



Research article

Solitary wave solutions of Sawada-Kotera equation using two efficient analytical methods

Muhammad Bilal Riaz^{1,2,3}, Faiza Naseer⁴, Muhammad Abbas^{4,*}, Magda Abd El-Rahman⁵, Tahir Nazir⁴ and Choon Kit Chan³

¹ IT4Innovations, VSB–Technical University of Ostrava, Ostrava, Czech Republic

² Department of Computer Science and Mathematics, Lebanese American University, Byblos, Lebanon

³ Faculty of Engineering and Quantity Surveying, INTI International University, Putra Nilai, 71800 Negeri Sembilan, Malaysia

⁴ Department of Mathematics, University of Sargodha, 40100 Sargodha, Pakistan

⁵ Department of Physics, College of Science, King Khalid University, Abha, 61413, Saudi Arabia

* **Correspondence:** Email: muhammad.abbas@uos.edu.pk; Tel: +92-304-6282830.

Abstract: The soliton solutions are one of the stable solutions where nonlinearity and dispersion are perfectly balanced. They are used in a wide variety of physical fields, including plasma, solid state, neuronal, biological production, and diffusion processes. Different analytical methods have been used until now to obtain the soliton solutions of the Sawada-Kotera (SK) equation. The purpose of this study is to offer two successful analytical methods for solving the classical (1+1) dimensional Sawada-Kotera (SK) equation. In order to solve the partial differential equation (PDE), both the modified auxiliary equation method (MAEM) and the extended direct algebraic method are applied. The classical fifth-order SK equation is examined in this study, leading to a variety of precise soliton solutions, including single, periodic, and dark soliton, which are obtained analytically. To illustrate the effect of the parameters, the results are shown in graphical form.

Keywords: modified auxiliary equation method; Sawada-Kotera equation; trigonometric solutions; hyperbolic solutions; extended direct algebraic method

Mathematics Subject Classification: 26A48, 26A51, 33B10, 39A1, 39B62

1. Introduction

Since the dawn of mankind, humans have shown a keen interest in comprehending natural phenomena, starting with fire, lightning, thunder, earthquakes and volcanoes, extending all the way

down to the nano particle. Many scientists have expressed interest in researching these events from the start of the 18th century. Until the middle of the 19th century, these studies are still being developed. The study of various scientific fields has benefited greatly from the use of PDEs, which have been used to describe a wide variety of events. Since the discovery of partial differential equations and their types, methods for solving them, and methods for determining their precise solutions, the world has been shrouded in the darkness of ignorance for a very long time, this darkness has started to dissipate. This has made it easier to comprehend these models and how to use them for global advancement, for keeping up with development, and for keeping up with the scientific renaissance.

Since then, mathematicians have investigated the forms of nonlinear partial differential equations. Scientists and academics have devised several methods for solving PDEs in an effort to comprehend these phenomena. The precise and approximate solutions to these models can be found using a variety of techniques. The key findings in identifying explicit solutions to nonlinear partial differential equations (NLPDEs) were deduced in [1]. In the scientific and social sciences, NLPDEs are frequently used to describe complicated events. Modern science and engineering have produced several well-known models to demonstrate the dynamics of nonlinearity, including the Kortewegde Vries (KdV) equation [2], the KdV Burgers equation [3], the modified KdV (mKdV) equation [4, 5], the modified KdV Kadomtsev Petviashvili (mKdV KP) equation [6] and others.

The SK equation was created by Wazwaz [7] from the mKdV equation, and a few multiple solitons results were found using the condensed Hirota method. Later, utilizing the sine-cosine function and Kudryashov approaches, many researchers explored the mKdV model and discovered several kink, brilliant and periodic solutions in singular form. Dual wave solutions were created by the authors using modified Kudryashov and auxiliary equation approaches. The SK equation is a special instance of the mKdV equation [8]. In optical fibre, the variable-coefficients in NLPDEs can indicate group velocity dispersion, Kerr nonlinearity, third-order dispersion, time-delay related to the cubic term, tidal and wave energy and external potential [9–12]. To the best of our knowledge, the (1+1)-dimensional time-dependent variable-coefficient SK equation has not been researched before. The literature shows that there is potential for more research into the SK equation using different approaches, as well as demonstrating how their physical explanation works. The SK equation is one of the well-known partial differential equations. It is a significant mathematical model that is applied to numerous physical situations, including quantum physics, nonlinear optics and the travel of long waves in shallow water under gravity. Sawada and Kotera produced the (1+1)-dimensional SK equation, which established the presence of multiple solutions in [13]. The classical (1+1) dimensional SK equation is given as [14]

$$u_t + 45u^2u_r - 15u_r u_{rr} - 15uu_{rrr} + u_{rrrrr} = 0, \quad (1.1)$$

which represents a nonlinear physical model equation. Different approaches were examined with this important model equation [13, 15–20]. The SK equation has a vast range from classical to bidirectional and dual-wave forms, which are used by a lot of researchers in their articles. Zeng used the Hirota bilinear method to construct the N-soliton solution of the (1+1)-dimensional variable-coefficient Sawada–Kotera equation [21], lump-type solutions to the bidirectional Sawada–Kotera equation [22], modified Kudryashov and new auxiliary equation methods to investigate the abundant solutions to the two-mode Sawada–Kotera equation [23], the trial function method to find the solitary wave solutions of bidirectional SK and variable-coefficient SK

equations [24], $\exp(-\psi(z))$ -expansion method to find meromorphic solutions to the KdV-SK equation [25], exact traveling wave solutions of the CDGSK equation using subequations method [26], Lie symmetry classification of the generalized PDE [27] and the lump solution of the CDGSK using bilinear operator [28].

Soliton theory is essential for finding these nonlinear models' exact solutions because it can reveal important underlying details about phenomena like optical fibres, plasma physics, fluid mechanics, biophysics, solid-state physics, control theory, systems identification and many more. In numerous branches of applied research and engineering, the age of analytical solutions is becoming increasingly important to gain a deeper understanding of nonlinear processes and their potential practical uses. To realize the wave structures more effectively, the traveling wave structures plays a significant role.

Because of this, eminent researchers of this decade are more eager to create any cutting-edge analytical answers for such models. Both fundamental theory and applications of nonlinear science heavily utilize NLPDE solitons. The soliton and solitary wave solutions give physical details and enhance the problem's mathematical aspects [29–34]. The extended direct algebraic method has been used previously for finding solutions of many equations, like to find optical solitons for quadratic-cubic nonlinearity [35], to find the solutions of cubic nonlinear Schrodinger equation [36] and for finding the solitary wave solutions of nonlinear evolution equations [37]. The extended direct algebraic method has not yet been applied to the SK equation. By displaying the discovered solutions in two- and three-dimensional graphs with various parameter values, this approach offers insightful information regarding the internal formation of the traveling wave phenomenon.

According to the literature, there is need for additional research on the SK equation in order to develop new dual-wave solutions using the modified Kudryashov method (mKM) and the auxiliary equation method (AEM), as well as to derive their physical clarifications. The modified auxiliary equation method is an interesting special case of auxiliary equation method. In this study, a modest effort has been made to create some soliton solutions to the SK equation using the modified auxiliary equation method and the extended direct algebraic method. This effort has been prompted by the current literature. The suggested methods are among the best direct methods for creating precise solitary wave solutions to nonlinear partial differential equations. The strategy in this paper is summarized as follows: Section 2 offers a description of MAEM. In Section 3, the MAEM technique is explained in detail. The extended direct algebraic method is given in Section 4. Section 5 provides an explanation of the graphical representations. Section 6 contains the closing comments.

2. Governing techniques

2.1. The algorithm of the MAEM technique

Let $u = u(r, t)$ satisfy the general equation given as

$$G(u, u_r, u_t, u_{rr}, u_{rt}, u_{tt}, \dots) = 0, \quad (2.1)$$

which can be converted to an ODE of the form

$$H(u, u', u'', \dots) = 0. \quad (2.2)$$

The provided expansion is present in the general solution

$$u(\zeta) = \alpha_0 + \sum_{j=1}^k [\alpha_j(\phi^g)^j + \beta_j(\phi^g)^{-j}], \quad (2.3)$$

where α and β are arbitrary constants.

The nature of the general solution $g(\zeta)$ to Eq (2.1) is determined by using an auxiliary equation given as

$$g'(\zeta) = \frac{a + b\phi^{-j} + c\phi^j}{\ln \phi}, \quad (2.4)$$

where $\phi > 0, \phi \neq 1$. Also, a, b, c are arbitrary values. The roots of Eq (2.4) are given below.

