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Research article

Common fixed point, Baire's and Cantor's theorems in neutrosophic 2-metric spaces

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Abstract: These fundamental Theorems of classical analysis, namely Baire's Theorem and Cantor's Intersection Theorem in the context of Neutrosophic 2-metric spaces, are demonstrated in this article. Naschie discussed high energy physics in relation to the Baire's Theorem and the Cantor space in descriptive set theory. We describe, how to demonstrate the validity and uniqueness of the common fixed-point theorem for four mappings in Neutrosophic 2-metric spaces.

Keywords: fuzzy metric spaces; fuzzy 2-metric spaces; neutrosophic metric spaces; common fixed point
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

Among the other improvements of the Zadeh-originally-proposed theory of fuzzy sets [1], progress has been made in discovering the fuzzy counterparts of the classical set theory. In actuality, over the past forty years, the fuzzy theory has developed into a topic of active study. Numerous scientific and technical fields have used it, including population dynamics [2], chaos control [3], computer programming [4], non-linear dynamical systems [5], and medicine [6]. Naschie [7–13] described the relationship of fuzzy Kähler interpolation of e^{∞} to the latest work on Cosmo-topology and the Poincaré dodecahedral conjecture and gave different applications and results of e^{∞}–theory from nanotechnology to brain research. This is, where the most fascinating application of fuzzy topology in quantum physics arises. Atanassov [14,15] presented the idea of intuitionistic fuzzy sets, and Oker [16] explored it in more details. We refer to [17,18] for intuitionistic fuzzy topological features. Recently, Park [19] presented the idea of intuitionistic fuzzy metric spaces. Kirişci and Simsek [20] introduced Neutrosophic metric spaces (NMSs). The concepts of intuitionistic fuzzy 2-normed spaces and intuitionistic fuzzy 2-metric spaces were introduced in [21,22], respectively. Schweizer and Sklar [23] worked on statistical metric spaces and Gähler [24] did work on 2-metric spaces. Certainly, there are some circumstances when the conventional metric is ineffective, and in these circumstances the intuitionistic fuzzy metric notion seems to be more appropriate. In other words, we may handle these circumstances by simulating the imperfection of the norm in some circumstances. In intuitionistic fuzzy 2-metric spaces (IF2MS), Mursaleen and Lohani [25] demonstrate Baire's and Cantor's Theorems ($(B\&C)$-Theorems). On IF2MS Bakry [26] established the Common fixed-point Theorem, see [27–29] for more details.

The main objectives of this manuscript are:

- To describe the notion of Neutrosophic 2-metric spaces (N2MSs), which would offer a more practical tool to address the inexactness of the metric or 2-metric in particular circumstances.
- To present $(B\&C)$-Theorems.
- In N2MS, we establish the common fixed-point theorem.

This article has four parts, in first section we will discuss some relevant definitions and examples. In second section, we will introduce the definition of N2MSs and prove some theorems in sense of N2MSs, in third section, we will prove $(B\&C)$-Theorem in the sense of N2MSs, and in last section, we find common fixed point for contraction mappings in the context of N2MS.

2. Preliminaries

In this section, we provide some basic definitions that are helpful for readers to understand the main section.

Definition 2.1. [23] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ be a continuous t-norm if it met the conditions listed below:

(a) $*$ is associative and commutative;
(b) $*$ is continuous;
(c) $\tau * 1 = \tau$ for all $\tau \in [0,1]$;
(d) $\tau * \sigma \leq c * d$ whenever $\tau \leq c$ and $\sigma \leq d$ for each $\tau, \sigma, c, d \in [0,1]$.

Definition 2.2. A binary operation $\lozenge: [0,1] \times [0,1] \to [0,1]$ be continuous t-conorm if it satisfies the above circumstance (a), (b), (d) and

(c′) $\tau \lozenge 0 = \tau$ for all $\tau \in [0,1]$.

Definition 2.3. [20] Let $\Xi \neq \emptyset$. Given a six tuple $(\Xi, \Psi, \Phi, \psi, *, \lozenge)$, where * is a CTN, ◊ is a CTCN, Ψ, Φ and ψ are neutrosophic sets on $\Xi \times \Xi \times (0, \infty)$. If $(\Xi, \Psi, \Phi, \psi, *, \lozenge)$ satisfies the following conditions that are given below for all $\varpi, \theta, \omega, \in \Xi$ and $t, s > 0$:

(N1) $\Psi(\varpi, \theta, t) + \Phi(\varpi, \theta, t) + \psi(\varpi, \theta, t) \leq 3$,
(N2) $0 \leq \Psi(\varpi, \theta, t) \leq 1$,
(N3) $\Psi(\varpi, \theta, t) = 1$ if and only if $\varpi = \theta$,
(N4) $\Psi(\varpi, \theta, t) = \Psi(\theta, \varpi, t)$,
(N5) $\Psi(\varpi, \omega, t + s) \geq \Psi(\varpi, \theta, t) * \Psi(\theta, \omega, s)$,
(N6) $\Psi(\varpi, \theta, \cdot): [0, \infty) \to [0,1]$ is continuous,
(N7) $\lim_{t \to \infty} \Psi(\varpi, \theta, t) = 1$,
(N8) $0 \leq \Phi(\varpi, \theta, t) \leq 1$,
(N9) $\Phi(\varpi, \theta, t) = 0$ if and only if $\varpi = \theta$,
(N10) $\Phi(\varpi, \theta, t) = \Phi(\theta, \varpi, t)$,
(N11) $\Phi(\varpi, \omega, t + s) \leq \Phi(\varpi, \theta, t) \lozenge \Phi(\theta, \omega, s)$,
(N12) $\Phi(\varpi, \theta, \cdot): [0, \infty) \to [0,1]$ is continuous,
(N13) $\lim_{t \to \infty} \Phi(\varpi, \theta, t) = 0$,
(N14) $0 \leq \psi(\varpi, \theta, t) \leq 1$,
(N15) $\psi(\varpi, \theta, t) = 0$ if and only if $\varpi = \theta$,
(N16) $\psi(\varpi, \theta, t) = \psi(\theta, \varpi, t)$,
(N17) $\psi(\varpi, \omega, (t + s)) \leq \psi(\varpi, \theta, t) \lozenge \psi(\theta, \omega, s)$,
(N18) $\psi(\varpi, \theta, \cdot): [0, \infty) \to [0,1]$ is continuous,
(N19) $\lim_{t \to \infty} \psi(\varpi, \theta, t) = 0$,
(N20) if $t \leq 0$, then $\Psi(\varpi, \theta, t) = 0, \Phi(\varpi, \theta, t) = 1, \psi(\varpi, \theta, t) = 1$.

Then, (Ψ, Φ, ψ) is a neutrosophic metric and $(\Xi, \Psi, \Phi, \psi, *, \lozenge)$ is a NMS.

Definition 2.4. [25] The 5-tuple $(\Xi, \Psi, \Phi, *, \lozenge)$ is said to be an IF2MS if Ξ is any non-empty set, $*$ is a continuous t-norm, ◊ is a continuous t-conorm, and Ψ, Φ are fuzzy sets on $\Xi \times \Xi \times \Xi \times (0, \infty)$, the following criteria, as listed below, must be met, for each $\varpi, \theta, \omega, w \in \Xi$ and $s, t > 0$:

(a) $\Psi(\varpi, \theta, \omega, t) + \Phi(\varpi, \theta, \omega, t) \leq 1$,
(b) Given distinct elements ϖ, θ of Ξ, there exist an element ω of Ξ such that $\Psi(\varpi, \theta, \omega, t) > 0$,
(c) $\Psi(\varpi, \theta, \omega, t) = 1$ if at least two of ϖ, θ, ω are equal,
(d) $\Psi(\varpi, \theta, \omega, t) = \Psi(\varpi, \omega, \theta, t) = \Psi(\theta, \omega, \varpi, t)$ for all ϖ, θ, ω in Ξ,
(e) $\Psi(\varpi, \theta, w, t) * \Psi(\varpi, w, \omega, s) * \Psi(w, \theta, \omega, r) \leq \Psi(\varpi, \theta, \omega, t + s + r)$ for all $\varpi, \theta, \omega, w \in \Xi$,
(f) $\Psi(\varpi, \theta, \omega, \cdot): (0, \infty) \to (0,1]$ is continuous,
(g) $\Phi(\varpi, \theta, \omega, t) < 1$,
(h) $\Phi(\varpi, \theta, \omega, t) = 0$ if at least two of ϖ, θ, ω are equal,
(i) $\Phi(\varpi, \theta, \omega, t) = \Phi(\varpi, \omega, \theta, t) = \Phi(\theta, \omega, \varpi, t)$ for all ϖ, θ, ω in Ξ,
(j) $\Phi(\varpi, \theta, w, t) \lozenge \Phi(\varpi, w, \omega, s) \lozenge \Phi(w, \theta, \omega, r) \geq \Phi(\varpi, \theta, \omega, t + s + r)$,
(k) $\Phi(\varpi, \theta, \omega, \cdot): (0, \infty) \to (0,1]$ is continuous.

Now, we define the notion of N2MSs and several topological notions in the context of N2MSs.

Definition 2.5. The 6-tuple $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ is said to be a N2MS if Ξ is any non-empty set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm, and Ψ, Φ, ψ are neutrosophic sets on $\Xi \times \Xi \times \Xi \times (0, \infty)$, satisfying the following conditions for each $\varpi, \theta, \omega, w \in \Xi$ and $s, t > 0$:

(N2M1) $\Psi(\varpi, \theta, \omega, t) + \Phi(\varpi, \theta, \omega, t) + \psi(\varpi, \theta, \omega, t) \leq 3$,

(N2M2) Given different elements ϖ, θ of Ξ,there exist an element ω of Ξ such that $\Psi(\varpi, \theta, \omega, t) > 0$,

(N2M3) $\Psi(\varpi, \theta, \omega, t) = 1$ if at least two of ϖ, θ, ω are equal,

(N2M4) $\Psi(\varpi, \theta, \omega, t) = \Psi(\varpi, \omega, \theta, t) = \Psi(\theta, \omega, \varpi, t)$ for all ϖ, θ, ω in Ξ,

(N2M5)

$$\Psi(\varpi, \theta, w, t) * \Psi(\varpi, w, \omega, s) * \Psi(w, \theta, \omega, r) \leq \Psi(\varpi, \theta, \omega, t + s + r)$$

for all $\varpi, \theta, \omega, w$ in Ξ,

(N2M6) $\Psi(\varpi, \theta, \omega \cdot): (0, \infty) \to (0,1]$ is continuous,

(N2M7) $\Phi(\varpi, \theta, \omega, t) \leq 1$,

(N2M8) $\Phi(\varpi, \theta, \omega, t) = 0$ if at least two of ϖ, θ, ω are equal,

(N2M9) $\Phi(\varpi, \theta, \omega, t) = \Phi(\varpi, \omega, \theta, t) = \Phi(\theta, \omega, \varpi, t)$ for all ϖ, θ, ω in Ξ,

(N2M10) $\Phi(\varpi, \theta, w, t) \Diamond \Phi(\varpi, w, \omega, s) \Diamond \Phi(w, \theta, \omega, r) \geq \Phi(\varpi, \theta, \omega, t + s + r)$,

(N2M11) $\Phi(\varpi, \theta, \omega, \cdot): (0, \infty) \to (0,1]$ is continuous,

(N2M12) $\psi(\varpi, \theta, \omega, t) \leq 1$,

(N2M13) $\psi(\varpi, \theta, \omega, t) = 0$ if at least two of ϖ, θ, ω are equal,

(N2M14) $\psi(\varpi, \theta, \omega, t) = \psi(\varpi, \omega, \theta, t) = \psi(\theta, \omega, \varpi, t)$ for all ϖ, θ, ω in Ξ,

(N2M15) $\psi(\varpi, \theta, w, t) \Diamond \psi(\varpi, w, \omega, s) \Diamond \psi(w, \theta, \omega, r) \geq \psi(\varpi, \theta, \omega, t + s + r)$,

(N2M16) $\psi(\varpi, \theta, \omega, \cdot): (0, \infty) \to (0,1]$ is continuous.

Here, the functions $\Psi(\varpi, \theta, \omega, t)$, $\Phi(\varpi, \theta, \omega, t)$ and $\psi(\varpi, \theta, \omega, t)$ denotes the degree of nearness, the degree of non-nearness and the degree of naturalness between ϖ, θ and ω with respect to t, respectively.

Example 2.1. Let (Ξ, d) be a 2-metric space. Suppose $\tau * \sigma = \tau \cdot \sigma$ and $\tau \Diamond \sigma = \max\{\tau, \sigma\}$ for all $\tau, \sigma \in [0,1]$ and let Ψ, Φ and ψ be neutrosophic sets on $\Xi^3 \times (0, \infty)$, defined by

$$\Psi(\varpi, \theta, \omega, t) = \frac{t}{t + md(\varpi, \theta, \omega)}, \qquad \Phi(\varpi, \theta, \omega, t) = \frac{md(\varpi, \theta, \omega)}{t + md(\varpi, \theta, \omega)}$$

and

$$\psi(\varpi, \theta, \omega, t) = \frac{md(\varpi, \theta, \omega)}{t},$$

for all $m \in R^+$. Then, $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ is an N2MS.

