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Computational comparative analysis of fixed point approximations of generalized α-nonexpansive mappings in hyperbolic spaces

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Abstract: In this article, we use the Picard-Thakur hybrid iterative scheme to approximate the fixed points of generalized α-nonexpansive mappings. For generalized α-nonexpansive mappings in hyperbolic spaces, we show several weak and strong convergence results. It is proved numerically and graphically that the Picard-Thakur hybrid iterative scheme converges more faster than other well-known hybrid iterative methods for generalized α-nonexpansive mappings. We also present an application to Fredholm integral equation.

Keywords: Picard-Thakur hybrid iteration; generalized α-nonexpansive mapping; fixed point; hyperbolic space
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1. Introduction

In a variety of general metric space contexts, the nonexpansive mappings can be defined. The α-nonexpansive mappings was first introduced by Aoyama and Kohsaka in 2011 [1]. For this type of mappings, they obtained certain fixed point and convergence results in Banach spaces, which making it more general than the Suzuki's condition (C) [2]. In 2017, Pant and Shukla introduced in [3] a new class of mappings, the generalized α-nonexpansive mappings. In the setting of uniformly convex hyperbolic spaces, Mebawondu and Izuchukwu [4] investigated some fixed point properties and the demiclosedness principle for generalized α-nonexpansive mappings.

Definition 1.1. *Let $\mathcal{J}$ be a nonempty subset of the hyperbolic space $\mathcal{H}$. The mapping $\zeta : \mathcal{J} \to \mathcal{J}$ is* *called generalized α-nonexpansive mapping or condition (C_α), if*

$$\begin{aligned} &\frac{1}{2}\hat{d}(s,\zeta(s)) \le \hat{d}(s,t) \\ \Rightarrow\ &\hat{d}(\zeta(s),\zeta(t)) \le \alpha\ \hat{d}(t,\zeta(s)) + \alpha\ \hat{d}(s,\zeta(t)) + (1-2\alpha)\ \hat{d}(s,t), \end{aligned} \tag{1.1}$$

for all $s,t \in \mathcal{J}$, $0 \le \alpha < 1$ *and* $\hat{d}(s,\zeta(t)) = \inf\{\hat{d}(s,r) : r \in \zeta(t)\}$.

In 1922, Banach [5] proposed the Banach Contraction Principle, which states that the Picard iterative scheme [6] can be used to approximate the fixed points of a contraction mapping. The Picard scheme is defined as follows

$$\begin{cases} s_1 \in \mathcal{J}, \\ s_{i+1} = \zeta(s_i), \quad i \in \mathbb{Z}^+. \end{cases}$$

The nonexpansive mapping does not converge to a fixed point in Picard iterations. For more details, we can see [7].

In 1953, Mann [8] approximated the fixed point for nonexpansive mappings using the iterative scheme defined below as

$$\begin{cases} s_1 \in \mathcal{J}, \\ s_{i+1} = (1-\eta_i)s_i + \eta_i\zeta(s_i), \ i \in \mathbb{Z}^+, \end{cases}$$

where $\{\eta_i\} \in (0,1)$.

Agarwal et al. [9] introduced the following S-iterative scheme, defined as

$$\begin{cases} s_1 \in \mathcal{J}, \\ s_{i+1} = (1-\eta_i)\zeta(s_i) + \eta_i\zeta(t_i), t_i = (1-\rho_i)s_i + \rho_i\zeta(s_i), \quad i \in \mathbb{Z}^+, \end{cases}$$

where $\{\eta_i\}$, $\{\rho_i\}$ are the sequences of parameters in $(0,1)$.

In 2017, Thakur et al. [10], introduced three step iterative scheme and the sequences of this scheme are given as

$$\begin{cases} s_1 \in \mathcal{J}, \\ s_{i+1} = (1-\eta_i)\zeta(u_i) + \eta_i\zeta(t_i), \\ t_i = (1-\rho_i)u_i + \rho_i\zeta(u_i), \quad i \in \mathbb{Z}^+, \\ u_i = (1-\sigma_i)s_i + \sigma_i\zeta(s_i), \end{cases}$$

where $\{\eta_i\}$, $\{\rho_i\}$ and $\{\sigma_i\}$ are the sequences of parameters in $(0,1)$.

Recently, the researchers in this area have introduced the hybrid iterative process obtained by the hybridization of Picard with Mann, Ishikawa, S, S^* and Thakur et al., iterative schemes which results faster convergence as compared with many others schemes. Khan [11], introduced Picard-Mann iterative scheme which is given by the sequence $\{s_i\}$ as

$$\begin{cases} s_1 \in \mathcal{J}, \\ s_{i+1} = \zeta(t_i), t_i = (1-\eta_i)s_i + \eta_i\zeta(s_i), \quad i \in \mathbb{Z}^+, \end{cases} \tag{1.2}$$

where $\{\eta_i\} \in [0,1)$.

The Picard-Ishikawa hybrid iterative scheme was given by Okeke [12], which is given by the sequence $\{s_i\}$ as

$$\begin{cases} s_1 \in \mathcal{J}, \\ s_{i+1} = \zeta(t_i), \\ t_i = (1-\eta_i)s_i + \eta_i\zeta(u_i), u_i = (1-\rho_i)s_i + \rho_i\zeta(s_i), \quad i \in \mathbb{Z}^+, \end{cases} \tag{1.3}$$

where $\{\eta_i\}, \{\rho_i\} \in [0, 1)$.

Similarly, Gursoy and Karakaya (see [13]) proposed Picard-S hybrid iterative scheme given by the sequence $\{s_i\}$ as

$$\begin{cases} s_1 \in \mathcal{J}, \\ s_{i+1} = \zeta(t_i), \\ t_i = (1-\eta_i)\zeta(s_i) + \eta_i\zeta(u_i), u_i = (1-\rho_i)s_i + \rho_i\zeta(s_i), \quad i \in \mathbb{Z}^+, \end{cases} \tag{1.4}$$

where $\{\eta_i\}, \{\rho_i\} \in [0, 1)$.

Also, Lamba and Panwar [14] gave the idea and proved the convergence results of Picard-S^* iterative scheme given by the sequence $\{s_i\}$ as

$$\begin{cases} s_1 \in \mathcal{J}, \\ s_{i+1} = \zeta(t_i) \\ t_i = (1-\eta_i)\zeta(s_i) + \eta_i\zeta(u_i), u_i = (1-\rho_i)s_i + \rho_i\zeta(v_i) \\ v_i = (1-\sigma_i)s_i + \sigma_i\zeta(s_i), \quad i \in \mathbb{Z}^+, \end{cases} \tag{1.5}$$

where $\{\eta_i\}, \{\rho_i\}, \{\sigma_i\} \in [0, 1)$.

Recently, motivated by hybridization behavior of iterative schemes Jie Jia et al. [15] proposed the Picard-Thakur hybrid iterative scheme. They proved analytically and numerically that their scheme has better rate of convergence than all above hybrid schemes and the sequence $\{s_i\}$ of this scheme is defined as

$$\begin{cases} s_1 \in \mathcal{J}, \\ s_{i+1} = \zeta(t_i) \\ t_i = (1-\eta_i)\zeta(v_i) + \eta_i\zeta(u_i), u_i = (1-\rho_i)v_i + \rho_i\zeta(v_i), \\ v_i = (1-\sigma_i)s_i + \sigma_i\zeta(s_i), \quad i \in \mathbb{Z}^+, \end{cases} \tag{1.6}$$

where $\{\eta_i\}, \{\rho_i\}, \{\sigma_i\} \in [0, 1)$.

For any other spaces algorithms for approximations of fixed points of α-nonexpansive mappings, using different type of iterative schemes were recently studied in [16–19].

Motivated by the convergence behavior of the scheme (1.6), we extend these results in order to verify the Δ and strong convergence results for a generalized α-nonexpansive mapping in hyperbolic space. Our results extend the results in Banach space to hyperbolic space. We performed a computational experiment to illustrate that the Picard-Thakur iterative scheme (1.6) converges more quickly than a number of other hybrid iterative schemes. We provide an application to Fredholm integral equation.

2. Preliminaries

Mebawondu and Izuchukwu gave the definition of generalized α-nonexpansive mappings in the framework of uniformly convex hyperbolic spaces (UCHS) [4].