2.2. Special cases

For Eq (2.4), the following situations occurred as

- When $b^2 - 4ac < 0, c \neq 0$,

$$\phi^{g(\zeta)} = \frac{-b + \sqrt{4ac - b^2} \tan\left(\frac{\sqrt{4ac - b^2}}{2}\zeta\right)}{2c},$$

or

$$\phi^{g(\zeta)} = \frac{-b + \sqrt{4ac - b^2} \cot\left(\frac{\sqrt{4ac - b^2}}{2}\zeta\right)}{2c}.$$

- When $b^2 - 4ac > 0, c \neq 0$,

$$\phi^{g(\zeta)} = \frac{-b + \sqrt{b^2 - 4ac} \tanh\left(\frac{\sqrt{b^2 - 4ac}}{2}\zeta\right)}{2c},$$

or

$$\phi^{g(\zeta)} = \frac{-b + \sqrt{b^2 - 4ac} \coth\left(\frac{\sqrt{b^2 - 4ac}}{2}\zeta\right)}{2c}.$$

- When $b^2 - 4ac = 0, c \neq 0$,

$$\phi^{g(\zeta)} = -\frac{2 + b\zeta}{2c\zeta}.$$

3. Exposition of the procedure

Consider the wave transformation for the SK equation

$$u(r, t) = u(\zeta), \zeta = r - d t, \quad (3.1)$$

where d is a constant. Substituting Eq (3.1) into Eq (1.1) and integrating, we get the ODE

$$-d u + 15u^3 - 15uu'' + u'''' = 0, \quad (3.2)$$

where $u' = \frac{du}{d\zeta}$ and the constants of integration are equal to zero. Balancing the highest order derivative u'''' and the highest order nonlinear term u^3 in Eq (3.2), we get $k = 2$ in accordance with the homogeneous balance principle. As a result, (2.3) produces the differential equation:

$$u(\zeta) = \alpha_0 + \alpha_1\phi^g + \beta_1\phi^{-g} + \alpha_2\phi^{2g} + \beta_2\phi^{-2g}. \quad (3.3)$$

The following set of algebraic equations is constructed using Eqs (3.3) and (2.4) in Eq (3.2), and balancing the coefficients of each power of ϕ^g to zero.

$$\left\{ \begin{array}{l} (\phi^g)^{-6} : 15\beta_2(8a^4 - 6a^2\beta_2 + \beta_2^2) = 0, \\ (\phi^g)^{-5} : 3(8a^4\beta_1 + 8a^2(14ab - 5\beta_1)\beta_2 + 5(-10ab + 3\beta_1)\beta_2^2) = 0, \\ (\phi^g)^{-4} : 15(ab\beta_1(4a^2 - 13\beta_2) + \beta_1^2(-2a^2 + 3\beta_2) + \beta_2(2a^2(11b^2 + 8ac - 3\alpha_0) + (-4(b^2 + 2ac) + 3\alpha_0)\beta_2)) = 0, \\ (\phi^g)^{-3} : 5(-9ab\beta_1^2 + 3\beta_1^3 + \beta_1(2a^2(5b^2 + 4ac - 3\alpha_0) - 3(5(b^2 + 2ac) - 6\alpha_0)\beta_2) \\ + \beta_2(2a(13b^3 + 44abc - 15b\alpha_0 - 9a\alpha_1) + 9(-2bc + \alpha_1)\beta_2)) = 0, \\ (\phi^g)^{-2} : -15(b^2 + 2ac - 3\alpha_0)\beta_1^2 + 15\beta_1(a(b^3 + 4abc - 3b\alpha_0 - 2a\alpha_1) + (-7bc + 6\alpha_1)\beta_2) \\ - \beta_2(d - 8(2b^4 + 29ab^2c + 17a^2c^2) + 15(4(b^2 + 2ac)\alpha_0 - 3\alpha_0^2 + 11ab\alpha_1 + \alpha_2(8a^2 - 3\beta_2) + 2c^2\beta_2)) = 0, \\ (\phi^g)^{-1} : \beta_1(-d + b^4 + 22ab^2c + 16a^2c^2 - 15((b^2 + 2ac)\alpha_0 - 3\alpha_0^2 + 4a^2\alpha_2 \\ + \alpha_1(4ab - 3\beta_1) + bc\beta_1)) + 15(-5(b^2 + 2ac)\alpha_1 + 6\alpha_0(-bc + \alpha_1) \\ + 2\alpha_2(-8ab + 3\beta_1) + 2c(b^3 + 4abc - c\beta_1))\beta_2 = 0, \\ (\phi^g)^0 : -d\alpha_0 + 15\alpha_0^3 + ab^3\alpha_1 + 8a^2bc\alpha_1 - 15ab\alpha_0\alpha_1 + 14a^2b^2\alpha_2 + 16a^3c\alpha_2 - 30a^2\alpha_0\alpha_2 + b^3c\beta_1 \\ + 8abc^2\beta_1 - 15bc\alpha_0\beta_1 - 30b^2\alpha_1\beta_1 - 60aca_1\beta_1 - 90\alpha_0\alpha_1\beta_1 - 135ab\alpha_2\beta_1 + 45\alpha_2\beta_1^2 + 14b^2c^2\beta_2 + 16ac^3\beta_2 \\ - 30c^2\alpha_0\beta_2 - 135bc\alpha_1\beta_2 + 45\alpha_1^2\beta_2 - 120b^2\alpha_2\beta_2 - 240aca_2\beta_2 + 90\alpha_0\alpha_2\beta_2 = 0, \\ (\phi^g)^1 : -15\alpha_1^2(ab - 3\beta_1) + 15\alpha_2(2ab(b^2 + 4ac) - 5(b^2 + 2ac)\beta_1 + 6\alpha_0(-ab + \beta_1) - 16bc\beta_2) \\ + \alpha_1(-d + b^4 + 22ab^2c + 16a^2c^2 - 15((b^2 + 2ac)\alpha_0 - 3\alpha_0^2 + 2\alpha_2(b^2 - 3\beta_2) + 4c(b\beta_1 + c\beta_2))) = 0, \\ (\phi^g)^2 : -15(b^2 + 2ac - 3\alpha_0)\alpha_1^2 + 15\alpha_1(-3aca_0 + \alpha_2(-7ab + 6\beta_1) + r(b^3 + 4abc - 2c\beta_1)) \\ - \alpha_2(d - 8(2b^4 + 29ab^2c + 17a^2c^2) + 15(4(b^2 + 2ac)\alpha_0 - 3\alpha_0^2 + \alpha_2(2a^2 - 3\beta_2) + c(11b\beta_1 + 8c\beta_2))) = 0, \\ (\phi^g)^3 : 5(-9bc\alpha_1^2 + 3\alpha_1^3 + \alpha_1(2c^2(5b^2 + 4ac - 3\alpha_0) - 3(5(b^2 + 2ac) - 6\alpha_0)\alpha_2) \\ + \alpha_2(-30bc\alpha_0 + 9\alpha_2(-2ab + \beta_1) + 2c(13b^3 + 44abc - 9c\beta_1))) = 0, \\ (\phi^g)^4 : 15(bc\alpha_1(4c^2 - 13\alpha_2) + \alpha_1^2(-2c^2 + 3\alpha_2) + \alpha_2(2c^2(11b^2 + 8ac - 3\alpha_0) + (-4(b^2 + 2ac) + 3\alpha_0)\alpha_2)) = 0, \\ (\phi^g)^5 : 3(8c^4\alpha_1 + 8c^2(14bc - 5\alpha_1)\alpha_2 + 5(-10bc + 3\alpha_1)\alpha_2^2) = 0, \\ (\phi^g)^6 : 15\alpha_2(8c^4 - 6c^2\alpha_2 + \alpha_2^2) = 0. \end{array} \right.$$

The system yields the following families.

Family 1

When $\alpha_0 = \frac{1}{30}(5b^2 + 40ac - \sqrt{5}\sqrt{4d + b^4 - 8ab^2c + 16a^2c^2})$, $\alpha_1 = 2bc$, $\beta_1 = 0$, $\alpha_2 = 2c^2$, $\beta_2 = 0$, the following cases are obtained.

For $b^2 - 4ac > 0$ and $n \neq 0$,

$$u_{1,1}(r, t) = \frac{1}{30}(5b^2 + 40ac - \sqrt{5}\sqrt{4d + b^4 - 8ab^2c + 16a^2c^2}) \quad (3.4)$$

$$+2bc\left(\frac{-b + \sqrt{b^2 - 4ac} \tanh\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c}\right) \\ +2c^2\left(\frac{-b + \sqrt{b^2 - 4ac} \tanh\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c}\right)^2,$$

or

$$u_{1,2}(r, t) = \frac{1}{30} \left(5b^2 + 40ac - \sqrt{5} \sqrt{4d + b^4 - 8ab^2c + 16a^2c^2} \right) \quad (3.5) \\ +2bc\left(\frac{-b + \sqrt{b^2 - 4ac} \coth\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c}\right) \\ +2c^2\left(\frac{-b + \sqrt{b^2 - 4ac} \coth\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c}\right)^2,$$

which are the hyperbolic solutions.

For $b^2 - 4ac < 0$ and $n \neq 0$,

$$u_{1,3}(r, t) = \frac{1}{30} \left(5b^2 + 40ac - \sqrt{5} \sqrt{4d + b^4 - 8ab^2c + 16a^2c^2} \right) \quad (3.6) \\ +2bc\left(\frac{-b + \sqrt{4ac - b^2} \tan\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c}\right) \\ +2c^2\left(\frac{-b + \sqrt{4ac - b^2} \tan\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c}\right)^2,$$

or

$$u_{1,4}(r, t) = \frac{1}{30} \left(5b^2 + 40ac - \sqrt{5} \sqrt{4d + b^4 - 8ab^2c + 16a^2c^2} \right) \quad (3.7) \\ +2bc\left(\frac{-b + \sqrt{4ac - b^2} \cot\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c}\right) \\ +2c^2\left(\frac{-b + \sqrt{4ac - b^2} \cot\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c}\right)^2,$$

which are trigonometric solutions.

Family 2

When $\alpha_0 = \frac{1}{30} \left(5b^2 + 40ac + \sqrt{5} \sqrt{4d + b^4 - 8ab^2c + 16a^2c^2} \right)$, $\alpha_1 = 2bc$, $\beta_1 = 0$, $\alpha_2 = 2c^2$, $\beta_2 = 0$, the following cases occurred.

For $b^2 - 4ac > 0$ and $n \neq 0$,

$$u_{2,1}(r, t) = \frac{1}{30} \left(5b^2 + 40ac + \sqrt{5} \sqrt{4d + b^4 - 8ab^2c + 16a^2c^2} \right) \quad (3.8) \\ +2bc\left(\frac{-b + \sqrt{b^2 - 4ac} \tanh\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c}\right) \\ +2c^2\left(\frac{-b + \sqrt{b^2 - 4ac} \tanh\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c}\right)^2,$$

or

$$\begin{aligned}
 u_{2,2}(r, t) = & \frac{1}{30} (5b^2 + 40ac + \sqrt{5} \sqrt{4d + b^4 - 8ab^2c + 16a^2c^2}) \\
 & + 2bc \left(\frac{-b + \sqrt{b^2 - 4ac} \coth\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c} \right) \\
 & + 2c^2 \left(\frac{-b + \sqrt{b^2 - 4ac} \coth\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c} \right)^2,
 \end{aligned} \tag{3.9}$$

which are the hyperbolic solutions.