Proof. Conditions $(\text{N2M1}) - (\text{N2M4}), (\text{N2M6}) - (\text{N2M9}), (\text{N2M10}) - (\text{N2M14})$ and (N2M16) are trivial, here we examine (N2M5), (N2M10) and (N2M15).

N2M5. From the definition of 2-metric space, we have

$$d(\varpi, \theta, \omega) \leq d(\varpi, \theta, w) + d(\varpi, w, \omega) + d(w, \theta, \omega).$$

Therefore,

$$tsr\,md(\varpi,\theta,\omega)$$
$$\leq (rts+rs^2+r^2s)md(\varpi,\theta,w)+(rts+rt^2+tr^2)md(\varpi,w,\omega)$$
$$+(t^2s+ts^2+rts)md(w,\theta,\omega)$$
$$\Rightarrow tsrmd(\varpi,\theta,\omega)$$
$$\leq (t+s+r)rsmd(\varpi,\theta,w)+(s+t+r)rtmd(\varpi,w,\omega)+(t+s+r)tsmd(w,\theta,\omega)$$
$$\Rightarrow tsr(t+s+r)+tsrmd(\varpi,\theta,\omega)$$
$$\leq tsr(t+s+r)+(t+s+r)rsmd(\varpi,\theta,w)+(s+t+r)rtmd(\varpi,w,\omega)$$
$$+(t+s+r)tsmd(w,\theta,\omega)$$
$$\Rightarrow tsr[(t+s+r)+md(\varpi,\theta,\omega)]$$
$$\leq (t+s+r)[tsr+rsmd(\varpi,\theta,w)+rtmd(\varpi,w,\omega)+tsmd(w,\theta,\omega)].$$

That is,

$$\Rightarrow tsr[(t+s+r)+md(\varpi,\theta,\omega)]$$
$$\begin{aligned}\leq (t+s+r)[&tsr+rsmd(\varpi,\theta,w)+rtmd(\varpi,w,\omega)+tsmd(w,\theta,\omega)\\&+rm^2d(\varpi,\theta,w)d(\varpi,w,\omega)+tm^2d(w,\theta,\omega)d(\varpi,w,\omega)\\&+sm^2d(w,\theta,\omega)d(\varpi,\theta,w)+m^3d(\varpi,\theta,w)d(\varpi,w,\omega)d(w,\theta,\omega)]\end{aligned}$$
$$\Rightarrow tsr[(t+s+r)+md(\varpi,\theta,\omega)]$$
$$\leq (t+s+r)\big[\big(t+md(\varpi,\theta,w)\big)\big(s+md(\varpi,w,\omega)\big)\big(r+md(w,\theta,\omega)\big)\big]$$
$$\Rightarrow \frac{(t+s+r)}{(t+s+r)+md(\varpi,\theta,\omega)}$$
$$\geq \frac{tsr}{\big(t+md(\varpi,\theta,w)\big)\big(s+md(\varpi,w,\omega)\big)\big(r+md(w,\theta,\omega)\big)}$$
$$\Rightarrow \frac{(t+s+r)}{(t+s+r)+md(\varpi,\theta,\omega)}$$
$$\geq \frac{t}{t+md(\varpi,\theta,w)}\cdot\frac{s}{s+md(\varpi,w,\omega)}\cdot\frac{r}{r+md(w,\theta,\omega)}.$$

We have continuous t-norm $\tau * \sigma = \tau\sigma$. Hence

$$\Psi(\varpi,\theta,w,t)*\Psi(\varpi,w,\omega,s)*\Psi(w,\theta,\omega,r)\leq \Psi(\varpi,\theta,\omega,t+s+r).$$

N2M10. $\Phi(\varpi,\theta,w,t)\ \lozenge\ \Phi(\varpi,w,\omega,s)\ \lozenge\ \Phi(w,\theta,\omega,r)\geq \Phi(\varpi,\theta,\omega,t+s+r)$. Observe the fact that

$$md(\varpi,\theta,\omega)$$
$$\leq [t+s+r+md(\varpi,\theta,\omega)]\max\left\{\frac{md(\varpi,\theta,w)}{t+md(\varpi,\theta,w)},\frac{md(\varpi,w,\omega)}{s+md(\varpi,w,\omega)},\frac{md(w,\theta,\omega)}{r+dm(w,\theta,\omega)}\right\}.$$

This implies

$$\frac{md(\varpi,\theta,\omega)}{t+s+r+md(\varpi,\theta,\omega)} \leq \max\left\{\frac{md(\varpi,\theta,w)}{t+md(\varpi,\theta,w)},\frac{md(\varpi,w,\omega)}{s+md(\varpi,w,\omega)},\frac{md(w,\theta,\omega)}{r+md(w,\theta,\omega)}\right\}.$$

Then

$$\Phi(\varpi,\theta,\omega,t+s+r) \leq \max\{\Phi(\varpi,\theta,w,t),\Phi(\varpi,w,\omega,s),\Phi(w,\theta,\omega,r)\}.$$

Hence,

$$\Phi(\varpi,\theta,w,t) \Diamond \Phi(\varpi,w,\omega,s) \Diamond \Phi(w,\theta,\omega,r) \geq \Phi(\varpi,\theta,\omega,t+s+r).$$

N2M15. $\psi(\varpi,\theta,w,t) \Diamond \psi(\varpi,w,\omega,s) \Diamond \psi(w,\theta,\omega,r) \geq \psi(\varpi,\theta,\omega,t+s+r)$. Observe that,

$$md(\varpi,\theta,\omega) \leq [t+s+r+md(\varpi,\theta,\omega)]\max\left\{\frac{md(\varpi,\theta,w)}{t},\frac{md(\varpi,w,\omega)}{s},\frac{md(w,\theta,\omega)}{r}\right\}.$$

This implies

$$\frac{md(\varpi,\theta,\omega)}{t+s+r+md(\varpi,\theta,\omega)} \leq \max\left\{\frac{md(\varpi,\theta,w)}{t},\frac{md(\varpi,w,\omega)}{s},\frac{md(w,\theta,\omega)}{r}\right\}.$$

Then

$$\psi(\varpi,\theta,\omega,t+s+r) \leq \max\{\psi(\varpi,\theta,w,t),\psi(\varpi,w,\omega,s),\psi(w,\theta,\omega,r)\}.$$

Hence

$$\psi(\varpi,\theta,w,t) \Diamond \psi(\varpi,w,\omega,s) \Diamond \psi(w,\theta,\omega,r) \geq \psi(\varpi,\theta,\omega,t+s+r).$$

Therefore, $(\Xi,\Psi,\Phi,\psi,*,\Diamond)$ is an N2MS.

Definition 2.6. Suppose $(\Xi,\Psi,\Phi,\psi,*,\Diamond)$ is a N2MS. Suppose $r \in (0,1)$, $t > 0$ and $\varpi \in \Xi$. The set $\mathbb{B}(\varpi,r,t) = \{\theta \in \Xi: \Psi(\varpi,\theta,\omega,t) > 1-r, \Phi(\varpi,\theta,\omega,t) < r \text{ and } \psi(\varpi,\theta,\omega,t) < r, \text{for all } \omega \in \Xi\}$ is called the open ball with center ϖ and radius r with respect to t.

Example 2.2. Let $\Xi = \{1,2,3\}$ and (Ξ,d) be a 2-metric space defined by $d(\varpi,\theta,\omega) = |\varpi-\theta-\omega|$. Suppose $\tau * \sigma = \tau \cdot \sigma$ and $\tau \Diamond \sigma = \max\{\tau,\sigma\}$ for all $\tau,\sigma \in [0,1]$ and let Ψ, Φ and ψ be neutrosophic sets on $\Xi^3 \times (0,\infty)$, defined by

$$\Psi(\varpi,\theta,\omega,t) = \frac{t}{t+|\varpi-\theta-\omega|}, \Phi(\varpi,\theta,\omega,t) = \frac{|\varpi-\theta-\omega|}{t+|\varpi-\theta-\omega|}$$

and

$$\psi(\varpi,\theta,\omega,t) = \frac{|\varpi-\theta-\omega|}{t}.$$

Then, $(\Xi,\Psi,\Phi,\psi,*,\Diamond)$ is a N2MS.

Let the center $\varpi = 1$, radius $r = 0.6, t = 6$, then $\mathbb{B}(1,0.6,6) = \{\theta \in \Xi: \Psi(\varpi,\theta,\omega,t) >$

$0.4, \Phi(\varpi, \theta, \omega, t) < 0.6$ and $\psi(\varpi, \theta, \omega, t) < 0.6$, for all $\omega \in \Xi\}$ is an open ball.

Definition 2.7. Suppose $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ is a N2MS. Then, a set $U \subset \Xi$ is open set if each of its points is the center of some open ball contained in U. The open set in a N2MS $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ is represented by $\mathbb{U}$.

Definition 2.8. Let $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ be a N2MS. A subset Ω of Ξ is said to be $\Phi\Psi\psi_2$ – bounded if there exits $t > 0$ and $r \in (0,1)$ such that $\Psi(\varpi, \theta, \omega, t) > 1 - r, \Phi(\varpi, \theta, \omega, t) < r$ and $\psi(\varpi, \theta, \omega, t) < r$ for all $\varpi, \theta \in \Omega$, and for all $\omega \in \Xi$.

Definition 2.9. Assume $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ is a N2MS. A sequence (ϖ_n) in Ξ is a Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists $n_\circ \in \mathbb{N}$ such that $\Psi(\varpi_n, \varpi_m, \omega, t) > 1 - r$, $\Phi(\varpi_n, \varpi_m, \omega, t) < r$ and $\psi(\varpi_n, \varpi_m, \omega, t) < r$ for all $n, m \geq n_\circ$ for all $\omega \in \Xi$.

Definition 2.10. Suppose $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ is a N2MS. A sequence $\varpi = (\varpi_k)$ is convergent to $l \in \Xi$, with respect to the N2MS if, for every $\epsilon > 0$ and $t > 0$, there exist $k_0 \in \mathbb{N}$ such that $\Psi(\varpi_k, l, \omega, t) > 1 - \epsilon, \Phi(\varpi_k, l, \omega, t) < \epsilon$ and $\psi(\varpi_k, l, \omega, t) < \epsilon$ for all $k \geq k_0$ and for all $\omega \in \Xi$. In this case, we write $(\Psi, \Phi, \psi)_2 - \lim \varpi = l$ or $\varpi_k \xrightarrow{(\Psi,\Phi,\psi)_2} l$ as $k \to \infty$.

Definition 2.11. Let $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ be a N2MS. Define $\tau_{(\Psi,\Phi,\psi)_2} = \{\Omega \subset \Xi$: for each $\varpi \in \Xi$, there exist $t > 0$ and $r \in (0,1)$ such that $\mathbb{B}(\varpi, r, t) \subset \Xi\}$. Then, $\tau_{(\Psi,\Phi,\psi)_2}$ is a topology on $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$.

Definition 2.12. Let $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ be a N2MS. If each Cauchy sequence converges with respect to $\tau_{(\Psi,\Phi,\psi)_2}$ it is said to be complete.

Definition 2.13. Let $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ be a N2MS. A collection $(F_n)_{n\in\mathbb{N}}$ of non-empty sets is said to be have the neutrosophic diameter zero if for each $r \in (0,1)$, and each $t > 0$, there exists $n_\circ \in \mathbb{N}$ such that $\Psi(\varpi, \theta, \omega, t) > 1 - r$, $\Phi(\varpi, \theta, \omega, t) < r$ and $\psi(\varpi, \theta, \omega, t) < r$ for all $\varpi, \theta \in F_{n_\circ}$ and for all $\omega \in \Xi$.

Definition 2.14. Let Ξ be any non-empty set and $(Y, \Psi, \Phi, \psi, *, \Diamond)$ be a N2MS. Then, a sequence (f_n) of functions from Ξ to Y is assumed to converge uniformly to a function f from Ξ *to* Y if given $t > 0$ and $r \in (0,1)$, there exists $n_\circ \in \mathbb{N}$ such that

$$\Psi(f_n(\varpi), f(\varpi), \omega, t) > 1 - r, \Phi(f_n(\varpi), f(\varpi), \omega, t) < r \text{ and } \psi(f_n(\varpi), f(\varpi), \omega, t) < r$$

for all $n \geq n_\circ$ and for all $\varpi, \omega \in \Xi$.

3. Baire's and Cantor's intersection theorems

In this section, we establish Baire's Theorem and Cantor's Intersection Theorem in the context of N2MS.

Theorem 3.1. Every open ball $\mathbb{B}(\varpi, r, t)$ in N2MS is an open set.

