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Research article

Mönch's fixed point theorem in investigating the existence of a solution to a system of sequential fractional differential equations

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Abstract: In this article, the existence of a solution to a system of fractional equations of sequential type was investigated via Mönch's fixed point theorem. In addition, the stability of this solutions was verified by the Ulam-Hyers method. Finally, an applied example is presented to illustrate the theoretical results obtained from the existence results.

Keywords: Caputo derivatives; Mönch's fixed point; sequential fractional derivative; existence; stability
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1. Introduction

Fractional differential equations appear naturally in a number of fields, such as physics, engineering, biophysics, blood flow phenomena, aerodynamics, electro analytical chemistry, biology and economics. For more details, we refer the readers to [1–4, 39, 40, 42] and many other references therein.

Nowadays, academic researchers deal with many physical phenomena in plasma physics, physical chemistry, geophysics, fluid mechanics, nonlinear optics, electromagnetic theory and fluid motion, and their mathematical models are expressed by nonlinear fractional differential equations (NFDEs). These equations are commonly used in various scientific disciplines and have been investigated from different viewpoints. The exact solutions of these equations have gained more and more interest. For this reason, a lot of different techniques have been dealt with by researchers.

Several studies have been conducted over the years to investigate how stability concepts such as the Mittag-Leffler function and exponential and Lyapunov stability apply to various types of dynamical systems. Ulam and Hyers, identified previously unknown types of stability known as Ulam-stability [31, 41, 43]. The Hyers type of stability study contributes significantly to our understanding of chemical processes and fluid movement, as well as semiconductors, population dynamics, heat conduction and elasticity.

The study of boundary value problems for equations with nonlinear fractional differentials has a prominent and important role in the theory of fractional Calculus and in the study of physical phenomena through the physical interpretation of boundary conditions. To pass quickly to the practical applications of fractional derivatives in various applied sciences, some valuable works in this field can be found in [17–19, 21, 22, 26–30, 32, 33].

Through the in-depth and comprehensive study of fractional differential equations, the existence and uniqueness of solutions to fractional differential equations are proven using a set of fixed point theories, such as Banach's, the Leray- Schauder alternative, Darbo's theorem and Mönch's fixed point theorem.

In [5], the authors used Darbo's fixed point theorem to study the existence and the stability of the solution of the following fractional differential equation (FDE) which involves the Hadamard fractional derivative (H-FD) of variable order:

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}^{\alpha}_{1+}\mathcal{U}(\omega) = \mathcal{F}_1(\omega, \mathcal{U}(\omega)), \quad \omega \in [1, \mathcal{T}], \\ \mathcal{U}(1) = \mathcal{U}(\mathcal{T}) = 0, \end{cases}$$

where $1 < \alpha \leq 2, \mathcal{F}_1 : [1, \mathcal{T}] \times \mathcal{R} \to \mathcal{R}$ is a continuous function, and ${}^{\mathcal{H}}\mathcal{D}^{\alpha}_{1+}, {}^{\mathcal{H}}\mathcal{I}^{\alpha}_{1+}$ are the Hadamard fractional derivative and integral of variable-order $\mathcal{U}(\omega)$.

Recently, in 2022, the authors developed the existence theory for a new class of nonlinear coupled systems of sequential fractional differential equations supplemented with coupled, non-conjugate, Riemann-Stieltjes, integro-multipoint boundary conditions [6]:

$$\begin{cases} ({}^{c}\mathcal{D}^{\xi_1+1} + {}^{c}\mathcal{D}^{\xi_1})\Phi_1(\omega) = \mathcal{G}_1(\omega, \Phi_1(\omega), \Psi_1(\omega)), \quad 2 < \xi_1 < 3, \omega \in [0, 1], \\ ({}^{c}\mathcal{D}^{\zeta_1+1} + {}^{c}\mathcal{D}^{\zeta_1})\Psi_1(\omega) = \mathcal{G}_2(\omega, \Phi_1(\omega), \Psi_1(\omega)), \quad 2 < \zeta_1 < 3, \omega \in [0, 1], \end{cases} \tag{1.1}$$

subject to the coupled boundary conditions:

$$\begin{cases} \Phi_1(0) = 0, \Phi_1'(0) = o, \ \Phi_1'(0) = 0, \ \Phi_1(1) = k \int_0^{\rho} \Psi_1(s) dAs + \sum_{i=1}^{n-2} \alpha_i \Psi_1(\sigma_i) + k_1 \int_v^1 \Psi_1(s) dA(s), \\ \Psi_1(0) = 0, \Psi_1'(0) = o, \ \Psi_1'(0) = 0, \ \Psi_1(1) = h \int_0^{\rho} \Phi_1(s) dAs + \sum_{i=1}^{n-2} \beta_i \Phi_1(\sigma_i) + h_1 \int_v^1 \Phi_1(s) dA(s), \end{cases} \tag{1.2}$$

where ${}^{c}\mathcal{D}^{P}$ denotes the Caputo fractional derivative of order $P \in \xi_1, \zeta_1, 0 < \rho < \sigma_i < v < 1, \mathcal{G}_1, \mathcal{G}_2 :$ $[0, 1] \times \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ are given continuous functions, $k, k_1, h, h_1, \alpha_i, \beta_i \in \mathcal{R}, i = 1, 2, \cdots n-2$, and A is a function of bounded variation.

In [7], the authors studied the existence and uniqueness of a multipoint BVP with H-FD (sequential type):

$$\begin{cases} ({}^{\mathcal{H}}\mathcal{D}^{\alpha} + \lambda {}^{\mathcal{H}}\mathcal{D}^{\alpha-1})\mathcal{U}(\omega) = \mathcal{F}_1(\omega, \mathcal{U}(\omega)), \quad \omega \in [1, \mathcal{T}], \quad 1 < \alpha \leq 2, \\ \mathcal{U}(1) = 0, \ \mathcal{U}(\mathcal{T}) = \sum_{j=1}^{m} \delta_{1j} \mathcal{V}(\omega_j), \end{cases}$$

where ${}^{\mathcal{H}}\mathcal{D}^{\cdot}$ is the Hadamard fractional derivative of order $\alpha, \mathcal{F}_1 : [1, \mathcal{T}] \times \mathcal{R} \to \mathcal{R}$ is a continuous function , $\lambda \in \mathcal{R}^+, \omega_j, j = 1, 2, \cdots m$, are given points with $1 \le \omega_1 \le \cdots \le \omega_m < \mathcal{T}$, and δ_{1j} are appropriate real numbers.

The authors in [8] implemented the theorems of Banach and Schaefer to set sufficient conditions that guaranteed the existence of solutions and the stability for the following FDE with H-FD:

$$\begin{cases} \mathcal{D}^{\alpha}\mathcal{U}(\omega) = \mathcal{F}_1(\omega, \mathcal{U}(\omega), \mathcal{V}(\omega)), & \omega \in [1, \mathcal{T}], \ \ 0 < \alpha \le 1, \\ \mathcal{D}^{\alpha}\mathcal{V}(\omega) = \mathcal{F}_2(\omega, \mathcal{U}(\omega), \mathcal{V}(\omega)), & \omega \in [1, \mathcal{T}], \ \ 0 < \alpha \le 1, \end{cases}$$

with the following coupled boundary conditions:

$$\begin{cases} \mathcal{U}(1) = \delta_1 \mathcal{V}(\mathcal{T}), \\ \mathcal{V}(1) = \delta_2 \mathcal{U}(\mathcal{T}), \end{cases}$$

where $\mathcal{D}^{\theta}$ is the Hadamard fractional derivative of order $\theta \in \{\alpha, \beta\}, \mathcal{F}_1, \mathcal{F}_2 : [1, \mathcal{T}] \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ are appropriate functions, and δ_1, δ_2 are real number, with $\delta_1\delta_2 \neq 1$.

Due to the importance of the subject and the possibility of employing it in various scientific fields, many researchers in the field of fractional differential have studied the systems of fractional differentials equations with a variety of serious conditions accompanying them. For more information about, these scientific papers, the reader can see [9–16], and the stability of solutions was studied after the existence of them. To enrich the reader, it is possible to see [20, 23–25].

Motivated by the works mentioned above, the existence of the solution for the following couple of nonlinear sequential fractional differential equations is investigated:

$$\begin{cases} {}^{C}\mathcal{D}^{\xi_1}({}^{C}\mathcal{D}^1 + \mu_1)\Phi_1(\omega) = \mathcal{G}_1(\omega, \Phi_1(\omega), \Psi_1(\omega)), & 1 < \xi_1 < 2, \ \ \mu_1 > 0, \vartheta_1 < \omega < \varpi_1, \\ {}^{C}\mathcal{D}^{\zeta_1}({}^{C}\mathcal{D}^1 + \mu_2)\Psi_1(\omega) = \mathcal{G}_2(\omega, \Phi_1(\omega), \Psi_1(\omega)), & 1 < \zeta_1 < 2, \ \ \mu_2 > 0, \vartheta_2 < \omega < \varpi_2, \\ \Phi_1(\vartheta_1) = 0, \quad \Phi_1(\varrho_1) = 0, \quad \Phi_1(\varpi_1) = 0, \\ \Psi_1(\vartheta_2) = 0, \quad \Psi_1(\varrho_2) = 0, \quad \Psi_1(\varpi_2) = 0, \quad -\infty < \vartheta_i < \varrho_i < \varpi_i < \infty, \ \ i = 1, 2. \end{cases} \tag{1.3}$$

${}^{C}\mathcal{D}^{\mathcal{P}}$ is the Caputo fractional derivative of order $\mathcal{P} \in \{\xi_1, \zeta_1, 1\}$, $\mathcal{G}_i : [\vartheta, \varpi] \times \mathcal{R}^2 \to \mathcal{R}$ are given continuous functions, and $\mu_i, \varpi_i, \vartheta_i, \varrho_i, i = 1, 2.$ are real constants. ${}^{C}\mathcal{D}^1$ is the ordinary differential operator.