For $b^2 - 4ac < 0$ and $n \neq 0$,

$$\begin{aligned}
 u_{2,3}(r, t) = & \frac{1}{30} (5b^2 + 40ac + \sqrt{5} \sqrt{4d + b^4 - 8ab^2c + 16a^2c^2}) \\
 & + 2bc \left(\frac{-b + \sqrt{4ac - b^2} \tan\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c} \right) \\
 & + 2c^2 \left(\frac{-b + \sqrt{4ac - b^2} \tan\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c} \right)^2,
 \end{aligned} \tag{3.10}$$

or

$$\begin{aligned}
 u_{2,4}(r, t) = & \frac{1}{30} (5b^2 + 40ac + \sqrt{5} \sqrt{4d + b^4 - 8ab^2c + 16a^2c^2}) \\
 & + 2bc \left(\frac{-b + \sqrt{4ac - b^2} \cot\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c} \right) \\
 & + 2c^2 \left(\frac{-b + \sqrt{4ac - b^2} \cot\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c} \right)^2,
 \end{aligned} \tag{3.11}$$

which are trigonometric solutions.

Family 3

When $\alpha_0 = \frac{1}{30} (5b^2 + 40ac - \sqrt{5} \sqrt{4d + b^4 - 8ab^2r + 16a^2c^2})$, $\alpha_1 = 0, \beta_1 = 2ab, \alpha_2 = 0, \beta_2 = 2a^2$, the following cases are made.

For $b^2 - 4ac > 0$ and $n \neq 0$

$$\begin{aligned}
 u_{3,1}(r, t) = & \frac{1}{30} (5b^2 + 40ac - \sqrt{5} \sqrt{4d + b^4 - 8ab^2r + 16a^2c^2}) \\
 & + 2ab \left(\frac{-b + \sqrt{b^2 - 4ac} \tanh\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c} \right)^{-1} \\
 & + 2a^2 \left(\frac{-b + \sqrt{b^2 - 4ac} \tanh\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c} \right)^{-2},
 \end{aligned} \tag{3.12}$$

or

$$u_{3,2}(r, t) = \frac{1}{30} (5b^2 + 40ac - \sqrt{5} \sqrt{4d + b^4 - 8ab^2r + 16a^2c^2}) \tag{3.13}$$

$$\begin{aligned}
& +2ab\left(\frac{-b + \sqrt{b^2 - 4ac} \coth\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c}\right)^{-1} \\
& +2a^2\left(\frac{-b + \sqrt{b^2 - 4ac} \coth\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c}\right)^{-2},
\end{aligned}$$

which are the hyperbolic solutions.

For $b^2 - 4ac < 0$ and $n \neq 0$,

$$\begin{aligned}
u_{3,3}(r, t) = \frac{1}{30} & \left(5b^2 + 40ac - \sqrt{5} \sqrt{4d + b^4 - 8ab^2r + 16a^2c^2}\right) \\
& +2ab\left(\frac{-b + \sqrt{4ac - b^2} \tan\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c}\right)^{-1} \\
& +2a^2\left(\frac{-b + \sqrt{4ac - b^2} \tan\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c}\right)^{-2},
\end{aligned} \tag{3.14}$$

or

$$\begin{aligned}
u_{3,4}(r, t) = \frac{1}{30} & \left(5b^2 + 40ac - \sqrt{5} \sqrt{4d + b^4 - 8ab^2r + 16a^2c^2}\right) \\
& +2ab\left(\frac{-b + \sqrt{4ac - b^2} \cot\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c}\right)^{-1} \\
& +2a^2\left(\frac{-b + \sqrt{4ac - b^2} \cot\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c}\right)^{-2},
\end{aligned} \tag{3.15}$$

which are the trigonometric solutions.

Family 4

When $\alpha_0 = \frac{1}{30} (5b^2 + 40ac + \sqrt{5} \sqrt{4d + b^4 - 8ab^2r + 16a^2c^2})$, $\alpha_1 = 0$, $\beta_1 = 2ab$, $\alpha_2 = 0$, $\beta_2 = 2a^2$, the following cases are obtained.

For $b^2 - 4ac > 0$ and $n \neq 0$,

$$\begin{aligned}
u_{4,1}(r, t) = \frac{1}{30} & \left(5b^2 + 40ac + \sqrt{5} \sqrt{4d + b^4 - 8ab^2r + 16a^2c^2}\right) \\
& +2ab\left(\frac{-b + \sqrt{b^2 - 4ac} \tanh\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c}\right)^{-1} \\
& +2a^2\left(\frac{-b + \sqrt{b^2 - 4ac} \tanh\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c}\right)^{-2},
\end{aligned} \tag{3.16}$$

or

$$\begin{aligned}
u_{4,2}(r, t) = \frac{1}{30} & \left(5b^2 + 40ac + \sqrt{5} \sqrt{4d + b^4 - 8ab^2r + 16a^2c^2}\right) \\
& +2ab\left(\frac{-b + \sqrt{b^2 - 4ac} \coth\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c}\right)^{-1} \\
& +2a^2\left(\frac{-b + \sqrt{b^2 - 4ac} \coth\left(\frac{\sqrt{b^2 - 4ac}\zeta}{2}\right)}{2c}\right)^{-2},
\end{aligned} \tag{3.17}$$

which are the hyperbolic solutions.

For $b^2 - 4ac < 0$ and $n \neq 0$,

$$\begin{aligned}
 u_{4,3}(r, t) = & \frac{1}{30} (5b^2 + 40ac + \sqrt{5} \sqrt{4d + b^4 - 8ab^2r + 16a^2c^2}) \\
 & + 2ab \left(\frac{-b + \sqrt{4ac - b^2} \tan\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c} \right)^{-1} \\
 & + 2a^2 \left(\frac{-b + \sqrt{4ac - b^2} \tan\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c} \right)^{-2},
 \end{aligned} \tag{3.18}$$

or

$$\begin{aligned}
 u_{4,4}(r, t) = & \frac{1}{30} (5b^2 + 40ac + \sqrt{5} \sqrt{4d + b^4 - 8ab^2r + 16a^2c^2}) \\
 & + 2ab \left(\frac{-b + \sqrt{4ac - b^2} \cot\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c} \right)^{-1} \\
 & + 2a^2 \left(\frac{-b + \sqrt{4ac - b^2} \cot\left(\frac{\sqrt{4ac - b^2}\zeta}{2}\right)}{2c} \right)^{-2},
 \end{aligned} \tag{3.19}$$

which are the trigonometric solutions.

4. The extended direct algebraic method

Assume a general NLPDE of the type

$$X(u, u_t, u_r, u_{tt}, u_{rr}, u_{rt}, \dots) = 0, \tag{4.1}$$

which can be transformed as

$$F(R, R', R'', \dots) = 0, \tag{4.2}$$

by using following transformation as

$$U(r, t) = U(\zeta)e^{t\Phi}, \tag{4.3}$$

where $\zeta = r - dt$ and $\Phi = \zeta_1 r + \zeta_2 t$, and the order of differentiation for various properties in Eq (4.2) is represented by the prime sign. Assume that Eq (4.2) has a solution of the form

$$U(\zeta) = \sum_{j=1}^l a_j (P(\zeta))^j, \tag{4.4}$$

where

$$P'(\zeta) = \ln \chi \left(\nu + \mu P(\zeta) + \lambda (P(\zeta))^2 \right), \chi \neq 0, 1. \tag{4.5}$$

In addition, λ , μ and ν are real constants. The general form of the solutions of Eq (4.5) w.r.t real parameters λ , μ and ν are as follows:

1) For $\mu^2 - 4\nu\lambda < 0$ and $\lambda \neq 0$,

$$P_1(\zeta) = -\frac{\mu}{2\lambda} + \frac{\sqrt{-\delta}}{2\lambda} \tan_x \left(\frac{\sqrt{-\delta}}{2} \zeta \right), \quad (4.6)$$

$$P_2(\zeta) = -\frac{\mu}{2\lambda} - \frac{\sqrt{-\delta}}{2\lambda} \cot_x \left(\frac{\sqrt{-\delta}}{2} \zeta \right), \quad (4.7)$$

$$P_3(\zeta) = -\frac{\mu}{2\lambda} + \frac{\sqrt{-\delta}}{2\lambda} \left(\tan_x(\sqrt{-\delta}\zeta) \pm \sqrt{mn} \sec_x(\sqrt{-\delta}\zeta) \right),$$

$$P_4(\zeta) = -\frac{\mu}{2\lambda} + \frac{\sqrt{-\delta}}{2\lambda} \left(\cot_x(\sqrt{-\delta}\zeta) \pm \sqrt{mn} \csc_x(\sqrt{-\delta}\zeta) \right),$$

$$P_5(\zeta) = -\frac{\mu}{2\lambda} + \frac{\sqrt{-\delta}}{4\lambda} \left(\tan_x \left(\frac{\sqrt{-\delta}}{4} \zeta \right) - \cot_x \left(\frac{\sqrt{-\delta}}{4} \zeta \right) \right).$$

where $\delta = \mu^2 - 4\nu\lambda$.

2) For $\mu^2 - 4\nu\lambda > 0$ and $\lambda \neq 0$,

$$P_6(\zeta) = -\frac{\mu}{2\lambda} - \frac{\sqrt{\delta}}{2\lambda} \tanh_x \left(\frac{\sqrt{\delta}}{2} \zeta \right),$$

$$P_7(\zeta) = -\frac{\mu}{2\lambda} - \frac{\sqrt{\delta}}{2\lambda} \coth_x \left(\frac{\sqrt{\delta}}{2} \zeta \right),$$

$$P_8(\zeta) = -\frac{\mu}{2\lambda} + \frac{\sqrt{\delta}}{2\lambda} \left(-\tanh_x(\sqrt{\delta}\zeta) \pm \iota \sqrt{mn} \sec_x(\sqrt{\delta}\zeta) \right),$$

$$P_9(\zeta) = -\frac{\mu}{2\lambda} + \frac{\sqrt{\delta}}{2\lambda} \left(-\coth_x(\sqrt{\delta}\zeta) \pm \sqrt{mn} \csc_x(\sqrt{\delta}\zeta) \right),$$

$$P_{10}(\zeta) = -\frac{\mu}{2\lambda} - \frac{\sqrt{\delta}}{4\lambda} \left(\tanh_x \left(\frac{\sqrt{\delta}}{4} \zeta \right) + \coth_x \left(\frac{\sqrt{\delta}}{4} \zeta \right) \right).$$

where $\delta = \mu^2 - 4\nu\lambda$.