Proof. Consider $\mathbb{B}(\varpi, r, t)$ be an open ball with center ϖ and radius r. Assume $\theta \in \mathbb{B}(\varpi, r, t)$. Therefore, $\Psi(\varpi, \theta, \omega, t) > 1 - r, \Phi(\varpi, \theta, \omega, t) < r,$ and $\psi(\varpi, \theta, \omega, t) < r$ for each $\omega \in \Xi$. There exists $\frac{t}{3} \in (0, t)$ such that $\Psi\left(\varpi, \theta, w, \frac{t}{3}\right) > 1 - r, \Phi\left(\varpi, \theta, w, \frac{t}{3}\right) < r,$ and $\psi\left(\varpi, \theta, w, \frac{t}{3}\right) < r$, due to

$\Psi(\varpi,\theta,\omega,t) > 1-r$. If we take $r_0 = \Psi\left(\varpi,\theta,w,\frac{t}{3}\right)$, then for $r_0 > 1-r, \varepsilon \in (0,1)$ will exist such that $r_0 > 1-\varepsilon > 1-r$. Given r_0 and ε such that $r_0 > 1-\varepsilon$. Then, $\{r_i\}_{i=1}^{6} \in (0,1)$ such that $r_0 * r_1 * r_2 > 1-\varepsilon, (1-r_0) \lozenge (1-r_3) \lozenge (1-r_4) \leq \varepsilon$, and $(1-r_0) \lozenge (1-r_5) \lozenge (1-r_6) \leq \varepsilon$. Choose $r_7 = \max\{r_i\}_{i=1}^{6}$. Consider the open ball $\mathbb{B}\left(\theta, 1-r_7, \frac{t}{3}\right)$. We will show that $\mathbb{B}\left(\theta, 1-r_7, \frac{t}{3}\right) \subset \mathbb{B}(\varpi, r, t)$. If we take $v \in \mathbb{B}\left(\theta, 1-r_7, \frac{t}{3}\right)$, then $\Psi\left(\varpi, w, \omega, \frac{t}{3}\right) > r_7$, $\Phi\left(\varpi, w, \omega, \frac{t}{3}\right) < r_7$, and $\psi\left(\varpi, w, \omega, \frac{t}{3}\right) < r_7$ and $\Psi\left(w,\theta,\omega,\frac{t}{3}\right) > r_7$, $\Phi\left(w,\theta,\omega,\frac{t}{3}\right) < r_7$, and $\psi\left(w,\theta,\omega,\frac{t}{3}\right) < r_7$. Then,

$$\Psi(\varpi,\theta,\omega,t) \geq \Psi\left(\varpi,\theta,w,\frac{t}{3}\right) * \Psi\left(\varpi,w,\omega,\frac{t}{3}\right) * \Psi\left(w,\theta,\omega,\frac{t}{3}\right)$$

$$\geq r_0 * r_7 * r_7 \geq r_0 * r_1 * r_2 \geq 1-\varepsilon > 1-r,$$

$$\Phi(\varpi,\theta,\omega,t) \geq \Phi\left(\varpi,\theta,w,\frac{t}{3}\right) * \Phi\left(\varpi,w,\omega,\frac{t}{3}\right) * \Phi\left(w,\theta,\omega,\frac{t}{3}\right)$$

$$\leq (1-r_0) \lozenge (1-r_7) \lozenge (1-r_7)$$

$$\leq (1-r_0) \lozenge (1-r_1) \lozenge (1-r_2) \leq \varepsilon < r,$$

$$\psi(\varpi,\theta,\omega,t) \geq \psi\left(\varpi,\theta,w,\frac{t}{3}\right) * \psi\left(\varpi,w,\omega,\frac{t}{3}\right) * \psi\left(w,\theta,\omega,\frac{t}{3}\right)$$

$$\leq (1-r_0) \lozenge (1-r_7) \lozenge (1-r_7)$$

$$\leq (1-r_0) \lozenge (1-r_1) \lozenge (1-r_2) \leq \varepsilon < r.$$

It shows that $v \in \mathbb{B}(\varpi, r, t)$ and $\mathbb{B}\left(\theta, 1-r_7, \frac{t}{3}\right) \subset \mathbb{B}(\varpi, r, t)$.

Theorem 3.2 Every N2MS is Hausdorff.

Proof. Let $(\Xi, \Psi, \Phi, \psi, *, \lozenge)$ be a N2MS. Let ϖ and θ be any distinct points in Ξ. Then, $0 < \Psi(\varpi,\theta,\omega,t) < 1$, $0 < \Phi(\varpi,\theta,\omega,t) < 1$, and $0 < \psi(\varpi,\theta,\omega,t) < 1$ for every $\omega \in \Xi$. Put $r_1 = \Psi(\varpi,\theta,\omega_1,t)$, $1-r_2 = \Phi(\varpi,\theta,\omega_1,t)$, and $1-r_3 = \psi(\varpi,\theta,\omega_1,t)$, $r_4 = \Psi\left(\varpi,\theta,w,\frac{t}{3}\right)$, $1-r_5 = \Phi\left(\varpi,\theta,w,\frac{t}{3}\right)$, $1-r_6 = \psi\left(\varpi,\theta,w,\frac{t}{3}\right)$ and $r = \max\{r_1, 1-r_2, 1-r_3, r_4, 1-r_5, 1-r_6\}$. For each $r_\circ \in (r, 1)$ there exist r_7 and r_8 such that $r_4 * r_7 * r_7 \geq r_\circ$ and $(1-r_5) * (1-r_8) * (1-r_8) \leq 1-r_\circ$. Put $r_9 = \max\{r_7, r_8\}$ and consider the open balls $\mathbb{B}\left(\varpi, 1-r_9, \frac{t}{3}\right)$ and $\mathbb{B}\left(\theta, 1-r_9, \frac{t}{3}\right)$. Then, clearly

$$\mathrm{y}\mathbb{B}\left(\varpi, 1-r_9, \frac{t}{3}\right) \cap \mathbb{B}\left(\theta, 1-r_9, \frac{t}{3}\right) = \emptyset.$$

If there is $w \in \mathbb{B}\left(\varpi, 1-r_9, \frac{t}{3}\right) \cap \mathbb{B}\left(\theta, 1-r_9, \frac{t}{3}\right) = \emptyset$. Then,

$$r_1 = \Psi(\varpi,\theta,\omega_1,t) \geq \Psi\left(\varpi,w,\omega_1,\frac{t}{3}\right) * \Psi\left(w,\theta,\omega_1,\frac{t}{3}\right) * \Psi\left(\varpi,\theta,w,\frac{t}{3}\right)$$

$$\geq r_4 * r_9 * r_9 \geq r_4 * r_7 * r_7 \geq r_\circ > r_1$$

and similarly, $1 - r_2 < 1 - r_2$, which is a contradiction. Hence, $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ is Hausdorff.

Definition 3.1. Suppose $\Omega \subseteq \bigcup_{U \in O} U$, a collection O of open sets is called an open cover of Ω. A subspace Ω of a NM2S is compact, if every open cover of Ω has a finite subcover.

Theorem 3.3. Every compact subset Ω of a N2MS $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ is $\Phi\Psi\psi_2$–bounded.

Proof. Let Ω be a compact subset of a N2MS $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$. Suppose the open cover $\{O(\varpi, \varepsilon, t) : \varpi \in \Omega\}$ for $t > 0, \varepsilon \in (0,1)$. Since Ω is compact, then there exist $\varpi_1, \varpi_2, \cdots, \varpi_n \in \Omega$ such that $\Omega \subseteq \bigcup_{k=1}^{n} O(\varpi_k, \varepsilon, t)$. For some $\varpi, \omega \in \Omega$, there exist $k, m \leq n$ such that $\varpi \in O(\varpi_k, \varepsilon, t)$ and $\omega \in O(\varpi_m, \varepsilon, t)$. Then, we get

$$\begin{cases} \Psi(\varpi, \varpi_k, w, t) > 1 - \varepsilon, \Phi(\varpi, \varpi_k, w, t) < \varepsilon, \psi(\varpi, \varpi_k, w, t) < \varepsilon, \\ \Psi(\omega, \varpi_m, w, t) > 1 - \varepsilon, \Phi(\omega, \varpi_m, w, t) < \varepsilon, \psi(\omega, \varpi_m, w, t) < \varepsilon, \end{cases}$$

for each $\omega \in \Xi$. Let

$$\begin{cases} \rho = \min\{\Psi(\varpi_k, \varpi_m, w, t) : 1 \leq k, m \leq n\}, \\ \sigma = \max\{\Phi(\varpi_k, \varpi_m, w, t) : 1 \leq k, m \leq n\}, \\ \gamma = \max\{\psi(\varpi_k, \varpi_m, w, t) : 1 \leq k, m \leq n\}. \end{cases}$$

Hence, for $0 < \xi_1, \xi_2, \xi_3 < 1$, we have

$$\begin{aligned} \Psi(\varpi, \omega, w, 5t) &\geq \Psi(\varpi, \varpi_k, w, t) * \Psi(\varpi, \omega, \varpi_k, t) * \Psi(\varpi_k, \omega, w, 3t) \\ &\geq \Psi(\varpi, \varpi_k, w, t) * \Psi(\varpi, \omega, \varpi_k, t) * \Psi(\varpi_k, \varpi_m, w, t) * \Psi(\varpi_k, \omega, \varpi_m, t) \\ &* \Psi(\varpi_m, \omega, w, t) \geq (1 - \varepsilon) * (1 - \varepsilon) * \rho * \rho * (1 - \varepsilon) > 1 - \xi_1, \end{aligned}$$

$$\begin{aligned} \Phi(\varpi, \omega, w, 5t) &\leq \Phi(\varpi, \varpi_k, w, t) \Diamond \Phi(\varpi, \omega, \varpi_k, t) \Diamond \Phi(\varpi_k, \omega, w, 3t) \\ &\leq \Phi(\varpi, \varpi_k, w, t) \Diamond \Phi(\varpi, \omega, \varpi_k, t) \Diamond \Phi(\varpi_k, \varpi_m, w, t) \Diamond \Phi(\varpi_k, \omega, \varpi_m, t) \\ &\Diamond \Phi(\varpi_m, \omega, w, t) \leq \varepsilon \Diamond \varepsilon \Diamond \sigma \Diamond \sigma \Diamond \varepsilon < \xi_2, \end{aligned}$$

$$\begin{aligned} \psi(\varpi, \omega, w, 5t) &\leq \psi(\varpi, \varpi_k, w, t) \Diamond \psi(\varpi, \omega, \varpi_k, t) \Diamond \psi(\varpi_k, \omega, w, 3t) \\ &\leq \psi(\varpi, \varpi_k, w, t) \Diamond \psi(\varpi, \omega, \varpi_k, t) \Diamond \psi(\varpi_k, \varpi_m, w, t) \Diamond \psi(\varpi_k, \omega, \varpi_m, t) \\ &\Diamond \psi(\varpi_m, \omega, w, t) \leq \varepsilon \Diamond \varepsilon \Diamond \sigma \Diamond \sigma \Diamond \varepsilon < \xi_3, \end{aligned}$$

If we take $\xi = \max\{\xi_1, \xi_2, \xi_3\}$ and $t_0 = 3t$, we have $\Psi(\varpi, \omega, w, t_0) > 1 - \xi, \Phi(\varpi, \omega, w, t_0) < \xi$ and $\psi(\varpi, \omega, w, t_0) < \xi$ for all $\varpi, \omega \in \Omega$. Hence, Ω is $\Phi\Psi\psi_2$–bounded.

Remark 3.1. Let $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ be a N2MS induced by a 2-metric d on Ξ. Then, $\Omega \subset \Xi$ is $\Phi\Psi\psi_2$ – bounded if and only if it is bounded.

Remark 3.2. In a N2MS every compact set is closed and bounded.

Theorem 3.4. Let $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ be a N2MS and $\tau_{(\Psi,\Phi,\psi)_2}$ be the topology on Ξ induced by the $(\Psi, \Phi, \psi)_2$. Then, for a sequence (ϖ_n) such that $\varpi_n \xrightarrow{(\Psi,\Phi,\psi)_2} \varpi$ if and only if $\Psi(\varpi_n, \varpi, \omega, t) \to 1$ and $\Phi(\varpi_n, \varpi, \omega, t) \to 0$ and $\psi(\varpi_n, \varpi, \omega, t) \to 0$ as $n \to \infty$ for all $\omega \in \Xi$ and $t > 0$.

Proof. Let $t > 0$. Suppose that $\varpi_n \xrightarrow{(\Psi,\Phi,\psi)_2} \varpi$. If $0 < \varepsilon < 1$, then there exist $N \in \mathbb{N}$ with $\varpi_n \in O(\varpi, \varepsilon, t)$, for all $n \geq N$. Therefore, $1 - \Psi(\varpi_n, \varpi, w, t) < \varepsilon, \Phi(\varpi_n, \varpi, w, t) < \varepsilon$ and $\psi(\varpi_n, \varpi, w, t) < \varepsilon$. So, we can write $\Psi(\varpi_n, \varpi, w, t) \to 1, \Phi(\varpi_n, \varpi, w, t) \to 0$, and $\psi(\varpi_n, \varpi, w, t) \to$

0 as $n \to \infty$.

