1.7) with initial data $\boldsymbol{u}_0$ in $L^\infty([0, T]; L^2(\Omega))$-norm for arbitrary $T > 0$, provided that $\varepsilon, \nu \to 0$, with the smallness condition (C) dropped out.

The remainder of this article is divided into four sections. In Section 2, we state our main result, namely, the convergence for small viscosity and small size of the obstacle. In Section 3, the proof of our main result is given. In Section 4, we validate the convergence hypothesis of the initial data. In Section 5, some comments and discussion are given.

2. Notations and main results

In this section, some notations will be introduced, and then we state our main results. $H^s(\Omega_\varepsilon)$ stands for the usual L^2-based Sobolev spaces of order s, and $H_0^s(\Omega_\varepsilon)$ denotes the closure of C_0^∞ under the H^s-norm.

For a scalar function ψ, we denote $(-\partial_2\psi, \partial_1\psi)$ by $\nabla^\perp\psi$, while for a vector field $\boldsymbol{u}$, we will use the notation $\nabla^\perp \cdot \boldsymbol{u} := -\partial_2\boldsymbol{u}_1 + \partial_1\boldsymbol{u}_2$. Moreover, $\boldsymbol{u}^\perp$ denotes $(-\boldsymbol{u}_2, \boldsymbol{u}_1)$.

Throughout the paper, if we denote by K a positive constant with neither any subscript nor superscript, then K is considered as a generic constant whose value can change from line to line in the inequalities. On the other hand, we denote by K_T a positive constant that may depend on parameter T. Also, We will use bold characters to denote vector valued functions and the usual characters for scalar functions.

We next state our main result. Let $\boldsymbol{u}_0$ be smooth, be divergence free and belong to $H^3(\mathbb{R}^2)$. We know there exists a smooth and global solution $\boldsymbol{u}$ of Euler Eqs (1.5)–(1.7) with initial data $\boldsymbol{u}_0$, see [9]. Furthermore, the authors in [9] proved that $\boldsymbol{u} \in L^\infty([0,T]; H^3(\mathbb{R}^2))$ for arbitrary fixed $T > 0$. Let $\boldsymbol{u}_0^\varepsilon \in L^2(\Omega_\varepsilon)$ be divergence free and satisfy the Dirichlet boundary conditions; then, Kozono and Yamazaki in [6] proved that there exists a unique global solution of Eqs (1.1)–(1.4) with initial data $\boldsymbol{u}_0^\varepsilon$. Both $\boldsymbol{u}_0^\varepsilon$ and $\boldsymbol{u}^{\nu,\varepsilon}$ are defined only in Ω_ε, but we will consider them on the whole space by setting them as zero in $\overline{O_\varepsilon}$.

Theorem 2.1. Suppose that the initial data $\boldsymbol{u}_0$ is antisymmetric and belongs to $H^3(\mathbb{R}^2)$, and there exists a family of approximations $\{\boldsymbol{u}_0^\varepsilon\}$ of $\boldsymbol{u}_0$ that satisfies the following hypothesis:

$$\|\boldsymbol{u}_0^\varepsilon - \boldsymbol{u}_0\|_{L^2(\mathbb{R}^2)} \to 0 \text{ as } \varepsilon \to 0. \tag{H}$$

Then, there exists a constant K_T depending only on the time interval $T > 0$ such that

$$\sup_{t\in[0,T]} \|\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}^\varepsilon\|_{L^2(\mathbb{R}^2)} \le K_T(\varepsilon + \sqrt{\nu} + \|\boldsymbol{u}_0^\varepsilon - \boldsymbol{u}_0\|_{L^2(\Omega_\varepsilon)}). \tag{2.1}$$

Remark 2.1. If the initial data satisfies $\|\boldsymbol{u}_0^\varepsilon - \boldsymbol{u}_0\|_{L^2(\Omega_\varepsilon)} \le K\varepsilon$, then we have that

$$\sup_{t\in[0,T]} \|\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}^\varepsilon\|_{L^2(\mathbb{R}^2)} \le K_T(\varepsilon + \sqrt{\nu}). \tag{2.2}$$