3) For $\nu\lambda > 0$ and $\mu = 0$,

$$P_{11}(\zeta) = \sqrt{\frac{\nu}{\lambda}} \tan_x(\sqrt{\nu\lambda}\zeta),$$

$$P_{12}(\zeta) = -\sqrt{\frac{\nu}{\lambda}} \cot_x(\sqrt{\nu\lambda}\zeta),$$

$$P_{13}(\zeta) = \sqrt{\frac{\nu}{\lambda}} \left(\tan_x(2\sqrt{\nu\lambda}\zeta) \pm \sqrt{mn} \sec_x(2\sqrt{\nu\lambda}\zeta) \right),$$

$$P_{14}(\zeta) = \sqrt{\frac{\nu}{\lambda}} \left(-\cot_x(2\sqrt{\nu\lambda}\zeta) \pm \sqrt{mn} \csc_x(2\sqrt{\nu\lambda}\zeta) \right),$$

$$P_{15}(\zeta) = \frac{1}{2} \sqrt{\frac{\nu}{\lambda}} \left(\tan_x \left(\frac{\sqrt{\nu\lambda}}{2} \zeta \right) + \cot_x \left(\frac{\sqrt{\nu\lambda}}{2} \zeta \right) \right).$$

4) For $\nu\lambda < 0$ and $\mu = 0$,

$$P_{16}(\zeta) = -\sqrt{-\frac{\nu}{\lambda}} \tanh_{\chi}(\sqrt{-\nu\lambda}\zeta),$$

$$P_{17}(\zeta) = -\sqrt{-\frac{\nu}{\lambda}} \coth_{\chi}(\sqrt{-\nu\lambda}\zeta),$$

$$P_{18}(\zeta) = \sqrt{-\frac{\nu}{\lambda}} \left(-\tanh_{\chi}(2\sqrt{-\nu\lambda}\zeta) \pm \iota \sqrt{mn}_{\chi}(2\sqrt{-\nu\lambda}\zeta) \right),$$

$$P_{19}(\zeta) = \sqrt{-\frac{\nu}{\lambda}} \left(-\coth_{\chi}(2\sqrt{-\nu\lambda}\zeta) \pm \sqrt{mn}_{\chi}(2\sqrt{-\nu\lambda}\zeta) \right),$$

$$P_{20}(\zeta) = -\frac{1}{2} \sqrt{-\frac{\nu}{\lambda}} \left(\tanh_{\chi}\left(\frac{\sqrt{-\nu\lambda}}{2}\zeta\right) + \coth_{\chi}\left(\frac{\sqrt{-\nu\lambda}}{2}\zeta\right) \right).$$

5) For $\mu = 0$ and $\nu = \lambda$,

$$P_{21}(\zeta) = \tan_{\chi}(\nu\zeta),$$

$$P_{22}(\zeta) = -\cot_{\chi}(\nu\zeta),$$

$$P_{23}(\zeta) = \tan_{\chi}(2\nu\zeta) \pm \sqrt{mn} \sec_{\chi}(2\nu\zeta),$$

$$P_{24}(\zeta) = -\cot_{\chi}(2\nu\zeta) \pm \sqrt{mn} \csc_{\chi}(2\nu\zeta),$$

$$P_{25}(\zeta) = \frac{1}{2} \left(\tan_{\chi}\left(\frac{\nu}{2}\zeta\right) - \cot_{\chi}\left(\frac{\nu}{2}\zeta\right) \right).$$

6) For $\mu = 0$ and $\lambda = -\nu$,

$$P_{26}(\zeta) = -\tanh_{\chi}(\nu\zeta),$$

$$P_{27}(\zeta) = -\coth_{\chi}(\nu\zeta),$$

$$P_{28}(\zeta) = -\tanh_{\chi}(2\nu\zeta) \pm \iota \sqrt{mn}_{\chi}(2\nu\zeta),$$

$$P_{29}(\zeta) = -\coth_{\chi}(2\nu\zeta) \pm \sqrt{mn}_{\chi}(2\nu\zeta),$$

$$P_{30}(\zeta) = -\frac{1}{2} \left(\tanh_{\chi}\left(\frac{\nu}{2}\zeta\right) + \coth_{\chi}\left(\frac{\nu}{2}\zeta\right) \right).$$

7) For $\mu^2 = 4\nu\lambda$,

$$P_{31}(\zeta) = \frac{-2\nu(\mu\zeta \ln(\chi) + 2)}{\mu^2\zeta \ln(\chi)}.$$

8) For $\mu = p$, $\nu = pq$, ($q \neq 0$), and $\lambda = 0$,

$$P_{32}(\zeta) = \chi^{p\zeta} - q.$$

9) For $\mu = \lambda = 0$,

$$P_{33}(\zeta) = \nu\zeta \ln(\chi).$$

10) For $\mu = \nu = 0$,

$$P_{34}(\zeta) = \frac{-1}{\lambda\zeta \ln(\chi)}.$$

11) For $\mu \neq 0$ and $\nu = 0$,

$$P_{35}(\zeta) = -\frac{m\mu}{\lambda(\cosh_{\chi}(\mu\zeta) - \sinh_{\chi}(\mu\zeta) + m)},$$

$$P_{36}(\zeta) = -\frac{\mu(\sinh_{\chi}(\mu\zeta) + \cosh_{\chi}(\mu\zeta))}{\xi(\sinh_{\chi}(\mu\zeta) + \cosh_{\chi}(\mu\zeta) + n)}.$$

12) For $\mu = p$, $\lambda = pq$, ($q \neq 0$ and $\nu = 0$),

$$P_{37}(\zeta) = -\frac{m\chi^{p\zeta}}{m - qn\chi^{p\zeta}}.$$

$$\sinh_{\chi}(\zeta) = \frac{m\chi^{\zeta} - n\chi^{-\zeta}}{2}, \quad \cosh_{\chi}(\zeta) = \frac{m\chi^{\zeta} + n\chi^{-\zeta}}{2},$$

$$\tanh_{\chi}(\zeta) = \frac{m\chi^{\zeta} - n\chi^{-\zeta}}{m\chi^{\zeta} + n\chi^{-\zeta}},$$

$$\operatorname{csc}_{\chi}(\zeta) = \frac{2}{m\chi^{\zeta} - n\chi^{-\zeta}}, \quad \operatorname{sec}_{\chi}(\zeta) = \frac{2}{m\chi^{\zeta} + n\chi^{-\zeta}},$$

$$\operatorname{coth}_{\chi}(\zeta) = \frac{m\chi^{\zeta} + n\chi^{-\zeta}}{m\chi^{\zeta} - n\chi^{-\zeta}},$$

$$\sin_{\chi}(\zeta) = \frac{m\chi^{i\zeta} - n\chi^{-i\zeta}}{2i}, \quad \cos_{\chi}(\zeta) = \frac{m\chi^{i\zeta} + n\chi^{-i\zeta}}{2},$$

$$\tan_{\chi}(\zeta) = -i \frac{m\chi^{i\zeta} - n\chi^{-i\zeta}}{m\chi^{i\zeta} + n\chi^{-i\zeta}},$$

$$\operatorname{csc}_{\chi}(\zeta) = \frac{2i}{m\chi^{i\zeta} - n\chi^{-i\zeta}}, \quad \operatorname{sec}_{\chi}(\zeta) = \frac{2}{m\chi^{i\zeta} + n\chi^{-i\zeta}},$$

$$\operatorname{cot}_{\chi}(\zeta) = i \frac{m\chi^{i\zeta} + n\chi^{-i\zeta}}{m\chi^{i\zeta} - n\chi^{-i\zeta}},$$

where the terms “deformation parameters” m and n refer to arbitrary constants with values larger than 0.

4.1. Imposition of technique on Eq (4.2)

We employ an ansatz in Eq (1.1) to determine the solution

$$U(r, t) = U(\zeta), \quad (4.8)$$

and $\zeta = r - dt$, where the coefficient of r is the wave number and d is the speed of soliton. By imposing the considered transformation in Eq (4.8) on Eq (1.1), we produce the following real parts:

$$-dU + 15U^3 - 15UU'' + U'''' = 0. \quad (4.9)$$

The homogeneous balancing constant in Eq (4.9) is 2, and thus the solution is given as follows:

$$U(\zeta) = \alpha_0 + \alpha_1 P(\zeta) + \alpha_2 (P(\zeta))^2, \quad (4.10)$$

where

$$P'(\zeta) = \ln(\chi) \left(\nu + \mu P + \lambda (P(\zeta))^2 \right). \quad (4.11)$$

By entering in the solution of Eq (4.10) along with Eqs (4.9)–(4.11) and by calculating the coefficients of powers of $P(\zeta)$, we get the following system of equations. This system of algebraic equations was solved using Wolfram Mathematica 12 software. The results are as follows:

$$\begin{cases} \alpha_0 = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}\nu\Delta, \\ \alpha_1 = 4\mu\Delta, \\ \alpha_2 = 4\lambda\Delta, \end{cases} \quad (4.12)$$

where $\Delta = \frac{1}{3}\lambda \log \chi^2$.

We get the general solution of Eq (1.1) by plugging Eq (4.12) into Eq (4.10):

$$U(r, t) = \frac{\Delta}{3\lambda} (\mu^2 + 8\lambda\nu + 12\lambda(\mu P_i(\zeta) + \lambda P_i(\zeta)^2)). \quad (4.13)$$

It should be seen that by choosing different values of P_j for the above equations, we get many solutions.