Definition 2.1. *Let $\mathcal{J}$ be a nonempty subset of a hyperbolic space $\mathcal{H}$. A self-mapping ζ from $\mathcal{J}$ to $\mathcal{J}$ ($\zeta : \mathcal{J} \to \mathcal{J}$) is called generalized α-nonexpansive if, for all $s, t \in \mathcal{J}$, there exists $\alpha \in [0, 1)$ such that*

$$\begin{aligned}\frac{1}{2}\hat{d}(s, \zeta(s)) \leq& \hat{d}(s, t)\\ \Rightarrow \hat{d}(\zeta(s), \zeta(t)) \leq& \alpha\, \hat{d}(t, \zeta(s)) + \alpha\, \hat{d}(s, \zeta(t)) + (1 - 2\alpha)\, \hat{d}(s, t),\end{aligned} \tag{2.1}$$

where $\hat{d}(s, \zeta(t)) = \inf\{\hat{d}(s, r) : r \in \zeta(t)\}$.

It is also known in related literature as condition (C_α).

In this study, we investigate some results in the framework of Kohlenbach's hyperbolic spaces [20]. These results concern normed linear spaces, convex subsets, and Hadamard manifolds [21], as well as $CAT(0)$ spaces in the Gromovian sense [22] and the Hilbert ball endowed with the hyperbolic metric [21].

Let $(\mathcal{H}, \hat{d})$ be a metric space and $Q : X^2 \times [0, 1]$. A triplet $(\mathcal{H}, \hat{d}, Q)$ is called a hyperbolic space, if

(H_1) $\hat{d}(s, Q(t, u, \eta)) \leq (1 - \eta)\hat{d}(s, t) + \eta\hat{d}(s, u)$,
(H_2) $\hat{d}(Q(s, u, \eta), Q(s, t, \rho)) \leq |\eta - \rho|\hat{d}(s, t)$,
(H_3) $Q(s, t, \eta) = Q(s, t, (1 - \eta))$,
(H_4) $\hat{d}(Q(s, v, \eta), Q(t, u, \eta)) \leq (1 - \eta)\hat{d}(s, t) + \eta\hat{d}(v, u)$.

for all $s, t, u, v \in \mathcal{H}$ and $\eta, \rho \in [0, 1]$.

The space $(\mathcal{H}, \hat{d}, Q)$ is called UCHS [23], if for all $s, t, u \in \mathcal{H}, \xi > 0$, and $0 < \epsilon \leq 2$ there exist $\delta \in (0, 1)$ such that $\hat{d}(s, t) \leq \xi$, $\hat{d}(s, u) \leq \xi$ & $\hat{d}(t, u) \geq \epsilon\xi$. Then we have

$$\hat{d}\left(Q(t, u, \frac{1}{2}), s\right) \leq (1 - \delta)\epsilon.$$

A mapping $\varphi : (0, \infty) \times (0, 2] \to (0, 1]$ is called modulus of uniform convexity of X if $\delta = \varphi(\xi, \epsilon)$, for a given $\epsilon \in (0, 2]$. If φ decreases with ξ (for a fixed ϵ) (i.e., for a given $\epsilon > 0$ and $\xi_2 \geq \xi_1 > 0$, we have $\varphi(\xi_2, \epsilon) \leq \varphi(\xi_1, \epsilon)$), then φ is monotone.

A subset $\mathcal{J}$ of a hyperbolic space $\mathcal{H}$ is convex if $Q(s, t, \sigma) \in \mathcal{J}$, for any $s, t \in \mathcal{J}$ and $\sigma \in [0, 1]$.

We use the notation $(1 - \sigma)s \oplus \sigma t$ for $Q(s, t, \sigma)$ where $s, t \in \mathcal{H}$ and $0 \leq \sigma \leq 1$. Leuştean remarked in [23] that $\mathcal{H}$ endowed with a normed function, i.e. $(\mathcal{H}, \|.\|)$, is a hyperbolic space, with $(1-\sigma)s \oplus \sigma t = (1 - \sigma)s + \sigma t$. As a result, uniformly convex Banach spaces are a generalization of the class of UCHS.

Let $\zeta : \mathcal{J} \to \mathcal{J}$ be a mapping and $s^* \in \mathcal{J} \subset \mathcal{H}$. If $\zeta(s^*) = s^*$ then s^* is a fixed point of ζ and $F(\zeta)$ represents the set of all fixed points of the mapping ζ.

Further, let us recall some fundamental definitions and results that will be used in our research. Dhompongsa et al. [24], gave the concept of Δ-convergence in hyperbolic spaces and he used it as in the following.

Let $\{s_i\}$ be a bounded sequence in hyperbolic space $\mathcal{H}$. Consider a mapping $r_s(., \{s_i\}) : \mathcal{H} \to [0, \infty)$ defined by

$$r_s(s, \{s_i\}) = \limsup_{i\to\infty} \hat{d}(s_i, s).$$

For each $s \in \mathcal{H}$, the value $r_s(s, \{s_i\})$ is called the asymptotic radius of $\{s_i\}$ at s.

The asymptotic radius of $\{s_i\}$ relative to $\mathcal{J} \subset \mathcal{H}$ is defined as

$$r_s(\mathcal{J}, \{s_i\}) = \inf\{r_s(s, \{s_i\}) : s \in \mathcal{J}\}.$$

The asymptotic center of $\{s_i\}$ relative to $\mathcal{J}$ is the set

$$A_s(\mathcal{J}, \{s_i\}) = \{s \in \mathcal{J} : r_s(\mathcal{J}, \{s_i\}) = r_s(s, \{s_i\})\}.$$

Definition 2.2. *[25] The sequence $\{s_i\}$ in $\mathcal{H}$. $\{s_i\}$ is Δ-convergent to $s \in \mathcal{H}$ if, for every subsequence $\{s_{i_j}\}$ of $\{s_i\}$, s is the asymptotic center. So, $\Delta - \lim_{i\to\infty} s_i = s$ and s denotes the $\Delta - \lim$ of $\{s_i\}$.*

Lemma 2.1. *[26] If $\mathcal{J}$ is a nonempty closed convex subset of a complete UCHS $\mathcal{H}$ with a monotone modulus of uniform convexity φ, then every bounded sequence $\{s_i\}$ in $\mathcal{H}$ has a unique asymptotic centre with respect to $\mathcal{J}$.*

Definition 2.3. *[27] A hyperbolic space $\mathcal{H}$ is said to satisfy Opial's condition if, for any sequence $\{s_i\}$ with $\{s_i\} \rightharpoonup s \in \mathcal{H}$, the following inequality holds*

$$\liminf_{i\to\infty} \hat{d}(s_i, s) < \liminf_{i\to\infty} \hat{d}(s_i, t),$$

for $t \in \mathcal{H}$ with $s \neq t$.

The concept of *Condition (I)* was introduced by Sentor and Dotson in [28] and is described as follows.

Definition 2.4. *Let Θ be an increasing function from $[0, \infty)$ to $[0, \infty)$ with $\Theta(0) = 0$ and $\Theta(c) > 0$, for all $c > 0$. Let $\zeta : \mathcal{J} \to \mathcal{J}$ be a self-mapping such that*

$$\hat{d}(s, \zeta(s)) \geq \Theta(\hat{d}(s, F(\zeta))),\ \forall\ s \in \mathcal{J},$$

where $\hat{d}(s, F(\zeta)) = \inf\{\hat{d}(s, s^*) : s^* \in F(\zeta)\}$.