3. Proof of main theorem

Proof of Theorem 2.1. Since $\boldsymbol{u}^{\nu,\varepsilon}$ is defined in Ω_ε, we extend $\boldsymbol{u}^{\nu,\varepsilon}$ to the whole plane by setting $\boldsymbol{u}^{\nu,\varepsilon} = 0$ in $\overline{O_\varepsilon}$. We then begin to estimate $\|\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}\|_{L^2(\mathbb{R}^2)}$, which can be divided into two parts:

$$\|\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}\|^2_{L^2(\mathbb{R}^2)} = \|\boldsymbol{u}\|^2_{L^2(O_\varepsilon)} + \|\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}\|^2_{L^2(\Omega_\varepsilon)}. \tag{3.1}$$

We consider the first part. Since $\boldsymbol{u} \in L^\infty([0,T]; H^3(\mathbb{R}^3))$, we get that $\|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^3)}$ is uniformly bounded in $[0,T]$. Therefore

$$\begin{aligned} \|\boldsymbol{u}\|_{L^2(O_\varepsilon)} &\le \|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^2)}|O_\varepsilon|^{\frac{1}{2}} \\ &\le K\varepsilon. \end{aligned} \tag{3.2}$$

We then focus on the estimate of $\|\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}\|_{L^2(\Omega_\varepsilon)}$. Observe that $\boldsymbol{u}$ is defined in the whole plane and consequently does not satisfy non-slip boundary conditions on Γ_ε. Therefore, we could not obtain the estimates of $\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}$ from (1.1)–(1.4) directly. We here consider instead a smooth vector field $\boldsymbol{u}^\varepsilon$ that is approximate to $\boldsymbol{u}$, while it vanishes on the boundary Γ_ε, thus allowing energy estimates.

Observe that $\boldsymbol{u}$ is divergence free in $\mathbb{R}^2$; there exists a stream function ψ of $\boldsymbol{u}$ such that $\boldsymbol{u} = \nabla^\perp\psi$. Moreover, we can choose ψ such that it vanishes at the origin. Notice that the initial data $\boldsymbol{u}_0$ of the Euler equations is antisymmetric, and it follows that $\boldsymbol{u}(0, t)$ vanishes at the origin for all $t \in [0, T]$. As a result, $\psi(x, t) = O(x^2)$ as $x \to 0$. Without loss of generality, we assume that the obstacle O is contained in the unit disk $B(0, 1)$. Let φ be an arbitrary smooth function in $\mathbb{R}^+$ such that

$$\varphi(x) = 1 \text{ for } x \in [2, \infty), \quad \varphi(x) = 0 \text{ for } x \in [0, \frac{3}{2}]. \tag{3.3}$$

We set $\varphi^\varepsilon(x) = \varphi(\frac{|x|}{\varepsilon})$. We then define the approximate sequence $\boldsymbol{u}^\varepsilon$ of $\boldsymbol{u}$ as follows:

$$\boldsymbol{u}^\varepsilon = \nabla^\perp(\varphi^\varepsilon\psi). \tag{3.4}$$

The approximate sequence $\boldsymbol{u}^\varepsilon$ has many properties, which are stated in the following lemma.

Lemma 3.1. Fix $T > 0$. There exists a constant K independent of ε such that

(1) $\boldsymbol{u}^\varepsilon$ is divergence free and vanishes on Γ_ε,
(2) $\|\boldsymbol{u}^\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq K$,
(3) $\|\boldsymbol{u}^\varepsilon\|_{W^{1,\infty}(\Omega_\varepsilon)} \leq K$,
(4) $\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}\|_{H^1(\Omega_\varepsilon)} + \|\boldsymbol{u}^\varepsilon - \varphi^\varepsilon\boldsymbol{u}\|_{H^1(\Omega_\varepsilon)} \leq K\varepsilon$.

Proof. From the definition of φ^ε, we know that it vanishes in a neighborhood of Γ_ε. Therefore $\boldsymbol{u}^\varepsilon$ vanishes on Γ_ε. Moreover,

$$\nabla \cdot \boldsymbol{u}^\varepsilon = \nabla \cdot [\nabla^\perp(\varphi^\varepsilon\psi)] = 0, \tag{3.5}$$

and consequently, $\boldsymbol{u}^\varepsilon$ is divergence free. We conclude that item (1) is verified. We begin to check item (2). Using the Minkowski inequality, we obtain

$$\begin{aligned} \|\boldsymbol{u}^\varepsilon\|^2_{L^2(\Omega)} &= \int_{\Omega_\varepsilon} |\varphi^\varepsilon\boldsymbol{u} + \nabla^\perp\varphi^\varepsilon\psi|^2 \\ &\leq \|\varphi^\varepsilon\boldsymbol{u}\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla^\perp\varphi^\varepsilon\psi\|^2_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{3.6}$$