(1) For $\mu^2 - 4\nu\lambda < 0$ and $\lambda \neq 0$,

$$\begin{aligned} U_1(r, t) &= \frac{8}{3\lambda}\nu\Delta - \frac{2\mu^2\Delta}{3\lambda} - \frac{\delta\Delta}{\lambda} \tan^2_{\chi} \left(\frac{\sqrt{-\delta}}{2}\zeta \right), \\ U_2(r, t) &= \frac{2}{3\lambda}\mu^2\Delta + \frac{8}{3}\nu\Delta - \frac{\Delta\delta}{\lambda} \cot^2_{\chi} \left(\frac{\sqrt{-\delta}}{2}\zeta \right), \\ U_3(r, t) &= -\frac{2\mu^2\Delta}{3\lambda} + \frac{8\nu\Delta}{3} - \frac{\delta\Delta}{\lambda} \left(\tan_{\chi} \left(\sqrt{-\delta}\zeta \right) \pm \sqrt{mn} \operatorname{sech}_{\chi} \left(\sqrt{-\delta}\zeta \right) \right)^2, \\ U_4(r, t) &= -\frac{2\mu^2\Delta}{3\lambda} + \frac{8\nu\Delta}{3} - \frac{\delta\Delta}{\lambda} \left(\cot_{\chi} \left(\sqrt{-\delta}\zeta \right) \pm \sqrt{mn} \operatorname{csc}_{\chi} \left(\sqrt{-\delta}\zeta \right) \right)^2, \\ U_5(r, t) &= -\frac{2\mu^2\Delta}{3\lambda} + \frac{8\nu\Delta}{3} - \frac{\delta\Delta}{4\lambda} \left(\tan_{\chi} \left(\frac{\sqrt{-\delta}}{4}\zeta \right) - \cot_{\chi} \left(\frac{\sqrt{-\delta}}{4}\zeta \right) \right)^2. \end{aligned} \quad (4.14)$$

(2) For $\mu^2 - 4\nu\lambda > 0$ and $\lambda \neq 0$,

$$\begin{aligned} U_6(r, t) &= -\frac{2\mu^2\Delta}{3\lambda} + \frac{8\nu\Delta}{3} + \frac{\delta\Delta}{\lambda} \tanh^2_{\chi} \left(\frac{\sqrt{\delta}}{2}\zeta \right), \\ U_7(r, t) &= -\frac{2\mu^2\Delta}{3\lambda} + \frac{8\nu\Delta}{3} + \frac{\delta\Delta}{\lambda} \coth^2_{\chi} \left(\frac{\sqrt{\delta}}{2}\zeta \right), \\ U_8(r, t) &= -\frac{2\mu^2\Delta}{3\lambda} + \frac{8\nu\Delta}{3} + \frac{\delta\Delta}{\lambda} \left(-\tanh_{\chi} \left(\sqrt{\delta}\zeta \right) \pm \iota \sqrt{mn} \operatorname{sech}_{\chi} \left(\sqrt{\delta}\zeta \right) \right)^2, \end{aligned}$$

$$U_9(r, t) = -\frac{2\mu^2\Delta}{3\lambda} + \frac{8v\Delta}{3} + \frac{\delta\Delta}{\lambda}(-\coth_\chi(\sqrt{\delta}\zeta) \pm \iota\sqrt{mn}\operatorname{csch}_\chi(\sqrt{\delta}\zeta))^2,$$

$$U_{10}(r, t) = -\frac{2\mu^2\Delta}{3\lambda} + \frac{8v\Delta}{3} + \frac{\delta\Delta}{\lambda}(\tanh_\chi\left(\frac{\sqrt{\delta}}{4}\zeta\right) + \coth_\chi\left(\frac{\sqrt{\delta}}{4}\zeta\right))^2.$$

(3) For $v\lambda > 0$ and $\mu = 0$,

$$U_{11}(r, t) = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}v\Delta + 4\mu\Delta\sqrt{\frac{v}{\lambda}}\tan_\chi(\sqrt{v\lambda}\zeta) + 4\lambda v\tan^2_\chi(\sqrt{v\lambda}\zeta),$$

$$U_{12}(r, t) = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}v\Delta - 4\mu\Delta\sqrt{\frac{v}{\lambda}}\cot_\chi(\sqrt{v\lambda}\zeta) + 4\lambda v\cot^2_\chi(\sqrt{v\lambda}\zeta),$$

$$U_{13}(r, t) = \Delta\left(\frac{\mu^2}{3\lambda} + \frac{8v}{3} + \sqrt{\frac{2\mu^2v}{\lambda}}(\tan_\chi(2\sqrt{v\lambda}\zeta) \pm \sqrt{mn}\operatorname{sec}_\chi(2\sqrt{v\lambda}\zeta))\right. \\ \left.+ 4v(\tan_\chi(2\sqrt{v\lambda}\zeta) \pm \sqrt{mn}\operatorname{sec}_\chi(2\sqrt{v\lambda}\zeta))^2\right),$$

$$U_{14}(r, t) = \frac{\mu^2\Delta}{3\lambda} + \frac{8v\Delta}{3} + 4\mu\Delta\sqrt{\frac{v}{\lambda}}(-\cot_\chi(2\sqrt{v\lambda}\zeta) \pm \sqrt{mn}\operatorname{csc}_\chi(2\sqrt{v\lambda}\zeta)) \\ + 4v\Delta(-\cot_\chi(2\sqrt{v\lambda}\zeta) \pm \sqrt{mn}\operatorname{csc}_\chi(2\sqrt{v\lambda}\zeta))^2,$$

$$U_{15}(r, t) = \frac{\mu^2\Delta}{3\lambda} + \frac{8v\Delta}{3} + 2\mu\Delta\sqrt{\frac{v}{\lambda}}\left(\tan_\chi\left(\frac{\sqrt{v\lambda}}{2}\zeta\right) + \cot_\chi\left(\frac{\sqrt{v\lambda}}{2}\zeta\right)\right) \\ + 2v\Delta\left(\tan_\chi\left(\frac{\sqrt{v\lambda}}{2}\zeta\right) + \cot_\chi\left(\frac{\sqrt{v\lambda}}{2}\zeta\right)\right)^2.$$

(4) For $v\lambda < 0$ and $\mu = 0$,

$$U_{16}(r, t) = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}v\Delta - 4\mu\Delta\left(\sqrt{-\frac{v}{\lambda}}\tanh_\chi(\sqrt{-v\lambda}\zeta)\right) - 4v\Delta(\tanh_\chi(\sqrt{-v\lambda}\zeta))^2,$$

$$U_{17}(r, t) = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}v\Delta - 4\mu\Delta\left(\sqrt{-\frac{v}{\lambda}}\coth_\chi(\sqrt{-v\lambda}\zeta)\right) - 4v\Delta(\coth_\chi(\sqrt{-v\lambda}\zeta))^2,$$

$$U_{18}(r, t) = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}v\Delta + 4\mu\Delta\sqrt{-\frac{v}{\lambda}}(-\tanh_\chi(2\sqrt{-v\lambda}\zeta) \pm \iota\sqrt{mn}\operatorname{sech}_\chi(2\sqrt{-v\lambda}\zeta)) \\ - 4v\Delta(\tanh_\chi(2\sqrt{-v\lambda}\zeta) \pm \iota\sqrt{mn}\operatorname{sech}_\chi(2\sqrt{-v\lambda}\zeta))^2,$$

$$U_{19}(r, t) = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}v\Delta + 4\mu\Delta\sqrt{-\frac{v}{\lambda}}(-\coth_\chi(2\sqrt{-v\lambda}\zeta) \pm \iota\sqrt{mn}\operatorname{csch}_\chi(2\sqrt{-v\lambda}\zeta))$$

$$-4\nu\Delta\left(\coth_{\chi}\left(2\sqrt{-\nu\lambda}\zeta\right)\pm\iota\sqrt{mn}\operatorname{csch}_{\chi}\left(2\sqrt{-\nu\lambda}\zeta\right)\right)^2,$$

$$U_{20}(r,t)=\frac{1}{3\lambda}\mu^2\Delta+\frac{8}{3}\nu\Delta+4\mu\Delta\left(-\frac{1}{2}\sqrt{-\frac{\nu}{\lambda}}\left(\tanh_{\chi}\left(\frac{\sqrt{-\nu\lambda}}{2}\zeta\right)+\coth_{\chi}\left(\frac{\sqrt{-\nu\lambda}}{2}\zeta\right)\right)\right) \\ +4\lambda\Delta\left(-\frac{1}{2}\sqrt{-\frac{\nu}{\lambda}}\left(\tanh_{\chi}\left(\frac{\sqrt{-\nu\lambda}}{2}\zeta\right)+\coth_{\chi}\left(\frac{\sqrt{-\nu\lambda}}{2}\zeta\right)\right)\right)^2.$$

(5) For $\mu = 0$ and $\nu = \lambda$,

$$U_{21}(r,t)=\frac{1}{3\lambda}\mu^2\Delta+\frac{8}{3}\nu\Delta+4\mu\Delta(\tan_{\chi}(\nu\zeta))+4\lambda\Delta(\tan_{\chi}(\nu\zeta))^2, \\ U_{22}(r,t)=\frac{1}{3\lambda}\mu^2\Delta+\frac{8}{3}\nu\Delta+4\mu\Delta(-\cot_{\chi}(\nu\zeta))+4\lambda\Delta(-\cot_{\chi}(\nu\zeta))^2, \\ U_{23}(r,t)=\frac{1}{3\lambda}\mu^2\Delta+\frac{8}{3}\nu\Delta+4\mu\Delta(\tan_{\chi}(2\nu\zeta)\pm\sqrt{mn}\sec_{\chi}(2\nu\zeta)) \\ +4\lambda\Delta(\tan_{\chi}(2\nu\zeta)\pm\sqrt{mn}\sec_{\chi}(2\nu\zeta))^2, \\ U_{24}(r,t)=\frac{1}{3\lambda}\mu^2\Delta+\frac{8}{3}\nu\Delta+4\mu\Delta(-\cot_{\chi}(2\nu\zeta)\pm\sqrt{mn}\csc_{\chi}(2\nu\zeta)) \\ +4\lambda\Delta(-\cot_{\chi}(2\nu\zeta)\pm\sqrt{mn}\csc_{\chi}(2\nu\zeta))^2, \\ U_{25}(r,t)=\frac{1}{3\lambda}\mu^2\Delta+\frac{8}{3}\nu\Delta+4\mu\Delta\left(\frac{1}{2}\left(\tan_{\chi}\left(\frac{\nu}{2}\zeta\right)-\cot_{\chi}\left(\frac{\nu}{2}\zeta\right)\right)\right) \\ +4\lambda\Delta\left(\frac{1}{2}\left(\tan_{\chi}\left(\frac{\nu}{2}\zeta\right)-\cot_{\chi}\left(\frac{\nu}{2}\zeta\right)\right)\right)^2.$$