Proposition 2.1. *[3] Let $\mathcal{J}$ be a nonempty subset of the hyperbolic space $\mathcal{H}$ and $\zeta : \mathcal{J} \to \mathcal{J}$ be a generalized α-nonexpansive mapping. Then, for all $s, t \in \mathcal{J}$, we have true*

$$\hat{d}(s, \zeta(t)) \leq \frac{(3 + \alpha)}{(1 - \alpha)} \hat{d}(s, \zeta(s)) + \hat{d}(s, t).$$

Lemma 2.2. *[29] Let $\mathcal{H}$ be a complete UCHS with a monotone modulus of convexity φ. Let $s \in \mathcal{H}$ and $\{\eta_i\}$ be a sequence in $[c, d]$ for some $c, d \in (0, 1)$. If $\{s_i\}$ and $\{t_i\}$ are sequences in $\mathcal{H}$, such that $\limsup_{i\to\infty} \hat{d}(s_i, s) \leq \omega$, $\limsup_{i\to\infty} \hat{d}(t_i, s) \leq \omega$ and $\limsup_{i\to\infty} \hat{d}(Q(s_i, t_i, \eta_i) = \omega$ hold for some $\omega \geq 0$, then $\lim_{n\to\infty} \hat{d}(s_i, t_i) = 0$.*

Lemma 2.3. *[3] Let $\mathcal{J}$ be a nonempty subset of $\mathcal{H}$. Let $\zeta : \mathcal{J} \to \mathcal{J}$ be a generalized α-nonexpansive mapping with $F(\zeta) \neq \emptyset$. Then ζ is quasi-nonexpansive.*

Definition 2.5. *A sequence $\{s_i\}$ is called a Fejér monotone sequence with respect to $\mathcal{J}$, subset of hyperbolic space $\mathcal{H}$, if*

$$\hat{d}(s_{i+1}, s) \leq \hat{d}(s_i, s) \textit{ for all } s \in \mathcal{J} \,\&\, i \geq 1.$$

Proposition 2.2. *[30] Let $\{s_i\}$, $\mathcal{H}$, $\mathcal{J}$ and ζ be defined as in Lemma 2.3 and $\{s_i\}$ is Fejér monotone w.r.t. $\mathcal{J}$. Then*

i. $\{s_i\}$ *is bounded.*
ii. The sequence $\{\hat{d}(s_i, s^*)\}$ *is decreasing and converges to* $s^* \in F(\zeta)$.
iii. $\lim_{i\to\infty} \hat{d}(s_i, F(\zeta))$ *exists.*

Theorem 2.1. *[31] A monontone sequence of real numbers is convergent if and only if it is bounded. Further,*

(i) If $\{\alpha_i\}$ *is bounded decreasing then* $\lim_{i\to\infty}\{\alpha_i\} = \inf\{\alpha_i : i \in \mathbb{N}\}$.
(ii) If $\{\alpha_i\}$ *is bounded increasing then* $\lim_{i\to\infty}\{\alpha_i\} = \sup\{\alpha_i : i \in \mathbb{N}\}$.

3. Rate of convergence

In the following, let us construct the iterative scheme (1.6) in a hyperbolic space as follows

$$\begin{cases} s_1 \in \mathcal{J}, \\ s_{i+1} = Q(\zeta(t_i), 0, 0), \\ t_i = Q(\zeta(v_i), \zeta(u_i), \eta_i), \quad i \in \mathbb{Z}^+, \\ u_i = Q(v_i, \zeta(v_i), \rho_i), \\ v_i = Q(s_i, \zeta(s_i), \sigma_i), \end{cases} \tag{3.1}$$

where $\{\eta_i\}, \{\rho_i\}, \{\sigma_i\} \in [0, 1)$.

Lemma 3.1. *Let $\mathcal{J}$ be a nonempty closed convex subset of a complete UCHS $\mathcal{H}$ and $\zeta : \mathcal{J} \to \mathcal{J}$ a generalized α-nonexpansive mapping with $F(\mathcal{J}) \neq \emptyset$. Let $\{s_i\}$ be a sequence defined by* (3.1). *Then* $\lim_{i\to\infty} \hat{d}(s_i, s^*)$ *exists for all* $s^* \in F(\zeta)$.

Proof. Let $s^* \in F(\zeta)$. Since ζ satisfies the condition (C_α) we have,

$$\frac{1}{2}\hat{d}(s, \zeta(s)) \leq \hat{d}(s, t) \text{ implies } \hat{d}(\zeta(s), \zeta(t)) \leq \alpha\hat{d}(t, \zeta(s)) + \alpha\hat{d}(s, \zeta(t)) + (1 - 2\alpha)\hat{d}(s, t).$$

Thus, by iterative scheme (3.1), we get

$$\begin{aligned} \hat{d}(v_i, s^*) =& \hat{d}(Q(s_i, \zeta(s_i), \sigma_i), s^*) \\ \leq& (1 - \sigma_i)\hat{d}(s_i, s^*) + \sigma_i\hat{d}(\zeta(s_i), s^*), \\ \hat{d}(\zeta(s_i), s^*) =& \hat{d}(\zeta(s_i), \zeta(s^*)) \leq \alpha\hat{d}(s^*, \zeta(s_i)) + \alpha\hat{d}(s_i, \zeta(s^*)) + (1 - 2\alpha)\hat{d}(s_i, s^*) \\ \leq& \alpha(\hat{d}(s^*, \zeta(s^*)) + \zeta\hat{d}(\zeta(s^*), \zeta(s_i))) + \alpha(\hat{d}(s_i, s^*) + \hat{d}(s^*, \zeta(s^*))) + (1 - 2\alpha)\hat{d}(s_i, s^*), \\ \hat{d}(\zeta(s_i), \zeta(s^*)) \leq& \hat{d}(s_i, s^*), \\ \hat{d}(v_i, s^*) \leq& (1 - \sigma_i)\hat{d}(s_i, s^*) + \sigma_i\hat{d}(s_i, s^*), \\ \hat{d}(v_i, s^*) \leq& \hat{d}(s_i, s^*). \end{aligned} \tag{3.2}$$

Using (3.1) together with (3.2), we get

$$\begin{aligned}
\hat{d}(u_i, s^*) =&\hat{d}(Q(v_i, \zeta(v_i), \rho_i), s^*) \le (1-\rho_i)\hat{d}(v_i, s^*) + \rho_i\hat{d}(\zeta(v_i), s^*),\\
\hat{d}(\zeta(v_i), s^*) \le&\alpha\hat{d}(s^*, \zeta(v_i)) + \alpha\hat{d}(v_i, \zeta s^*) + (1-2\alpha)\hat{d}(v_i, s^*) \le \hat{d}(v_i, s^*)\\
\text{implies}\quad \hat{d}(u_i, s^*) \le&(1-\rho_i)\hat{d}(v_i, s^*) + \rho_i\hat{d}(v_i, s^*) \le \hat{d}(v_i, s^*) \le \hat{d}(s_i, s^*). \qquad (3.3)
\end{aligned}$$

Using (3.1) together with (3.3) and (3.2), we get

$$\begin{aligned}
\hat{d}(t_i, s^*) =&\hat{d}(Q(\zeta(v_i), \zeta(u_i), \eta_i), s^*)\\
\le&(1-\eta_i)\hat{d}(\zeta(v_i), s^*) + \eta_i\hat{d}(\zeta(u_i), s^*),\\
\hat{d}(\zeta(u_i), s^*) \le&\alpha\hat{d}(s^*, \zeta(u_i)) + \alpha\hat{d}(u_i, \zeta s^*) + (1-2\alpha)\hat{d}(u_i, s^*) \le \hat{d}(u_i, s^*)\\
\text{implies}\quad \hat{d}(t_i, s^*) \le&(1-\eta_i)\hat{d}(v_i, s^*) + \eta_i\hat{d}(u_i, s^*)\\
\le&(1-\eta_i)\hat{d}(s_i, s^*) + \eta_i\hat{d}(s_i, s^*) \le \hat{d}(s_i, s^*). \qquad (3.4)
\end{aligned}$$

Using (3.1) together with (3.4), we get

$$\begin{aligned}
\hat{d}(s_{i+1}, s^*) =&\hat{d}(Q(\zeta(t_i), 0, 0), s^*)\\
\le&\hat{d}(\zeta(t_i), s^*) \le \hat{d}(s_i, s^*). \qquad (3.5)
\end{aligned}$$

This shows that $\{\hat{d}(s_i, s^*)\}$ is Fejér monotone w.r.t. $F(\zeta)$. Then, $\lim_{i\to\infty}\hat{d}(s_i, s^*)$ exists. □

Lemma 3.2. *Let $\mathcal{J}, \mathcal{H}, \zeta$ be defined as in Lemma* 3.1. *For an arbitrary $s_0 \in \mathcal{J}$, a sequence $\{s_i\}$ is defined by* (3.1). *Then $\{s_i\}$ is bounded and $\lim_{i\to\infty}\hat{d}(s_i, \zeta(s_i)) = 0$ if and only if $F(\zeta) \neq \emptyset$.*

Proof. Let $\{s_i\}$ be a sequence defined by (3.1) and $s^* \in F(\zeta)$. From above Lemma 3.1, $\lim_{i\to\infty}\hat{d}(s_i, s^*)$ exists. Let $\lim_{i\to\infty}\hat{d}(s_i, s^*) = \omega > 0$. If $\omega = 0$, then $\lim_{i\to\infty}\hat{d}(s_i, \zeta(s_i)) = 0$.