Since $\boldsymbol{u}$ is smooth and bounded in $H^3(\mathbb{R}^2)$ for $t \in [0, T]$, we see that the first term on the right side of (3.6) is uniformly bounded with respect to ε. Meanwhile, we observe that $\psi(x) = O(x^2)$ when $x \to 0$, and $\nabla^\perp\varphi^\varepsilon$ is supported in the annulus C_ε, which is

$$C_\varepsilon = \{x \in \mathbb{R}^2 \mid \frac{3}{2}\varepsilon \leq |x| \leq 2\varepsilon\}. \tag{3.7}$$

It immediately follow that

$$\begin{aligned} \|\nabla^\perp\varphi^\varepsilon\psi\|_{L^2(\Omega_\varepsilon)} &\leq \|\nabla\varphi^\varepsilon\|_{L^\infty(C_\varepsilon)}\|\psi\|_{L^2(C_\varepsilon)} \\ &\leq K\varepsilon^2. \end{aligned} \tag{3.8}$$

Therefore, we have the following estimate:

$$\|\boldsymbol{u}^\varepsilon\|_{L^2(\Omega)} \leq K. \tag{3.9}$$

We next handle $\|\nabla \boldsymbol{u}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)}$ similarly. We first rewrite $\nabla \boldsymbol{u}^{\varepsilon}$ as

$$\nabla \boldsymbol{u}^{\varepsilon} = \nabla\nabla^{\perp}\varphi^{\varepsilon}\psi + \nabla\varphi^{\varepsilon}\boldsymbol{u} + \nabla^{\perp}\varphi^{\varepsilon}\boldsymbol{u}^{\perp} + \varphi^{\varepsilon}\nabla\boldsymbol{u}. \tag{3.10}$$

We see that $\nabla\nabla^{\perp}\varphi^{\varepsilon}$ vanishes outside of the annulus C_ε, and $\psi = O(\varepsilon^2)$ in C_ε. Therefore, the first term on the right side of (3.10) obeys

$$\|\nabla\nabla^{\perp}\varphi^{\varepsilon}\psi\|_{L^2(\Omega_\varepsilon)} = \|\nabla\nabla^{\perp}\varphi^{\varepsilon}\psi\|_{L^2(C_\varepsilon)} \le K\varepsilon. \tag{3.11}$$

The second and third terms on the right side of (3.10) can be handled together. Using again the property that $\nabla^{\perp}\varphi^{\varepsilon}$ is supported in the annulus C_ε and $\boldsymbol{u} = O(\varepsilon)$ in C_ε , it follows that

$$\|\nabla\varphi^{\varepsilon}\boldsymbol{u} + \nabla^{\perp}\varphi^{\varepsilon}\boldsymbol{u}^{\perp}\|_{L^2(\Omega_\varepsilon)} \le K\varepsilon. \tag{3.12}$$

The fourth term on the right side of (3.10) is uniformly bounded with respect to ε. We therefore conclude that

$$\|\nabla \boldsymbol{u}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)} \le K. \tag{3.13}$$

Combining (3.9) and (3.13), we immediately establish item (2). Now, we begin to estimate item (3), which can be handled similarly. From the definition of $\boldsymbol{u}^{\varepsilon}$, we know that

$$\begin{aligned}\|\boldsymbol{u}^{\varepsilon}\|_{W^{1,\infty}(\Omega_\varepsilon)} = &\|\varphi^{\varepsilon}\boldsymbol{u} + \nabla^{\perp}\varphi^{\varepsilon}\psi\|_{L^\infty(\Omega_\varepsilon)} \\ &+ \|\nabla\nabla^{\perp}\varphi^{\varepsilon}\psi + \nabla\varphi^{\varepsilon}\boldsymbol{u} + \nabla^{\perp}\varphi^{\varepsilon}\boldsymbol{u}^{\perp} + \varphi^{\varepsilon}\nabla\boldsymbol{u}\|_{L^\infty(\Omega_\varepsilon)}.\end{aligned} \tag{3.14}$$

Observe that φ^{ε} and $\boldsymbol{u}$ are uniformly bounded in $[0, T]$, and we have

$$\|\varphi^{\varepsilon}\boldsymbol{u}\|_{L^\infty(\Omega_\varepsilon)} + \|\varphi^{\varepsilon}\nabla\boldsymbol{u}\|_{L^\infty(\Omega_\varepsilon)} \le K. \tag{3.15}$$