The originality and distinction of this work is summarized in employing Mönch's fixed point theorem with the aid of the Kuratowski measure of non-compactness and Carathéodory's conditions, to verify the necessary conditions for the existence of the solution to the system of fractional and nonlinear equations of sequential type. This work also examines the stability of the solution for the proposed system of equations.

The second Section of this study contains useful preliminaries needed in the next sections. In Section 4, the stability of this solution using the Ulam-Hyers stability technique is verified, and the fifth Section will represent an applied numerical example of the system of equations mentioned above. Finally, Conclusions are obtained in the sixth Section.

2. Preliminaries

This section introduces fundooamental fractional calculus concepts, principles and initial results [1–3].

Definition 2.1. *[34] The fractional integral of order α with the lower limit zero for a function f is defined as*

$$I^{\alpha} f(\omega) = \frac{1}{\Gamma(\alpha)} \int_0^{\omega} \frac{f(\upsilon)}{(\omega-\upsilon)^{1-\alpha}} d\upsilon, \omega > 0, \alpha > 0,$$

provided the right-hand side is point-wise defined on [0,∞), where $\Gamma(.)$ is the gamma function, which is defined by $\Gamma(\alpha) = \int_0^{\infty} \omega^{\alpha-1} e^{-\omega} d\omega$.

Definition 2.2. *[34] The (R-L) fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in \mathbb{N}$, is defined as*

$$D^{\alpha}_{0+} f(\omega) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{d\omega}\right)^n \int_0^{\omega} (\omega-\upsilon)^{n-\alpha-1} f(\upsilon) d\upsilon, \omega > 0,$$

where the function k has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3. *[34] The Caputo derivative of order $r \in [n-1, n)$ for a function $f : [0, \infty) \to (\mathbb{R})$ can be written as*

$$^{c}D^{r}_{0+} f(\omega) = D^{r}_{0+} \left(f(\omega) - \sum_{k=0}^{n-1} \frac{\omega^k}{k!} f^{(k)}(0) \right), \omega > 0, n-1 < r < n.$$

Note that the CFDs of order $r \in [n-1, n)$ exist almost everywhere on $[0, \infty)$ if $f \in AC^n([0, \infty), (\mathbb{R}))$.

Definition 2.4. *It has been shown in [34] that*

$$\mathcal{I}^{\xi_1 c}\mathcal{D}^{\xi_1} \Phi_1(\omega) = \Phi_0(\omega) - c_0 - c_1(\omega - t) - \cdots - c_{n-1}(\omega - t)^{n-1}, \;\; \omega - t, n-1 < \xi_1 < n.$$

Remark 2.5. *If $k \in C^n[0, \infty)$, then*

$$^{c}D^{\psi}_{0+} k(\omega) = \frac{1}{\Gamma(n-\psi)} \int_0^{\omega} \frac{k^n(\upsilon)}{(\omega-\upsilon)^{\psi+1-n}} d\upsilon = I^{n-\psi} k^(n)(\omega), \omega > 0, n-1 < \psi < n.$$

Denote the Banach space of all continuous functions z from $[\vartheta, \varpi]$ into $\hat{\mathcal{M}}^*$ by $C([\vartheta, \varpi], \hat{\mathcal{M}}^*)$, accompanied by the norm: $\|Z\| = \sup_{\vartheta \le \omega \le \varpi} \{z(\omega)\}$.

Definition 2.6. *[35] The Kuratowski measure of non compactness k defined on bounded set ψ of Banach space $\hat{\mathcal{M}}^*$ is:*

$$k(\psi) := \inf\{r > 0 : \psi = \psi_i \text{ and diam } (\psi_i) \le r \text{ for } 1 \le i \le m\}.$$

The following lemma dealing with the linear variant of the system (1.3) plays a key role in the forthcoming analysis.

Lemma 2.7. *[35] Given the Banach space $\hat{\mathcal{M}}^*$ were $\psi, \mathcal{V}$ are two bounded proper subsets of $\hat{\mathcal{M}}^*$, the following characteristics are true.*

(1) If $\psi \subset \mathcal{V}$, *then* $k(\psi) \leq k(\mathcal{V})$*;*

(2) $k(\psi) = k(\bar{\psi}) = k(\overline{conv}\psi)$*;*

(3) ψ *is relatively compact* $k(\psi) = 0$*;*

(4) $k(\delta\psi) = |\delta| k(\psi), \delta \in \mathcal{R}$*;*

(5) $k(\psi \cup \mathcal{V}) = \max\{k(\psi), k(\mathcal{V})\}$*;*

(6) $k(\psi + \mathcal{V}) = k(\psi) + k(\mathcal{V}), \psi + \mathcal{V} = \{x | x = \mathfrak{u} + \mathfrak{v}, \mathfrak{u} \in \psi, \mathfrak{v} \in \mathcal{V}\}$;

(7) $k(\psi + y) = k(\psi), \forall y \in \hat{\mathcal{M}}^*$.

Lemma 2.8. *[36] Given an equicontinuous and bounded set* $\mathcal{W}^* \subset C([\vartheta, \varpi], \hat{\mathcal{M}}^*)$*, the function* $\varpi \rightarrowtail k(\mathcal{W}^*(\varpi))$ *is continuous on* $[\vartheta, \varpi], k_C(\mathcal{W}^*) = \max_{\varpi \in [\vartheta, \varpi]} k(\mathcal{W}^*(\varpi))$*, and*

$$k\left(\int_{\mathfrak{a}}^{\mathcal{T}} x(\omega) d\omega\right) \leq \left(\int_{\mathfrak{a}}^{\mathcal{T}} (x(\omega)) d\omega\right), \mathcal{W}^*(\omega) = \{x(\omega) : x \in \mathcal{W}^*\}. \tag{2.1}$$

Definition 2.9. *[37] Given the function* $\Psi : [\vartheta, \varpi] \times \hat{\mathcal{M}}^* \to \hat{\mathcal{M}}^*$, Ψ *satisfies Carathéodory's conditions, if the following conditions apply:*

$\Psi(\varpi, z)$ *is measurable in* ϖ *for* $z \in \hat{\mathcal{M}}^*$*;*

$\Psi(\varpi, z)$ *is continuous in* $z \in \hat{\mathcal{M}}^*$ *for* $\varpi \in [\vartheta, \varpi]$.

Theorem 2.10. *[38] (Mönch's fixed point theorem) Given a bounded, closed, and convex subset* $\Omega \subset \hat{\mathcal{M}}^*$*, such that* $0 \in \Omega$*, let also* $\mathcal{T}$ *be a continuous mapping of* Ω *into itself.*
If $\mathcal{W}^* = \overline{conv}\mathcal{T}(\mathcal{W}^*)$*, or* $\mathcal{W}^* = \mathcal{T}(\mathcal{W}^*) \cup \{0\}$*, and* $k(\mathcal{W}^*) = 0$*, satisfied* $\forall\ \mathcal{W}^* \subset \Omega$*, then* $\mathcal{T}$ *has a fixed point.*

Lemma 2.11. *Assume that* $\mathcal{H}_1$ *and* $\mathcal{H}_2 \in C([\vartheta, \varpi], \mathcal{R})$*. The solution for the system*

$$\begin{cases} {}^C\mathcal{D}^{\xi_1}({}^C\mathcal{D}^1 + \mu_1)\Phi_1(\omega) = \mathcal{H}_1(\omega), \\ {}^C\mathcal{D}^{\zeta_1}({}^C\mathcal{D}^1 + \mu_1)\Psi_1(\omega) = \mathcal{H}_2(\omega), \\ \Phi_1(\vartheta_1) = 0, \quad \Phi_1(\varrho_1) = 0, \quad \Phi_1(\varpi_1) = 0, \\ \Psi_1(\vartheta_2) = 0, \quad \Psi_1(\varrho_2) = 0, \quad \Psi_1(\varpi_2) = 0, \quad -\infty < \vartheta_i < \varrho_i < \varpi_i < \infty, \ \ i = 1, 2, \end{cases} \tag{2.2}$$

is

$$\begin{aligned} \Phi_1(\omega) = & \int_{\vartheta_1}^{\omega} e^{-\mu_1(\omega - \upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon - \rho)^{\xi_1 - 1}}{\Gamma(\xi_1)} \mathcal{H}_1(\rho) d\rho\right) d\upsilon \\ & + \chi_1(\omega) \int_{\vartheta_1}^{\varrho_1} e^{-\mu_1(\varrho_1 - \upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon - \rho)^{\xi_1 - 1}}{\Gamma(\xi_1)} \mathcal{H}_1(\rho) d\rho\right) d\upsilon \\ & + \chi_2(\omega) \int_{\vartheta_1}^{\varpi_1} e^{-\mu_1(\varpi_1 - \upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon - \rho)^{\xi_1 - 1}}{\Gamma(\xi_1)} \mathcal{H}_1(\rho) d\rho\right) d\upsilon, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\Psi_1(\omega) = &\int_{\vartheta_2}^{\omega} e^{-\mu_2(\omega-\upsilon)}\left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\mathcal{H}_2(\rho)d\rho\right)d\upsilon \\ &+\chi_3(\omega)\int_{\vartheta_2}^{\varrho_2} e^{-\mu_2(\varrho_2-\upsilon)}\left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\mathcal{H}_2(\rho)d\rho\right)d\upsilon \\ &+\chi_4(\omega)\int_{\vartheta_2}^{\varpi_2} e^{-\mu_2(\varpi_2-\upsilon)}\left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\mathcal{H}_2(\rho)d\rho\right)d\upsilon,\end{aligned} \tag{2.4}$$

