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Research article

General stability for a system of coupled quasi-linear and linear wave equations and with memory term

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Abstract: In this paper, a system of coupled quasi-linear and linear wave equations with a finite memory term is concerned. By constructing an appropriate Lyapunov function, we prove that the total energy associated with the system is stable under suitable conditions on memory kernel.

Keywords: coupled wave equations; finite memory; exponential stability; nonlinearity
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1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$ be a bounded open set with a regular boundary $\Gamma = \partial\Omega$. A coupled wave equation, via laplacian and with just one memory term is considered:

$$\begin{cases} |y_t|^\rho y_{tt}(x,t) - a\Delta y(x,t) - c\Delta y_{tt} + c\Delta z(x,t) + \displaystyle\int_0^t g(t-s)\Delta y(x,s)ds = 0, \quad \text{in } \Omega \times (0,\infty), \\ z_{tt}(x,t) - \Delta z(x,t) - \frac{1}{c}\Delta z_{tt} + c\Delta y(x,t) = 0, \quad \text{in } \Omega \times (0,\infty), \\ y = z = 0, \quad \text{on } \Gamma \times (0,\infty), \\ y(x,0) = y_0(x),\ z(x,0) = z_0(x),\ y_t(x,0) = y_1(x),\ z_t(x,0) = z_1(x), \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where $a > 0$, $c \in \mathbb{R}^*$ such that $a > c^2$, and

$$a = b + c^2, \tag{1.2}$$

where b is a positive constant satisfying

$$l = b - \int_0^\infty g(s)ds > 0. \tag{1.3}$$

Throughout this paper, we assume that ρ is a positive constant that verifies

$$\rho > 0 \text{ if } n = 2 \text{ or } 0 < \rho \leq \frac{2}{n-2} \text{ if } n \geq 3.$$

Morris and Özer [17, 18] proposed the following piezoelectric beam model

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0, & \text{in } (0,\ell)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, & \text{in } (0,\ell)\times(0,\infty), \\ v(0) = p(0) = \alpha v_x(\ell) - \gamma\beta p_x(\ell) = 0, \\ \beta p_x(\ell) - \gamma\beta v_x(\ell) = -\frac{V(t)}{h}, \end{cases} \tag{1.4}$$

where the coefficients ρ, α, γ, μ, β, ℓ and $h > 0$ are the mass density per unit volume, elastic stiffness, piezoelectric coefficient, magnetic permeability, impermeability coefficient of the beam and Euler-Bernoulli beam of length and thickness, respectively. $V(t)$ denotes the voltage directed to the electrodes that included full magnetic effects. They obtained that for a dense set of system parameters with $V(t) = p_t(\ell, t)$, the system (1.4) is strongly controllable in the energy space. Ramos, Gonçalves and Corrêa Neto [22] added a damping term δv_t with $\delta > 0$ in the first equation of problem (1.4) and set $V(t) = 0$. They analyzed the exponential stability of the total energy of the continuous problem and showed a numerical counterpart in a totally discrete domain. Ramos, Freitas and Almeida et al. [23] replaced δv_t by $\xi_1 v_t + \xi_2 v_t(x, t-\tau)$; that is, they considered a system with time delay in the internal state feedback, where $\xi_2 v_t(x, t-\tau)$ with $\xi_2 > 0$ represents the time delay on the vertical displacement and $\tau > 0$ represents the respective retardation time. By using an energy-based approach, the exponential stability of solutions was also proved in [23]. Soufyane, Afilal and Santos [24] generalized their results and established an energy decay rate for piezoelectric beams with magnetic effect, nonlinear damping and nonlinear delay terms by using a perturbed energy method and some properties of convex functions. Recently, Akil [1] investigated the stabilization of a system of piezoelectric beams under (Coleman or Pipkin)-Gurtin thermal law with magnetic effect. It is certainly not the object of the present paper to consider the evolution equations like (1.4) with nonlinear damping and/or time-delay terms. In this paper, we mainly consider the effect of the viscoelastic memory damping $\int_0^t g(t-s)\Delta y(x,s)ds$, which is presented only in the first equation of the evolution equations like (1.4) and with Dirichlet conditions on the whole boundary. Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. Generally, one makes full use of the memory term (infinite memory $\int_0^\infty g(s)\Delta y(x,t-s)ds$ or finite memory $\int_0^t g(t-s)\Delta y(x,s)ds$) to describe the viscoelastic damping effect. The aforementioned model can be used to describe the motion of two elastic membranes subject to an elastic force that pulls one membrane toward the other. We note that one of these membranes possesses a rigid surface and that has an interior that is somehow permissive to slight deformations, such that the material density varies according to the velocity. The study of viscoelastic problems has attracted the attention of many authors and a flurry works have been published. It is certainly beyond the scope of the present paper to give a comprehensive review for only one viscoelastic equation. In this regard, we would like to mention some references regarding the energy decay in the presence of viscoelastic effects, for instance, [2, 4–7, 10, 15, 21] and references therein. It is not difficult to find that with the analysis of exponential stability for models consisting of two coupled wave equations, one of them with a memory effect is a subject of great importance. Dos Santos, Fortes and Cardoso [9] first investigated the issue of exponential stability of the following two coupled wave equations:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \int_{-\infty}^{t} g(t-s)v_{xx}(s)ds = 0, & \text{in } (0,\ell)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, & \text{in } (0,\ell)\times(0,\infty), \end{cases}$$