Conversely, $\Psi(\varpi_n, \varpi, w, t) \to 1, \Phi(\varpi_n, \varpi, w, t) \to 0$ and $\psi(\varpi_n, \varpi, w, t) \to 0$ as $n \to \infty$, for all $t > 0$. Then, for $0 < \varepsilon < 1$, there exist $N \in \mathbb{N}$ such that $1 - \Psi(\varpi_n, \varpi, w, t) < \varepsilon, \Phi(\varpi_n, \varpi, w, t) < \varepsilon$ and $\psi(\varpi_n, \varpi, w, t) < \varepsilon$ for all $N \in \mathbb{N}$. Then, $\Psi(\varpi_n, \varpi, w, t) > 1 - \varepsilon, \Phi(\varpi_n, \varpi, w, t) < \varepsilon$ and $\psi(\varpi_n, \varpi, w, t) < \varepsilon$ for all $N \in \mathbb{N}$. Then, $\varpi_n \in O(\varpi, \varepsilon, t)$, for all $n \geq N$. This completes the proof.

Theorem 3.5. Let $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ be a NS2MS such that every Cauchy sequence in Ξ has a convergent subsequence. Then, $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ is complete.

Proof. Let the sequence (ϖ_n) be a Cauchy and let (ϖ_{n_i}) be a subsequence of (ϖ_n) and $\varpi_{n_i} \to \varpi$. Let $t > 0$ and $\mu \in (0,1)$. Consider $0 < \varepsilon < 1$ such that $(1-\varepsilon) * (1-\varepsilon) * (1-\varepsilon) \geq 1 - \mu$, $\varepsilon \Diamond \varepsilon \Diamond \varepsilon \leq \mu$. Since (ϖ_n) is a Cauchy sequence, there exist $N \in \mathbb{N}$ such that $\Psi\left(\varpi_m, \varpi_n, w, \frac{t}{3}\right) > 1 - \varepsilon, \Phi\left(\varpi_m, \varpi_n, w, \frac{t}{3}\right) < \varepsilon$ and $\psi\left(\varpi_m, \varpi_n, w, \frac{t}{3}\right) < \varepsilon$ for all $m, n \geq N$. Since $\varpi_{n_i} \to \varpi$, there is positive integer i_p such that $i_p > N$, $\Psi\left(\varpi_{i_p}, \varpi, w, \frac{t}{3}\right) > 1 - \varepsilon, \Phi\left(\varpi_{i_p}, \varpi, w, \frac{t}{3}\right) < \varepsilon$ and $\psi\left(\varpi_{i_p}, \varpi, w, \frac{t}{3}\right) < \varepsilon$. Therefore, if $n \geq N$,

$$\Psi(\varpi_n, \varpi, w, t) \geq \Psi\left(\varpi_{i_p}, \varpi, w, \frac{t}{3}\right) * \Psi\left(\varpi_n, \varpi_{i_p}, w, \frac{t}{3}\right) * \Psi\left(\varpi_n, \varpi, \varpi_{i_p}, \frac{t}{3}\right)$$

$$> (1-\varepsilon) * (1-\varepsilon) * (1-\varepsilon) \geq 1 - \mu,$$

$$\Phi(\varpi_n, \varpi, w, t) \leq \Phi\left(\varpi_{i_p}, \varpi, w, \frac{t}{3}\right) \Diamond \Phi\left(\varpi_n, \varpi_{i_p}, w, \frac{t}{3}\right) \Diamond \Phi\left(\varpi_n, \varpi, \varpi_{i_p}, \frac{t}{3}\right) < \varepsilon \Diamond \varepsilon \Diamond \varepsilon \leq \mu,$$

$$\psi(\varpi_n, \varpi, w, t) \leq \psi\left(\varpi_{i_p}, \varpi, w, \frac{t}{3}\right) \Diamond \psi\left(\varpi_n, \varpi_{i_p}, w, \frac{t}{3}\right) \Diamond \psi\left(\varpi_n, \varpi, \varpi_{i_p}, \frac{t}{3}\right) < \varepsilon \Diamond \varepsilon \Diamond \varepsilon \leq \mu.$$

Thus, we have $\varpi_n \to \varpi$. This completes the proof.

Theorem 3.6. Let $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ be a N2MS and let Ω be subset of Ξ with the subspace N2MS $(\Psi_\Omega, \Phi_\Omega, \psi_\Omega)_2 = \left(\Psi|_{\Omega^3 \times (0,\infty)}, \ \Phi|_{\Omega^3 \times (0,\infty)}, \psi|_{\Omega^3 \times (0,\infty)}\right)_2$. Then, $(\Omega, \Psi_\Omega, \Phi_\Omega, \psi_\Omega, *, \Diamond)$ is complete if and only if Ω is closed subset of Ξ.

Proof. Assume Ω is a closed subset of Ξ and let ϖ_n be a Cauchy sequence in $(\Omega, \Psi_\Omega, \Phi_\Omega, \psi_\Omega, *, \Diamond)$. Then, $(\varpi)_n$ be a Cauchy sequence in Ξ and hence there is a point $\varpi \in \Xi$ such that $\varpi_n \xrightarrow{(\Psi,\Phi,\psi)_2} \varpi$. Then, $\varpi \in \bar{\Omega} = \Omega$ and thus $(\varpi)_n$ converges in Ω. Hence $(\Omega, \Psi_\Omega, \Phi_\Omega, \psi_\Omega, *, \Diamond)$ is complete.

Conversely, let $(\Omega, \Psi_\Omega, \Phi_\Omega, \psi_\Omega, *, \Diamond)$ is a complete and Ω is not closed. Let $\varpi \in \bar{\Omega} \backslash \Omega$. Then, there is a sequence $(\varpi)_n$ of points in Ω that converges to ϖ and thus $(\varpi)_n$ is a Cauchy sequence. Thus, for each $0 < \epsilon < 1$ and each > 0, there is $k_\circ \in \mathbb{N}$ such that $\Psi(\varpi_k, \varpi_l, \theta, t) > 1 - \epsilon, \Phi(\varpi_k, \ \varpi_l, \theta, t) < \epsilon$ and $\psi(\varpi_k, \ \varpi_l, \theta, t) < \epsilon$ for all $k, l \geq k_\circ$ and for all $\theta \in \Xi$. Since, $(\varpi)_n$ is a sequence in Ω,

$$\Psi(\varpi_k, \varpi_l, \theta, t) = \Psi_\Omega(\varpi_k, \varpi_l, \theta, t), \Phi(\varpi_k, \varpi_l, \theta, t) = \Phi_\Omega(\varpi_k, \varpi_l, \theta, t)$$

and

$$\psi(\varpi_k, \varpi_l, \theta, t) = \psi_\Omega(\varpi_k, \varpi_l, \theta, t)$$

for all $\theta \in \Xi$. Therefore $(\varpi)_n$ is a Cauchy sequence in Ω. Since $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ is complete, there is a $\omega \in \Omega$ such that $\varpi_n \to \omega$. That is, for each $0 < \epsilon < 1$ and each $t > 0$, there is $k_\circ \in \mathbb{N}$ such that $\Psi(\varpi_l, \omega, \theta, t) > 1 - \epsilon$, $\Phi(\varpi_l, \omega, \theta, t) < \epsilon$ and $\psi(\varpi_l, \omega, \theta, t) < \epsilon$ for all $l \geq k_\circ$ and for all $\theta \in \Xi$. But since $(\varpi)_n$ is a sequence in Ω and $\omega \in \Omega$, $\Psi(\varpi_l, \omega, \theta, t) = \Psi_\Omega(\varpi_l, \omega, \theta, t)$, $\Phi(\varpi_l, \omega, \theta, t) = \Phi_\Omega(\varpi_l, \omega, \theta, t)$ and $\psi(\varpi_l, \omega, \theta, t) = \psi_\Omega(\varpi_l, \omega, \theta, t)$ we see that $(\varpi)_n$ converges in $(\Omega, \Psi, \Phi, \psi, *, \Diamond)$ to both ϖ and ω. Since, $\varpi \notin \Omega$ and $\omega \in \Omega$, $\varpi \neq \omega$, that results in a contradiction.

Lemma 3.1. Let $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ be a N2MS. If $t > 0$ and $r, s \in (0,1)$ such that $(1-s) * (1-s) * \alpha \geq (1-r)$, $s \Diamond s \Diamond \alpha' \leq r$, then $\overline{\mathbb{B}\left(\varpi, s, \frac{t}{3}\right)} \subset \mathbb{B}(\varpi, r, t)$, where $\alpha = \Psi\left(\varpi, \theta, w, \frac{t}{3}\right)$, $\alpha' = \Phi\left(\varpi, \theta, w, \frac{t}{3}\right)$ and $\alpha' = \psi\left(\varpi, \theta, w, \frac{t}{3}\right)$.

Theorem 3.7. A subset Ω of a N2MS $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ is nowhere dense if and only if every non-empty open set in Ξ contains an open ball whose closure is disjoint from Ω.

Proof. Let $\mathbb{U}$ be a non-empty open subset of Ξ. Then, there exist a non-empty open subset of $\mathbb{V} \subset \mathbb{U}$ and $\mathbb{V} \cap \bar{\Omega} \neq \emptyset$. Let $\varpi \in \mathbb{V}$. Then, there exist $r \in (0,1)$ and $t > 0$ such that $\mathbb{B}(\varpi, r, t) \subset \mathbb{V}$. Choose $s \in (0,1)$ such that $(1-s) * (1-s) * \alpha \geq (1-r)$ and $s \Diamond s \Diamond \alpha' \leq r$ for some fixed $\alpha \in (0,1)$. By Lemma 3.1, we have $\overline{\mathbb{B}\left(\varpi, s, \frac{t}{3}\right)} \subset \mathbb{B}(\varpi, r, t)$. Thus $\overline{\mathbb{B}\left(\varpi, s, \frac{t}{3}\right)} \subset \mathbb{U}$ and $\overline{\mathbb{B}\left(\varpi, s, \frac{t}{3}\right)} \cap \Omega = \emptyset$.

Conversely suppose Ω is not nowhere dense. Then, $\operatorname{int} \bar{\Omega} \neq \emptyset$, so there exists a non-empty open set $\mathbb{U}$ such that $\mathbb{U} \subset \bar{\Omega}$. Let $\mathbb{B}(\varpi, r, t)$ be an open ball such that $\mathbb{B}(\varpi, r, t) \subset \mathbb{U}$. Then, $\overline{\mathbb{B}(\varpi, r, t)} \cap \Omega \neq \emptyset$. This is a contradiction.

Theorem 3.8. Let $(\mathbb{U}_n : n \in \mathbb{N})$ be a sequence of dense open subsets of a complete N2MS $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$. Then, $\cap_{n \in \mathbb{N}} \mathbb{U}_n$ is also dense in $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$.

Proof. Let $\mathbb{V}$ be a non-empty open set of Ξ. Since, $\mathbb{U}_1$ is dense in Ξ, $\mathbb{V} \cap \mathbb{U}_1 \neq \emptyset$. Let $\varpi_1 \in \mathbb{V} \cap \mathbb{U}_1$. Since $\mathbb{V} \cap \mathbb{U}_1$ is open, there exists $r_1 \in (0,1)$ and $t_1 > 0$ such that $\mathbb{B}(\varpi_1, r_1, t_1) \subset \mathbb{V} \cap \mathbb{U}_1$. Choose $r_1' < r_1$ and $t_1' = \min(t_1, 1)$ such that $\overline{\mathbb{B}(\varpi_1, r_1', t_1')} \subset \mathbb{V} \cap \mathbb{U}_1$. Since, $\mathbb{U}_2$ is dense in Ξ. By Theorem 3.1 $\mathbb{B}(\varpi_1, r_1', t_1') \cap \mathbb{U}_2 \neq \emptyset$. Let $\varpi_2 \in \mathbb{B}(\varpi_1, r_1', t_1') \cap \mathbb{U}_2$. Since, $\mathbb{B}(\varpi_1, r_1', t_1') \cap \mathbb{U}_2$ is open, there exists $r_2 \in \left(0, \frac{1}{2}\right)$ and $t_2 > 0$ such that $\mathbb{B}(\varpi_2, r_2, t_2) \subset \mathbb{B}(\varpi_1, r_1', t_1') \cap \mathbb{U}_2$. Choose $r_2' < r_2$ and $t_2' = \min\left(t_2, \frac{1}{2}\right)$ such that $\overline{\mathbb{B}(\varpi_2, r_2', t_2')} \subset \mathbb{B}(\varpi_1, r_1', t_1') \cap \mathbb{U}_2$. Continuing in this manner, we obtain a sequence $(\varpi)_n$ in Ξ and a sequence (t_n') such that

$$0 < t_n' < \frac{1}{n} \text{ and } \overline{\mathbb{B}(\varpi_{n+1}, r_{n+1}', t_{n+1}')} \subset \mathbb{B}(\varpi_n, r_n', t_n') \cap \mathbb{U}_{n+1}.$$

Now, it's simple to observe that $(\varpi)_n$ is a Cauchy sequence. Since Ξ is complete, there exists $\varpi \in \Xi$ such that $\varpi_n \xrightarrow{(\Psi, \Phi, \psi)_2} \varpi$. Since $\varpi_k \in \mathbb{B}(\varpi_n, r_n', t_n')$ for $k \geq n$, we obtain $\varpi \in \overline{\mathbb{B}(\varpi_n, r_n', t_n')}$. Hence $\varpi \in \overline{\mathbb{B}(\varpi_n, r_n', t_n')} \subset \mathbb{B}(\varpi_{n-1}, r_{n-1}', t_{n-1}') \cap \mathbb{U}_n$ for all n. Therefore, $V \cap (\cap_{n \in \mathbb{N}} \mathbb{U}_n \neq \emptyset$. Hence $\cap_{n \in \mathbb{N}} \mathbb{U}_n$ is dense in $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$.