where

$$\chi_1(\omega) = \frac{\Lambda_4\varepsilon_1(\omega) - \Lambda_3\varepsilon_2(\omega)}{\Delta_1}, \quad \chi_2(\omega) = \frac{\Lambda_1\varepsilon_2(\omega) - \Lambda_2\varepsilon_1(\omega)}{\Delta_1}, \tag{2.5}$$

$$\varepsilon_1(\omega) = \mu_1(1 - e^{-\mu_1(\omega-\vartheta_1)}), \quad \varepsilon_2(\omega) = \mu_1(\omega - \vartheta_1) + e^{-\mu_1(\omega-\vartheta_1)} - 1, \tag{2.6}$$

$$\chi_3(\omega) = \frac{\Lambda_8\varepsilon_3(\omega) - \Lambda_7\varepsilon_4(\omega)}{\Delta_2}, \quad \chi_4(\omega) = \frac{\Lambda_5\varepsilon_4(\omega) - \Lambda_6\varepsilon_3(\omega)}{\Delta_2}, \tag{2.7}$$

$$\varepsilon_3(\omega) = \mu_2(1 - e^{-\mu_2(\omega-\vartheta_2)}), \quad \varepsilon_4(\omega) = \mu_2(\omega - \vartheta_2) + e^{-\mu_2(\omega-\vartheta_2)} - 1, \tag{2.8}$$

$$\Delta_1 = \Lambda_2\Lambda_3 - \Lambda_1\Lambda_4 \neq 0, \tag{2.9}$$

$$\Delta_2 = \Lambda_6\Lambda_7 - \Lambda_5\Lambda_8 \neq 0, \tag{2.10}$$

$$\begin{aligned}\Lambda_1 &= \mu_1(1 - e^{-\mu_1(\varrho_1-\vartheta_1)}), \quad \Lambda_2 = \mu_1(\varrho_1 - \vartheta_1) + e^{-\mu_1(\varrho_1-\vartheta_1)} - 1, \\ \Lambda_3 &= \mu_1(1 - e^{-\mu_1(\varpi_1-\vartheta_1)}), \quad \Lambda_4 = \mu_1(\varpi_1 - \vartheta_1) + e^{-\mu_1(\varpi_1-\vartheta_1)} - 1,\end{aligned} \tag{2.11}$$

$$\begin{aligned}\Lambda_5 &= \mu_2(1 - e^{-\mu_2(\varrho_2-\vartheta_2)}), \quad \Lambda_6 = \mu_2(\varrho_2 - \vartheta_2) + e^{-\mu_2(\varrho_2-\vartheta_2)} - 1, \\ \Lambda_7 &= \mu_2(1 - e^{-\mu_2(\varpi_2-\vartheta_2)}), \quad \Lambda_8 = \mu_2(\varpi_2 - \vartheta_2) + e^{-\mu_2(\varpi_2-\vartheta_2)} - 1.\end{aligned} \tag{2.12}$$

Proof. Applying ${}^{C}\mathcal{I}^{\xi_1}$ and ${}^{C}\mathcal{I}^{\zeta_1}$ on (2.2) and using the definition (2.4), we get

$$\begin{aligned}{}^{C}\mathcal{D}^{\xi_1}({}^{C}\mathcal{D}^{1} + \mu_1)\Phi_1(\omega) = \mathcal{H}_1, \\ {}^{C}\mathcal{D}^{\zeta_1}({}^{C}\mathcal{D}^{1} + \mu_2)\Psi_1(\omega) = \mathcal{H}_2.\end{aligned}$$

Using $\Phi_1(\vartheta_1) = 0,\ \Psi_1(\vartheta_2) = 0$ and evaluating the integration, we get

$$\Phi_1(\omega) = \int_{\vartheta_1}^{\omega} e^{-\mu_1(\omega-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{H}_1(\rho) d\rho \right) d\upsilon + \frac{c_0}{\mu_1} [1 - e^{-\mu(\omega-\upsilon)}] + \frac{c_1}{\mu_1} [\mu_1(\omega - \vartheta_1) + e^{-\mu_2(\omega-\vartheta_1)} - 1]. \tag{2.13}$$

$$\Psi_1(\omega) = \int_{\vartheta_2}^{\omega} e^{-\mu_2(\omega-\upsilon)} \left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)} \mathcal{H}_2(\rho) d\rho \right) d\upsilon + \frac{d_0}{\mu_2} [1 - e^{-\mu(\omega-\upsilon)}] + \frac{d_1}{\mu_2} [\mu_2(\omega - \vartheta_2) + e^{-\mu_2(\omega-\vartheta_2)} - 1]. \tag{2.14}$$

Making use of the conditions $\Phi_1(\varrho_1) = 0, \Psi_1(\varrho_2) = 0, \Phi_1(\varpi_1) = 0$ and $\Phi_1(\varpi_1) = 0$ in (2.13) and (2.14), we obtain

$$\Lambda_1 c_0 + \Lambda_2 c_1 = -\mu_1^2 \int_{\vartheta_1}^{\varrho_1} e^{-\mu_1(\varrho_1-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{H}_1(\rho) d\rho \right) d\upsilon, \tag{2.15}$$

$$\Lambda_3 c_0 + \Lambda_4 c_1 = -\mu_1^2 \int_{\vartheta_1}^{\varpi_1} e^{-\mu_1(\varpi_1-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{H}_1(\rho) d\rho \right) d\upsilon, \tag{2.16}$$

and

$$\Lambda_5 d_0 + \Lambda_6 d_1 = -\mu_2^2 \int_{\vartheta_2}^{\varrho_2} e^{-\mu_2(\varrho_2-\upsilon)} \left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)} \mathcal{H}_2(\rho) d\rho \right) d\upsilon, \tag{2.17}$$

$$\Lambda_7 d_0 + \Lambda_8 d_1 = -\mu_2^2 \int_{\vartheta_2}^{\varpi_2} e^{-\mu_2(\varpi_2-\upsilon)} \left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)} \mathcal{H}_2(\rho) d\rho \right) d\upsilon. \tag{2.18}$$

Solving the system (2.15)–(2.18), we find that

$$c_0 = \frac{\mu_1^2}{\Delta_1} \Bigg[\Lambda_4 \int_{\vartheta_1}^{\varrho_1} e^{-\mu_1(\varrho_1-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{H}_1(\rho) d\rho \right) d\upsilon - \Lambda_2 \int_{\vartheta_1}^{\varpi_1} e^{-\mu_1(\varpi_1-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{H}_1(\rho) d\rho \right) d\upsilon \Bigg],$$

and

$$c_1 = \frac{\mu_1^2}{\Delta_1} \Bigg[\Lambda_1 \int_{\vartheta_1}^{\varpi_1} e^{-\mu_1(\varpi_1-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{H}_1(\rho) d\rho \right) d\upsilon - \Lambda_3 \int_{\vartheta_1}^{\varrho_1} e^{-\mu_1(\varrho_1-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{H}_1(\rho) d\rho \right) d\upsilon \Bigg],$$

$$d_0 = \frac{\mu_2^2}{\Delta_2} \Bigg[\Lambda_4 \int_{\vartheta_2}^{\varrho_2} e^{-\mu_2(\varrho_2-\upsilon)} \left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)} \mathcal{H}_2(\rho) d\rho \right) d\upsilon - \Lambda_2 \int_{\vartheta_2}^{\varpi_2} e^{-\mu_2(\varpi_2-\upsilon)} \left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)} \mathcal{H}_2(\rho) d\rho \right) d\upsilon \Bigg],$$

and

$$d_1 = \frac{\mu_2^2}{\Delta_2}\left[\Lambda_2 \int_{\vartheta_2}^{\varpi_2} e^{-\mu_2(\varpi_2-\upsilon)}\left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\mathcal{H}_2(\rho)d\rho\right)d\upsilon \right.$$
$$\left. -\Lambda_3 \int_{\vartheta_2}^{\varrho_2} e^{-\mu_2(\varrho_2-\upsilon)}\left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\mathcal{H}_2(\rho)d\rho\right)d\upsilon\right].$$

Inserting the values of c_0, c_1, c_0, and c_1 in (2.13) and (2.14) and using the notations (2.5)–(2.10), we obtain (2.3) and (2.4). This completes the proof. □

3. Existence results via Mönch's fixed point theorem

Let $\widehat{O} = \{(\Phi_1(\omega), \Psi_1(\omega))|(\Phi_1, \Psi_1) \in C([\vartheta, \varpi], \mathcal{R}) \times C([\vartheta, \varpi], \mathcal{R})\}$. Clearly, the aforementioned set $\widehat{O}$ is a Banach space endowed with norm

$$\|(\Phi_1, \Psi_1)\|_{\widehat{O}} = \|\Phi_1\|_\infty + \|\Psi_1\|_\infty.$$

To show that our system (1.3) has a solution, we set the following Assumptions,

($\widehat{\mathcal{W}}_1$) Suppose that $\mathcal{G}_1, \mathcal{G}_2 : [\vartheta, \varpi] \times (\mathcal{R})^2 \to \mathcal{R}$ satisfy the Carathéodory conditions.