Moreover, notice that $\nabla\varphi^{\varepsilon}$ vanishes outside of C_ε, and $\psi = O(\varepsilon^2)$ in C_ε. We are ready to check that

$$\|\nabla^{\perp}\varphi^{\varepsilon}\psi\|_{L^\infty(\Omega_\varepsilon)} + \|\nabla\nabla^{\perp}\varphi^{\varepsilon}\psi + \nabla\varphi^{\varepsilon}\boldsymbol{u} + \nabla^{\perp}\varphi^{\varepsilon}\boldsymbol{u}^{\perp}\|_{L^\infty(\Omega_\varepsilon)} \le K. \tag{3.16}$$

Combining the estimates (3.14)–(3.16), we conclude that item (3) holds. We at last handle item (4). From the definition of $\boldsymbol{u}^{\varepsilon}$, we have

$$\begin{aligned}\|\boldsymbol{u}^{\varepsilon} - \boldsymbol{u}\|_{H^1(\Omega_\varepsilon)} &= \|\nabla^{\perp}\varphi^{\varepsilon}\psi + \varphi^{\varepsilon}\boldsymbol{u} - \boldsymbol{u}\|_{H^1(\Omega_\varepsilon)} \\ &\le \|\nabla^{\perp}\varphi^{\varepsilon}\psi\|_{H^1(\Omega_\varepsilon)} + \|\varphi^{\varepsilon}\boldsymbol{u} - \boldsymbol{u}\|_{H^1(\Omega_\varepsilon)}.\end{aligned} \tag{3.17}$$

Using again the property that $\nabla\varphi^{\varepsilon}$ is supported in the annulus C_ε and $\psi = O(\varepsilon^2)$ in this annulus, it follows that

$$\begin{aligned}\|\nabla^{\perp}\varphi^{\varepsilon}\psi\|_{H^1(\Omega_\varepsilon)} &\le \|\nabla\varphi^{\varepsilon}\psi\|_{L^2(\Omega_\varepsilon)} + \|D^2\varphi^{\varepsilon}\psi\|_{L^2(\Omega_\varepsilon)} + \|\nabla\varphi^{\varepsilon}\boldsymbol{u}\|_{L^2(\Omega_\varepsilon)} \\ &\le K\varepsilon.\end{aligned} \tag{3.18}$$

It is easy to see that $\varphi^{\varepsilon}\boldsymbol{u} - \boldsymbol{u}$ vanishes outside of $\varepsilon B(0, 2)$ and is uniformly bounded with respect to ε, and we thus obtain

$$\begin{aligned}\|\varphi^{\varepsilon}\boldsymbol{u} - \boldsymbol{u}\|_{H^1(\Omega_\varepsilon)} &\le |\nabla\varphi^{\varepsilon}\boldsymbol{u}\|_{L^2(\Omega_\varepsilon)} + \|\varphi^{\varepsilon}\nabla\boldsymbol{u} - \nabla\boldsymbol{u}\|_{L^2(\Omega_\varepsilon)} \\ &\le K\varepsilon.\end{aligned} \tag{3.19}$$

Collecting (3.17)–(3.19) gives

$$\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{u}\|_{H^1(\Omega_\varepsilon)} \leq K\varepsilon. \tag{3.20}$$

We have left to show that

$$\|\boldsymbol{u}^{\varepsilon}-\varphi^{\varepsilon}\boldsymbol{u}\|_{H^1(\Omega_\varepsilon)} \leq K\varepsilon. \tag{3.21}$$

We first rewrite this term as

$$\|\boldsymbol{u}^{\varepsilon}-\varphi^{\varepsilon}\boldsymbol{u}\|_{H^1(\Omega_\varepsilon)} = \|\nabla^{\perp}\varphi^{\varepsilon}\psi+\varphi^{\varepsilon}\boldsymbol{u}-\varphi^{\varepsilon}\boldsymbol{u}\|_{H^1(\Omega_\varepsilon)} = \|\nabla^{\perp}\varphi^{\varepsilon}\psi\|_{H^1(\Omega_\varepsilon)}. \tag{3.22}$$

The previous estimate (3.18) yields that (3.21) holds. Therefore, item (4) is verified, and the proof of Lemma 3.1 is completed.