Next, consider $\omega > 0$. By Lemma 3.1 we obtain that

$$\hat{d}(s_{i+1}, s^*) \le \hat{d}(t_i, s^*) \le \hat{d}(u_i, s^*) \le \hat{d}(v_i, s^*) \le \hat{d}(s_i, s^*). \qquad (3.6)$$

Taking lim sup in (3.6), we get

$$\limsup_{i\to\infty}\hat{d}(t_i, s^*) \le \limsup_{i\to\infty}\hat{d}(u_i, s^*) \le \limsup_{i\to\infty}\hat{d}(v_i, s^*) \le \omega. \qquad (3.7)$$

Since,

$$\begin{aligned}
\hat{d}(\zeta(s_i), s^*) \le&\alpha\hat{d}(s^*, \zeta(s_i)) + \alpha\hat{d}(s_i, \zeta(s^*)) + (1-2\alpha)\hat{d}(s_i, s^*)\\
\le&\alpha\hat{d}(\zeta(s_i), s^*) + (1-\alpha)\hat{d}(s_i, s^*) \le \hat{d}(s_i, s^*),
\end{aligned}$$

we get that

$$\limsup_{i\to\infty}\hat{d}(\zeta(s_i), s^*) \le \omega. \qquad (3.8)$$

Then, we have

$$\hat{d}(s_{i+1}, s^*) \leq \hat{d}(v_i, s^*),$$
$$\omega = \liminf_{i\to\infty} \hat{d}(s_{i+1}, s^*) \leq \liminf_{i\to\infty} \hat{d}(v_i, s^*) \leq \limsup_{i\to\infty} \hat{d}(v_i, s^*) \leq \omega. \tag{3.9}$$

By Eqs (3.7) and (3.9), we get

$$\lim_{i\to\infty} \hat{d}(v_i, s^*) = \omega,$$

which implies that

$$\lim_{i\to\infty} \hat{d}(Q(s_i, \zeta(s_i), \sigma_i) = \omega.$$

By Lemma 2.2, we have

$$\lim_{i\to\infty} \hat{d}(s_i, \zeta(s_i)) = 0.$$

On the other hand, if $\{s_i\}$ is bounded in $\mathcal{H}$, then by Lemma 2.1, $A_s\mathcal{J}, \{s_i\}) = \{s\}$ is a singleton (i.e, has a unique asymptotic center in $\mathcal{H}$) and $\lim_{i\to\infty} \hat{d}(s_i, \zeta(s_i)) = 0$. Then, ζ satisfies the condition (C_α) on $\mathcal{J}$.

Using Lemma 2.1, we have

$$\hat{d}(s_i, \zeta(s)) \leq \frac{(3+\alpha)}{(1-\alpha)} \hat{d}(s_i, \zeta(s_i)) + \hat{d}(s_i, s).$$

Taking $\limsup$ as $i \to \infty$ on both sides we get

$$\begin{aligned} r(\zeta(s^*), \{s_i\}) &= \limsup_{i\to\infty} \hat{d}(s^*, \zeta(s^*)) \\ &\leq \limsup_{i\to\infty} [\frac{(3+\alpha)}{(1-\alpha)} \hat{d}(s^*, \zeta(s_i)) + \hat{d}(s_i, s^*)] \\ &\leq \limsup_{i\to\infty} \hat{d}(s_i, s^*) = r_s(s^*, \{s_i\}). \end{aligned}$$

By utilizing the special property of asymptotic center, we get $\zeta(s^*) = s^*$; so $s \in F(\zeta)$. Hence, $F(\zeta)$ is nonempty. □

Theorem 3.1. *Let $\mathcal{J}, \mathcal{H}, \zeta$ and $\{s_i\}$ be a sequence defined as in Lemma* 3.1 *with monotone modulus of uniform convexity φ. Let $F(\zeta) \neq \emptyset$. Then $\{s_i\}$ Δ- converges to a fixed of ζ.*

Proof. From Lemma 3.2, the sequence $\{s_i\}$ is bounded. Therefore $\{s_i\}$ has a Δ-convergent subsequence. We have to prove that for every Δ-convergent subsequence there is a unique Δ-limit, which is the fixed point of ζ.

Let λ and μ be Δ-limits of the subsequences $\{\lambda_i\}$ & $\{\mu_i\}$ and respectively, of the sequence $\{s_i\}$. By Lemma 2.1 we have $r_s(\mathcal{J}, \{\lambda_i\}) = \{\lambda\}$ and $r_s(\mathcal{J}, \{\mu_i\}) = \{\mu\}$.

By Lemma 3.2, $\lim_{i\to\infty} \hat{d}(s_i, \zeta(s_i)) = 0$. We claim that λ and μ are two fixed points of ζ. Since, ζ satisfies the condition (C_α) and by Lemma 2.1, we have

$$\hat{d}(s_i, \zeta(s)) \leq \frac{(3+\alpha)}{(1-\alpha)} \hat{d}(s_i, \zeta(s_i)) + \hat{d}(s_i, s).$$

By taking lim sup, we get

$$
\begin{aligned}
r_s(\zeta(\lambda), \{\lambda_i\}) &= \limsup_{i\to\infty} \hat{d}(\lambda_i, \zeta(\lambda)) \\
&\leq \limsup_{i\to\infty} [\frac{(3+\alpha)}{(1-\alpha)} \hat{d}(\lambda_i, \zeta(\lambda_i)) + \hat{d}(\lambda_i, \lambda)] \\
&\leq \limsup_{i\to\infty} \hat{d}(\lambda_i, \lambda) = r_s(\lambda, \{\lambda_i\}).
\end{aligned}
$$

By uniqueness of the asymptotic center, $\zeta(\lambda) = \lambda$, so $\lambda \in F(\zeta)$. Similarly, we also have $\mu \in F(\zeta)$. Further, we show that $\lambda = \mu$. Let $\lambda \neq \mu$; then by uniqueness of the asymptotic center, we have

$$
\begin{aligned}
\limsup_{i\to\infty} \hat{d}(s_i, \lambda) &= \limsup_{i\to\infty} \hat{d}(\lambda_i, \lambda) \\
&\leq \limsup_{i\to\infty} \hat{d}(s_i, \mu) = \limsup_{i\to\infty} \hat{d}(s_i, \mu) = \limsup_{i\to\infty} \hat{d}(\mu_i, \mu) \\
&\leq \limsup_{i\to\infty} \hat{d}(\mu_i, \lambda) = \limsup_{i\to\infty} \hat{d}(s_i, \lambda).
\end{aligned}
$$

Contradiction. Hence $\lambda = \mu$. Then the sequence $\{s_i\}$ Δ-converges to $F(\zeta)$. □

Further, we give some strong convergence theorems.

Theorem 3.2. *Let $\mathcal{J}, \mathcal{H}, \zeta$ and $\{s_i\}$ be as in Lemma* 3.1 *which satisfy condition* (C_α) *and* $F(\zeta) \neq \emptyset$, *then $\{s_i\}$ converges strongly to $F(\zeta)$ or ($\{s_i\} \to s^* \in F(\zeta)$) if and only if* $\liminf_{i\to\infty} \hat{d}(s_i, F(\zeta)) = 0$.

Proof. Let $\{s_i\} \to s^* \in F(\zeta)$. Then, $\lim_{i\to\infty} \hat{d}(s_i, s^*) = 0$. Because $0 \leq \liminf_{i\to\infty} \hat{d}(s_i, F(\zeta)) \leq \hat{d}(s_i, s^*)$, therefore $\liminf_{i\to\infty} \hat{d}(s_i, F(\zeta)) = 0$.

Conversely, assume that $\liminf_{i\to\infty} \hat{d}(s_i, F(\zeta)) = 0$. From Lemma 3.1 we get $\hat{d}(s_{i+1}, s^*) \leq \hat{d}(s_i, s^*)$ for all $s^* \in F(\zeta)$.