($\widehat{\mathcal{W}}_2$) $\exists\ \mathcal{K}_{\mathcal{G}_1}, \mathcal{K}_{\mathcal{G}_2} \in \mathcal{L}^1[\vartheta, \varpi] \times (\mathcal{R})_+$, and $\exists\ \mathfrak{H}_{\mathcal{G}_1}, \mathfrak{H}_{\mathcal{G}_2} : (\mathcal{R})_+ \to (\mathcal{R})_+$ such that $\forall\ \omega \in [\vartheta, \varpi], \forall(\Phi_1, \Psi_1 \in \widehat{O})$ we have

$$\|\mathcal{G}_1(\omega, \Phi_1, \Psi_1)\|_\infty \le \mathcal{K}_{\mathcal{G}_1}(\omega)\mathfrak{H}_{\mathcal{G}_1}(\|\Phi_1\|_\infty + \|\Psi_1\|_\infty),$$
$$\|\mathcal{G}_2(\omega, \Phi_1, \Psi_1)\|_\infty \le \mathcal{K}_{\mathcal{G}_2}(\omega)\mathfrak{H}_{\mathcal{G}_2}(\|\Phi_1\|_\infty + \|\Psi_1\|_\infty),$$

where $\mathfrak{H}_{\mathcal{G}_1}$, $\mathfrak{H}_{\mathcal{G}_2}$are non-decreasing continuous functions.

($\widehat{\mathcal{W}}_3$) Let $\mathcal{S} \subset \widehat{O} \times \widehat{O}$, be assumed to be bounded, and

$$\mathcal{K}(\mathcal{G}_1, (\omega, \mathcal{S})) \le \mathcal{K}_{\mathcal{G}_1}(\omega), \mathcal{K}(\mathcal{S}),$$
$$\mathcal{K}(\mathcal{G}_2, (\omega, \mathcal{S})) \le \mathcal{K}_{\mathcal{G}_2}(\omega), \mathcal{K}(\mathcal{S}).$$

For easy computations, we let

$$\widehat{\chi_1} = \max_{\omega\in[\vartheta,\varpi]} |\chi_1(\omega)|, \quad \widehat{\chi_2} = \max_{\omega\in[\vartheta,\varpi]} |\chi_2(\omega)|, \tag{3.1}$$

$$\widehat{\chi_3} = \max_{\omega\in[\vartheta,\varpi]} |\chi_3(\omega)|, \quad \widehat{\chi_4} = \max_{\omega\in[\vartheta,\varpi]} |\chi_4(\omega)|, \tag{3.2}$$

and

$$\widehat{\Upsilon}_1 = \frac{1}{\mu_1\Gamma(\xi_1+1)}\left\{(1+\widehat{\chi_2})(\varpi_1-\vartheta_1)^{\xi_1}(1-e^{-\mu_1(\varpi_1-\vartheta_1)}) + \widehat{\chi_1}(\varrho_1-\vartheta_1)^{\xi_1}(1-e^{-\mu_1(\varrho_1-\vartheta_1)})\right\}$$
$$-\frac{(\varpi_1-\vartheta_1)^{\xi_1}(1-e^{-\mu_1(\varpi_1-\vartheta_1)})}{\mu_1\Gamma(\xi_1+1)}, \tag{3.3}$$

and

$$\widehat{\Upsilon}_2 = \frac{1}{\mu_2 \Gamma(\zeta_1+1)} \left\{ (1+\widehat{\chi_4})(\varpi_2-\vartheta_2)^{\zeta_1}(1-e^{-\mu_2(\varpi_2-\vartheta_2)}) + \widehat{\chi_3}(\varrho_2-\vartheta_2)^{\zeta_1}(1-e^{-\mu_2(\varrho_2-\vartheta_2)}) \right\} - \frac{(\varpi_2-\vartheta_2)^{\zeta_1}(1-e^{-\mu_2(\varpi_2-\vartheta_2)})}{\mu_2 \Gamma(\zeta_1+1)}. \tag{3.4}$$

Theorem 3.1. *Assume that the Assumptions* $(\widehat{\mathcal{W}}_1)$, $(\widehat{\mathcal{W}}_2)$, *and* $(\widehat{\mathcal{W}}_3)$ *hold. If*

$$\max\{\mathcal{K}^*_{\mathcal{G}_1}\widehat{\Upsilon}_1, \mathcal{K}^*_{\mathcal{G}_2}\widehat{\Upsilon}_2\} < 1, \tag{3.5}$$

where $\mathcal{K}^*_{\mathcal{G}_i} = \sup_{1\le\omega\le e} \mathcal{K}^*_{\mathcal{G}_i}(\omega), \ \ \forall_i = 1,2,$ *then the system of fractional differential equations given by (1.3) has at least one solution on* $[\vartheta, \varpi]$.

Proof. Define the continuous operator $\widehat{\Xi} : \widehat{\mathcal{O}} \to \widehat{\mathcal{O}}$ as

$$\widehat{\Xi} = \widehat{\Xi}_1(\Phi_1,\Psi_1)(\omega), \widehat{\Xi}_2(\Phi_1,\Psi_1)(\omega),$$

where

$$\begin{aligned}\widehat{\Xi}_1 = & \int_{\vartheta_1}^{\omega} e^{-\mu_1(\omega-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{G}_1(u,\Phi_1(\rho),\Psi_1(\rho))d\rho \right) d\upsilon \\ & + \chi_1(\omega) \int_{\vartheta_1}^{\varrho_1} e^{-\mu_1(\varrho_1-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{G}_1(u,\Phi_1(\rho),\Psi_1(\rho))d\rho \right) d\upsilon \\ & + \chi_2(\omega) \int_{\vartheta_1}^{\varpi_1} e^{-\mu_1(\varpi_1-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{G}_1(u,\Phi_1(\rho),\Psi_1(\rho))d\rho \right) d\upsilon, \quad \omega \in [\vartheta,\varpi],\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\widehat{\Xi}_2 = & \int_{\vartheta_2}^{\omega} e^{-\mu_2(\omega-\upsilon)} \left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)} \mathcal{G}_2(u,\Phi_1(\rho),\Psi_1(\rho))d\rho \right) d\upsilon \\ & + \chi_3(\omega) \int_{\vartheta_2}^{\varrho_2} e^{-\mu_2(\varrho_2-\upsilon)} \left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)} \mathcal{G}_2(u,\Phi_1(\rho),\Psi_1(\rho))d\rho \right) d\upsilon \\ & + \chi_4(\omega) \int_{\vartheta_2}^{\varpi_2} e^{-\mu_2(\varpi_2-\upsilon)} \left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)} \mathcal{G}_2(u,\Phi_1(\rho),\Psi_1(\rho))d\rho \right) d\upsilon, \quad \omega \in [\vartheta,\varpi].\end{aligned} \tag{3.7}$$

Based on $(\widehat{\mathcal{W}}_1)$ and $(\widehat{\mathcal{W}}_2)$, the operator $\widehat{\Xi}$ is well defined.

Now, the operator equation

$$(\Phi_1,\Psi_1) = \widehat{\Xi}(\Phi_1,\Psi_1), \tag{3.8}$$

is equivalent to the fractional Eqs (2.3) and (2.4). Keep in mind that showing the existence of a solution for (3.8) is equivalent to showing the existence of a solution for (1.3).

Next, we define $\mathcal{S}_\Theta = \{(\Phi_1,\Psi_1) \in \widehat{\mathcal{O}} : \|(\Phi_1,\Psi_1)\|_{\widehat{\mathcal{O}}} \le \Theta, \Theta > 0\}$ to be a closed bounded convex ball in $\widehat{\mathcal{O}}$ with

$$\Theta \ge \mathcal{K}^*_{\mathcal{G}_1}\widehat{\Upsilon}_1\mathfrak{H}_{\mathcal{G}_1}(\Theta) + \mathcal{K}^*_{\mathcal{G}_2}\widehat{\Upsilon}_2\mathfrak{H}_{\mathcal{G}_2}(\Theta).$$

Now, to satisfy Mönch's fixed point theorem conditions, we split our proof into four steps.

Step 1. We show that $\widehat{\Xi}\mathcal{S}_\Theta \subset \mathcal{S}_\Theta$, we let $\omega \in [\vartheta, \varpi]$ and $\forall(\Phi_1, \Psi_1) \in \mathcal{S}_\Theta$, and we have

$$\begin{aligned}\|\widehat{\Xi}_1(\Phi_1, \Psi_1)\|_\infty = &\int_{\vartheta_1}^{\omega} e^{-\mu_1(\omega-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\|\mathcal{G}_1(u, \Phi_1(\rho), \Psi_1(\rho))\|_\infty d\rho\right)d\upsilon\\ &+\chi_1(\omega)\int_{\vartheta_1}^{\varrho_1} e^{-\mu_1(\varrho_1-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\|\mathcal{G}_1(u, \Phi_1(\rho), \Psi_1(\rho))\|_\infty d\rho\right)d\upsilon\\ &+\chi_2(\omega)\int_{\vartheta_1}^{\varpi_1} e^{-\mu_1(\varpi_1-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\|\mathcal{G}_1(u, \Phi_1(\rho), \Psi_1(\rho))\|_\infty d\rho\right)d\upsilon, \quad \omega \in [\vartheta, \varpi].\end{aligned} \tag{3.9}$$