with boundary condition

$$v(0,t) = p(0,t) = v_x(\ell,t) = p_x(\ell,t) = 0, \quad t > 0,$$

and initial data

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x), p(x,0) = p_0(x), p_t(x,0) = p_1(x), \quad x \in (0,\ell),$$

$$v(x,-t) = v_2(x,t), (x,t) \in (0,\ell)\times(0,\infty),$$

where $v_0,\ v_1,\ v_2,\ p_0$ and p_1 are known functions belonging to appropriate spaces and $\alpha = \alpha_1 + \gamma^2\beta$ with α_1 positive constant satisfies $\kappa := \alpha_1 - \int_0^\infty g(s)ds > 0$. They deduced that the past history term acting on the longitudinal motion equation is sufficient to cause the exponential decay of the semigroup associated with the system, independent of any relation involving the model coefficients. Zhang, Xu and Han [25] considered a kind of fully magnetic effected nonlinear multidimensional piezoelectric beam with viscoelastic infinite memory; that is, they studied the following problem

$$\begin{cases} \rho v_{tt}(x,t) = \alpha\Delta v(x,t) - \gamma\beta\Delta p(x,t) - \int_0^\infty g(s)\Delta v(x,t-s)ds + f_1(v,p), & x\in\Omega,\ t>0 \\ \mu p_{tt} x, t = \beta\Delta p(x,t) - \gamma\beta\Delta v(x,t) + f_2(v,p), & x\in\Omega,\ t>0 \\ v(x,t) = p(x,t) = 0, & x\in\Gamma_0,\ t>0 \\ \alpha\frac{\partial v}{\partial\vec{n}}(x,t) - \gamma\beta\frac{\partial p}{\partial\vec{n}}(x,t) = \beta\frac{\partial p}{\partial\vec{n}}(x,t) - \gamma\beta\frac{\partial v}{\partial\vec{n}}(x,t) = 0, & x\in\Gamma_1,\ t>0 \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ p(x,0) = p_0(x),\ p_t(x,0) = p_1(x), & x\in\Omega, \\ v(x,-s) = h(x,s), & x\in\Omega,\ s>0, \end{cases}$$

where $\partial\Omega = \Gamma_0 \cup \Gamma_1,\ \Gamma_0 \cap \Gamma_1 = \emptyset,\ \vec{n}$ is the unit outward normal vector of Γ_1 and the functions $f_i(v,p), i = 1,2$ and $h(x,s)$ are nonlinear source terms and memory history function, respectively. Based on frequency-domain analysis, they proved that the corresponding coupled linear system can be indirectly stabilized exponentially by only one viscoelastic infinite memory term. Moreover, by the energy estimation method under certain conditions, they obtained the exponential decay of the solution to the nonlinear coupled PDE's (partial differential equations) system.

We also recall the works [12, 13, 19, 20], where the authors studied the wellposedness and the asymptotic behavior of a linear (and quasi-linear) system of two coupled nonlinear viscoelastic wave equations. We also cite the recent works [3, 11], where the authors studied a similar problem to (1.1) with $\rho = 0$, without dispersion terms and under different types of damping (localized frictional and past history damping). Through our review of the literature, we found that no prior studies have explored this type of coupling (one equation is quasi-linear and the other one is linear) with the presence of a memory term (or a past history term). Consequently, the significance of our work is that it pioneers the impact of memory term in this context and, furthermore, our main result extends exponential decay outcomes, which have previously been established for the coupling of two viscoelastic wave equations via zero-order or first-order terms to the realm of coupling by second-order terms. Also, our result removes the assumption of equal wave propagation speeds, a common feature in numerous prior studies.