Thus $\hat{d}(s_{i+1}, F(\zeta)) \leq \hat{d}(s_i, F(\zeta))$. Therefore, $\lim_{i\to\infty} \hat{d}(s_i, F(\zeta))$ exists. From the assumption of our theorem, $\liminf_{i\to\infty} \hat{d}(s_i, F(\zeta)) = 0$. So, we have $\lim_{i\to\infty} \hat{d}(s_i, F(\zeta)) = 0$.

Next, we prove that $\{s_i\}$ is a Cauchy sequence in $\mathcal{J}$. Let $\upsilon > 0$. Since, $\lim_{i\to\infty} \hat{d}(s_i, F(\zeta)) = 0$, for any given $\upsilon > 0$, there is $i_0 \in \mathbb{N}$ such that

$$
\hat{d}(s_i, F(\zeta)) < \frac{\upsilon}{2}, \quad i \geq i_0.
$$

In particular, $\inf\{\hat{d}(s_i, s^*) : s^* \in F(\zeta)\} < (\upsilon/2)$. Then, there exists $s^{**} \in F(\zeta)$ such that $\hat{d}(s_{n_0}, s^{**}) < (\upsilon/2)$. For any $i, j \geq i_0$, we get

$$
\begin{aligned}
\hat{d}(s_{i+j}, s^i) &\leq \hat{d}(s_{i+j}, s^{**}) + \hat{d}(s^{**}, s_i) \\
&\leq \hat{d}(s_{i_0}, s^{**}) + \hat{d}(s^{**}, s_i) \\
&\leq \frac{\upsilon}{2} + \frac{\upsilon}{2} = \upsilon.
\end{aligned}
$$

This implies that $\{s_i\}$ is a Cauchy sequence in $\mathcal{J}$. Then, $\mathcal{J}$ is a closed and complete subset of a $\mathcal{H}$. Then the sequence $\{s_i\}$ must to converge to a point from $\mathcal{J}$. Let $\lim_{i\to\infty} s_i = \omega \in \mathcal{J}$.

Next, we prove $\omega \in F(\zeta)$. Since, $F(\zeta)$ satisfies the condition (C_α) and using Lemma 2.1, we have

$$\hat{d}(s_i, \zeta(\omega)) \leq \frac{(3+\alpha)}{(1-\alpha)}\hat{d}(s_i, \zeta(s_i)) + \hat{d}(s_i, \omega).$$

Letting $i \to \infty$, and using Lemma 3.2 we get $\hat{d}(s_i, \zeta(s_i)) = 0$. Then, we have $\hat{d}(\omega, \zeta(\omega)) = 0$, which means ω is a fixed point of ζ. Hence, $\{s_i\}$ converges strongly to $F(\zeta)$. □

Theorem 3.3. *Let $\mathcal{J}, \mathcal{H}, \zeta$ and $\{s_i\}$ be a sequence defined as in Lemma* 3.1 *with condition (C_α), and $F(\zeta) \neq \emptyset$. If ζ satisfies the Sentor and Dotson condition (I), then the sequence $\{s_i\}$ converges strongly to a fixed of ζ.*

Proof. From Lemma 3.2, we have

$$\lim_{i\to\infty} \hat{d}(s_i, \zeta(s_i)) = 0. \tag{3.10}$$

From *Condition (I)* and (3.10), we get

$$0 \leq \lim_{i\to\infty} \Theta(\hat{d}(s_i, F(\zeta))) \leq \lim_{i\to\infty} \hat{d}(s_i, \zeta(s_i)),$$

which implies

$$\lim_{i\to\infty} \Theta(\hat{d}(s_i, F(\zeta))) = 0.$$

Since $\Theta : [0,\infty) \to [0,\infty)$ is an increasing function satisfying $\Theta(0) = 0$, $\Theta(c) > 0$ for all $c > 0$, we have

$$\lim_{i\to\infty} \hat{d}(s_i, F(\zeta)) = 0.$$

Then, all conditions of the Theorem 3.2 are satisfied. Hence, $\{s_i\}$ converges strongly to $F(\zeta)$. □

4. Numerical example

This section deals with the numerical and analytical analysis of the Picard-Thakur hybrid iterative scheme (1.6). We provide a numerical example for the convergence of a mapping which satisfying the condition C_α) but fails to satisfy Suzuki's *Condition (C)*.

Example 4.1. *Let $\mathcal{J} = [0,\infty)$ endowed with usual norm $|.|$. Let ζ be a self map on $\mathcal{J}$ as $\zeta : [0,\infty) \to [0,\infty)$ and defined as*

$$\zeta(s) = \begin{cases} \frac{s+5}{2} & if\ \ s \geq 3 \\ 5 & if\ s \in [0,3). \end{cases}$$

First, we have to show that ζ does not satisfy the Suzuki's condition (C). For this, let $s = \frac{5}{2}$ and $t = \frac{7}{2}$. Since

$$\frac{1}{2}\hat{d}(s, \zeta(s)) = \frac{1}{2}|\frac{5}{2} - \frac{15}{4}| = \frac{1}{2}|\frac{5}{4}| = \frac{5}{8} = 0.65$$

and

$$\hat{d}(s,t) = |\frac{5}{2} - \frac{7}{2}| = \frac{3}{2} = 1.50,$$

which implies $\frac{1}{2}\hat{d}(s,\zeta(s)) < \hat{d}(s,t)$.

Now,

$$\hat{d}(\zeta(s),\zeta(t)) = |5 - \frac{t+5}{2}| = |5 + \frac{3}{4}| = \frac{23}{4} = 5.75 > 1.50 = \hat{d}(s,t).$$

So, $\frac{1}{2}\hat{d}(s,\zeta(s)) < \hat{d}(s,t)$ *but* $\hat{d}(\zeta(s),\zeta(t)) \geq \hat{d}(s,t)$. *Hence,* ζ *does not satisfy the Suzuki's* Condition (C).

Next, we have to show that ζ *is generalized* α*-nonexpansive mapping i.e.,*

$$\hat{d}(\zeta(s),\zeta(t)) \leq \alpha\hat{d}(t,\zeta(s)) + \alpha\hat{d}(s,\zeta(t)) + (1-2\alpha)\hat{d}(s,t).$$

We take $\alpha = \frac{1}{3}$ *and we consider the following cases:*

Case 1: When $s \geq 3$ *and* $t \in [0,3)$ *then*

$$\hat{d}(\zeta(s),\zeta(t)) = |\frac{s+5}{2} - 5| = \frac{1}{2}|s-5|$$

and

$$\begin{aligned}\alpha\hat{d}(t,\zeta(s)) + \alpha\hat{d}(s,\zeta(t)) + (1-2\alpha)\hat{d}(s,t) =& \frac{1}{3}|t - \frac{s+5}{2}| + \frac{1}{3}|s-5| + \frac{1}{3}|s-t| \\ \geq& \frac{1}{3}|(t - \frac{s+5}{2}) - (s-5)| + \frac{1}{3}|s-t| \\ \geq& \frac{1}{3}|\frac{-s}{2} + t + \frac{5}{2}| + \frac{1}{3}|s-t| \\ \geq& \frac{1}{2}|s-5| = \hat{d}(\zeta(s),\zeta(t)).\end{aligned}$$

Case 2: Let $s \in [0,3)$ *and* $t \geq 3$*, then it is obvious that*

$$\hat{d}(\zeta(s) - \zeta(t)) \leq \frac{1}{3}\hat{d}(t,\zeta(s)) + \frac{1}{3}\hat{d}(s,\zeta(t)) + (\frac{1}{3})\hat{d}(s,t).$$

Case 3: Let $s \in [0,3)$ *and* $t \in [0,3)$*; then*

$$\begin{aligned}\alpha\hat{d}(l,\zeta(s)) + \alpha\hat{d}(s,\zeta(t)) + (1-2\alpha)\hat{d}(s,t) =& \frac{1}{3}|t-5| + \frac{1}{3}|s-5| + \frac{1}{3}|s-t| \\ \geq& \frac{1}{3}|(t-5) - (s-5)| + \frac{1}{3}|s-t| \\ \geq& \frac{2}{3}|s-t| \\ \geq& 0 = \hat{d}(\zeta(s),\zeta(t)).\end{aligned}$$

So, ζ *is generalized* $\frac{1}{3}$*−nonexpansive mapping.*

Further, we take different choices of parameters η_i, ρ_i and σ_i and set the stoping criterion as $\|s_i - s^*\| \leq 10^{-15}$, where $s^* = 5$ is fixed point of the problem. In Table 1, we obtain the convergence behavior of the iterative scheme (1.6) compare with other schemes discussed in literature by choosing different initial point and fixed parameters as: $\eta_i = \frac{i}{(i^3+6)^2}$, $\rho_i = \sqrt{\frac{i}{(6i+5)^7}}$ and $\sigma_i = 1 - \frac{i}{(i+5)^2}$. In Table 2, we compare the iterative scheme (1.6) by choosing different choices of parameters.