Using $(\widehat{\mathcal{W}}_2)$, $\forall\omega \in [\vartheta, \varpi]$ we have

$$\begin{aligned}\|\mathcal{G}_1(\omega), \Phi_1(\omega), \Psi_1(\omega)\|_\infty \leq &\mathcal{K}^*_{\mathcal{G}_1}(\omega)\mathfrak{H}_{\mathcal{G}_1}(\|\Phi_1(\omega)\|_\infty + \|\Psi_1(\omega)\|_\infty)\\ \leq\ &\mathcal{K}^*_{\mathcal{G}_1}\mathfrak{H}_{\mathcal{G}_1}(\Theta).\end{aligned}$$

$$\begin{aligned}\|\widehat{\Xi}_1(\Phi_1, \Psi_1)\|_\infty = &\int_{\vartheta_1}^{\omega} e^{-\mu_1(\omega-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\mathcal{K}^*_{\mathcal{G}_1}\mathfrak{H}_{\mathcal{G}_1}\mathcal{G}_1\|\Phi_1(\rho)\|_\infty + \|\Psi_1(\rho))\|_\infty d\rho\right)d\upsilon\\ &+\chi_1(\omega)\int_{\vartheta_1}^{\varrho_1} e^{-\mu_1(\varrho_1-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\mathcal{K}^*_{\mathcal{G}_1}\mathfrak{H}_{\mathcal{G}_1}\mathcal{G}_1\|\Phi_1(\rho)\|_\infty + \|\Psi_1(\rho))\|_\infty d\rho\right)d\upsilon\\ &+\chi_2(\omega)\int_{\vartheta_1}^{\varpi_1} e^{-\mu_1(\varpi_1-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\mathcal{K}^*_{\mathcal{G}_1}\mathfrak{H}_{\mathcal{G}_1}\mathcal{G}_1\|\mathcal{X}_1(\rho)\|_\infty + \|\Psi_1(\rho))\|_\infty d\rho\right)d\upsilon,\\ &\leq \mathcal{K}^*_{\mathcal{G}_1}\mathfrak{H}_{\mathcal{G}_1}(\Theta),\end{aligned} \tag{3.10}$$

and similarly,

$$\begin{aligned}\|\widehat{\Xi}_2(\Phi_1, \Psi_1)\|_\infty = &\int_{\vartheta_2}^{\omega} e^{-\mu_2(\omega-\upsilon)}\left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\mathcal{K}^*_{\mathcal{G}_2}\mathfrak{H}_{\mathcal{G}_2}\mathcal{G}_2\|\Phi_1(\rho)\|_\infty + \|\Psi_1(\rho))\|_\infty d\rho\right)d\upsilon\\ &+\chi_3(\omega)\int_{\vartheta_2}^{\varrho_2} e^{-\mu_2(\varrho_2-\upsilon)}\left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\mathcal{K}^*_{\mathcal{G}_2}\mathfrak{H}_{\mathcal{G}_2}\mathcal{G}_2\|\Phi_1(\rho)\|_\infty + \|\Psi_1(\rho))\|_\infty d\rho\right)d\upsilon\\ &+\chi_4(\omega)\int_{\vartheta_2}^{\varpi_2} e^{-\mu_2(\varpi_2-\upsilon)}\left(\int_{\vartheta_2}^{\upsilon} \frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\mathcal{K}^*_{\mathcal{G}_2}\mathfrak{H}_{\mathcal{G}_2}\mathcal{G}_2\|\Phi_1(\rho)\|_\infty + \|\Psi_1(\rho))\|_\infty d\rho\right)d\upsilon,\\ &\leq \mathcal{K}^*_{\mathcal{G}_2}\mathfrak{H}_{\mathcal{G}_2}(\Theta).\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) yields

$$\begin{aligned}\|\widehat{\Xi}(\Phi_1, \Psi_1)\|_{\widehat{\mathcal{O}}} = &\|\widehat{\Xi}_1(\Phi_1, \Psi_1)\|_\infty + \|\widehat{\Xi}_2(\Phi_1, \Psi_1)\|_\infty\\ \leq &\mathcal{K}^*_{\mathcal{G}_1}\widehat{\Upsilon}_1\mathfrak{H}_{\mathcal{G}_1}(\Theta) + \mathcal{K}^*_{\mathcal{G}_2}\widehat{\Upsilon}_2\mathfrak{H}_{\mathcal{G}_2}(\Theta)\\ \leq &\Theta,\end{aligned} \tag{3.12}$$

$\widehat{\Xi}\mathcal{S}_\Theta \subset \mathcal{S}_\Theta$.

Step 2. We show the continuity of the operator $\widehat{\Xi}$, for this, we define the sequence

$$\{\mathcal{V}_n = (\Phi_{1n}, \Psi_{1n})\} \in \mathcal{S}_\Theta, \quad \text{then show that } \mathcal{V}_n \to \mathcal{V} = (\Phi_1, \Psi_1) \text{ as } n \to \infty.$$

Because of Carathéodory continuity of $\mathcal{G}_1$, it is clear that

$$\mathcal{G}_1(\cdot, \Phi_{1n}(\cdot), \Psi_{1n}(\cdot)) \to \mathcal{G}_1(\cdot, \Phi_1(\cdot), \Psi_1(\cdot)) \text{ as } n \to \infty.$$

Recalling $(\widehat{\mathcal{W}_2})$, we deduce that

$$\left(\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma\xi_1}\right)\|\mathcal{G}_1(r, \Phi_{1n}(r), \Psi_{1n}(r)) - \mathcal{G}_1(r, \Phi_1(r), \Psi_1(r))\|_\infty \le \mathcal{K}^*_{\mathcal{G}_1}\mathfrak{H}_{\mathcal{G}_1}(\Theta)\left(\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma\xi_1}\right). \tag{3.13}$$

Together with the Lebesgue dominated convergence theorem and the fact that the function

$$\mathcal{M} \to \mathcal{K}^*_{\mathcal{G}_1}\mathfrak{H}_{\mathcal{G}_1}(\Theta)\left(\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma\xi_1}\right) \tag{3.14}$$

is Lebesgue integrable on $[\vartheta, \varpi]$, we get

$$\begin{aligned}
&\|\widehat{\Xi}_1(\Phi_1, \Psi_1)\|_\infty \qquad (3.15)\\
&\le \int_{\vartheta_1}^{\omega} e^{-\mu_1(\omega-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon}\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\|\mathcal{G}_1(r, \Phi_{1n}(r), \Psi_{1n}(r)) - \mathcal{G}_1(r, \Phi_1(r), \Psi_1(r))\|_\infty d\rho\right)d\upsilon\\
&\quad + \chi_1(\omega)\int_{\vartheta_1}^{\varrho_1} e^{-\mu_1(\varrho_1-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon}\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\|\mathcal{G}_1(r, \Phi_{1n}(r), \Psi_{1n}(r)) - \mathcal{G}_1(r, \Phi_1(r), \Psi_1(r))\|_\infty d\rho\right)d\upsilon\\
&\quad + \chi_2(\omega)\int_{\vartheta_1}^{\varpi_1} e^{-\mu_1(\varpi_1-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon}\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\|\mathcal{G}_1(r, \Phi_{1n}(r), \Psi_{1n}(r)) - \mathcal{G}_1(r, \Phi_1(r), \Psi_1(r))\|_\infty d\rho\right)d\upsilon,
\end{aligned}$$

that is,

$$\|\widehat{\Xi}_1(\Phi_{1n}, \Psi_{1n})(\omega) - \widehat{\Xi}_1(\Phi_1, \Psi_1)(\omega)\|_\infty \to 0 \text{ as } n \to \infty \ \forall \ \omega \in [\vartheta, \varpi].$$

Then,

$$\|\widehat{\Xi}_1(\Phi_{1n}, \Psi_{1n}) - \widehat{\Xi}_1(\Phi_1, \Psi_1)\|_\infty \to 0 \text{ as } n \to \infty, \tag{3.16}$$

which means that the operator $\widehat{\Xi}_1$ is continuous. In a similar way, we get

$$\|\widehat{\Xi}_2(\Phi_{1n}, \Psi_{1n}) - \widehat{\Xi}_2(\Phi_1, \Psi_1)\|_\infty \to 0 \text{ as } n \to \infty. \tag{3.17}$$

(3.16) and (3.17) yield

$$\|\widehat{\Xi}(\Phi_{1n}, \Psi_{1n}) - \widehat{\Xi}(\Phi_1, \Psi_1)\|_{\widehat{\mathcal{O}}} \to 0 \text{ as } n \to \infty. \tag{3.18}$$

By getting (3.18), we conclude that the operator is continuous.

Step 3. We show that the operator is equicontinuous.

Let $\omega_1, \omega_2 \in [\vartheta, \varpi]$ and $\forall(\Phi_1, \Psi_1) \in \mathcal{S}_\Theta$. Then,

$$\begin{aligned}
&\|\widehat{\Xi}_1(\Phi_1, \Psi_1)(\omega_2) - \widehat{\Xi}_1(\Phi_1, \Psi_1)(\omega_1)\|_\infty\\
&= \left|\int_{\vartheta_1}^{\omega_1}[e^{-\mu_1(\omega_2-\upsilon)} - e^{-\mu_1(\omega_1-\upsilon)}]\left(\int_{\vartheta_1}^{\upsilon}\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\mathcal{G}_1(u, \Phi_1(\rho), \Psi_1(\rho))d\rho\right)d\upsilon\right.
\end{aligned}$$

$$+\int_{\omega_1}^{\omega_2}[e^{-\mu_1(\omega_1-\upsilon)}]\left(\int_{\vartheta_1}^{\upsilon}\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\mathcal{G}_1(u,\Phi_1(\rho),\Psi_1(\rho))d\rho\right)d\upsilon\Bigg| \tag{3.19}$$
$$\leq\left\{\mathcal{K}^*_{\mathcal{G}_1}\mathfrak{H}_{\mathcal{G}_1}(\Theta)\frac{1}{\mu_1\Gamma(\xi_1+1)}\{2(1-e^{\mu_1(\omega_2-\omega_2)})+|e^{\mu_2(\omega_2-\vartheta_1)}-e^{\mu_2(\omega_1-\vartheta_1)}|\}\right\}$$
$$\to 0 \;\; as \;\; \omega_1\to\omega_2.$$