Remark 3.3. A non-empty subset $F$of a N2MS Ξ has Neutrosophic diameter zero if and only if F is a singleton set.

Theorem 3.9. A N2MS $(\varXi,\varPsi,\varPhi,\psi,*,\Diamond)$ is complete if and only if every nested sequence $(F_n)_{n\in\mathbb{N}}$ of non-empty closed sets with Neutrosophic diameter zero have non-empty intersection.

Proof. Firstly, we prove that $(\varXi,\varPsi,\varPhi,\psi,*,\Diamond)$ is complete under the given hypothesis. Let (ϖ_n) be a Cauchy sequence in $\varXi$. Set $B_n=(\varpi_k: k\geq n)$ and $F_n=\bar{B}_n$, then we claim that F_n has Neutrosophic diameter zero. For given $s\in(0,1)$ and $t>0$, we choose $r\in(0,1)$ such that $(1-r)*(1-r)*(1-r)*(1-r)*(1-r)>1-s$ and $r\Diamond r\Diamond r\Diamond r\Diamond r<s$. Since (ϖ_n) is a Cauchy, there exists $n_\circ\in\mathbb{N}$ such that $\varPsi\left(\varpi_n,\varpi_m,\omega,\frac{t}{9}\right)>1-r$, $\varPhi\left(\varpi_n,\varpi_m,\omega,\frac{t}{9}\right)<r$ and $\psi\left(\varpi_n,\varpi_m,\omega,\frac{t}{9}\right)<r$ for all $n,m\geq n_\circ$ and for all $\omega\in\varXi$. Therefore $\varPsi\left(\varpi,\theta,\omega,\frac{t}{9}\right)>1-r$, $\varPhi\left(\varpi,\theta,\omega,\frac{t}{9}\right)<r$ and $\psi\left(\varpi,\theta,\omega,\frac{t}{9}\right)<r$ for all $\varpi,\theta\in B_{n_\circ}$ and for all $\omega\in\varXi$. Let $\varpi,\theta\in F_{n_\circ}$. Then, there exist sequences (ϖ_n') and (θ_n') in $B_{n_\circ}$ such that $\varpi_n'\xrightarrow{(\varPsi,\varPhi,\psi)_2}\varpi$ and $\theta_n'\xrightarrow{(\varPsi,\varPhi,\psi)_2}\theta$. Hence, $\varpi_n'\in\mathbb{B}\left(\varpi,r,\frac{t}{9}\right)$ and $\theta_n'\in\mathbb{B}\left(\theta,r,\frac{t}{9}\right)$ for sufficiently large n. Now $\varpi_n'\in\mathbb{B}\left(\varpi,r,\frac{t}{9}\right)$ implies that $\varPsi\left(\varpi,\varpi_n',\omega,\frac{t}{9}\right)>1-r$, $\varPhi\left(\varpi,\varpi_n',\omega,\frac{t}{9}\right)<r$ and $\psi\left(\varpi,\varpi_n',\omega,\frac{t}{9}\right)<r$ for all $\omega\in\varXi$, therefore in particular for some $\theta\in\varXi$ we have $\varPsi\left(\varpi,\varpi_n',\theta,\frac{t}{9}\right)>1-r$, $\varPhi\left(\varpi,\varpi_n',\theta,\frac{t}{9}\right)<r$ and $\psi\left(\varpi,\varpi_n',\theta,\frac{t}{9}\right)<r$, similarly for $\theta_n'\in\mathbb{B}\left(\theta,r,\frac{t}{9}\right)$. Now, we have

$$\varPsi(\varpi,\theta,\omega,t)\geq\varPsi\left(\varpi_n',\theta,\omega,\frac{t}{3}\right)*\varPsi\left(\varpi,\varpi_n',\omega,\frac{t}{3}\right)*\varPsi\left(\varpi,\theta,\varpi_n',\frac{t}{3}\right),$$

$$\varPsi\left(\varpi_n',\theta,\omega,\frac{t}{3}\right)\geq\varPsi\left(\theta_n',\theta,\omega,\frac{t}{9}\right)*\varPsi\left(\varpi_n',\theta_n',\omega,\frac{t}{9}\right)*\varPsi\left(\varpi_n',\theta,\theta_n',,\frac{t}{9}\right).$$

Hence

$$\varPsi(\varpi,\theta,\omega,t)\geq(1-r)*(1-r)*(1-r)*(1-r)*(1-r)>1-s,$$

$$\varPhi(\varpi,\theta,\omega,t)\leq r\Diamond r\Diamond r\Diamond r\Diamond r<s,$$

and similarly

$$\psi(\varpi,\theta,\omega,t)\leq r\Diamond r\Diamond r\Diamond r\Diamond r<s,$$

for all $\varpi,\theta\in F_{n_\circ}$ and for all $\omega\in\varXi$. Thus (F_n) has Neutrosophic diameter zero and hence by hypothesis $\cap_{n\in\mathbb{N}}F_n$ is non-empty. Take $\varpi\in\cap_{n\in\mathbb{N}}F_n$. We see that $\varpi_n\xrightarrow{(\varPsi,\varPhi,\psi)_2}\varpi$. Then, for each $t\to 0$, there exist $n_\circ\in\mathbb{N}$ such that $\varPsi(\varpi_n,\varpi,\omega,t)>1-r$, $\varPhi(\varpi_n,\varpi,\omega,t)<r$ and $\psi(\varpi_n,\varpi,\omega,t)<r$ for all $n\geq n_\circ$ and for all $\omega\in\varXi$. Therefore, for each $t>0$, $\varPsi(\varpi_n,\varpi,\omega,t)\to 1$, $\varPhi(\varpi_n,\varpi,\omega,t)\to 0$ and $\psi(\varpi_n,\varpi,\omega,t)\to 0$ as $n\to\infty$ for all $\omega\in\varXi$. Hence, $(\varXi,\varPsi,\varPhi,\psi,*,\Diamond)$ is complete.

Conversely, suppose that $(\varXi,\varPsi,\varPhi,\psi,*,\Diamond)$ is complete and $(F_n)_{n\in\mathbb{N}}$ is nested sequence of non-empty closed sets with Neutrosophic diameter zero. For each $n\in\mathbb{N}$, choose a point $\varpi_n\in F_n$. We claim that (ϖ_n) is a Cauchy sequence. Since (F_n) has Neutrosophic diameter zero, for $r\in(0,1)$ and $t>0$, there exists $n_\circ\in\mathbb{N}$ such that $\varPsi(\varpi,\theta,\omega,t)>1-r$, $\varPhi(\varpi,\theta,\omega,t)<r$, and $\psi(\varpi,\theta,\omega,t)<r$ for all $n\geq n_\circ$, $\varpi,\theta\in F_n$, and $\omega\in\varXi$. Since (F_n) is nested sequence, $\varPsi(\varpi_n,\theta_n,\omega,t)>1-r$, $\varPhi(\varpi_n,\theta_n,\omega,t)<r$, and $\psi(\varpi_n,\theta_n,\omega,t)<r$ for all $n,m\in n_\circ$ and for all $\omega\in\varXi$. Hence (ϖ_n) is a

Cauchy sequence. Since $(\Xi,\Psi,\Phi,\psi,*,\Diamond)$ is complete, $\varpi_n \xrightarrow{(\Psi,\Phi,\psi)_2} \varpi$ for some $\varpi \in \Xi$. Therefore $\varpi \in \bar{F}_n = F_n$ for every n, and hence $\varpi \in \cap_{n\in\mathbb{N}} F_n$.

Remark 3.4. The element $\varpi \in \cap_{n\in\mathbb{N}} F_n$ is unique.

Note that the topologies induced by the standard N2MS and the corresponding 2-metric are same. So, we have the following result.

Corollary 3.1. A 2-metric space (Ξ, d) is complete if and only if every nested sequence $(F_n)_{n\in\mathbb{N}}$ of non-empty closed sets with diameter tending to zero have non-empty intersection.

4. Common fixed-point results in N2MS

Lemma 4.1. If $(\Xi,\Psi,\Phi,\psi,*,\Diamond)$ be a N2MS. Then, $\Psi(\varpi,\theta,\omega,t)$ is non-decreasing $\Phi(\varpi,\theta,\omega,t)$ non-increasing and $\psi(\varpi,\theta,\omega,t)$ is non-increasing for all $\varpi,\theta,\omega \in \Xi$.

Proof. Let $s,t > 0$ be any points such that $t > s. t = s + \frac{t-s}{2} + \frac{t-s}{2}$. Hence, we have

$$\Phi(\varpi,\theta,\omega,t) = \Phi\left(\varpi,\theta,\omega,s + \frac{t-s}{2} + \frac{t-s}{2}\right)$$

$$\leq \Phi(\varpi,\theta,\omega,s) \Diamond \Phi\left(\varpi,\omega,\omega,\frac{t-s}{2}\right) \Diamond \Phi\left(\omega,\theta,\omega,\frac{t-s}{2}\right) = \Phi(\varpi,\theta,\omega,s)$$

and

$$\psi(\varpi,\theta,\omega,t) = \psi\left(\varpi,\theta,\omega,s + \frac{t-s}{2} + \frac{t-s}{2}\right)$$

$$\leq \psi(\varpi,\theta,\omega,s) \Diamond \psi\left(\varpi,\omega,\omega,\frac{t-s}{2}\right) \Diamond \psi\left(\omega,\theta,\omega,\frac{t-s}{2}\right) = \psi(\varpi,\theta,\omega,s).$$

Similarly, $\Psi(\varpi,\theta,\omega,t) > \Psi(\varpi,\theta,\omega,s)$.

From Lemma 4.1, let $(\Xi,\Psi,\Phi,\psi,*,\Diamond)$ be a N2MS with the following conditions:

$$\lim_{t\to\infty} \Psi(\varpi,\theta,\omega,t) = 1, \lim_{t\to\infty} \Phi(\varpi,\theta,\omega,t) = 0 \text{ and } \lim_{t\to\infty} \psi(\varpi,\theta,\omega,t) = 0.$$

Lemma 4.2. Let $(\Xi,\Psi,\Phi,\psi,*,\Diamond)$ be a N2MS. If there exists $q \in (0,1)$ such that $\Psi(\varpi,\theta,\omega,qt+0) \geq \Psi(\varpi,\theta,\omega,t)$, $\Phi(\varpi,\theta,\omega,qt+0) \leq \Phi(\varpi,\theta,\omega,t)$ and $\psi(\varpi,\theta,\omega,qt+0) \leq \psi(\varpi,\theta,\omega,t)$ for all $\varpi,\theta,\omega \in \Xi$ with $\omega \neq \varpi, \omega \neq \theta$ and $t > 0$. Then, $\varpi = \theta$.

Proof. Since

$$\Psi(\varpi,\theta,\omega,t) \geq \Psi(\varpi,\theta,\omega,qt+0) \geq \Psi(\varpi,\theta,\omega,t),$$

$$\Phi(\varpi,\theta,\omega,t) \leq \Phi(\varpi,\theta,\omega,qt+0) \leq \Phi(\varpi,\theta,\omega,t),$$

and

$$\psi(\varpi,\theta,\omega,t) \leq \psi(\varpi,\theta,\omega,qt+0) \leq \psi(\varpi,\theta,\omega,t)$$

for all $t > 0$, $\Psi(\varpi, \theta, \omega, .)$, $\Phi(\varpi, \theta, \omega, .)$ and $\psi(\varpi, \theta, \omega)$ are constant. Since $\lim_{t\to\infty} \Psi(\varpi, \theta, \omega, t) = 1$, $\lim_{t\to\infty} \Phi(\varpi, \theta, \omega, t) = 0$ and $\lim_{t\to\infty} \psi(\varpi, \theta, \omega, t) = 0$. Then, $\Psi(\varpi, \theta, \omega, t) = 1$, $\Phi(\varpi, \theta, \omega, t) = 0$ and $\psi(\varpi, \theta, \omega, t) = 0$. Consequently, for all $t > 0$. Hence $\varpi = \theta$ because $\omega \neq \varpi$, $\omega \neq \theta$.