Table 1. Influence of initial points for different iterative schemes.

Initial points	Picard-Ishikawa hybrid Scheme	Picard-S hybrid Scheme	Picard-S^* hybrid Scheme	Picard-Thakur hybrid Scheme
3.16	53	26	25	17
4.90	49	24	23	17
5.55	50	25	25	17
6.25	52	27	26	17
10.00	54	28	27	17

Table 2. Comparison of iterative schemes for different choices of parameters.

	B: For $\eta_i = \sqrt{\frac{i+2}{i^2+3}}$,	$\rho_i = \sqrt{\frac{i+3}{2i+6}}$,	$\sigma_i = \sqrt{\frac{i^2+2}{i^2+i+3}}$	
Initial points	Picard-Ishikawa hybrid Scheme	Picard-S hybrid Scheme	Picard-S^* hybrid Scheme	Picard-Thakur hybrid Scheme
3.16	37	24	22	17
4.25	36	23	21	17
4.90	34	22	20	17
5.55	37	23	22	17
6.00	38	25	22	17
	C: For $\eta_i = \frac{\sqrt{i}}{(i^3+6)^6}$	$\rho_i = \frac{\sqrt{i}}{(6i+5)^7}$,	$\sigma_i = 1 - \frac{\sqrt{i}}{(i+5)^2}$	
3.16	50	27	26	17
4.25	49	25	25	17
4.90	45	25	24	17
5.55	47	26	25	17
6.00	48	26	25	17
	D: For $\eta_i = \frac{2i}{(9i+8)^2}$	$\rho_i = 1 - \frac{1}{(i+7)^3}$	$\sigma_i = 1 - \frac{6i}{(7i+5)^5}$	
3.16	50	27	26	17
4.25	49	25	25	17
4.90	46	25	25	17
5.55	51	26	26	17
6.00	52	27	26	17

Next, we will give a comparison of iterative schemes for different choices of parameters of the Tables 1 and 2, with initial points $s_0 = 3.16$ and 6.00.

Using the values of our tables we will generate in *Matlab*-version *R2021a* some figures proving that the iterative scheme (1.6) is stable and converges faster to the fixed point, w.r.t. different choices of initial points and parameters. Then, we will obtain Figures 1–3 for parameter used in Table 1, Figures 4 and 5 for parameters B, Figures 6 and 7 for parameters C and Figures 8 and 9 for parameters D used in Table 2, respectively, for initial points $s_0 = 3, 16$ and 6.00.

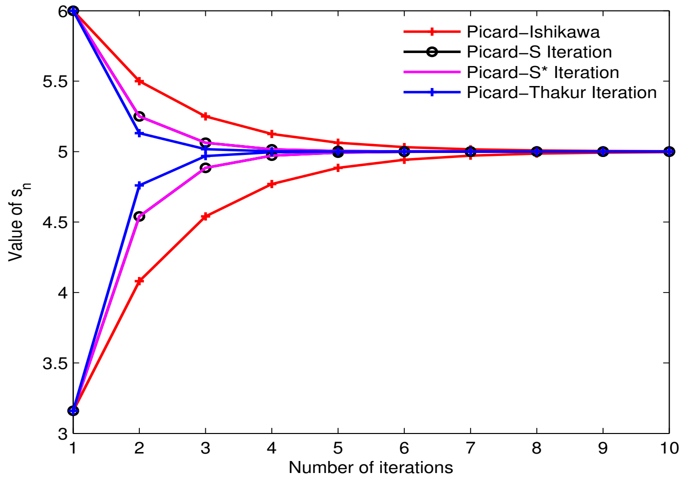

Figure 1. Comparison of iterative schemes for two initial values $s_0 = 3.16$ and $s_0 = 6.00$.

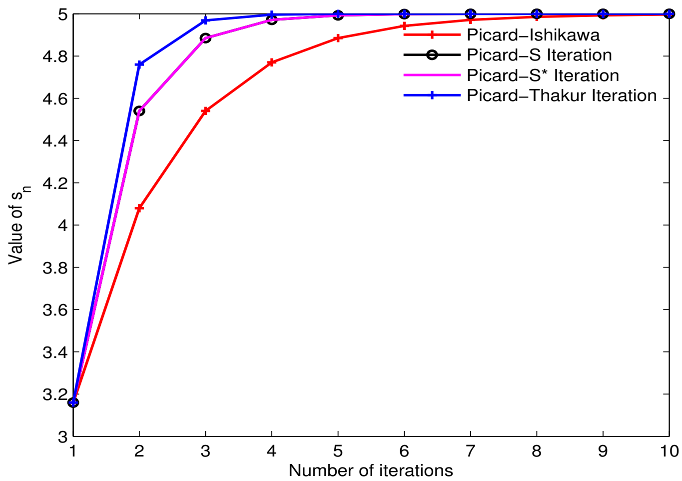

Figure 2. Graph for different parameters of Table 1.

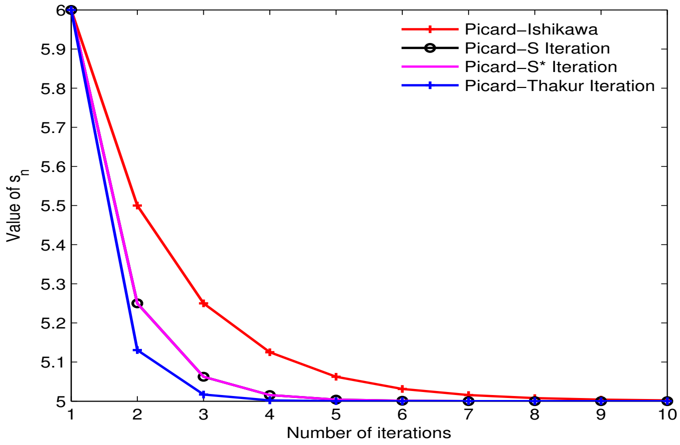

Figure 3. Graph for different parameters of Table 1.

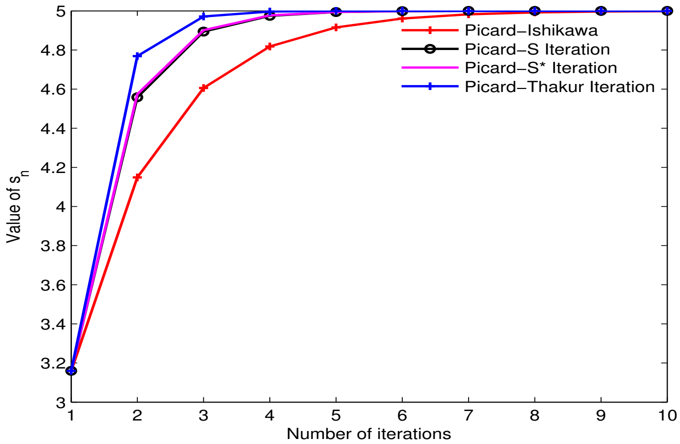

Figure 4. Graph for different parameters from B section of Table 2.

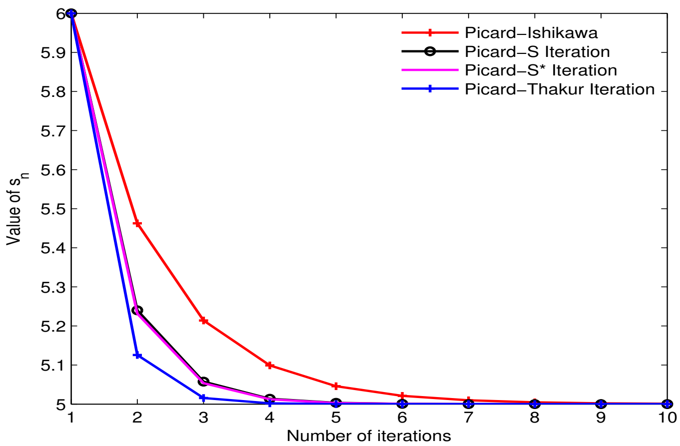

Figure 5. Graph for different parameters from B section of Table 2.