Motivated by the above works, we are concerned with the stability of a system of coupled quasi-linear and linear wave equations with only one viscoelastic finite memory involved. Different from the works in [9, 25], in this paper we focus on the finite memory damping and the system is quasi-linear. Some technical difficulties may be caused by the nonlinearity and the finite memory term. The remaining part of the paper is subdivided as follows: In section two, we give preliminaries and technical lemmas, which are crucial to establish the decay rates. By using the perturbed energy method, we prove the general decay of the energy associated with system (1.1) in the last section.

2. Preliminaries and technical lemmas

In this section, we give necessary assumptions and establish three lemmas needed for the proof of our main result.

We use the standard Lebesgue space $L^2(\Omega)$ with its usual norm $\|\cdot\|$. We denote, respectively, by C_p and C_s the embedding constants of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$, for $r > 0$ if $n = 2$ or $0 < r \leq \frac{2n}{n-2}$ if $n \geq 3$, i.e.,

$$\|y\| \leq C_p\|\nabla y\|, \quad \|y\|_r \leq C_s\|\nabla y\|, \quad \forall\ y \in H_0^1(\Omega),$$

where $\|z\|_r$ denotes the usual $L^r(\Omega)$-norm.

In this paper, we take into account the following conditions:

(H1): $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that $g(0) > 0$ and $g'(s) < 0$ for any $s \in \mathbb{R}_+$.

(H2): There exists a nonincreasing continuous function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$g'(t) \leq -\xi(t)g(t), \quad \forall\ t \geq 0. \tag{2.1}$$

The energy of solutions of system (1.1) is given by

$$\begin{aligned} E(t) &= \frac{1}{\rho+2}\int_\Omega |y_t|^{\rho+2}dx + \frac{1}{2}\left(b - \int_0^t g(s)ds\right)\int_\Omega |\nabla y|^2dx + \frac{1}{2}(g\circ\nabla y)(t) + \frac{c}{2}\int_\Omega |\nabla y_t|^2dx \\ &+ \frac{1}{2}\int_\Omega |z_t|^2dx + \frac{1}{2c}\int_\Omega |\nabla z_t|^2dx + \frac{1}{2}\int_\Omega |c\nabla y - \nabla z|^2dx, \end{aligned} \tag{2.2}$$

where

$$(g\circ\nabla y)(t) = \int_0^t g(t-s)\|\nabla y(t) - \nabla y(s)\|^2ds.$$

The energy satisfies the following dissipation law.

Proposition 2.1. *We have*

$$E'(t) = \frac{1}{2}(g'\circ\nabla y)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla y|^2dx \leq 0. \tag{2.3}$$

Proof. Multiplying $(1.1)_1$ by y_t and $(1.1)_2$ by z_t, we integrate by parts on Ω to obtain

$$\frac{d}{dt}\left(\frac{1}{\rho+2}\int_\Omega |y_t|^{\rho+2}dx + \frac{a}{2}\int_\Omega |\nabla y|^2 dx + \frac{c}{2}\int_\Omega |\nabla y_t|^2 dx\right) - c\left(\int_\Omega \nabla z\cdot\nabla y_t dx\right) - \int_0^t g(t-s)\int_\Omega \nabla y(s)\nabla y_t dxdt = 0 \tag{2.4}$$

and

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |z_t|^2dx + \int_\Omega |\nabla z|^2 dx + \frac{1}{c}\int_\Omega |\nabla z_t|^2 dx\right) - c\left(\int_\Omega \nabla y\cdot\nabla z_t dx\right) = 0. \tag{2.5}$$

Thus, a direct computation shows that

$$\begin{aligned}\int_0^t g(t-s)\int_\Omega \nabla y(s)\nabla y_t(t)dxds &= \frac{1}{2}(g'\circ\nabla y)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla y(t)|^2 dx \\ &- \frac{1}{2}\frac{d}{dt}\left\{(g\circ\nabla y)(t) - \left(\int_0^t g(s)ds\right)\int_\Omega |\nabla y(t)|^2dx\right\}.\end{aligned} \tag{2.6}$$