Lemma 4.3. Let $(\Xi, \Psi, \Phi, \psi, *, \Diamond)$ be a N2MS and let $\lim_{t\to\infty} \varpi_n = \varpi$, $\lim_{t\to\infty} \theta_n = \theta$. Then, the following conditions are satisfied for all $\tau \in \Xi$ and $t \geq 0$:

(1)

$$\lim_{n\to\infty} \inf \Psi(\varpi_n, \theta_n, \tau, t) \geq \Psi(\varpi, \theta, \tau, t),\ \lim_{n\to\infty} \sup \Phi(\varpi_n, \theta_n, \tau, t) \leq \Phi(\varpi, \theta, \tau, t)$$

and

$$\lim_{n\to\infty} \sup \psi(\varpi_n, \theta_n, \tau, t) \leq \psi(\varpi, \theta, \tau, t).$$

(2)

$$\Psi(\varpi, \theta, \tau, t) \geq \lim_{n\to\infty} \mathrm{Sup} \Psi(\varpi_n, \theta_n, \tau, t), \Phi(\varpi, \theta, \tau, t + 0) \leq \lim_{n\to\infty} \inf \Phi(\varpi_n, \theta_n, \tau, t)$$

and

$$\psi(\varpi, \theta, \tau, t + 0) \leq \lim_{n\to\infty} \inf \psi(\varpi_n, \theta_n, \tau, t).$$

Proof. For all $\tau \in \Xi$ and $t \geq 0$, we have

$$\begin{aligned} \Psi(\varpi_n, \theta_n, \tau, t) &\geq \Psi(\varpi_n, \theta_n, \varpi, t_1) * \Psi(\varpi_n, \varpi, \tau, t_2) * \Psi(\varpi, \theta_n, \tau, t), \qquad t_1 + t_2 = 0 \\ &\geq \Psi(\varpi_n, \theta_n, \varpi, t_1) * \Psi(\varpi_n, \varpi, \tau, t_2) * \Psi(\varpi, \theta_n, \theta, t_3) \\ &\quad * \Psi(\varpi, \theta, \tau, t_4) * \Psi(\theta, \theta_n, \tau, t), \qquad t_3 + t_4 = 0 \end{aligned}$$

which implies $\lim_{n\to\infty} \inf \Psi(\varpi_n, \theta_n, \tau, t) \geq 1 * 1 * 1 * \Psi(\varpi, \theta, \tau, t) * 1 = \Psi(\varpi, \theta, \tau, t)$, also

$$\begin{aligned} \Phi(\varpi_n, \theta_n, \tau, t) &\leq \Phi(\varpi_n, \theta_n, \varpi, t_1) \Diamond \Phi(\varpi_n, \varpi, \tau, t_2) \Diamond \Phi(\varpi, \theta_n, \tau, t), \qquad t_1 + t_2 = 0 \\ &\leq \Phi(\varpi_n, \theta_n, \varpi, t_1) \Diamond \Phi(\varpi_n, \varpi, \tau, t_2) \Diamond \Phi(\varpi, \theta_n, \theta, t_3) \\ &\quad \Diamond \Phi(\varpi, \theta, \tau, t_4) \Diamond \Phi(\theta, \theta_n, \tau, t), \qquad t_3 + t_4 = 0 \end{aligned}$$

which implies

$$\lim_{n\to\infty} \mathrm{Sup}\, \Phi(\varpi_n, \theta_n, \tau, t) \leq 0 \Diamond 0 \Diamond 0 \Diamond \Phi(\varpi, \theta, \tau, t) \Diamond 0 = \Phi(\varpi, \theta, \tau, t)$$

and

$$\begin{aligned} \psi(\varpi_n, \theta_n, \tau, t) &\leq \psi(\varpi_n, \theta_n, \varpi, t_1) \Diamond \psi(\varpi_n, \varpi, \tau, t_2) \Diamond \psi(\varpi, \theta_n, \tau, t), \qquad t_1 + t_2 = 0 \\ &\leq \psi(\varpi_n, \theta_n, \varpi, t_1) \Diamond \psi(\varpi_n, \varpi, \tau, t_2) \Diamond \psi(\varpi, \theta_n, \theta, t_3) \\ &\quad \Diamond \psi(\varpi, \theta, \tau, t_4) \Diamond \psi(\theta, \theta_n, \tau, t), \qquad t_3 + t_4 = 0 \end{aligned}$$

which implies $\lim_{n\to\infty} \text{Sup}\, \psi(\varpi_n, \theta_n, \tau, t) \leq 0 \lozenge 0 \lozenge 0 \lozenge \psi(\varpi, \theta, \tau, t) \lozenge 0 = \psi(\varpi, \theta, \tau, t)$.

(2) Let $\varepsilon > 0$ be given. For all $\tau \in \varpi$ and $t > 0$, we have

$$\Psi(\varpi, \theta, \tau, t + 2\varepsilon) \geq \Psi\left(\varpi, \theta, \varpi_n, \frac{\varepsilon}{2}\right) * \Psi\left(\varpi, \varpi_n, \tau, \frac{\varepsilon}{2}\right) * \Psi(\varpi_n, \theta, \tau, t + \varepsilon)$$

$$\geq \Psi\left(\varpi, \theta, \varpi_n, \frac{\varepsilon}{2}\right) * \Psi\left(\varpi, \varpi_n, \tau, \frac{\varepsilon}{2}\right) * \Psi\left(\varpi_n, \theta, \theta_n, \frac{\varepsilon}{2}\right) * \Psi(\varpi_n, \theta_n, \tau, t) * \Psi\left(\theta_n, \theta, \tau, \frac{\varepsilon}{2}\right).$$

Consequently,

$$\Psi(\varpi, \theta, \tau, t + 2\varepsilon) \geq \lim_{n\to\infty} \sup \Psi(\varpi_n, \theta_n, \tau, t).$$

Letting $\varepsilon \to 0$, we have

$$\Psi(\varpi, \theta, \tau, t + 0) \geq \lim_{n\to\infty} \sup \Psi(\varpi_n, \theta_n, \tau, t).$$

Also, we have

$$\Phi(\varpi, \theta, \tau, t + 2\varepsilon) \leq \Phi\left(\varpi, \theta, \varpi_n, \frac{\varepsilon}{2}\right) \lozenge \Phi\left(\varpi, \varpi_n, \tau, \frac{\varepsilon}{2}\right) \lozenge \Phi(\varpi_n, \theta, \tau, t + \varepsilon)$$

$$\geq \Phi\left(\varpi, \theta, \varpi_n, \frac{\varepsilon}{2}\right) \lozenge \Phi\left(\varpi, \varpi_n, \tau, \frac{\varepsilon}{2}\right) \lozenge \Phi\left(\varpi_n, \theta, \theta_n, \frac{\varepsilon}{2}\right) \lozenge \Phi(\varpi_n, \theta_n, \tau, t)$$

$$\lozenge \Phi\left(\theta_n, \theta, \tau, \frac{\varepsilon}{2}\right) \lozenge \Phi(\varpi_n, \theta_n, \tau, t) \lozenge \Phi\left(\theta_n, \theta, \tau, \frac{\varepsilon}{2}\right).$$

Consequently,

$$\Phi(\varpi, \theta, \tau, t + 2\varepsilon) \leq \lim_{n\to\infty} \inf \Phi(\varpi_n, \theta_n, \tau, t).$$

Letting $\varepsilon \to 0$, we have

$$\Phi(\varpi, \theta, \tau, t + 0) \leq \lim_{n\to\infty} \inf \Phi(\varpi_n, \theta_n, \tau, t)\,.$$

and

$$\psi(\varpi, \theta, \tau, t + 2\varepsilon) \leq \psi\left(\varpi, \theta, \varpi_n, \frac{\varepsilon}{2}\right) \lozenge \psi\left(\varpi, \varpi_n, \tau, \frac{\varepsilon}{2}\right) \lozenge \psi(\varpi_n, \theta, \tau, t + \varepsilon)$$

$$\geq \psi\left(\varpi, \theta, \varpi_n, \frac{\varepsilon}{2}\right) \lozenge \psi\left(\varpi, \varpi_n, \tau, \frac{\varepsilon}{2}\right) \lozenge \psi\left(\varpi_n, \theta, \theta_n, \frac{\varepsilon}{2}\right) \lozenge \psi(\varpi_n, \theta_n, \tau, t)$$

$$\lozenge \psi\left(\theta_n, \theta, \tau, \frac{\varepsilon}{2}\right) \lozenge \psi(\varpi_n, \theta_n, \tau, t) \lozenge \psi\left(\theta_n, \theta, \tau, \frac{\varepsilon}{2}\right).$$

Consequently,

$$\psi(\varpi,\theta,\tau,t+2\varepsilon)\le \lim_{n\to\infty}\inf\psi(\varpi_n,\theta_n,\tau,t).$$

Letting $\varepsilon\to 0$, we have

$$\psi(\varpi,\theta,\tau,t+0)\le \lim_{n\to\infty}\inf\psi(\varpi_n,\theta_n,\tau,t).$$

Lemma 4.4. Let $(\Xi,\Psi,\Phi,\psi,*,\lozenge)$ be a N2MS and let Ω and B continuous self-mappings of Ξ and $[\Omega,B]$ are compatible. Let ϖ_n be a sequence in X such that $\Omega\varpi_n\to\omega$ and $B\varpi_n\to\omega$. Then, $\Omega B\varpi_n\to B\omega$.

Proof. Since Ω,B are compatible maps, $\Omega B\varpi_n\to\Omega\omega, B\Omega\varpi_n\to B\omega$ and so, $\Psi\left(\Omega B\varpi_n,\Omega\omega,\tau,\frac{t}{3}\right)\to 1$, $\Phi\left(B\Omega\varpi_n,B\omega,\tau,\frac{t}{3}\right)\to 0$ and $\psi\left(B\Omega\varpi_n,B\omega,\tau,\frac{t}{3}\right)\to 0$ for all $\tau\in\Xi$ and $t>0$.

$$\Psi(\Omega B\varpi_n,B\omega,\tau,t)\ge\Psi\left(\Omega B\varpi_n,B\omega,B\Omega\varpi_n,\frac{t}{3}\right)*\Psi\left(\Omega B\varpi_n,B\Omega\varpi_n,\tau,\frac{t}{3}\right)*\Psi\left(B\Omega\varpi_n,B\omega,\tau,\frac{t}{3}\right)$$

$$\ge\Psi\left(B\Omega\varpi_n,B\omega,\Omega B\varpi_n,\frac{t}{3}\right)*\Psi\left(B\Omega\varpi_n,\Omega B\varpi_n,\tau,\frac{t}{3}\right)*\Psi\left(B\Omega\varpi_n,B\omega,\tau,\frac{t}{3}\right)\to 1.$$

Also, we have

$$\Phi(\Omega B\varpi_n,B\omega,\tau,t)\le\Phi\left(\Omega B\varpi_n,B\omega,B\Omega\varpi_n,\frac{t}{3}\right)\lozenge\Phi\left(\Omega B\varpi_n,B\Omega\varpi_n,\tau,\frac{t}{3}\right)\lozenge\Phi\left(B\Omega\varpi_n,B\omega,\tau,\frac{t}{3}\right)$$

$$\le\Phi\left(B\Omega\varpi_n,B\omega,\Omega B\varpi_n,\frac{t}{3}\right)\lozenge\Phi\left(B\Omega\varpi_n,\Omega B\varpi_n,\tau,\frac{t}{3}\right)\lozenge\Phi\left(B\Omega\varpi_n,B\omega,\tau,\frac{t}{3}\right)\to 0,$$

for all $\tau\in\Xi$ and $t>0$, and

$$\psi(\Omega B\varpi_n,B\omega,\tau,t)\le\psi\left(\Omega B\varpi_n,B\omega,B\Omega\varpi_n,\frac{t}{3}\right)\lozenge\psi\left(\Omega B\varpi_n,B\Omega\varpi_n,\tau,\frac{t}{3}\right)\lozenge\psi\left(B\Omega\varpi_n,B\omega,\tau,\frac{t}{3}\right)$$

$$\le\psi\left(B\Omega\varpi_n,B\omega,\Omega B\varpi_n,\frac{t}{3}\right)\lozenge\psi\left(B\Omega\varpi_n,\Omega B\varpi_n,\tau,\frac{t}{3}\right)\lozenge\psi\left(B\Omega\varpi_n,B\omega,\tau,\frac{t}{3}\right)\to 0,$$

for all $\tau\in\Xi$ and $t>0$. Hence, $\Omega B\varpi_n\to B\omega$.

Theorem 4.1. Let $(\Xi,\Psi,\Phi,\psi,*,\lozenge)$ be a complete N2MS with continuous t-norm * and continuous t-conorm $\lozenge$. Let S and T be continuous self-mapping of Ξ. Then, S and T have a unique common fixed point in Ξ if and only if there exist two self-mappings Ω,B of Ξ satisfying

(1) $\Omega\Xi\subset T\Xi, B\Xi\subset S\Xi$,

(2) The pair $\{\Omega,S\}$ and $\{B,T\}$ are compatible

(3) There exists $q\in(0,1)$ such that for every $\varpi,\theta,\tau\in\Xi$ and $t>0$,

$$\Psi(\Omega\varpi,B\theta,\tau,qt)\ge\min\{\Psi(S\varpi,T\varpi,\tau,t),\Psi(\Omega\varpi,S\varpi,\tau,t),\Psi(B\theta,T\theta,\tau,t),\Psi(\Omega\varpi,B\theta,\tau,qt)\},$$

$$\Phi(\Omega\varpi,B\theta,\tau,qt)\le\max\{\Phi(S\varpi,T\varpi,\tau,t),\Phi(\Omega\varpi,S\varpi,\tau,t),\Phi(B\theta,T\theta,\tau,t),\Phi(\Omega\varpi,B\theta,\tau,qt)\},$$

$$\psi(\Omega\varpi,B\theta,\tau,qt)\le\max\{\psi(S\varpi,T\varpi,\tau,t),\psi(\Omega\varpi,S\varpi,\tau,t),\psi(B\theta,T\theta,\tau,t),\psi(\Omega\varpi,B\theta,\tau,qt)\}.$$

Then, Ω, B, S and T have a unique common fixed point in Ξ.