We then proceed the proof of Theorem 2.1. We observe that the second term of (3.1) satisfies

$$\|\boldsymbol{u}^{\nu,\varepsilon}-\boldsymbol{u}\|_{L^2(\Omega_\varepsilon)} \leq \|\boldsymbol{u}^{\nu,\varepsilon}-\boldsymbol{u}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)}+\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{u}\|_{L^2(\Omega_\varepsilon)}. \tag{3.23}$$

It follows from item (3) of Lemma 3.1 that

$$\|\boldsymbol{u}^{\nu,\varepsilon}-\boldsymbol{u}\|_{L^2(\Omega_\varepsilon)} \leq \|\boldsymbol{u}^{\nu,\varepsilon}-\boldsymbol{u}^{\varepsilon}\|_{L^2(\Omega_\varepsilon)}+K\varepsilon. \tag{3.24}$$

We then begin to estimate the $L^2(\Omega_\varepsilon)$-norm of $\boldsymbol{u}^{\nu,\varepsilon}-\boldsymbol{u}^{\varepsilon}$, which is defined as $W^{\nu,\varepsilon}$. On one hand, from the definition, we know $\boldsymbol{u}^{\varepsilon}$ satisfies

$$\begin{aligned}\partial_t\boldsymbol{u}^{\varepsilon} &= \partial_t[\nabla^{\perp}(\varphi\psi)]\\ &= \varphi^{\varepsilon}\partial_t\boldsymbol{u}+\nabla^{\perp}\varphi^{\varepsilon}\psi\\ &= \varphi^{\varepsilon}(-\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\nabla p)+\nabla^{\perp}\varphi^{\varepsilon}\psi.\end{aligned} \tag{3.25}$$

Consequently, by subtraction of (3.25) from (1.1), we get the identity about $W^{\nu,\varepsilon}$

$$\partial_t W^{\nu,\varepsilon} = \nu\Delta\boldsymbol{u}^{\nu,\varepsilon}-\boldsymbol{u}^{\nu,\varepsilon}\cdot\nabla\boldsymbol{u}^{\nu,\varepsilon}-\nabla p^{\nu,\varepsilon}+\varphi^{\varepsilon}(\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p)-\nabla^{\perp}\varphi^{\varepsilon}\psi. \tag{3.26}$$

Multiplying (3.26) by $W^{\nu,\varepsilon}$ and integrating over Ω_ε, we can see that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)} &= \nu\int_{\Omega_\varepsilon}\boldsymbol{u}^{\nu,\varepsilon}\cdot W^{\nu,\varepsilon}-\int_{\Omega_\varepsilon}[\boldsymbol{u}^{\nu,\varepsilon}\cdot\nabla\boldsymbol{u}^{\nu,\varepsilon}-\varphi^{\varepsilon}\boldsymbol{u}\cdot\nabla\boldsymbol{u}]\cdot W^{\nu,\varepsilon}\\ &\quad+\int_{\Omega_\varepsilon}\varphi^{\varepsilon}\nabla p\cdot W^{\nu,\varepsilon}-\int_{\Omega_\varepsilon}\psi\nabla^{\perp}\varphi^{\varepsilon}\cdot W^{\nu,\varepsilon}\\ &=: I_1+I_2+I_3+I_4.\end{aligned} \tag{3.27}$$

We will examine each term on the right-hand-side of (3.27). We begin with I_1. Firstly, we rewrite it as

$$I_1 = \nu\int_{\Omega_\varepsilon}\Delta W^{\nu,\varepsilon}\cdot W^{\nu,\varepsilon}+\nu\int_{\Omega_\varepsilon}\Delta\boldsymbol{u}^{\varepsilon}\cdot W^{\nu,\varepsilon}. \tag{3.28}$$

Since $W^{\nu,\varepsilon}$ vanishes at the boundary, using integration by parts, we get that

$$\begin{aligned} I_1 &= -\nu\|\nabla W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)} - \nu\int_{\Omega_\varepsilon}\sum_{i,j}\partial_i \boldsymbol{u}^\varepsilon_j \partial_i W^{\nu,\varepsilon}_j \\ &\le -\frac{\nu}{2}\|\nabla W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)} + \frac{\nu}{2}\|\boldsymbol{u}^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}, \end{aligned} \tag{3.29}$$

where we have used the Cauchy-Schwarz and Young's inequalities. Thanks to Lemma 3.1, we arrive at

$$I_1 \le -\frac{\nu}{2}\|\nabla W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)} + K\nu. \tag{3.30}$$