Using (2.6) and the fact that $a = b + c^2$ in (2.4), we infer that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{\rho+2}\int_\Omega |y_t|^{\rho+2}dx + \frac{1}{2}\left(b - \int_0^t g(s)ds\right)\int_\Omega |\nabla y|^2 dx + \frac{1}{2}(g\circ\nabla y)(t) + \frac{c}{2}\int_\Omega |\nabla y_t|^2dx\right) \\ &+\frac{c^2}{2}\frac{d}{dt}\int_\Omega |\nabla y|^2 dx - c\left(\int_\Omega \nabla z\cdot\nabla y_t dx\right) - \frac{1}{2}(g'\circ\nabla y)(t) + \frac{1}{2}g(t)\int_\Omega |\nabla y(t)|^2dx = 0.\end{aligned} \tag{2.7}$$

By adding (2.5) and (2.7), (2.3) holds true. □

(2.3) implies that system (1.1) is dissipative, and so $E(t) \leq E(0)$.

Using the Faedo-Galerkin method, for instance, Liu [12] and Mustafa [19], we obtain the following local existence result:

Proposition 2.2. *Let* $(y_0, y_1), (z_0, z_1) \in H_0^1(\Omega)\times H_0^1(\Omega)$ *be given. Assume that g satisfies* **(H1)** *and* **(H2)**, *then problem* (1.1) *has a unique local solution* (y, z) *satisfying*

$$y, y_t, z, z_t \in C\left([0,T); H_0^1(\Omega)\right),$$

for some $T > 0$.

Thus, it is easy to see that

$$\frac{l}{2}\int_\Omega |\nabla y|^2dx + \frac{c}{2}\int_\Omega |\nabla y_t|^2dx + \frac{1}{4}\int_\Omega |\nabla z|^2dx + \frac{1}{2c}\int_\Omega |\nabla z_t|^2dx \leq \left(2+\frac{c^2}{l}\right)E(t) \leq \left(2+\frac{c^2}{l}\right)E(0),$$

which gives that the solution of problem (1.1) is bounded and global in time.

Lemma 2.3. *Under assumptions* **(H1)** *and* **(H2)**, *the functional*

$$A(t) = \frac{1}{\rho+1}\int_\Omega y|y_t|^\rho y_t dx + c\int_\Omega \nabla y\nabla y_t dx + \int_\Omega zz_t dx + \frac{1}{c}\int_\Omega \nabla z\nabla z_t dx$$

satisfies along the solution and the estimate:

$$\begin{aligned} A'(t) &\leq \frac{1}{\rho+1}\int_\Omega |y_t|^{\rho+2}dx - \frac{l}{2}\int_\Omega |\nabla y|^2 dx + c\int_\Omega |\nabla y_t|^2 dx + \int_\Omega |z_t|^2 dx + \frac{1}{c}\int_\Omega |\nabla z_t|^2 dx \\ &- \int_\Omega |c\nabla y - \nabla z|^2 dx + \frac{b-l}{2l}(g\circ\nabla y)(t). \end{aligned} \tag{2.8}$$

Proof. Multiplying $(1.1)_1$ by y and integrating by parts over Ω, we obtain

$$\begin{aligned} &\frac{d}{dt}\frac{1}{\rho+1}\int_\Omega y|y_t|^\rho y_t dx - \frac{1}{\rho+1}\int_\Omega |y_t|^{\rho+2}dx + b\int_\Omega |\nabla y|^2 dx + c\int_\Omega \nabla y\,(c\nabla y - \nabla z)\,dx \\ &+\frac{d}{dt}\int_\Omega c\nabla y_t \nabla y dx - c\int_\Omega |\nabla y_t|^2 dx - \int_\Omega \nabla y(t)\int_0^t g(t-s)\nabla y(s)dsdx = 0. \end{aligned} \tag{2.9}$$

Therefore, multiplying $(1.1)_2$ by z and integrating by parts over Ω, we infer that

$$\frac{d}{dt}\int_\Omega z z_t dx - \int_\Omega |z_t|^2 dx + \int_\Omega |\nabla z|^2 dx - c\int_\Omega \nabla y \nabla z dx + \frac{d}{dt}\frac{1}{c}\int_\Omega \nabla z \nabla z_t dx - \frac{1}{c}\int_\Omega |\nabla z_t|^2 dx = 0. \tag{2.10}$$