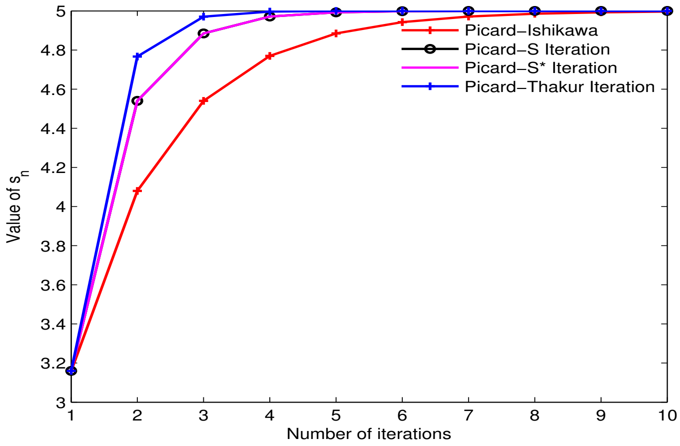

Figure 6. Graph for different parameters from C section of Table 2.

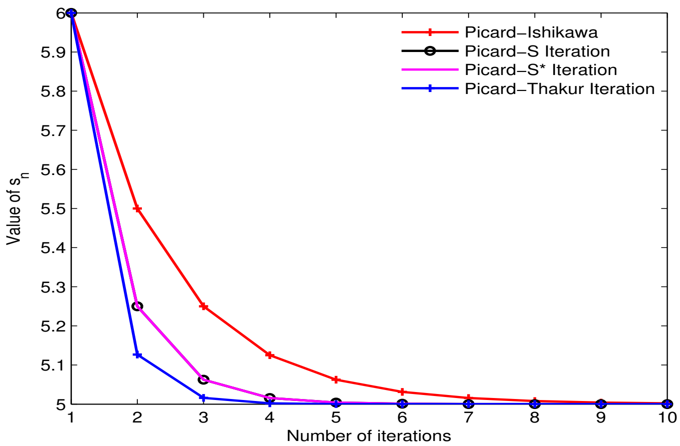

Figure 7. Graph for different parameters from C section of Table 2.

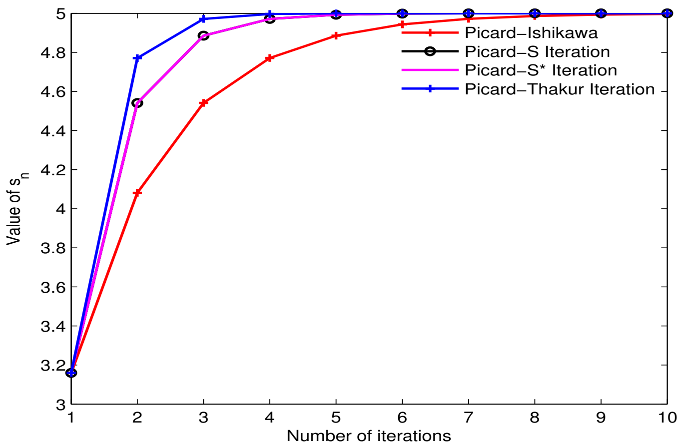

Figure 8. Graph for different parameters from D section of Table 2.

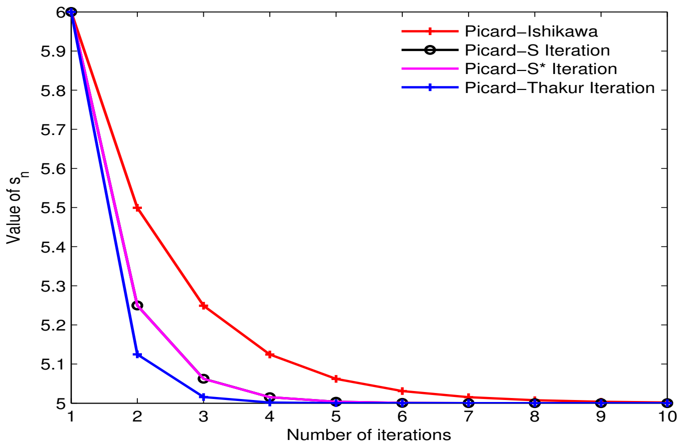

Figure 9. Graph for different parameters from D section of Table 2.

Studying the figures and tables previous presented here we can conclude that the iterative scheme (1.6) has better rate of convergence than other hybrid schemes.

5. Application to Fredholm integral equations

This section deals with the application to Fredholm integral equations. Let us consider the following integral equation

$$s(\bar{a}) = \mathcal{F}(\bar{a}) + \mu \int_g^h \mathcal{W}(\bar{a}, \bar{b}) f(\bar{b}, s(\bar{b})) d\bar{b}, \bar{a} \in [g, h],\ \mu \geq 0. \tag{5.1}$$

Suppose that $\mathcal{J}$ be nonempty compact subset of a Banach space, which is special case of hyperbolic space $\mathcal{H} = C[g, h]$, where $C[g, h]$ denotes the space of all continuous real-valued functions defined on an interval $[g, h]$, with norm $\|s - t\| = \max_{\bar{a} \in [g,h]} |s(\bar{a}) - t(\bar{a})|$. Obviously, $(C[g, h], \|.\|)$ is a Banach space.

Consider the following conditions hold:

C_1) $\mathcal{F} : [g, h] \to R$ is continuous.

C_2) $f : [g, h] \times \mathcal{J} \to \mathcal{J}$ is continuous, $f(\bar{a}, s) \geq 0$ and satisfy the Lipschitzian condition w.r.t. second variable, that is, there exist $\mathcal{L} \geq 0$ such that for all $s, t \in \mathcal{J}$

$$\|f(\bar{a}, s) - f(\bar{a}, t)\| \leq \mathcal{L}\|s - t\|.$$

C_3) $\mathcal{F} : [g, h] \times [g, h] \to R$ is continuous such that $\mathcal{W}(\bar{a}, \bar{b}) \geq 0$ for all $\mathcal{W}(\bar{a}, \bar{b}) \in [g, h] \times [g, h]$ and $\int_g^h \mathcal{W}(\bar{a}, \bar{b}) d\bar{b} \leq \mathcal{M}$ where $\mathcal{M} \in R$ is fixed, such that $\mu = \frac{1}{\mathcal{L}\mathcal{M}}$.

Theorem 5.1. *Suppose that the conditions* (C_1)–(C_3) *are satisfied. Then the problem* (5.1) *has a solution* s^* *in* $C[g, h]$ *and the iteration scheme* (1.6) *converges to* s^*.

Proof. Let $\{s_i\}$ be a sequence generated by (1.6). Then we get

$$\zeta(s(\bar{a})) = \mathcal{F}(\bar{a}) + \mu \int_g^h \mathcal{W}(\bar{a}, \bar{b}) f(\bar{b}, s(\bar{b})) d\bar{b}, \bar{a} \in [g, h], \mu \geq 0.$$

Let s^* be fixed point of ζ. We prove that $s_i \to s^*$ as $i \to \infty$. Define $v_i = (1-\sigma_i)s_i + \sigma_i\zeta(s_i)$. Then, the following estimation holds

$$\begin{aligned}
\|v_i - s^*\| =& \|(1-\sigma_i)s_i + \sigma_i\zeta(s_i) - s^*\| \\
\leq& (1-\sigma_i)\|s_i - s^*\| + \sigma_i\|\zeta(s_i) - s^*\| \\
=& (1-\sigma_i)\|s_i - s^*\| + \sigma_i\|\mathcal{F}(\bar{a}) + \mu\int_g^h \mathcal{W}(\bar{a},\bar{b})f(\bar{b}, s_i(\bar{b}))d\bar{b} - \mathcal{F}(\bar{a}) \\
&- \mu\int_g^h \mathcal{W}(\bar{a},\bar{b})f(\bar{b}, s^*(\bar{b}))d\bar{b}\| \\
\leq& (1-\sigma_i)\|s_i - s^*\| + \sigma_i\mu\int_g^h \mathcal{W}(\bar{a},\bar{b})\|f(\bar{b}, s_i(\bar{b})) - f(\bar{b}, s^*(\bar{b}))\|d\bar{b} \\
\leq& (1-\sigma_i)\|s_i - s^*\| + \sigma_i\mu\int_g^h \mathcal{W}(\bar{a},\bar{b})\mathcal{L}\|s_i - s^*\|d\bar{b} \\
\leq& (1-\sigma_i)\|s_i - s^*\| + \sigma_i\mu\mathcal{L}\mathcal{M}\|s_i - s^*\| \\
\leq& (1-\sigma_i)\|s_i - s^*\| + \sigma_i\|s_i - s^*\| \\
=& \|s_i - s^*\|.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|u_i - s^*\| &= \|(1-\rho_i)v_i + \rho_i\zeta(u_i) - s^*\| \leq \|v_i - s^*\| \leq \|s_i - s^*\|, \\
\|t_i - s^*\| &= \|(1-\eta_i)\zeta(v_i) + \eta_i\zeta(u_i)\| \leq \|v_i - s^*\| \leq \|s_i - s^*\|, \\
\|s_{i+1} - s^*\| &= \|\zeta(t_i) - s^*\| \leq \|t_i - s^*\| \leq \|s_i - s^*\|.
\end{aligned}$$

Let $\|s_i - s^*\| = a_i$. Then, we have $a_{i+1} \leq a_i$.