We then consider the second term. Observing $\int W^{\nu,\varepsilon}\cdot[(W^{\nu,\varepsilon}+\boldsymbol{u}^\varepsilon)\cdot\nabla W^{\nu,\varepsilon}] = 0$, it follows that

$$\begin{aligned} I_2 &= -\int_{\Omega_\varepsilon}[\boldsymbol{u}^{\nu,\varepsilon}\cdot\nabla\boldsymbol{u}^{\nu,\varepsilon} - \varphi^\varepsilon\boldsymbol{u}\cdot\nabla\boldsymbol{u}]\cdot W^{\nu,\varepsilon} \\ &= -\int_{\Omega_\varepsilon}[(W^{\nu,\varepsilon}+\boldsymbol{u}^\varepsilon)\cdot\nabla(W^{\nu,\varepsilon}+\boldsymbol{u}^\varepsilon) - \varphi^\varepsilon\boldsymbol{u}\cdot\nabla\boldsymbol{u}]\cdot W^{\nu,\varepsilon} \\ &= -\int_{\Omega_\varepsilon}[(W^{\nu,\varepsilon}+\boldsymbol{u}^\varepsilon)\cdot\nabla\boldsymbol{u}^\varepsilon - \varphi^\varepsilon\boldsymbol{u}\cdot\nabla\boldsymbol{u}]\cdot W^{\nu,\varepsilon} \\ &= -\int_{\Omega_\varepsilon}[W^{\nu,\varepsilon}\cdot\nabla\boldsymbol{u}^\varepsilon]\cdot W^{\nu,\varepsilon} - \int_{\Omega_\varepsilon}[\boldsymbol{u}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon - \varphi^\varepsilon\boldsymbol{u}\cdot\nabla\boldsymbol{u}]\cdot W^{\nu,\varepsilon} \\ &=: I_{21} + I_{22}. \end{aligned} \tag{3.31}$$

The term I_{21} satisfies the following estimate:

$$\begin{aligned} |I_{21}| &= \left|-\int_{\Omega_\varepsilon}[W^{\nu,\varepsilon}\cdot\nabla\boldsymbol{u}^\varepsilon]\cdot W^{\nu,\varepsilon}\right| \\ &\le \|\boldsymbol{u}^\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\|W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)}. \end{aligned} \tag{3.32}$$

From Lemma 3.1, we conclude that

$$|I_{21}| \le K\|W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)}. \tag{3.33}$$

Before we handle I_{22}, we first rewrite it as

$$I_{22} = -\int_{\Omega_\varepsilon}[(\boldsymbol{u}^\varepsilon - \varphi^\varepsilon\boldsymbol{u})\cdot\nabla\boldsymbol{u}^\varepsilon]\cdot W^{\nu,\varepsilon} + \int_{\Omega_\varepsilon}[\varphi^\varepsilon\boldsymbol{u}\cdot\nabla(\boldsymbol{u}-\boldsymbol{u}^\varepsilon)]\cdot W^{\nu,\varepsilon}. \tag{3.34}$$

Using the Cauchy-Schwarz and Young's inequalities, we get that

$$\begin{aligned} |I_{22}| \le \|W\|^2_{L^2(\Omega_\varepsilon)} &+ \|\boldsymbol{u}^\varepsilon - \varphi^\varepsilon\boldsymbol{u}\|^2_{L^2(\Omega_\varepsilon)}\|\nabla\boldsymbol{u}^\varepsilon\|^2_{L^\infty(\Omega_\varepsilon)} \\ &+ \|\nabla(\boldsymbol{u}-\boldsymbol{u}^\varepsilon)\|^2_{L^2(\Omega_\varepsilon)}\|\varphi^\varepsilon\boldsymbol{u}\|^2_{L^\infty(\Omega_\varepsilon)}. \end{aligned} \tag{3.35}$$

It follows from Lemma 3.1 that

$$|I_{22}| \le K\varepsilon^2. \tag{3.36}$$

Combining (3.33) and (3.36) gives

$$|I_2| \le K(\|W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^2). \tag{3.37}$$

Next, we begin to estimate I_3. To this aim, we use integration by parts to rewrite it as

$$I_3 = \int_{\Omega_\varepsilon} \nabla(p\varphi^\varepsilon) \cdot W^{\nu,\varepsilon} - \int_{\Omega_\varepsilon} p\nabla\varphi^\varepsilon \cdot W^{\nu,\varepsilon} = -\int_{\Omega_\varepsilon} p\nabla\varphi^\varepsilon \cdot W^{\nu,\varepsilon}. \tag{3.38}$$