Combining (2.9) and (2.10), we find

$$\begin{aligned} A'(t) &= \frac{1}{\rho+1}\int_\Omega |y_t|^{\rho+2}dx - b\int_\Omega |\nabla y|^2 dx - \int_\Omega |c\nabla y - \nabla z|^2 dx + c\int_\Omega |\nabla y_t|^2 dx \\ &+ \int_\Omega \nabla y(t)\int_0^t g(t-s)\nabla y(s)dsdx + \int_\Omega |z_t|^2 dx + \frac{1}{c}\int_\Omega |\nabla z_t|^2 dx. \end{aligned} \tag{2.11}$$

It is easy to check that [14]

$$\begin{aligned} &\int_\Omega \nabla y(t)\int_0^t g(t-s)\nabla y(s)dsdx \\ &\leq (b-\frac{l}{2})\int_\Omega |\nabla y|^2\,dx + \frac{b-l}{2l}(g\circ\nabla y)(t). \end{aligned} \tag{2.12}$$

Inserting (2.12) in (2.11), the inequality (2.8) holds true. □

Lemma 2.4. *Assume that* **(H1)** *and* **(H2)** *hold and* (y, y_t, z, z_t) *is a solution of* (1.1)*, then the functional*

$$B(t) = \int_\Omega \left(\Delta y_t - \frac{1}{\rho+1}|y_t|^\rho y_t\right)\int_0^t g(t-s)\,(y(t)-y(s))\,dsdx$$

satisfies

$$\begin{aligned} B'(t) &\leq -\frac{1}{\rho+1}\left(\int_0^t g(s)ds\right)\int_\Omega |y_t|^{\rho+2}dx + \frac{c\delta_1}{2}\int_\Omega |c\nabla y - \nabla z|^2 dx + \left(\frac{b^2\delta_2}{2} + 2(b-l)^2\delta_2\right)\int_\Omega |\nabla y|^2\,dx \\ &+ \left(\delta_2 + \frac{\delta_2 C_s^{2(\rho+1)}}{\rho+1}\left(\frac{2}{c}E(0)\right)^\rho - \int_0^t g(s)ds\right)\int_\Omega |\nabla y_t|^2\,dx \\ &+ \left(\frac{c(b-l)}{2\delta_1} + \frac{b-l}{2\delta_2} + (b-l)(2\delta_2 + \frac{1}{4\delta_2})\right)(g\circ\nabla y)(t) \\ &+ \frac{g(0)}{4\delta_2}\left(1 + \frac{C_p^2}{\rho+1}\right)(-g'\circ\nabla y)(t), \end{aligned} \tag{2.13}$$

for any $\delta_1, \delta_2 > 0$.

Proof. By exploiting Eq (1.1) and integrating by parts, we have

$$
\begin{aligned}
B'(t) &= -\frac{1}{\rho+1}\left(\int_0^t g(s)ds\right)\int_\Omega |y_t|^{\rho+2}dx + b\int_\Omega \nabla y(t)\int_0^t g(t-s)\nabla(y(t)-y(s))dsdx \\
&+ c\int_\Omega (c\nabla y-\nabla z)\int_0^t g(t-s)\nabla(y(t)-y(s))dsdx - \left(\int_0^t g(s)ds\right)\int_\Omega |\nabla y_t|^2dx \\
&- \int_\Omega\left(\int_0^t g(t-s)\nabla y(s)ds\right)\left(\int_0^t g(t-s)\nabla(y(t)-y(s))ds\right)dx \\
&- \int_\Omega \nabla y_t(t)\int_0^t g'(t-s)\nabla(y(t)-y(s))dsdx \\
&- \frac{1}{\rho+1}\int_\Omega |y_t|^\rho y_t\int_0^t g'(t-s)(y(t)-y(s))dsdx. \qquad (2.14)
\end{aligned}
$$

By the Young inequality and Cauchy Schwarz inequality, we infer for any $\delta_1>0$ that

$$
c\int_\Omega (c\nabla y-\nabla z)\int_0^t g(t-s)\nabla(y(t)-y(s))dsdx \le \frac{c\delta_1}{2}\int_\Omega |c\nabla y-\nabla z|^2dx + \frac{c(b-l)}{2\delta_1}(g\circ\nabla y)(t). \quad (2.15)
$$