(6) For $\mu = 0$ and $\nu = -\lambda$,

$$U_{26}(r,t)=\frac{1}{3\lambda}\mu^2\Delta+\frac{8}{3}\nu\Delta+4\mu\Delta(-\tanh_{\chi}(\nu\zeta))+4\lambda\Delta(-\tanh_{\chi}(\nu\zeta))^2, \\ U_{27}(r,t)=\frac{1}{3\lambda}\mu^2\Delta+\frac{8}{3}\nu\Delta+4\mu\Delta(-\coth_{\chi}(\nu\zeta))+4\lambda\Delta(-\coth_{\chi}(\nu\zeta))^2, \\ U_{28}(r,t)=\frac{1}{3\lambda}\mu^2\Delta+\frac{8}{3}\nu\Delta+4\mu\Delta(-\tanh_{\chi}(2\nu\zeta)\pm\iota\sqrt{mn}\operatorname{coth}_{\chi}(2\nu\zeta)) \\ +4\lambda\Delta(-\tanh_{\chi}(2\nu\zeta)\pm\iota\sqrt{mn}\operatorname{coth}_{\chi}(2\nu\zeta))^2, \\ U_{29}(r,t)=\frac{1}{3\lambda}\mu^2\Delta+\frac{8}{3}\nu\Delta+4\mu\Delta(-\coth_{\chi}(2\nu\zeta)\pm\iota\sqrt{mn}\operatorname{csc}h_{\chi}(2\nu\zeta)) \\ +4\lambda\Delta(-\coth_{\chi}(2\nu\zeta)\pm\iota\sqrt{mn}\operatorname{csc}h_{\chi}(2\nu\zeta))^2, \\ U_{30}(r,t)=\frac{1}{3\lambda}\mu^2\Delta+\frac{8}{3}\nu\Delta+4\mu\Delta\left(-\frac{1}{2}\left(\tanh_{\chi}\left(\frac{\nu}{2}\zeta\right)+\coth_{\chi}\left(\frac{\nu}{2}\zeta\right)\right)\right) \\ +4\lambda\Delta\left(-\frac{1}{2}\left(\tanh_{\chi}\left(\frac{\nu}{2}\zeta\right)+\coth_{\chi}\left(\frac{\nu}{2}\zeta\right)\right)\right)^2.$$

(7) For $\mu^2 = 4\nu\lambda$,

$$U_{31}(r, t) = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}\nu\Delta + 4\mu\Delta\left(\frac{-2\nu(\mu\zeta \ln(\chi) + 2)}{\mu^2\zeta \ln(\chi)}\right) + 4\lambda\Delta\left(\frac{-2\nu(\mu\zeta \ln(\chi) + 2)}{\mu^2\zeta \ln(\chi)}\right)^2.$$

(8) For $\mu = p, \nu = pq, (q \neq 0)$, and $\lambda = 0$,

$$U_{32}(r, t) = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}\nu\Delta + 4\mu\Delta(\chi^{p\zeta} - q) + 4\lambda\Delta(\chi^{p\zeta} - q)^2.$$

(9) For $\mu = \lambda = 0$,

$$U_{33}(r, t) = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}\nu\Delta + 4\mu\Delta(\nu\zeta \ln(\chi)) + 4\lambda\Delta(\nu\zeta \ln(\chi))^2.$$

(10) For $\mu = \nu = 0$,

$$U_{34}(r, t) = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}\nu\Delta + 4\mu\Delta\left(\frac{-1}{\lambda\zeta \ln(\chi)}\right) + 4\lambda\Delta\left(\frac{-1}{\lambda\zeta \ln(\chi)}\right)^2.$$

(11) For $\mu \neq 0$ and $\nu = 0$,

$$U_{35}(r, t) = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}\nu\Delta + 4\mu\Delta\left(-\frac{m\mu}{\lambda(\cosh_\chi(\mu\zeta) - \sinh_\chi(\mu\zeta) + m)}\right) + 4\lambda\Delta\left(-\frac{m\mu}{\lambda(\cosh_\chi(\mu\zeta) - \sinh_\chi(\mu\zeta) + m)}\right)^2,$$

$$U_{36}(r, t) = \frac{1}{3\lambda}\mu^2\Delta + \frac{8}{3}\nu\Delta + 4\mu\Delta\left(-\frac{\mu(\sinh_\chi(\mu\zeta) + \cosh_\chi(\mu\zeta))}{\lambda(\sinh_\chi(\mu\zeta) + \cosh_\chi(\mu\zeta) + n)}\right) + 4\lambda\Delta\left(-\frac{\mu(\sinh_\chi(\mu\zeta) + \cosh_\chi(\mu\zeta))}{\lambda(\sinh_\chi(\mu\zeta) + \cosh_\chi(\mu\zeta) + n)}\right)^2.$$

(12) For $\mu = p, \lambda = pq, (q \neq 0$ and $\nu = 0)$,

$$U_{37}(r, t) = \frac{\mu^2\Delta}{3\lambda} + \frac{8\nu\Delta}{3} + \left(\frac{4\mu\Delta m\chi^{p\zeta}}{qn\chi^{p\zeta} - m}\right) + \left(\frac{2\sqrt{\lambda\Delta}m\chi^{p\zeta}}{m - qn\chi^{p\zeta}}\right)^2.$$

5. Findings and discussion

The SK equation is a nonlinear equation. The wave behaviour connected to the SK equation is shown using the MAEM, which provides the equation's solitary wave solutions. Several of the answers that have been found are shown graphically in figures to more fully understand the dynamics of the solutions created. Figure 1 depicts the solitary waves associated with $|\mu_{1,1}|$. The 3D plot shows that the variation in the fractional order has a significant influence on the exchange of amplitude and soliton nature. It demonstrates the spatial and temporal behavior of $U(r, t)$ in the situation of a fractional order system. The behavior indicates that the solitary wave solution's wave nature varies as the fractional parameter is increased.

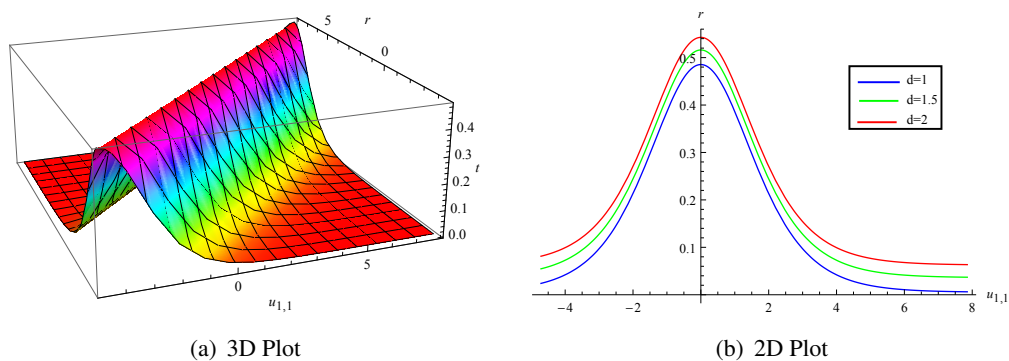


Figure 1. 3D and 2D plots for $|u_{1,1}|$ when $d = 1, a = 0.1, b = 1, c = 0.1$.

Figure 2 shows the solitary waves associated with $|u_{1,3}|$ with parametric values $d = 1, a = 0.1, b = 0.01, c = 0.1$, while Figure 3 illustrates the solution for $|u_{2,1}|$ having parametric values $d = 1, a = 0.1, b = 1, c = 0.1$.

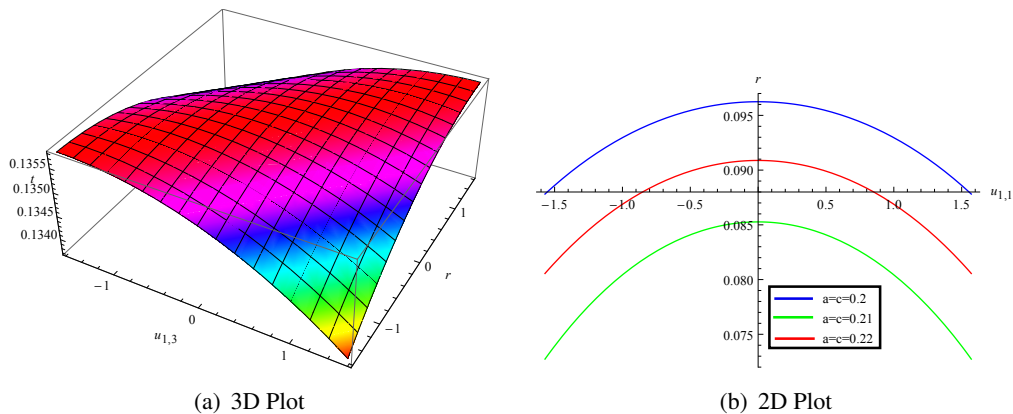


Figure 2. 3D and 2D plots for $|u_{1,3}|$ when $d = 1, a = 0.1, b = 0.01, c = 0.1$.