From the Cauchy-Schwarz and Young's inequalities, we see that

$$|I_3| \le \frac{1}{2}\|W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{2}\|p\nabla\varphi^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}. \tag{3.39}$$

Observing that we could choose the pressure p such that it vanishes at the origin, and recalling that $\nabla\varphi^\varepsilon$ vanishes outside of C_ε, it immediately follows that

$$|I_3| \le \frac{1}{2}\|W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)} + K\varepsilon^2. \tag{3.40}$$

We then estimate I_4. Using again the property that $\nabla\varphi^\varepsilon$ vanishes outside of C_ε, we get

$$|I_4| \le \frac{1}{2}\|W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)} + K\varepsilon^2. \tag{3.41}$$

Collecting the estimates about I_1, I_2, I_3, I_4, we conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{2}\|\nabla W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)} \le K\|W^{\nu,\varepsilon}\|^2_{L^2(\Omega_\varepsilon)} + K\varepsilon^2 + K\nu. \tag{3.42}$$

It follows from Grönwall's inequality that

$$\sup_{t\in[0,T]} \|\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}^\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le K_T(\varepsilon^2 + \nu + \|\boldsymbol{u}_0^\varepsilon - \boldsymbol{u}_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}). \tag{3.43}$$

Collecting (3.1), (3.2), (3.24) and (3.43) yields

$$\sup_{t\in[0,T]} \|\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}\|^2_{L^2(\mathbb{R}^2)} \le K_T(\varepsilon^2 + \nu + \|\boldsymbol{u}_0^\varepsilon - \boldsymbol{u}_0^\varepsilon\|^2_{L^2(\Omega_\varepsilon)}). \tag{3.44}$$

The proof of Theorem 2.1 is completed.

4. Examination of hypothesis (H)

In this section, we will examine the validation of hypothesis (H). We will make use of the stream function of velocity field $\boldsymbol{u}$ to construct an approximate family $\{\boldsymbol{u}_0^\varepsilon\}_{\varepsilon\le 1}$ of $\boldsymbol{u}_0$ such that $\boldsymbol{u}_0^\varepsilon$ satisfies the Dirichlet boundary conditions on O_ε and converges to $\boldsymbol{u}_0$ in L^2 space as $\varepsilon \to 0$.

Lemma 4.1. Suppose that $\boldsymbol{u}_0$ is smooth and belongs to $H^3(\mathbb{R}^2)$. Then, there exists an approximate family $\{\boldsymbol{u}_0^\varepsilon\}$ of $\boldsymbol{u}_0$ that satisfies the hypothesis (H).

Proof. Let ψ_0 be the stream function of $\boldsymbol{u}_0$, which is defined as

$$\psi_0(x) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp \cdot \boldsymbol{u}_0(y)}{2\pi|x-y|^2}\,\mathrm{d}y + \int_{\mathbb{R}^2} \frac{y^\perp \cdot \boldsymbol{u}_0(y)}{2\pi|y|^2}\,\mathrm{d}y. \tag{4.1}$$

The constant $\int_{\mathbb{R}^2} \frac{y^\perp \cdot \boldsymbol{u}_0(y)}{2\pi|y|^2}\,\mathrm{d}y$ in the above identity is to guarantee that ψ_0 vanishes at the origin. We are ready to define the approximate sequence $\{\boldsymbol{u}_0^\varepsilon\}$. Without loss of generality, we assume that obstacle O is contained in the unit disk at the origin. Let $\eta : \mathbb{R}^+ \to [0,1]$ be a smooth function such that

$$\eta(x) = 0 \text{ for } x \in [0, \frac{3}{2}], \;\; \eta(x) = 1 \text{ for } x \in [2, \infty). \tag{4.2}$$

For $x \in \mathbb{R}^2$, we set $\eta^\varepsilon = \eta^\varepsilon(x) = \eta(\frac{|x|}{\varepsilon})$. We can see that η^ε vanishes in a neighbourhood of the boundary Γ_ε, and we define $\boldsymbol{u}_0^\varepsilon$ as

$$\boldsymbol{u}_0^\varepsilon = \nabla^\perp(\eta^\varepsilon \psi_0). \tag{4.3}$$