Let $\{a_i\}$ be the sequence of positive real numbers and monotone decreasing and hence, bounded. By Theorem 2.1 we have $\lim\limits_{i\to\infty} a_i = 0$; so $s_i \to s^*$. □

6. Conclusions

In this paper, we investigated the fixed point approximation for generalized α-nonexpansive mappings in hyperbolic space utilizing the Picard-Thakur hybrid iteration scheme and some strong and Δ-convergence results were established. Through numerical example we have shown that the proposed the Picard-Thakur hybrid iterative scheme (1.6) converges faster than (1.3)–(1.5) for different type of initial points and parameters.

We extend various results in hyperbolic space as in literature. Additionally, we also provide an application to Fredholm integral equation.

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Conflict of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

References

1. K. Aoyama, F. Kohsaka, Fixed point theorem for α-nonexpansive mappings in Banach spaces, *Nonlinear Anal.: Theory Methods Appl.*, **74** (2011), 4387–4391. https://doi.org/10.1016/j.na.2011.03.057
2. T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, *J. Math. Anal. Appl.*, **340** (2008), 1088–1095.
3. R. Pant, R. Shukla, Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces, *Numer. Func. Anal. Opt.*, **38** (2017), 248–266. https://doi.org/10.1080/01630563.2016.1276075
4. A. A. Mebawondu, C. Izuchukwu, Some fixed points properties, strong and Δ-convergence results for generalized α-nonexpansive mappings in hyperbolic spaces, *Adv. Fixed Point Theory*, **8** (2018), 1–20. https://doi.org/10.28919/afpt/3415
5. S. Banach, Sur les opérations dans les ensembles abstraites et leurs applications, *Fund. Math.*, **3** (1922), 133–187.
6. E. Picard, Mémoire sur la théorie des équations aux dérivées partielles et la méthode des approximations successives, *J. Math. Pures Appl.*, **6** (1980), 145–210.
7. V. Berinde, *Iterative approximation of fixed points*, Springer-Verlag Berlin, Heidelberg, 2007. https://doi.org/10.1007/978-3-540-72234-2
8. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510. https://doi.org/10.2307/2032162
9. R. P. Agarwal, D. O. Regan, D. R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *J. Nonlinear Convex Anal.*, **8** (2007), 61.
10. D. Thakur, B. S. Thakur, M. Postolache, New iteration scheme for approximating fixed points of nonexpansive mappings, *Filomat*, **30** (2016), 2711–2720. https://doi.org/10.2298/FIL1610711T
11. S. H. Khan, A Picard-Mann hybrid iterative process, *Fixed Point Theory Appl.*, **1** (2013), 1–10. https://doi.org/10.1186/1687-1812-2013-69
12. G. A. Okeke, Convergence analysis of the Picard-Ishikawa hybrid iterative process with applications, *Afr. Mat.*, **30** (2019), 817–835. https://doi.org/10.1007/s13370-019-00686-z
13. F. Gürsoy, V. Karakaya, A Picard-S hybrid type iteration method for solving a differential equation with retarded argument, *arXiv*, 2014. https://doi.org/10.48550/arXiv.1403.2546
14. P. Lamba, A. Panwar, A Picard S* iterative algorithm for approximating fixed points of generalized α-nonexpansive mappings, *J. Math. Comput. Sci.*, **11** (2021), 2874–2892. https://doi.org/10.28919/jmcs/5624
15. J. Jia, K. Shabbir, K. Ahmad, N. A. Shah, T. Botmart, Strong convergence of a new hybrid iterative scheme for nonexpensive mappings and applications, *J. Funct. Spaces*, **2022** (2022), 4855173. https://doi.org/10.1155/2022/4855173

16. S. Temir, Ö. Korkut, Approximating fixed points of generalized α-nonexpansive mappings by the new iteration process, *J. Math. Sci. Model.*, **5** (2022), 35–39. http://dx.doi.org/10.33187/jmsm.993823
17. B. A. B. Dehaish, R. K. Alharbi, On convergence theorems for generalized alpha nonexpansive mappings in Banach spaces, *J. Funct. Spaces*, **2021** (2021), 1031–1050. https://doi.org/10.1155/2021/6652741
18. K. Ibrahim, J. L. Olakunle, A three steps iterative process for approximating the fixed points of multivalued generalized α-nonexpansive mappings in uniformly convex hyperbolic spaces, *Sigma J. Eng. Nat. Sci.*, **38** (2021), 1031–1050.
19. S. Cholatis, S. Kittikorn, K. E. Chakkrid, K Wongvisarut, The generalized α-nonexpansive mappings and related convergence theorems in hyperbolic spaces, *J. Inf. Math. Sci.*, **11** (2019), 1–17. https://doi.org/10.26713/jims.v11i1.1147
20. U. Kohlenbach, Some logical metatheorems with applications in functional analysis, *Trans. Amer. Math. Soc.*, **357** (2005), 89–128. https://doi.org/10.1090/S0002-9947-04-03515-9
21. S. Reich, I. Shafrir, Nonexpansive iterations in hyperbolic spaces, *Nonlinear Anal.: Theory Methods Appl.*, **15** (1990), 537–558. https://doi.org/10.1016/0362-546X(90)90058-O
22. M. R. Bridson, A. Haefliger, *Metric spaces of non-positive curvature,* Springer Berlin, Heidelberg, 2013. https://doi.org/10.1007/978-3-662-12494-9
23. L. Leustean, A quadratic rate of asymptotic regularity for CAT (0)-spaces, *J. Math. Anal. Appl.*, **325** (2007), 386–399. https://doi.org/10.1016/j.jmaa.2006.01.081
24. S. Dhompongsa, B. Panyanak, On Δ-convergence theorems in CAT (0) spaces, *Comput. Math. Appl.*, **56** (2008), 2572–2579. https://doi.org/10.1016/j.camwa.2008.05.036
25. W. A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, *Nonlinear Anal.: Theory Methods Appl.*, **68** (2008), 3689–3696. https://doi.org/10.1016/j.na.2007.04.011
26. L. Leuştean, Nonexpansive iterations in uniformly convex W-hyperbolic spaces, *Nonlinear Anal. Opt. I: Nonlinear Anal.*, **513** (2010), 193–209. https://dx.doi.org/10.1090/conm/513/10084
27. T. Kuczumow, An almost convergence and its applications, *Ann. Univ. Mariae Curie-Skłodowska, Sect. A.,* **32** (1978), 79–88.
28. H. F. Senter, W. G. Dotson, Approximating fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.*, **44** (1974), 375–380. https://doi.org/10.1090/S0002-9939-1974-0346608-8
29. A. R. Khan, H. Fukhar-ud-din, M. A. A. Khan, An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces, *Fixed Point Theory Appl.*, **2012** (2012), 1–12. https://doi.org/10.1186/1687-1812-2012-54
30. M. Imdad, S. Dashputre, Fixed point approximation of Picard normal S-iteration process for generalized nonexpansive mappings in hyperbolic spaces, *Math. Sci.,* **10** (2016), 131–138. https://doi.org/10.1007/s40096-016-0187-8
31. R. G. Bartle, D. R. Sherbert, *Introduction to real analysis*, Wiley, New York, 2000.

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