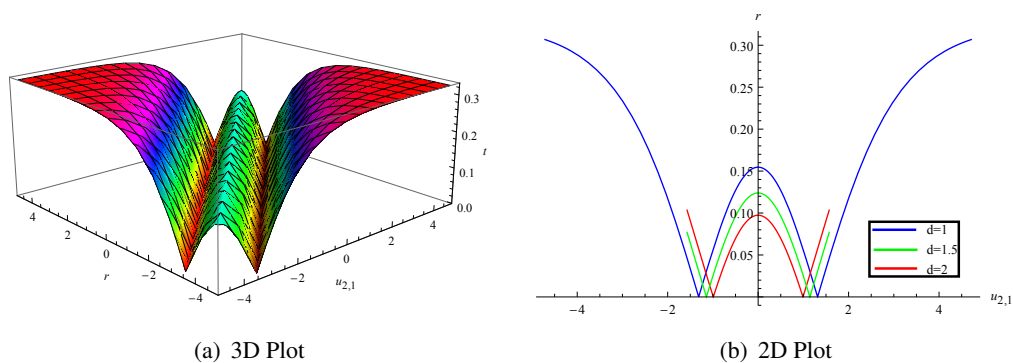


Figure 3. 3D and 2D plots for $|u_{2,1}|$ when $d = 1, a = 0.1, b = 1, c = 0.1$.

Figure 4 depicts the solitary waves associated with $|u_{2,3}|$ having parametric values $d = 1, a = 0.1,$

$b = 0.01, c = 0.1$. Figure 5 shows the solitary waves associated with $|u_{3,1}|$ with parametric values $d = 1, a = 0.1, b = 1, c = 0.1$, while Figure 6 illustrates the solution for $|u_{4,1}|$ having parametric values $d = 1, a = 0.1, b = 1, c = 0.1$.

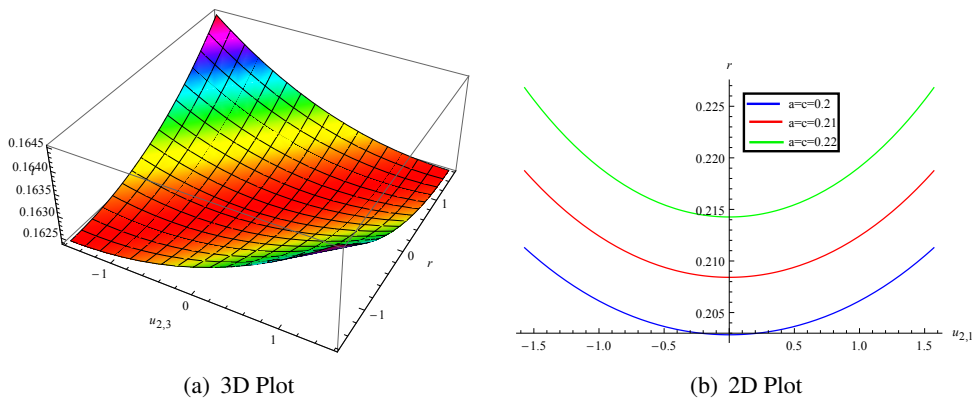


Figure 4. 3D and 2D plots for $|u_{2,3}|$ when $d = 1, a = 0.1, b = 0.01, c = 0.1$.

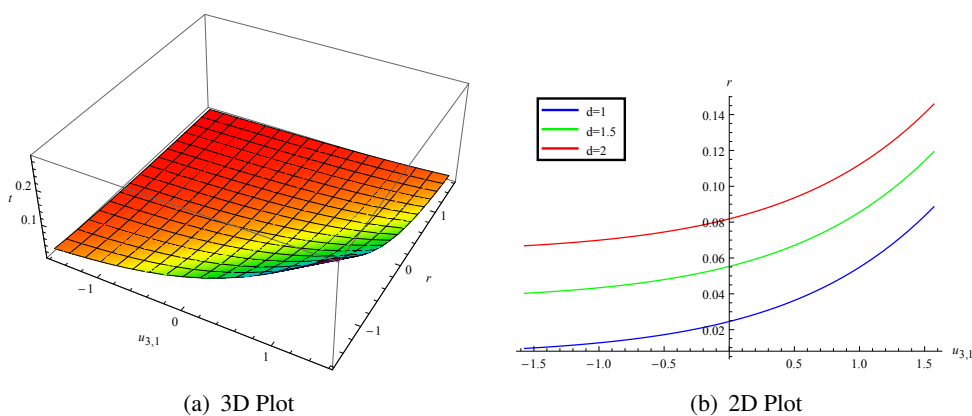


Figure 5. 3D and 2D plots for $|u_{3,1}|$ when $d = 1, a = 0.1, b = 1, c = 0.1$.

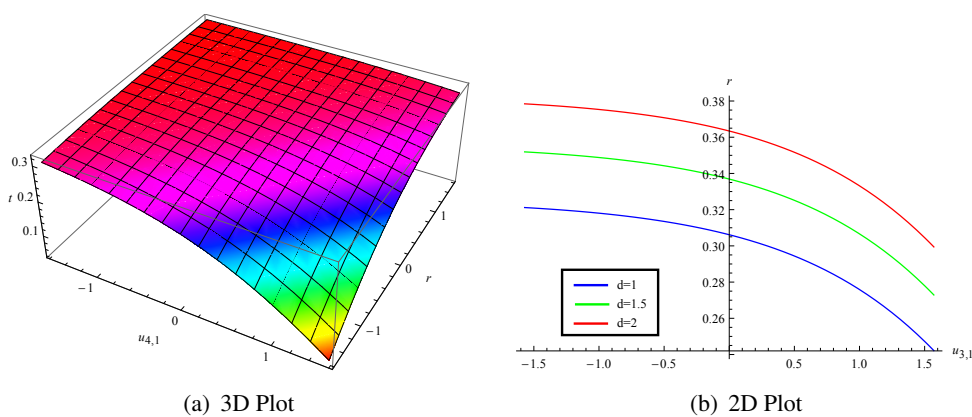


Figure 6. 3D and 2D plots for $|u_{4,1}|$ when $d = 1, a = 0.1, b = 1, c = 0.1$.

The propagation of waves and the contour graphs are also shown in the results. Solutions obtained by the extended direct algebraic method are also shown by 3D, 2D and contour graphs. For Eq (4.9), the real part of the solution is presented in Figure 7 by using the parameters $\chi = 3, m = 9, n = 1, a = 0.1, b = 1, \lambda = 0.3, \nu = 0.2, \zeta_1 = 2, \mu = 0.3, \xi = 5$, for $\alpha = 0.002, \alpha = 0.08$ and $\alpha = 0.2$ respectively. For Eq (4.9), the imaginary part of the solution is presented in Figure 8 by using the parameters $\chi = 3, m = 9, n = 1, a = 0.1, b = 1, \lambda = 0.3, \nu = 0.2, \zeta_1 = 2, \mu = 0.3, \xi = 5$, for $\alpha = 0.002, \alpha = 0.08$ and $\alpha = 0.2$ respectively. For Eq (4.9), the imaginary part of the solution is presented in Figure 9 by using the parameters $\chi = 3, m = 9, n = 1, a = 0.1, b = 1, \lambda = 0.3, \nu = 0.2, \zeta_1 = 2, \mu = 0.3, \xi = 5$, for $\alpha = 1, \alpha = 0.5$ and $\alpha = 0.7$ respectively.

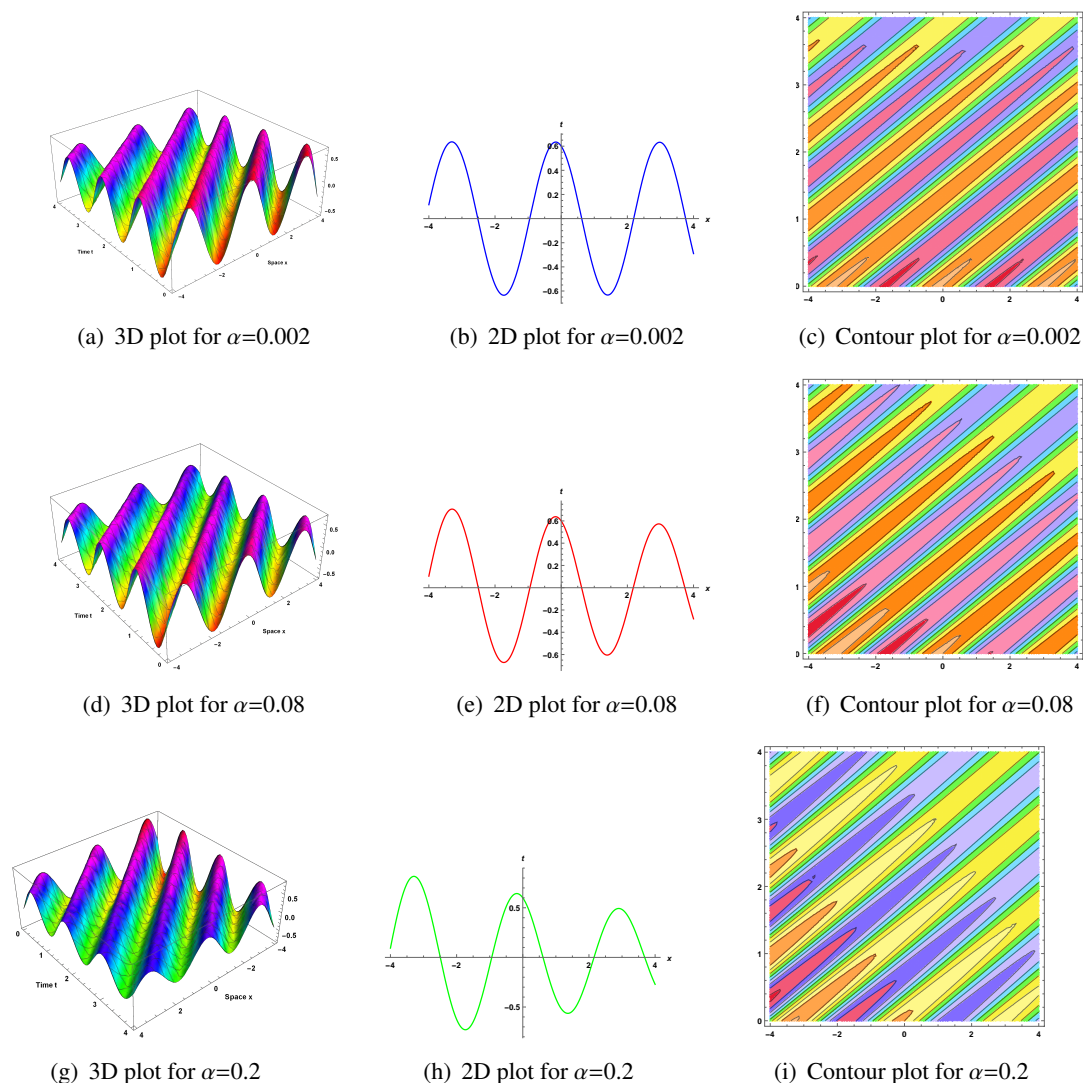


Figure 7. For the real part of solution U_3 , 3D, contour and 2D plots with different values of α .