In a like manner, we have

$$\begin{aligned}
&\|\widehat{\Xi}_2(\Phi_1,\Psi_1)(\omega_2)-\widehat{\Xi}_2(\Phi_1,\Psi_1)(\omega_1)\|_\infty\\
&=\left|\int_{\vartheta_2}^{\omega_1}[e^{-\mu_2(\omega_2-\upsilon)}-e^{-\mu_2(\omega_2-\upsilon)}]\left(\int_{\vartheta_2}^{\upsilon}\frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\mathcal{G}_2(u,\Phi_1(\rho),\Psi_1(\rho))d\rho\right)d\upsilon\right.\\
&\quad\left.+\int_{\omega_1}^{\omega_2}[e^{-\mu_2(\omega_2-\upsilon)}]\left(\int_{\vartheta_2}^{\upsilon}\frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\mathcal{G}_2(u,\Phi_1(\rho),\Psi_1(\rho))d\rho\right)d\upsilon\right|, \qquad (3.20)\\
&\leq\left\{\mathcal{K}^*_{\mathcal{G}_2}\mathfrak{H}_{\mathcal{G}_2}(\Theta)\frac{1}{\mu_2\Gamma(\zeta_1+1)}\left[2(1-e^{\mu_2(\omega_2-\omega_1)})+|e^{\mu_2(\omega_2-\vartheta_2)}-e^{\mu_2(\omega_1-\vartheta_2)}|\right]\right\}\to 0 \;\; as \;\; \omega_1\to\omega_2.
\end{aligned}$$

From (3.19) and (3.20), it is clear that both inequalities are independent of $(\Phi_1,\Psi_1)\in\mathcal{S}_\Theta$, which means that the operator $\widehat{\Xi}$ is bounded and equicontinuous.

Step 4. To satisfy all conditions of Mönch's fixed point theorem, finally, we let

$$\mathcal{U}=\mathcal{U}_1\cap\mathcal{U}_2,\mathcal{U}_1,\mathcal{U}_2\subset\mathcal{S}_\Theta.$$

Furthermore, $\mathcal{U}_1$ and $\mathcal{U}_2$ are assumed to be bounded and equicontinuous.

Show that

$$\mathcal{U}_1\subset\overline{conv}(\widehat{\Xi}_1(\mathcal{U}_1)\cup\{o\}), \;\text{ and }\; \mathcal{U}_2\subset\overline{conv}(\widehat{\Xi}_1(\mathcal{U}_1)\cup\{o\}).$$

Thus, the functions

$$\Pi_1(\omega)=k(\mathcal{U}_1(\omega)),$$
$$\Pi_2(\omega)=k(\mathcal{U}_2(\omega)),$$

are continuous on $[\vartheta,\varpi]$. By Kuratowski Lemma (2.7) and $(\widehat{\mathcal{W}}_3)$, we write

$$\begin{aligned}
\Pi_1(\omega)&=k(\mathcal{U}_1(\omega))\\
&\leq k(\overline{conv}(\widehat{\Xi}_1(\mathcal{U}_1)\cup\{o\}))\\
&\leq k(\widehat{\Xi}_1\mathcal{U}_1(\omega))\\
&\leq k\left\{\int_{\vartheta_1}^{\omega}e^{-\mu_1(\omega-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon}\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\|\mathcal{G}_1(u,\Phi_1(\rho),\Psi_1(\rho))\|\,d\rho\right)d\upsilon\right.\\
&\quad+\chi_1(\omega)\int_{\vartheta_1}^{\varrho_1}e^{-\mu_1(\varrho_1-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon}\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\|\mathcal{G}_1(u,\Phi_1(\rho),\Psi_1(\rho))\|\,d\rho\right)d\upsilon\\
&\quad\left.+\chi_2(\omega)\int_{\vartheta_1}^{\varpi_1}e^{-\mu_1(\varpi_1-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon}\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\|\mathcal{G}_1(u,\Phi_1(\rho),\Psi_1(\rho))\|\,d\rho\right)d\upsilon:(\Phi_1,\Psi_1)\in\mathcal{U}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq k\left\{\int_{\vartheta_1}^{\omega} e^{-\mu_1(\omega-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon}\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\mathcal{G}_1(u,\mathcal{U}_1(\rho))d\rho\right)d\upsilon\right.\\
&+\chi_1(\omega)\int_{\vartheta_1}^{\varrho_1} e^{-\mu_1(\varrho_1-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon}\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\mathcal{G}_1(u,\mathcal{U}_1(\rho))d\rho\right)d\upsilon\\
&\left.+\chi_2(\omega)\int_{\vartheta_1}^{\varpi_1} e^{-\mu_1(\varpi_1-\upsilon)}\left(\int_{\vartheta_1}^{\upsilon}\frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)}\mathcal{G}_1(u,\mathcal{U}_1(\rho))d\rho\right)d\upsilon:(\Phi_1,\Psi_1)\in\mathcal{U}\right\}\\
&\leq \mathcal{K}^*_{\mathcal{G}_1}\frac{1}{\mu_1\Gamma(\xi_1+1)}\left\{(1+\widehat{\chi_2})(\varpi_1-\vartheta_1)^{\xi_1}(1-e^{-\mu_1(\varpi_1-\vartheta_1)})+\widehat{\chi_1}(\varrho_1-\vartheta_1)^{\xi_1}(1-e^{-\mu_1(\varrho_1-\vartheta_1)})\right\}\\
&-\frac{(\varpi_1-\vartheta_1)^{\xi_1}(1-e^{-\mu_1(\varpi_1-\vartheta_1)})}{\mu_1\Gamma(\xi_1+1)}\|\Pi_1\|_\infty\\
&\leq \mathcal{K}^*_{\mathcal{G}_1}\widehat{\Upsilon}_1\|\Pi_1\|_\infty,
\end{aligned}
\tag{3.21}
$$

that is

$$\|\Pi_1\|_\infty \leq \mathcal{K}^*_{\mathcal{G}_1}\widehat{\Upsilon}_1\|\Pi_1\|_\infty,$$

it is also supposed that $\max\{\mathcal{K}^*_{\mathcal{G}_1}\widehat{\Upsilon}_1,\mathcal{K}^*_{\mathcal{G}_2}\widehat{\Upsilon}_2\}<1$, which implies $\|\Pi_1\|_\infty=0$, so $\Pi_1(\omega)=0,\ \forall\ \omega\in[\vartheta,\varpi]$.

In a like manner, we get $\Pi_2(\omega)=0,\ \forall\ \omega\in[\vartheta,\varpi]$.

Consequently, $k(\mathcal{U}(\omega))\leq k(\mathcal{U}_1(\omega))=0$, and $k(\mathcal{U}(\omega))\leq k(\mathcal{U}_1(\omega))=0$, implying $\mathcal{U}(\omega)$ is relatively compact in $\widehat{\mathcal{O}}\times\widehat{\mathcal{O}}$. Based on Arzelá-Ascoli theorem, we obtain that $\mathcal{U}$ is relatively compact in $\mathcal{S}_\Theta$.

Now, all conditions of Mönch's fixed point theorem apply; therefore $\widehat{\Xi}$ has fixed point (Φ_1,Ψ_1) on $\mathcal{S}_n$. □

4. Hyers-Ulam stability of system

Let us define nonlinear operator $\mathcal{Z}_1,\mathcal{Z}_2\in C([\vartheta,\varpi],\mathcal{R})\times C([\vartheta,\varpi],\mathcal{R})\to C([\vartheta,\varpi],\mathcal{R})$, where $\widehat{\Xi}_1$ and $\widehat{\Xi}_2$ are defined by (3.6) and (3.7).

$$
\begin{cases}
|{}^C\mathcal{D}^{\xi_1}({}^C\mathcal{D}^1+\mu_1)\Phi_1(\omega)-\mathcal{G}_1(\omega,\Phi_1(\omega),\Psi_1(\omega))|\leq\mathcal{Z}_1(\Phi_1,\Psi_1)(\omega), & \omega\in(\vartheta,\varpi),\\
|{}^C\mathcal{D}^{\zeta_1}({}^C\mathcal{D}^1+\mu_2)\Psi_1(\omega)-\mathcal{G}_2(\omega,\Phi_1(\omega),\Psi_1(\omega))|\leq\mathcal{Z}_2(\Phi_1,\Psi_1)(\omega),
\end{cases}
\tag{4.1}
$$

for $\omega\in[\vartheta,\varpi]$. For some $\varsigma_1,\varsigma_2>0$, we consider the following inequal:

$$\|\mathcal{Z}_1,(\Phi_1,\Psi_1)\|\leq\varsigma_1,\quad \|\mathcal{Z}_2,(\Phi_1,\Psi_1)\|\leq\varsigma_2. \tag{4.2}$$

Definition 4.1. *Problem (1.3) is Hyers-Ulam stable if there exist $\mathcal{M}_i>0, i=1,2,3,4$, such that for given $\varsigma_1,\varsigma_2>0$ and for each solution $(\Phi_1,\Psi_1)\in C([\vartheta,\varpi],\mathcal{R})\times C([\vartheta,\varpi],\mathcal{R})$ of inequality 4.1, there exists a solution $(\Phi_1^*,\Psi_1^*)\in C([\vartheta,\varpi],\mathcal{R})\times C([\vartheta,\varpi],\mathcal{R})$ of problem (1.3) with*

$$\|(\Phi_1)-(\hat{\Phi}_1)\|\leq\mathcal{M}_1\varsigma_1+\mathcal{M}_2\varsigma_2,$$

$$\|(\Psi_1)-(\hat{\Psi}_1)\|\leq\mathcal{M}_3\varsigma_1+\mathcal{M}_4\varsigma_2.$$