Likewise, for (2.15) it is easy to check that for every $\delta_2>0$,

$$
b\int_\Omega \nabla y(t)\int_0^t g(t-s)\nabla(y(t)-y(s))dsdx \le \frac{b^2\delta_2}{2}\int_\Omega |\nabla y|^2dx + \frac{(b-l)}{2\delta_2}(g\circ\nabla y)(t) \quad (2.16)
$$

and

$$
\int_\Omega \nabla y_t(t)\int_0^t g'(t-s)\nabla(y(t)-y(s))dsdx \le \delta_2\int_\Omega |\nabla y_t|^2dx - \frac{g(0)}{4\delta_2}(g'\circ\nabla y)(t). \quad (2.17)
$$

Now, the remaining terms can be estimated as estimates (3.11) and (3.15) in [16]:

$$
\begin{aligned}
&-\int_\Omega\left(\int_0^t g(t-s)\nabla y(s)ds\right)\left(\int_0^t g(t-s)\nabla(y(t)-y(s))ds\right) \\
&\le (2\delta_2+\frac{1}{4\delta_2})(b-l)(g\circ\nabla y)(t) + 2\delta_2(b-l)^2\int_\Omega |\nabla y|^2dx \qquad (2.18)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\rho+1}\int_\Omega |y_t|^\rho y_t\int_0^t g'(t-s)(y(t)-y(s))dsdx \\
&\le \frac{C_s^{2(\rho+1)}\delta_2}{\rho+1}(\frac{2}{c}E(0))^\rho\int_\Omega |\nabla y_t|^2dx - \frac{g(0)C_p^2}{4(\rho+1)\delta_2}(g'\circ\nabla y)(t). \qquad (2.19)
\end{aligned}
$$

The combination of (2.14)–(2.19) yields to the desired inequality (2.13). □

Lemma 2.5. *Let $Z=(y,y_t,z,z_t)$ be a solution of* (1.1)*, then under the assumptions* **(H1)** *and* **(H2)** *the functional*

$$
D(t) = \frac{1}{\rho+1}\int_\Omega |y_t|^\rho y_t(cy-z)dx + c\int_\Omega z_t(cy-z)dx + c\int_\Omega \nabla y_t(c\nabla y-\nabla z)dx + \int_\Omega \nabla z_t(c\nabla y-\nabla z)dx
$$

satisfies

$$\begin{aligned} D'(t) &\le \delta_3 \int_\Omega |c\nabla y - \nabla z|^2 dx + \frac{b^2}{2\delta_3} \int_\Omega |\nabla y|^2 dx + \frac{b-l}{2\delta_3}(g \circ \nabla y)(t) \\ &+ (c^2 + c^3 C_p^2) \int_\Omega |\nabla y_t|^2 dx - \frac{3c}{4} \int_\Omega |z_t|^2 dx + \left(\frac{C_s^2 \delta_4}{2(\rho+1)} - 1 \right) \int_\Omega |\nabla z_t|^2 dx \\ &+ \left(\frac{c}{\rho+1} + \frac{((\rho+2)E(0))^{\frac{2}{\rho+2}}}{2(\rho+1)\delta_4} \right) \int_\Omega |y_t|^{\rho+2} dx \end{aligned} \tag{2.20}$$

for every $\delta_3, \delta_4 > 0$.

Proof. Multiplying $(1.1)_1$ by $cy - z$, using $(1.1)_2$ and integrating by parts over Ω, we obtain

$$\begin{aligned} &\frac{d}{dt}\frac{1}{\rho+1} \int_\Omega |y_t|^\rho y_t (cy - z) dx - \frac{1}{\rho+1} \int_\Omega |y_t|^\rho y_t (cy - z)_t dx \\ &+ b \int_\Omega \nabla y \nabla (cy - z) dx + \int_\Omega (cz_{tt} - \Delta z_{tt})(cy - z) dx + \frac{d}{dt} c \int_\Omega \nabla y_t \nabla (cy - z) dx \\ &- c \int_\Omega \nabla y_t \nabla (cy - z)_t dx - \int_\Omega (c\nabla y - \nabla z) \int_0^t g(t-s) \nabla y(s) ds dx = 0, \end{aligned}$$