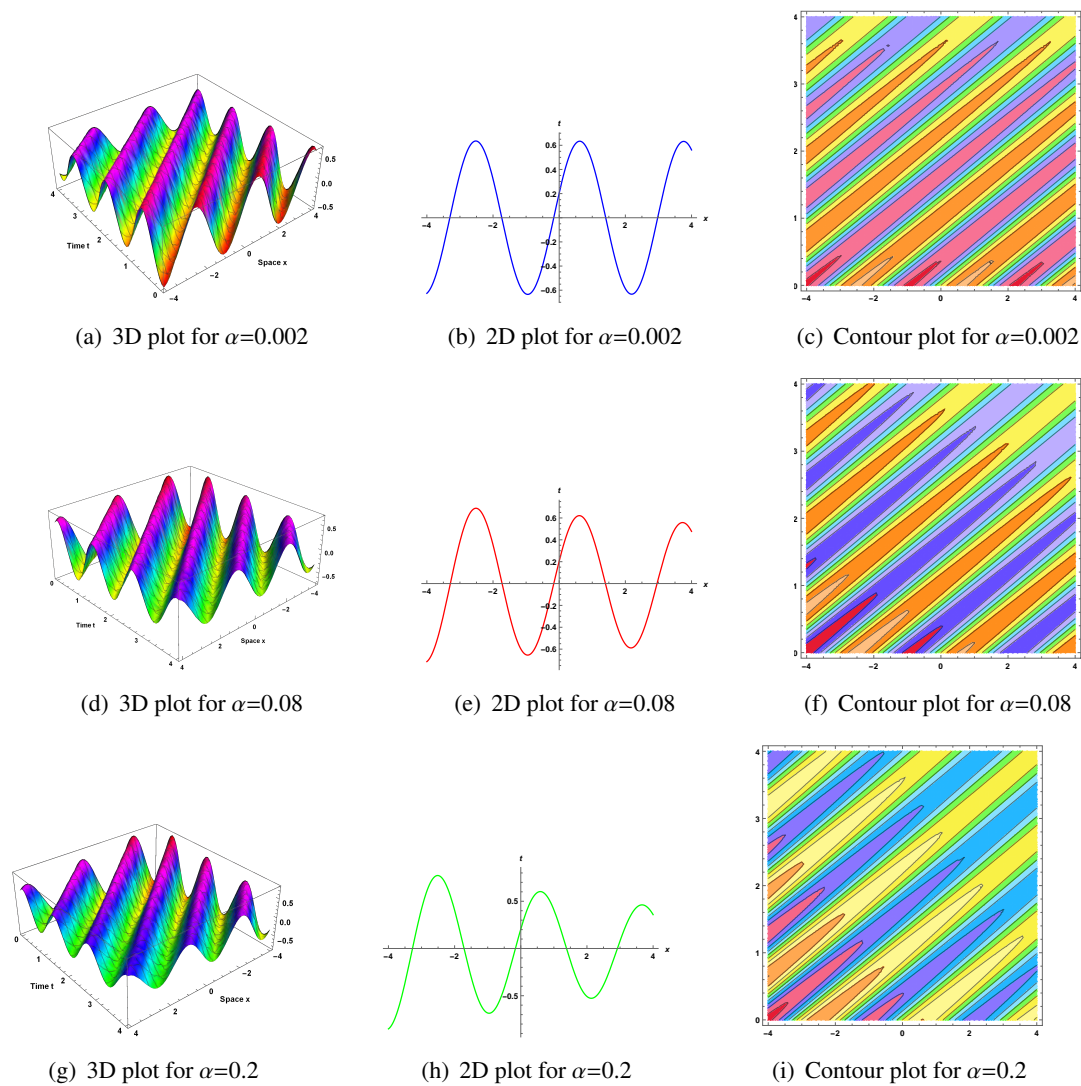


Figure 8. For the imaginary part of solution U_3 , 3D, contour and 2D plots with different values of α .

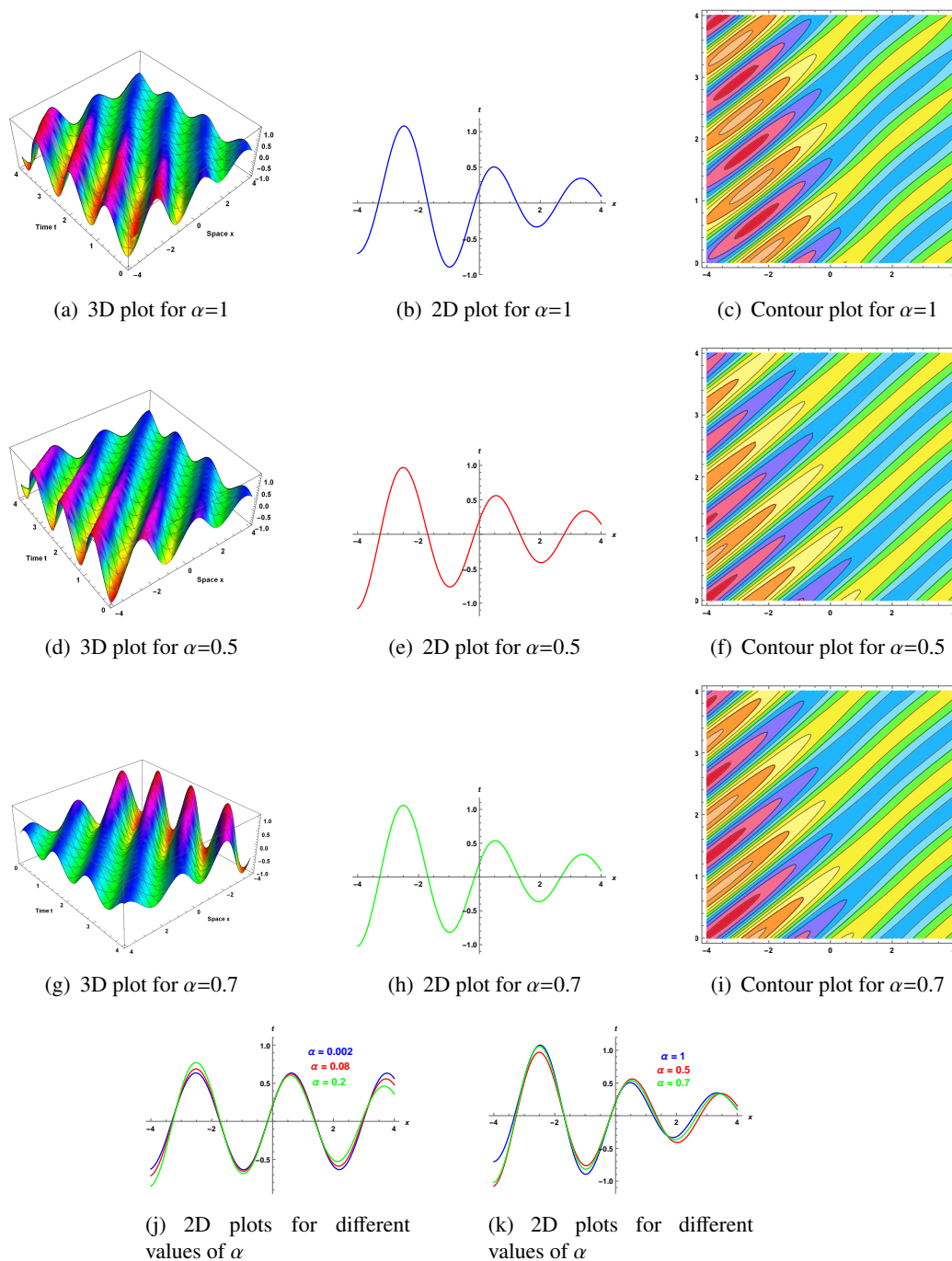


Figure 9. For the imaginary part of solution U_3 , 3D, contour and 2D plots with different values of α .

6. Conclusions

In this study, we used the modified auxiliary equation method and the extended direct algebraic method to locate distinct generalized solitary solutions to the SK equations. These results are validated

for accuracy using symbolic computation tools on a computer. In contrast to the majority of the already used methodologies, the modified auxiliary equation method can produce creative and more general solutions. This exemplifies the value and immense potential of the modified auxiliary equation method in resolving difficult solitary wave issues that arise in mathematical physics. Using a modified auxiliary equation method, the wave solutions to the Sawada-Kotera problem are examined. Brilliant periodic waves, steady bright soliton waves and symmetric waves all have physical qualities that have been identified. Graphics are used to show the effects of the parameters.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The author Magda Abd El-Rahman extends their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through large group Research Project under grant number RGP2/69/44. The author Muhammad Bilal Riaz thankful to Ministry of Education, Youth and Sports of the Czech Republic for their support through the e-INFRA CZ (ID: 90254).

Conflict of interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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