Remark 4.2. (Φ_1, Ψ_1) *is a solution of inequality (4.1) if there exist functions* $\mathcal{Z}_i \in C([\vartheta, \varpi], \mathcal{R})$*, i=1,2, which depend upon* Φ_1, Ψ_1 *respectively, such that*

$$|\mathcal{Z}_1(\omega)| \leq \varsigma_1,$$
$$|\mathcal{Z}_2(\omega)| \leq \varsigma_2.$$

$$\begin{cases} {}^C\mathcal{D}^{\xi_1}({}^C\mathcal{D}^1 + \mu_1)\Phi_1(\omega) = \mathcal{G}_1(\omega, \Phi_1(\omega), \Psi_1(\omega)) + \mathcal{Z}_1(\omega), & \omega \in (\vartheta, \varpi), \\ {}^C\mathcal{D}^{\zeta_1}({}^C\mathcal{D}^1 + \mu_2)\Psi_1(\omega) = \mathcal{G}_2(\omega, \Phi_1(\omega), \Psi_1(\omega)) + \mathcal{Z}_2(\omega). \end{cases}$$

Remark 4.3. *If* (Φ_1, Ψ_1) *represent a solution of inequality (4.1), then* (Φ_1, Ψ_1) *is a solution of the following inequality:*

$$\|(\Phi_1) - (\hat{\Phi}_1)\| \leq \mathcal{M}_1\varsigma_1 + \mathcal{M}_2\varsigma_2,$$
$$\|(\Psi_1) - (\hat{\Psi}_1)\| \leq \mathcal{M}_3\varsigma_1 + \mathcal{M}_4\varsigma_2.$$

From Remark 4.2, we have

$$\begin{cases} {}^C\mathcal{D}^{\xi_1}({}^C\mathcal{D}^1 + \mu_1)\Phi_1(\omega) = \mathcal{G}_1(\omega, \Phi_1(\omega), \Psi_1(\omega)) + \mathcal{Z}_1(\omega), & \omega \in (\vartheta, \varpi), \\ {}^C\mathcal{D}^{\zeta_1}({}^C\mathcal{D}^1 + \mu_2)\Psi_1(\omega) = \mathcal{G}_2(\omega, \Phi_1(\omega), \Psi_1(\omega)) + \mathcal{Z}_2(\omega). \end{cases}$$

With the help of Definition 4.1 and Remark 4.2, we verify Remark 4.3 in the following lines.

$$\begin{aligned} \hat{\Phi}_1(\omega) =& \widehat{\Xi}_1(\hat{\Phi}_1, \hat{\Psi}_1)(\omega) + \left\{ \int_{\vartheta_1}^{\omega} e^{-\mu_1(\omega-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{G}_1(u, \Phi_1(\rho), \Psi_1(\rho)) d\rho \right) d\upsilon \right. \\ &+ \chi_1(\omega) \int_{\vartheta_1}^{\varrho_1} e^{-\mu_1(\varrho_1-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{G}_1(u, \Phi_1(\rho), \Psi_1(\rho)) d\rho \right) d\upsilon \\ &\left. + \chi_2(\omega) \int_{\vartheta_1}^{\varpi_1} e^{-\mu_1(\varpi_1-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \mathcal{G}_1(u, \Phi_1(\rho), \Psi_1(\rho)) d\rho \right) d\upsilon \right\}, \end{aligned}$$

and it follows that

$$\begin{aligned} &|\widehat{\Xi}_1(\hat{\Phi}_1, \hat{\Psi}_1)(\omega) - \hat{\Phi}_1(\omega)| \\ &\leq \left\{ \int_{\vartheta_1}^{\omega} e^{-\mu_1(\omega-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \varsigma_1 d\rho \right) d\upsilon \right. \\ &+ \chi_1(\omega) \int_{\vartheta_1}^{\varrho_1} e^{-\mu_1(\varrho_1-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \varsigma_1 d\rho \right) d\upsilon \\ &\left. + \chi_2(\omega) \int_{\vartheta_1}^{\varpi_1} e^{-\mu_1(\varpi_1-\upsilon)} \left(\int_{\vartheta_1}^{\upsilon} \frac{(\upsilon-\rho)^{\xi_1-1}}{\Gamma(\xi_1)} \varsigma_1 d\rho \right) d\upsilon \right\} \\ &\leq \frac{1}{\mu_1 \Gamma(\xi_1+1)} \left\{ (1+\widehat{\chi_2})(\varpi_1-\vartheta_1)^{\xi_1}(1-e^{-\mu_1(\varpi_1-\vartheta_1)}) + \widehat{\chi_1}(\varrho_1-\vartheta_1)^{\xi_1}(1-e^{-\mu_1(\varrho_1-\vartheta_1)}) \right\} \\ &- \frac{(\varpi_1-\vartheta_1)^{\xi_1}(1-e^{-\mu_1(\varpi_1-\vartheta_1)})}{\mu_1 \Gamma(\xi_1+1)} \varsigma_1 \\ &\leq \widehat{\Upsilon}_1 \varsigma_1. \end{aligned}$$

In a like manner, we obtain

$$
\begin{aligned}
&|\widehat{\Xi}_2(\hat{\Phi}_1,\hat{\Psi}_1)(\omega)-\hat{\Psi}_1(\omega)|\\
&\qquad\le\left\{\int_{\vartheta_2}^{\omega}e^{-\mu_2(\omega-\upsilon)}\left(\int_{\vartheta_2}^{\upsilon}\frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\varsigma_2 d\rho\right)d\upsilon\right.\\
&\qquad+\chi_3(\omega)\int_{\vartheta_2}^{\varrho_2}e^{-\mu_2(\varrho_2-\upsilon)}\left(\int_{\vartheta_2}^{\upsilon}\frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\varsigma_2 d\rho\right)d\upsilon\\
&\qquad\left.+\chi_4(\omega)\int_{\vartheta_2}^{\varpi_2}e^{-\mu_2(\varpi_2-\upsilon)}\left(\int_{\vartheta_2}^{\upsilon}\frac{(\upsilon-\rho)^{\zeta_1-1}}{\Gamma(\zeta_1)}\varsigma_2 d\rho\right)d\upsilon\right\}\\
&\qquad\le\frac{1}{\mu_2\Gamma(\zeta_1+1)}\left\{(1+\widehat{\chi_4})(\varpi_2-\vartheta_2)^{\zeta_1}(1-e^{-\mu_2(\varpi_2-\vartheta_2)})+\widehat{\chi_3}(\varrho_2-\vartheta_2)^{\zeta_1}(1-e^{-\mu_2(\varrho_2-\vartheta_2)})\right\}\\
&\qquad-\frac{(\varpi_2-\vartheta_2)^{\zeta_1}(1-e^{-\mu_2(\varpi_2-\vartheta_2)})}{\mu_2\Gamma(\zeta_1+1)}\varsigma_2\\
&\qquad\le\widehat{\Upsilon}_2\varsigma_2.
\end{aligned}
$$

We obtain

$$|(\Phi_1,\Psi_1)-(\hat{\Phi}_1,\hat{\Psi}_1)|\le\widehat{\Upsilon}_1\varsigma_1+\widehat{\Upsilon}_2\varsigma_2,$$

where $\widehat{\Upsilon}_1$ and $\widehat{\Upsilon}_2$ are defined in (3.3) and (3.4). Thus, the operator $\widehat{\Xi}$, which is given by (3.6) and (3.7), can be extracted from the fixed point property, as follows:

$$
\begin{aligned}
|\Phi_1(\omega)-\Phi_1*(\omega)| =&|\Phi_1(\omega)-\widehat{\Xi}_1(\Phi_1*,\Psi_1*)(\omega)+\widehat{\Xi}_1(\Phi_1*,\Psi_1*)(\omega)-\Phi_1*(\omega)|\\
\le&|\widehat{\Xi}_1(\Phi_1,\Psi_1)(\omega)-\widehat{\Xi}_1(\Phi_1*,\Psi_1*)(\omega)|+|\widehat{\Xi}_1(\Phi_1*,\Psi_1*)(\omega)-\Phi_1*(\omega)|\\
\le&((\widehat{\Upsilon}_1\phi_1+\widehat{\Upsilon}_1\hat{\phi}_1)+(\widehat{\Upsilon}_1\phi_2+\widehat{\Upsilon}_1\hat{\phi}_2))\|(\Phi_1,\Psi_1)-(\Phi_1*,\Psi_1*)\|\\
&+\widehat{\Upsilon}_1\hat{\varsigma}_1+\widehat{\Upsilon}_1\hat{\varsigma}_2.
\end{aligned}\tag{4.3}
$$

$$
\begin{aligned}
|\Psi_1(\omega)-\Psi_1*(\omega)| =&|\Psi_1\omega)-\widehat{\Xi}_2(\Phi_1*,\Psi_1*)(\omega)+\widehat{\Xi}_2(\Phi_1*,\Psi_1*)(\omega)-\Psi_1*(\omega)|\\
\le&|\widehat{\Xi}_2(\Phi_1,\Psi_1)(\omega)-\widehat{\Xi}_2(\Phi_1*,\Psi_1*)(\omega)|+|\widehat{\Xi}_2(\Phi_1*,\Psi_1*)(\omega)-\Psi_1*(\omega)|\\
\le&((\widehat{\Upsilon}_2\phi_1+\widehat{\Upsilon}_2\hat{\phi}_1)+(\widehat{\Upsilon}_2\phi_2+\widehat{\Upsilon}_2\hat{\phi}_2))\|(\Phi_1,\Psi_1)-(\Phi_1*,\Psi_1*)\|\\
&+\widehat{\Upsilon}_2\hat{\varsigma}_1+\widehat{\Upsilon}_2\hat{\varsigma}_2.
\end{aligned}\tag{4.4}
$$

(4.3) and (4.4) yield

$$
\begin{aligned}
\|(\Phi_1,\Psi_1)-(\Phi_1*,\Psi_1*)\|\le&\frac{(\widehat{\Upsilon}_1+\widehat{\Upsilon}_2)\hat{\varsigma}_1+(\widehat{\Upsilon}_1+\widehat{\Upsilon}_2)\hat{\varsigma}_2}{1-((\widehat{\Upsilon}_1+\widehat{\Upsilon}_2)(\phi_1+\phi_2)+(\widehat{\Upsilon}_1+\widehat{\Upsilon}_2)(\hat{\phi}_1+\hat{\phi}_2))}\\
\le&\mathcal{V}_1\hat{\varsigma}_1+\mathcal{V}_2\hat{\varsigma}_2,
\end{aligned}
$$

with

$$\mathcal{V}_1=\frac{(\widehat{\Upsilon}_1+\widehat{\Upsilon}_2)}{1-((\widehat{\Upsilon}_1+\widehat{\Upsilon}_2)(\phi_1+\phi_2)+(\widehat{\Upsilon}_1+\widehat{\Upsilon}_2)(\hat{\phi}_1+\hat{\phi}_2))},$$

$$\mathcal{V}_2=\frac{(\widehat{\Upsilon}_1+\widehat{\Upsilon}_2)}{1-((\widehat{\Upsilon}_1+\widehat{\Upsilon}_2)(\phi_1+\phi_2)+(\widehat{\Upsilon}_1+\widehat{\Upsilon}_2)(\hat{\phi}_1+\hat{\phi}_2))}.$$

Hence, the problem (1.3) is U-H stable.