It is easy to check that $\boldsymbol{u}_0^\varepsilon$ satisfies the Dirichlet boundary conditions in Ω_ε and is divergence free. We now show that $\boldsymbol{u}_0^\varepsilon$ converges to $\boldsymbol{u}_0$ as $\varepsilon \to 0$ in $L^2(\mathbb{R}^2)$. In fact, observing that $\boldsymbol{u}_0^\varepsilon$ vanishes in O_ε, it follows that

$$\|\boldsymbol{u}_0^\varepsilon - \boldsymbol{u}_0\|_{L^2(\mathbb{R})}^2 = \|\boldsymbol{u}_0\|_{L^2(O_\varepsilon)}^2 + \|\nabla^\perp \eta^\varepsilon \psi + \eta^\varepsilon \boldsymbol{u}_0 - \boldsymbol{u}_0\|_{L^2(\Omega_\varepsilon)}^2. \tag{4.4}$$

As the Lebesgue measure of O is $O(\varepsilon^2)$, we get that

$$\|\boldsymbol{u}_0\|_{L^2(O_\varepsilon)}^2 \leq K\varepsilon^2. \tag{4.5}$$

Meanwhile, both $\nabla^\perp \eta^\varepsilon$ and $(\eta^\varepsilon - 1)$ are supported in the annulus C_ε, that is,

$$C_\varepsilon = \{x \in \mathbb{R}^2 \mid \frac{3}{2} \leq |x| \leq 2\}. \tag{4.6}$$

It follows that

$$\begin{aligned} |\nabla^\perp \eta^\varepsilon \psi + \eta^\varepsilon \boldsymbol{u}_0 - \boldsymbol{u}_0\|_{L^2(\Omega_\varepsilon)} &\leq \|\nabla^\perp \eta^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \|\psi\|_{L^2(C_\varepsilon)} \\ &\quad + \|1 - \eta^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \|\boldsymbol{u}_0\|_{L^2(C_\varepsilon)} \\ &\leq K\varepsilon. \end{aligned} \tag{4.7}$$

Combining (4.4), (4.5) and (4.7) yields immediately

$$\|\boldsymbol{u}_0^\varepsilon - \boldsymbol{u}_0\|_{L^2(\mathbb{R})} \leq K\varepsilon. \tag{4.8}$$

The proof of Lemma 4.1 is completed.

Collecting the results from Lemma 4.1, we have the following corollary.

Corollary 4.1. Let initial approximate data $\boldsymbol{u}_0^\varepsilon$ of $\boldsymbol{u}_0$ be constructed as in Lemma 4.1, and let $\boldsymbol{u}^{\nu,\varepsilon}$ be the solution of Navier-Stokes Eqs (1.1)–(1.4) with initial data $\boldsymbol{u}_0^\varepsilon$. Let $\boldsymbol{u}$ be the solution of Euler Eqs (1.5)–(1.7) with initial data $\boldsymbol{u}_0$. Then, we have that $\boldsymbol{u}^{\nu,\varepsilon}$ converges to $\boldsymbol{u}$ in the following sense:

$$\sup_{t\in[0,T]} \|\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}\|_{L^2(\mathbb{R}^2)} \leq K_T(\varepsilon + \sqrt{\nu}). \tag{4.9}$$

Proof. From inequality (3.44), it follows that

$$\sup_{t\in[0,T]} \|\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}^{\varepsilon}\|_{L^2(\mathbb{R}^2)} \leq K_T(\varepsilon + \sqrt{\nu} + \|\boldsymbol{u}_0^{\varepsilon} - \boldsymbol{u}_0^{\varepsilon}\|_{L^2(\mathbb{R}^2)}). \tag{4.10}$$

Combining with inequality (4.8), we conclude that

$$\sup_{t\in[0,T]} \|\boldsymbol{u}^{\nu,\varepsilon} - \boldsymbol{u}\|_{L^2(\mathbb{R}^2)} \leq K_T(\varepsilon + \sqrt{\nu}). \tag{4.11}$$

5. Conclusions

In [4], the authors established the convergence result by assuming the size ε of the obstacle is smaller than some constant K times the viscosity ν, and the main idea in the proof is to compensate for the mismatch between the slip boundary condition of ideal flows and the Dirichlet boundary conditions of viscous flows.

In this paper, we have established the convergence with the smallness condition $\varepsilon \leq K\nu$ eliminated, and the convergece rate is obtained. We want to remark that the convergence requires that $\varepsilon \to 0$, and one would like to study the vanishing viscosity limit problem with the size of the obstacle fixed, which is the most physically important problem.

Conflict of interest

The author declares no conflict of interest.

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