Proof. Suppose S and T have a (unique) common fixed point say $\omega \in \Xi$. Define $\Omega: \Xi \to \Xi$ be $\Omega\varpi = \omega$ for all $\varpi \in \Xi$, and $B: \Xi \to \Xi$ be $B\varpi = \omega$ for all $\varpi \in \Xi$. Then, one can see that (1)–(3) are satisfied. Conversely, assume that there exist two self-mapping Ω, B of ϖ satisfying condition (1)–(3). From condition (1) we can construct two sequences ϖ_n and θ_n of Ξ such that $\theta_{2n-1} = T\varpi_{2n-1}$and $\theta_{2n-1} = S\varpi_{2n} = B\varpi_{2n-1}$ for $n = 1,2,3, ...$, putting $\varpi = \varpi_{2n}$and $\varpi = \varpi_{2n+1}$ in condition (3), we have that for all $\tau \in \Xi$ and $t > 0$,

$$\Psi(\theta\varpi_{2n+1}, \theta\varpi_{2n+2}, \tau, qt) = \Psi(\Omega\varpi_{2n}, B\varpi_{2n+1}, \tau, qt)$$

$$\geq \min\{\Psi(S\varpi_{2n}, T\varpi_{2n+1}, \tau, t), \Psi(\Omega\varpi_{2n}, S\varpi_{2n}, \tau, t),$$

$$\Psi(B\varpi_{2n+1}, T\varpi_{2n+1}, \tau, t), \Psi(\Omega\varpi_{2n}, T\varpi_{2n+1}, \tau, t)\}$$

$$\geq \min\{\Psi(\theta\varpi_{2n}, \theta\varpi_{2n+1}, \tau, qt), \Psi(\theta\varpi_{2n+1}, \theta\varpi_{2n+1}, \tau, qt)\},$$

$$\Phi(\theta\varpi_{2n+1}, \theta\varpi_{2n+2}, \tau, qt) = \Phi(\Omega\varpi_{2n}, B\varpi_{2n+1}, \tau, qt)$$

$$\leq \max\{\Phi(S\varpi_{2n}, T\varpi_{2n+1}, \tau, t), \Phi(\Omega\varpi_{2n}, S\varpi_{2n}, \tau, t),$$

$$\Phi(B\varpi_{2n+1}, T\varpi_{2n+1}, \tau, t), \Phi(\Omega\varpi_{2n}, T\varpi_{2n+1}, \tau, t)\}$$

$$\leq \max\{\Phi(\theta\varpi_{2n}, \theta\varpi_{2n+1}, \tau, qt), \Phi(\theta\varpi_{2n+1}, \theta\varpi_{2n+1}, \tau, qt)\},$$

and

$$\psi(\theta\varpi_{2n+1}, \theta\varpi_{2n+2}, \tau, qt) = \psi(\Omega\varpi_{2n}, B\varpi_{2n+1}, \tau, qt)$$

$$\leq max\{\psi(S\varpi_{2n}, T\varpi_{2n+1}, \tau, t), \psi(\Omega\varpi_{2n}, S\varpi_{2n}, \tau, t),$$

$$\psi(B\varpi_{2n+1}, T\varpi_{2n+1}, \tau, t), \psi(\Omega\varpi_{2n}, T\varpi_{2n+1}, \tau, t)\}$$

$$\leq max\{\psi(\theta\varpi_{2n}, \theta\varpi_{2n+1}, \tau, qt), \psi(\theta\varpi_{2n+1}, \theta\varpi_{2n+1}, \tau, qt)\}.$$

Which implies that

$$\Psi(\theta\varpi_{2n+1}, \theta\varpi_{2n+2}, \tau, qt) \geq \Psi(\theta\varpi_{2n+1}, \theta\varpi_{2n+1}, \tau, qt)$$

$$\Phi(\theta\varpi_{2n+1}, \theta\varpi_{2n+2}, \tau, qt) \leq \Phi(\theta\varpi_{2n+1}, \theta\varpi_{2n+1}, \tau, qt),$$

and

$$\psi(\theta\varpi_{2n+1}, \theta\varpi_{2n+2}, \tau, qt) \leq \psi(\theta\varpi_{2n+1}, \theta\varpi_{2n+1}, \tau, qt).$$

By using Lemma 4.1 and letting $\varpi = \varpi_{2n+1}$ and $\theta = \varpi_{2n+1}$ in condition (3), we have that

$$\Psi(\theta_{2n+2}, \theta_{2n+3}, \tau, qt) \geq \Psi(\theta_{2n+1}, \theta_{2n+1}, \tau, t)$$

$$\Phi(\theta_{2n+2}, \theta_{2n+3}, \tau, qt) \geq \Phi(\theta_{2n+1}, \theta_{2n+1}, \tau, t),$$

and

$$\psi(\theta_{2n+2}, \theta_{2n+3}, \tau, qt) \geq \psi(\theta_{2n+1}, \theta_{2n+1}, \tau, t)$$

for all $\tau \in \Xi$ and $t > 0$.

In general, we obtain that for all $\tau \in \Xi$ and $t > 0$ and $n = 1,2,3, \cdots$, we have

$$\Psi(\theta_n, \theta_{n+1}, \tau, qt) \geq \Psi(\theta_{n-1}, \theta_n, \tau, t),$$

$$\Phi(\theta_n, \theta_{n+1}, \tau, qt) \leq \Phi(\theta_{n-1}, \theta_n, \tau, t),$$

and

$$\psi(\theta_n, \theta_{n+1}, \tau, qt) \leq \psi(\theta_{n-1}, \theta_n, \tau, t).$$

Thus, for all $\tau \in \Xi$ and $t > 0$ and $n = 1,2,3,\cdots$, we have

$$\Psi(\theta_n, \theta_{n+1}, \tau, t) \geq \Psi\left(\theta_0, \theta_1, \tau, \frac{t}{q^n}\right). \tag{4.1}$$

$$\Phi(\theta_n, \theta_{n+1}, \tau, t) \leq \Phi\left(\theta_0, \theta_1, \tau, \frac{t}{q^n}\right). \tag{4.2}$$

$$\psi(\theta_n, \theta_{n+1}, \tau, t) \leq \psi\left(\theta_0, \theta_1, \tau, \frac{t}{q^n}\right). \tag{4.3}$$

We now show that $\{\theta_n\}$ is a Cauchy sequence in Ξ, let $m > n$. Then, for all $\tau \in \Xi$ and $t > \varpi$, we have

$$\begin{aligned}\Psi(\theta_m, \theta_n, \tau, t) &\geq \Psi\left(\theta_m, \theta_n, \theta_{n+1}, \frac{t}{3}\right) * \Psi\left(\theta_{n+1}, \theta_n, \tau, \frac{t}{3}\right) * \Psi\left(\theta_m, \theta_{n+1}, \tau, \frac{t}{3}\right) \\ &\geq \Psi\left(\theta_m, \theta_n, \theta_{n+1}, \frac{t}{3}\right) * \Psi\left(\theta_{n+1}, \theta_n, \tau, \frac{t}{3}\right) * \Psi\left(\theta_m, \theta_{n+1}, \theta_{n+2}, \frac{t}{3}\right) \\ &* \Psi\left(\theta_{n+2}, \theta_{n+1}, \tau, \frac{t}{3^2}\right) * \Psi\left(\theta_m, \theta_{n+2}, \tau, \frac{t}{3^2}\right) * \cdots * \Psi\left(\theta_m, \theta_{m-1,}\tau, \frac{t}{3^{m-n}}\right),\end{aligned}$$

$$\begin{aligned}\Phi(\theta_m, \theta_n, \tau, t) &\leq \Phi\left(\theta_m, \theta_n, \theta_{n+1}, \frac{t}{3}\right) \Diamond \Phi\left(\theta_{n+1}, \theta_n, \tau, \frac{t}{3}\right) \Diamond \Phi\left(\theta_m, \theta_{n+1}, \tau, \frac{t}{3}\right) \\ &\leq \Phi\left(\theta_m, \theta_n, \theta_{n+1}, \frac{t}{3}\right) \Diamond \Phi\left(\theta_{n+1}, \theta_n, \tau, \frac{t}{3}\right) \Diamond \Phi\left(\theta_m, \theta_{n+1}, \theta_{n+2}, \frac{t}{3}\right) \\ &\Diamond \Phi\left(\theta_{n+2}, \theta_{n+1}, \tau, \frac{t}{3^2}\right) \Diamond \Phi\left(\theta_m, \theta_{n+2}, \tau, \frac{t}{3^2}\right) \Diamond \cdots \Diamond \Phi\left(\theta_m, \theta_{m-1,}\tau, \frac{t}{3^{m-n}}\right),\end{aligned}$$

and

$$\begin{aligned}\psi(\theta_m, \theta_n, \tau, t) &\leq \psi\left(\theta_m, \theta_n, \theta_{n+1}, \frac{t}{3}\right) \Diamond \psi\left(\theta_{n+1}, \theta_n, \tau, \frac{t}{3}\right) \Diamond \psi\left(\theta_m, \theta_{n+1}, \tau, \frac{t}{3}\right) \\ &\leq \psi\left(\theta_m, \theta_n, \theta_{n+1}, \frac{t}{3}\right) \Diamond \psi\left(\theta_{n+1}, \theta_n, \tau, \frac{t}{3}\right) \Diamond \psi\left(\theta_m, \theta_{n+1}, \theta_{n+2}, \frac{t}{3}\right) \\ &\Diamond \psi\left(\theta_{n+2}, \theta_{n+1}, \tau, \frac{t}{3^2}\right) \Diamond \psi\left(\theta_m, \theta_{n+2}, \tau, \frac{t}{3^2}\right) \Diamond \cdots \Diamond \psi\left(\theta_m, \theta_{m-1,}\tau, \frac{t}{3^{m-n}}\right).\end{aligned}$$

Letting $m, n \to \infty$, we have

$$\lim_{n\to\infty} \Psi(\theta_m, \theta_n, \tau, t) = 1, \lim_{n\to\infty} \Phi(\theta_m, \theta_n, \tau, t) = 0, \ and \ \lim_{n\to\infty} \psi(\theta_m, \theta_n, \tau, t) = 0.$$

Thus, $\{\theta_n\}$ is a Cauchy sequence in Ξ. It follows from completeness of Ξ that there exists $\omega \in \Xi$ such that

$$\lim_{n\to\infty} \theta_n = \omega, \lim_{n\to\infty} \theta_{2n-1} = \lim_{n\to\infty} \varpi_{2n-1} = \lim_{n\to\infty} \Omega\varpi_{2n-2} = \omega,$$

and

$$\lim_{n\to\infty} \theta_{2n} = \lim_{n\to\infty} S\varpi_{2n} = \lim_{n\to\infty} B\varpi_{2n-1} = \omega.$$

From Lemma 4.4, we have

$$\Omega S\varpi_{2n+1} = S\omega \text{ and } BT\varpi_{2n+1} = T\omega. \tag{4.4}$$

Meanwhile, for all $\tau \in \Xi$ with $\tau \neq S\omega$ and $\tau \neq T\omega$ and $t > 0$, we have

$$\Psi(\Omega S\varpi_{2n+1}, BT\varpi_{2n+1}, \tau, qt) \geq min\{\Psi(SS\varpi_{2n+1}, TT\varpi_{2n+1}, \tau, t),$$
$$\Psi(\Omega S\varpi_{2n+1}, SS\varpi_{2n+1}, \tau, t), \Psi(BT\varpi_{2n+1}, TT\varpi_{2n+1}, q\tau, t), \Psi(\Omega S\varpi_{2n+1}, TT\varpi_{2n+1}, \tau, t)\},$$
$$\Phi(\Omega S\varpi_{2n+1}, BT\varpi_{2n+1}, \tau, qt) \leq max\{\Phi(SS\varpi_{2n+1}, TT\varpi_{2n+1}, \tau, t),$$
$$\Phi(\Omega S\varpi_{2n+1}, SS\varpi_{2n+1}, \tau, t), \Phi(BT\varpi_{2n+1}, TT\varpi_{2n+1}, q\tau, t), \Phi(\Omega S\varpi_{2n+1}, TT\varpi_{2n+1}, \tau, t)\},$$

and

$$\psi(\Omega S\varpi_{2n+1}, BT\varpi_{2n+1}, \tau, qt) \leq max\{\psi(SS\varpi_{2n+1}, TT\varpi_{2n+1}, \tau, t),$$
$$\psi(\Omega S\varpi_{2n+1}, SS\varpi_{2n+1}, \tau, t), \psi(BT\varpi_{2n+1}, TT\varpi_{2n+1}, q\tau, t), \psi(\Omega S\varpi_{2n+1}, TT\varpi_{2n+1}, \tau, t)\}.$$