5. Example

Define $\Phi_0 = \{\psi = (\Phi_1, \Phi_2, \Phi_3 \cdots, \Phi_n, \cdots) : \lim_{n\to\infty} \Phi_n = 0\}$, and it is obvious that z_0 is a Banach space with $\|\Phi\|_\infty = \sup_{n\geq 1} |\Phi_n|$.

Example 5.1. *Consider the following system:*

$$\begin{cases} {}^C\mathcal{D}^{\xi_1}({}^C\mathcal{D}^1 + \mu_1)\Phi_1(\omega) = \mathcal{G}_1(\omega, \Phi_1(\omega), \Psi_1(\omega)), & 1 < \xi_1 < 2,\ \mu_1 > 0, \vartheta_1 < \omega < \varpi_1, \\ {}^C\mathcal{D}^{\zeta_1}({}^C\mathcal{D}^1 + \mu_2)\Psi_1(\omega) = \mathcal{G}_2(\omega, \Phi_1(\omega), \Psi_1(\omega)), & 1 < \zeta_1 < 2,\ \mu_2 > 0, \vartheta_2 < \omega < \varpi_2, \\ \Phi_1(\vartheta_1) = 0, \quad \Phi_1(\varrho_1) = 0, \quad \Phi_1(\varpi_1) = 0, & \\ \Psi_1(\vartheta_2) = 0, \quad \Psi_1(\varrho_2) = 0, \quad \Psi_1(\varpi_2) = 0, & -\infty < \vartheta_i < \varrho_i < \varpi_i < \infty,\ i = 1, 2. \end{cases} \tag{5.1}$$

Here, $\xi_1 = \frac{7}{4}$, $\zeta_1 = \frac{8}{5}$, $\vartheta_1 = 1, \vartheta_2 = 1$, $\varrho_1 = \frac{5}{3}$, $\varrho_2 = \frac{8}{5}$, $\varpi_1 = 2$, $\varpi_2 = 2$, $\mu_1 = \mu_2 = 1$, $\Delta_1 = 0.3583844$, $\Delta_2 = 0.173209134$,

$$\mathcal{G}_1(\omega, \Phi_1(\omega), \Psi_1(\omega)) = \left\{ \frac{|\Phi_1(\omega)|}{(\omega + 9)(1 + |\Phi_1(\omega)|)} + \frac{1}{27(1 + |\Psi_1^2(\omega)|)} + \frac{1}{81} \right\},$$
$$\mathcal{G}_2(\omega, \Phi_1(\omega), \Psi_1(\omega)) = \left\{ \frac{\sin(2\pi|\Phi_1(\omega))|}{40\pi} + \frac{1}{10\sqrt{\omega + 4}} + \frac{|\Psi_1(\omega)|}{10(1 + |\Phi_1(\omega)|)} \right\},$$

$\forall \omega \in [1, 2]$ *with* $\{\Phi_n\}_{n\geq 1}, \{\Psi_n\}_{n\geq 1} \in \Phi_0$*, and the hypothesis* $\mathcal{A}_2$ *of theorem 3.1 is verified. Also,*

$$\begin{aligned} \|\mathcal{G}_1(\omega, \Phi_1(\omega), \Psi_1(\omega))\|_\infty &\leq \left\| \left\{ \frac{|\Phi_1(\omega)|}{(\omega + 9)(1 + |\Phi_1(\omega)|)} + \frac{1}{27(1 + |\Psi_1^2(\omega)|)} + \frac{1}{81} \right\} \right\|_\infty \\ &\leq \frac{1}{(\omega + 9)}(\|\Phi_1\| + 1) \\ &= \mathcal{K}_{\mathcal{G}_1}(\omega)\mathfrak{H}_{\mathcal{G}_1}(\|\Phi_1\|_\infty). \end{aligned}$$

Similarly,

$$\begin{aligned} \|\mathcal{G}_2(\omega, \Phi_1(\omega), \Psi_1(\omega))\|_\infty &\leq \left\| \left\{ \frac{\sin(2\pi|\Phi_1(\omega))|}{40\pi} + \frac{1}{10\sqrt{\omega + 4}} + \frac{|\Psi_1(\omega)|}{10(1 + |\Psi_1(\omega)|)} \right\} \right\|_\infty \\ &\leq \frac{1}{10}(\|\Psi_1\| + 1) \\ &= \mathcal{K}_{\mathcal{G}_2}(\omega)\mathfrak{H}_{\mathcal{G}_2}(\|\Psi_1\|_\infty). \end{aligned}$$

As a result, Theorem 3.1's condition $\mathcal{A}_2$ *is also verified.*

Next, by relying on the bounded subset $\mathcal{S} \subset \widehat{\mathcal{E}} \times \widehat{\mathcal{E}}$*, we get*

$$\mathcal{K}(\mathcal{G}_1, (\omega, \mathcal{S})) \leq \mathcal{K}_{\mathcal{G}_1}(\omega)\mathcal{K}(\mathcal{S}),$$
$$\mathcal{K}(\mathcal{G}_2, (\omega, \mathcal{S})) \leq \mathcal{K}_{\mathcal{G}_2}(\omega)\mathcal{K}(\mathcal{S}),$$

where in our case, we have $\mathcal{K}_{\mathcal{G}_1}(\omega) = \frac{1}{\omega+9}$, $\mathcal{K}_{\mathcal{G}_2}(\omega) = \frac{\omega}{10}$*. The latter two inequalities show that the condition* ($\mathcal{A}_2$) *of Theorem 3.1 is satisfied.*

Finally, we calculate

$$\mathcal{K}^{*}_{\mathcal{G}_1}(\omega) = \frac{1}{10},$$

$$\widehat{\Upsilon}_1 \leq \frac{1}{\mu_1 \Gamma(\xi_1+1)} \left\{(1+\widehat{\chi_2})(\varpi_1-\vartheta_1)^{\xi_1}(1-e^{-\mu_1(\varpi_1-\vartheta_1)}) + \widehat{\chi_1}(\varrho_1-\vartheta_1)^{\xi_1}(1-e^{-\mu_1(\varrho_1-\vartheta_1)})\right\} - \frac{(\varpi_1-\vartheta_1)^{\xi_1}(1-e^{-\mu_1(\varpi_1-\vartheta_1)})}{\mu_1\Gamma(\xi_1+1)} \approx 0.268629314, \tag{5.2}$$

and

$$\mathcal{K}^{*}_{\mathcal{G}_2}(\omega) = \frac{2}{10},$$

$$\widehat{\Upsilon}_2 \leq \frac{1}{\mu_2 \Gamma(\zeta_1+1)} \left\{(1+\widehat{\chi_4})(\varpi_2-\vartheta_2)^{\zeta_1}(1-e^{-\mu_2(\varpi_2-\vartheta_2)}) + \widehat{\chi_3}(\varrho_2-\vartheta_2)^{\zeta_1}(1-e^{-\mu_2(\varrho_2-\vartheta_2)})\right\} - \frac{(\varpi_2-\vartheta_2)^{\zeta_1}(1-e^{-\mu_2(\varpi_2-\vartheta_2)})}{\mu_2\Gamma(\zeta_1+1)} \approx 0.3906796025. \tag{5.3}$$

Then, $\max\{\widehat{\Upsilon}_1\mathcal{K}_{\mathcal{G}_2}(\omega), \widehat{\Upsilon}_2\mathcal{K}_{\mathcal{G}_2}(\omega)\} = \max\{0.026862931, 0.078135920\} = 0.078135920 < 1$. *Thus, Theorem 3.1's requirements are all satisfied, that is, Eq (5.1) has at least one solution* $(\psi, \Psi_1) \in C(\psi_0) \times C([1,2], \psi_0)$.

6. Conclusions

Based on Mönch's fixed point theorem with the aid of the Kuratowski measure of non-compactness and Carathéodory's conditions, we have proved that there is a solution to the system of fractional differential equations given in (1.3). In addition, we verified the stability of the solutions for this system using the method of Ulam-Hyers. We concluded the work with an applied example that makes it easier for the reader to understand the theoretical results. For future work, those interested in the field can also investigate these solutions via other fractional derivatives, such as Caputo-Hadamard, Katugampola, Hilfer and ψ-Caputo. Also, there could possibility to discuss the existence according to the resolvents operators see [43].

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Conflict of interest

The authors declare no conflicts of interest.

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