which implies that

$$\begin{aligned} D'(t) &= \frac{1}{\rho+1} \int_\Omega |y_t|^\rho y_t (cy - z)_t dx - b \int_\Omega \nabla y \nabla (cy - z) dx + c \int_\Omega z_t (cy - z)_t dx \\ &\quad + c \int_\Omega \nabla y_t \nabla (cy - z)_t dx + \int_\Omega \nabla z_t \nabla (cy - z)_t dx + \int_\Omega (c\nabla y - \nabla z) \int_0^t g(t-s) \nabla y(s) ds dx \\ &= \frac{c}{\rho+1} \int_\Omega |y_t|^{\rho+2} dx - \frac{1}{\rho+1} \int_\Omega |y_t|^\rho y_t z_t dx - b \int_\Omega \nabla y \nabla (cy - z) dx + c^2 \int_\Omega z_t y_t dx - c \int_\Omega |z_t|^2 dx \\ &\quad + c^2 \int_\Omega |\nabla y_t|^2 dx - \int_\Omega |\nabla z_t|^2 dx + \int_\Omega (c\nabla y - \nabla z) \int_0^t g(t-s) \nabla y(s) ds dx. \end{aligned} \tag{2.21}$$

Thanks to Young's inequality and Cauchy Schwarz's inequality, we find for any $\delta_3 > 0$ that

$$-b \int_\Omega \nabla y \nabla (cy - z) dx \le \frac{\delta_3}{2} \int_\Omega |c\nabla y - \nabla z|^2 dx + \frac{b^2}{2\delta_3} \int_\Omega |\nabla y|^2 dx \tag{2.22}$$

and

$$\int_\Omega (c\nabla y - \nabla z) \int_0^t g(t-s) \nabla y(s) ds dx \le \frac{\delta_3}{2} \int_\Omega |c\nabla y - \nabla z|^2 dx + \frac{b-l}{2\delta_3}(g \circ \nabla y)(t). \tag{2.23}$$

Using Hölder's inequality, Young's inequality and Poincaré's inequality, we derive that

$$\begin{aligned} c^2 \int_\Omega z_t y_t dx &\le \frac{c}{4} \int_\Omega |z_t|^2 dx + c^3 \int_\Omega |y_t|^2 dx \\ &\le \frac{c}{4} \int_\Omega |z_t|^2 dx + c^3 C_p^2 \int_\Omega |\nabla y_t|^2 dx, \end{aligned} \tag{2.24}$$

$$
\begin{aligned}
-\frac{1}{\rho+1}\int_\Omega |y_t|^\rho y_t z_t dx &\le \frac{1}{\rho+1}\Big\{\int_\Omega |y_t|^{\rho+2}dx\Big\}^{\frac{\rho+1}{\rho+2}}\Big\{\int_\Omega |z_t|^{\rho+2}dx\Big\}^{\frac{1}{\rho+2}} \\
&\le \frac{\delta_4}{2(\rho+1)}\Big\{\int_\Omega |z_t|^{\rho+2}dx\Big\}^{\frac{2}{\rho+2}} + \frac{1}{2(\rho+1)\delta_4}\Big\{\int_\Omega |y_t|^{\rho+2}dx\Big\}^{\frac{2(\rho+1)}{\rho+2}} \\
&\le \frac{C_s^2\delta_4}{2(\rho+1)}\int_\Omega |\nabla z_t|^2 dx + \frac{\Big((\rho+2)E(0)\Big)^{\frac{2}{\rho+2}}}{2(\rho+1)\delta_4}\int_\Omega |y_t|^{\rho+2}dx, \qquad (2.25)
\end{aligned}
$$

for any $\delta_4 > 0$.

Inserting (2.22)–(2.25) in (2.21), we obtain (2.20). □

3. General stability

We define the functional $\mathcal{F}$ by

$$\mathcal{F}(t) = NE(t) + N_1A(t) + N_2B(t) + N_3D(t),$$

where N, N_1, N_2 and N_3 are positive constants that will be chosen later.

It is easy to check, for N sufficiently large, that $E(t) \sim \mathcal{F}(t)$, i.e.,

$$c_1E(t) \le \mathcal{F}(t) \le c_2E(t), \ \forall\ t \ge 0, \qquad (3.1)$$

for some constants $c_1, c_2 > 0$.

The main result of this paper reads as follows.