Taking limit as $n \to \infty$ and using (4.4), we have for all $\tau \in \Xi$ with $\tau \neq S\omega$ and $\tau \neq T\omega$ and $t > 0$.

$$\Psi(S\omega, T\omega, \tau, qt + 0) \geq min\{\Psi(S\omega, T\omega, \tau, t), \Psi(S\omega, S\omega, \tau, t),$$
$$\Psi(T\omega, T\omega, \tau, t), \Psi(S\omega, S\omega, \tau, t)\} = \Psi(S\omega, T\omega, \tau, t),$$
$$\Phi(S\omega, T\omega, \tau, qt + 0) \leq max\{\Phi(S\omega, T\omega, \tau, t), \Phi(S\omega, S\omega, \tau, t),$$
$$\Phi(T\omega, T\omega, \tau, t), \Phi(S\omega, S\omega, \tau, t)\} = \Phi(S\omega, T\omega, \tau, t),$$

and

$$\psi(S\omega, T\omega, \tau, qt + 0) \leq max\{\psi(S\omega, T\omega, \tau, t), \psi(S\omega, S\omega, \tau, t),$$
$$\psi(T\omega, T\omega, \tau, t), \psi(S\omega, S\omega, \tau, t)\} = \psi(S\omega, T\omega, \tau, t).$$

By Lemma 4.2, we have

$$S\omega = T\omega. \tag{4.5}$$

From condition (3), we get for all $\tau \in \Xi$ with $\tau \neq \Omega\omega, \tau \neq T\omega$ and $t > 0$

$$\Psi(\Omega\omega, BT\varpi_{2n+1}, \tau, qt) \geq min\{\Psi(S\omega, TT\varpi_{2n+1}, \tau, t), \Psi(\Omega_\omega, S_\omega, \tau, t),$$
$$\Psi(BT\varpi_{2n+1}, TT\varpi_{2n+1}, \tau, t), \Psi(\Omega\omega, TT\varpi_{2n+1}, \tau, t)\},$$
$$\Phi(\Omega\omega, BT\varpi_{2n+1}, \tau, qt) \leq max\{\Phi(S\omega, TT\varpi_{2n+1}, \tau, t), \Phi(\Omega_\omega, S_\omega, \tau, t),$$

$$\Phi(BT\varpi_{2n+1}, TT\varpi_{2n+1}, \tau, t), \Phi(\Omega\omega, TT\varpi_{2n+1}, \tau, t)\},$$

and

$$\psi(\Omega\omega, BT\varpi_{2n+1}, \tau, qt) \leq max\{\psi(S\omega, TT\varpi_{2n+1}, \tau, t), \psi(\Omega_\omega, S_\omega, \tau, t),$$
$$\psi(BT\varpi_{2n+1}, TT\varpi_{2n+1}, \tau, t), \psi(\Omega\omega, TT\varpi_{2n+1}, \tau, t)\}.$$

Taking limit as $n \to \infty$, using condition (3), and Lemma 4.3, we have for all $\tau \in \Xi$,

$$\Psi(\Omega\omega, T\omega, \tau, qt + 0) \geq min\{\Psi(S\omega, T\omega, \tau, t), \Psi(\Omega\omega, S\omega, \tau, t),$$
$$\Psi(T\omega, T\omega, \tau, t), \Psi(\Omega\omega, T\omega, \tau, t)\} = \Psi(\Omega\omega, T\omega, \tau, t),$$
$$\Phi(\Omega\omega, T\omega, \tau, qt + 0) \leq max\{\Phi(S\omega, T\omega, \tau, t), \Phi(\Omega\omega, S\omega, \tau, t),$$
$$\Phi(T\omega, T\omega, \tau, t), \Phi(\Omega\omega, T\omega, \tau, t)\} = \Phi(\Omega\omega, T\omega, \tau, t),$$

and

$$\psi(\Omega\omega, T\omega, \tau, qt + 0) \leq max\{\psi(S\omega, T\omega, \tau, t), \psi(\Omega\omega, S\omega, \tau, t),$$
$$\psi(T\omega, T\omega, \tau, t), \psi(\Omega\omega, T\omega, \tau, t)\} = \psi(\Omega\omega, T\omega, \tau, t).$$

By Lemma 4.2, we have

$$\Omega\omega = T\omega. \tag{4.6}$$

For all $\tau \in \Xi$ with $\tau \neq \Omega\omega$ and $\tau \neq B\omega$, and $t > 0$, we have

$$\Psi(\Omega\omega, B\omega, \tau, qt) \geq min\{\Psi(S\omega, T\omega, \tau, t), \Psi(\Omega\omega, S\omega, \tau, t),$$
$$\Psi(B\omega, T\omega, \tau, t), \Psi(\Omega\omega, T\omega, \tau, t)\},$$
$$\geq min\{\Psi(T\omega, T\omega, \tau, t), \Psi(\Omega\omega, T\omega, \tau, t), \Psi(B\omega, \Omega\omega, \tau, t), \Psi(T\omega, T\omega, \tau, t)\}$$
$$= \Psi(\Omega\omega, B\omega, \tau, t),$$
$$\Phi(\Omega\omega, B\omega, \tau, qt) \leq max\{\Phi(S\omega, T\omega, \tau, t), \Phi(\Omega\omega, S\omega, \tau, t),$$
$$\Phi(B\omega, T\omega, \tau, t), \Phi(\Omega\omega, T\omega, \tau, t)\},$$
$$\leq max\{\Phi(T\omega, T\omega, \tau, t), \Phi(\Omega\omega, T\omega, \tau, t), \Phi(B\omega, \Omega\omega, \tau, t), \Phi(T\omega, T\omega, \tau, t)\}$$
$$= \Phi(\Omega\omega, B\omega, \tau, t),$$

and

$$\psi(\Omega\omega, B\omega, \tau, qt) \leq max\{\psi(S\omega, T\omega, \tau, t), \psi(\Omega\omega, S\omega, \tau, t),$$
$$\psi(B\omega, T\omega, \tau, t), \psi(\Omega\omega, T\omega, \tau, t)\}$$
$$\leq max\{\psi(T\omega, T\omega, \tau, t), \psi(\Omega\omega, T\omega, \tau, t), \psi(B\omega, \Omega\omega, \tau, t), \psi(T\omega, T\omega, \tau, t)\}$$
$$= \psi(\Omega\omega, B\omega, \tau, t).$$

By Lemma 4.2, $\Omega\omega = B\omega$ and (4.7), it follows that $\Omega\omega = B\omega = S\omega = T\omega$. For all $\tau \in \Xi$ with $\tau \neq B\omega$ and $\tau \neq \omega$, and $t > 0$

$$\Psi(\Omega\varpi_{2n}, B\omega, \tau, qt) \geq min\{\Psi(S\varpi_{2n}, T\omega, \tau, t), \Psi(\Omega\varpi_{2n}, S\varpi_{2n}, \tau, t)$$
$$\Psi(B\omega, T\omega, \tau, t), \Psi(\Omega\varpi_{2n}, T_\omega, \tau, t)\},$$
$$\Phi(\Omega\varpi_{2n}, B\omega, \tau, qt) \leq max\{\Phi(S\varpi_{2n}, T\omega, \tau, t), \Phi(\Omega\varpi_{2n}, S\varpi_{2n}, \tau, t)$$
$$\Phi(B\omega, T\omega, \tau, t), \Phi(\Omega\varpi_{2n}, T_\omega, \tau, t)\},$$

and

$$\psi(\Omega\varpi_{2n}, B\omega, \tau, qt) \leq max\{\psi(S\varpi_{2n}, T\omega, \tau, t), \psi(\Omega\varpi_{2n}, S\varpi_{2n}, \tau, t)$$
$$\psi(B\omega, T\omega, \tau, t), \psi(\Omega\varpi_{2n}, T_\omega, \tau, t)\}.$$

Taking limit as $n \to \infty$ and using (4.3) and Lemma 4.3, we have for all $\tau \in \Xi$ we $\tau \neq B\omega, \tau \neq \omega$ and $t > 0$

$$\Psi(\omega, B\omega, \tau, qt + 0) \geq min\{\Psi(\omega, T\omega, \tau, t), \Psi(\omega, \omega, \omega, t), \Psi(B\omega, B\omega, \tau, t), \Psi(\omega, T\omega, \tau, t)\}$$
$$\geq \Psi(\omega, T\omega, \tau, t) \geq \Psi(\omega, B\omega, \tau, t),$$
$$\Phi(\omega, B\omega, \tau, qt + 0) \leq max\{\Phi(\omega, T\omega, \tau, t), \Phi(\omega, \omega, \omega, t), \Phi(B\omega, B\omega, \tau, t), \Phi(\omega, T\omega, \tau, t)\}$$
$$\leq \Phi(\omega, T\omega, \tau, t) \leq \Phi(\omega, B\omega, \tau, t),$$

and

$$\psi(\omega, B\omega, \tau, qt + 0) \leq max\{\psi(\omega, T\omega, \tau, t), \psi(\omega, \omega, \omega, t), \psi(B\omega, B\omega, \tau, t), \psi(\omega, T\omega, \tau, t)\}$$
$$\leq \psi(\omega, T\omega, \tau, t) \leq \psi(\omega, B\omega, \tau, t).$$

So, we have

$$\Psi(\omega, B\omega, \tau, qt) \geq \Psi(\omega, B\omega, \tau, t), \Phi(\omega, B\omega, \tau, qt) \leq \Phi(\omega, B\omega, \tau, t)$$

and

$$\Phi(\omega, B\omega, \tau, qt) \leq \Phi(\omega, B\omega, \tau, t),$$

here $B\omega = \omega$. Thus $\omega = \Omega\omega = B\omega = S\omega = T\omega$ and so, ω is a common fixed point of Ω, B, C and T. For uniqueness, let w be another common fixed point of Ω, B, S, T for all $\tau \in \Xi$ with $\tau \neq \omega, \tau \neq w$ and $t > 0$, we have

$$\Psi(\omega, w, \tau, qt) = \Psi(\Omega\omega, Bw, \tau, qt)$$
$$\geq min\{\Psi(S\omega, Tw, \tau, t), \Psi(\Omega\omega, S\omega, \tau, t), \Psi(Bw, Tw, \tau, t), \Psi(\Omega\omega, Tw, \tau, t)\}$$
$$\geq min\{\Psi(\omega, w, \tau, t), \Psi(\omega, w, \tau, t), \Psi(w, w, \tau, t), \Psi(\omega, w, \tau, t)\} \geq \Psi(\omega, w, \tau, t),$$
$$\Phi(\omega, w, \tau, qt) = \Phi(\Omega\omega, Bw, \tau, qt)$$
$$\leq max\{\Phi(S\omega, Tw, \tau, t), \Phi(\Omega\omega, S\omega, \tau, t), \Phi(Bw, Tw, \tau, t), \Phi(\Omega\omega, Tw, \tau, t)\}$$
$$\leq max\{\Phi(\omega, w, \tau, t), \Phi(\omega, w, \tau, t), \Phi(w, w, \tau, t), \Phi(\omega, w, \tau, t)\} \leq \Phi(\omega, w, \tau, t),$$

and

$$\psi(\omega, w, \tau, qt) = \psi(\Omega\omega, Bw, \tau, qt)$$
$$\leq max\{\psi(S\omega, Tw, \tau, t), \psi(\Omega\omega, S\omega, \tau, t), \psi(Bw, Tw, \tau, t), \psi(\Omega\omega, Tw, \tau, t)\}$$
$$\leq max\{\psi(\omega, w, \tau, t), \psi(\omega, w, \tau, t), \psi(w, w, \tau, t), \psi(\omega, w, \tau, t)\} \leq \psi(\omega, w, \tau, t).$$

Which implies that

$$\Psi(\omega, w, \tau, qt) \geq \Psi(\omega, w, \tau, t), \Phi(\omega, w, \tau, qt) \leq \Phi(\omega, w, \tau, t),$$

and

$$\psi(\omega, w, \tau, qt) \leq \psi(\omega, w, \tau, t),$$

hence $\omega = w$. This completes the proof.

5. Conclusions

The N2MS idea, which is an extension of the NMS, was investigated in this article because it offers a greater context for dealing with the ambiguity and uncertainty in natural problems that arises in many fields of research and engineering. In this new setting, we constructed the Baire's and Cantor's Theorems, which could be very helpful tools in the advancement of fuzzy set theory. We derived the common fixed-point theorem with respect to N2MS. This work can easily be extending in the context of neutrosophic b-2-metric spaces, neutrosophic controlled 2-metric spaces and neutrosophic partial 2-metric spaces.

Conflict of interest

There are no competing interests declared by the authors.

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