Theorem 3.1. *Let $(y_0, y_1), (z_0, z_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Assume that (**H1**) and (**H2**) hold true, then for any $t_1 > 0$, there exists positive constants β_1 and β_2 such that the energy $E(t)$ satisfies*

$$E(t) \le \beta_2 e^{-\beta_1 \int_{t_1}^t \xi(s)ds}. \qquad (3.2)$$

Proof. Set $g_0 = \int_0^{t_1} g(s)ds > 0$. By using (2.11), (2.13), (2.20) and (2.3), one obtains for all $t \ge t_1$

$$
\begin{aligned}
\mathcal{F}'(t) \le\ & \left\{\frac{N}{2} - \frac{N_2 g(0)}{4\delta_2}(1 + \frac{C_p^2}{\rho+1})\right\}(g' \circ \nabla y)(t) \\
&- \left\{\frac{N_2 g_0}{\rho+1} - \frac{N_1}{\rho+1} - N_3\left(\frac{c}{\rho+1} + \frac{\Big((\rho+2)E(0)\Big)^{\frac{2}{\rho+2}}}{2(\rho+1)\delta_4}\right)\right\}\int_\Omega |y_t|^{\rho+2}dx \\
&- \left\{\frac{N_1 l}{2} - N_2\left(\frac{b^2\delta_2}{2} + 2(b-l)^2\delta_2\right) - \frac{N_3 b^2}{2\delta_3}\right\}\int_\Omega |\nabla y|^2 dx \\
&- \left\{N_2 g_0 - N_1 c - N_2\left(\delta_2 + \frac{\delta_2 C_s^{2(\rho+1)}}{\rho+1}\left(\frac{2}{c}E(0)\right)^\rho\right) - N_3(c^2 + c^3C_p^2)\right\}\int_\Omega |\nabla y_t|^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&- \left\{\frac{3cN_3}{4} - N_1\right\}\int_\Omega |z_t|^2 dx \\
&- \left\{N_3 - \frac{N_1}{c} - \frac{N_3 C_s^2 \delta_4}{2(\rho+1)}\right\}\int_\Omega |\nabla z_t|^2 dx \\
&- \left\{N_1 - \frac{cN_2\delta_1}{2} - N_3\delta_3\right\}\int_\Omega |c\nabla y - \nabla z|^2 dx \\
&+ \left\{\frac{N_1(b-l)}{2l} + N_2\left(\frac{c(b-l)}{2\delta_1} + \frac{(b-l)}{2\delta_2} + (b-l)(2\delta_2 + \frac{1}{4\delta_2})\right) + \frac{N_3(b-l)}{2\delta_3}\right\}(g\circ\nabla y)(t). \quad (3.3)
\end{aligned}
$$

By choosing $\delta_1 = \frac{N_1}{cN_2}$, $\delta_2 = \frac{lN_1}{N_2(b^2+4(b-l)^2)}$, $\delta_3 = \frac{N_1}{4N_3}$ and $\delta_4 = \frac{2(\rho+1)N_1}{3cC_s^2 N_3}$, (3.3) becomes

$$
\begin{aligned}
\mathcal{F}'(t) \le\ & \left\{N_1(b-l)\left(\frac{1}{2l} + \frac{2l}{b^2+4(b-l)^2}\right) + N_2^2\left(\frac{c^2(b-l)}{2N_1} + \frac{3(b-l)(b^2+4(b-l)^2)}{4lN_1}\right) + \frac{2(b-l)N_3^2}{N_1}\right\} \\
\times\ & (g\circ\nabla y)(t) + \left\{\frac{N}{2} - \frac{N_2^2 g(0)(b^2+4(b-l)^2)}{4lN_1}(1 + \frac{C_p^2}{\rho+1})\right\}(g'\circ\nabla y)(t) \\
-\ & \left\{\frac{N_2 g_0}{\rho+1} - \frac{N_1}{\rho+1} - N_3\left(\frac{c}{\rho+1} + \frac{3cC_s^2 N_3\left((\rho+2)E(0)\right)^{\frac{2}{\rho+2}}}{4N_1(\rho+1)^2}\right)\right\}\int_\Omega |y_t|^{\rho+2} dx \\
-\ & \frac{2N_3^2 b^2}{N_1}\int_\Omega |\nabla y|^2 dx - \left\{\frac{3cN_3}{4} - N_1\right\}\int_\Omega |z_t|^2 dx - \left\{N_3 - \frac{4N_1}{3c}\right\}\int_\Omega |\nabla z_t|^2 dx \\
-\ & \left\{N_2 g_0 - N_1 c - \frac{N_1 l}{b^2+4(b-l)^2}\left(1 + \frac{C_s^{2(\rho+1)}}{\rho+1}\left(\frac{2}{c}E(0)\right)^\rho\right) - N_3(c^2 + c^3 C_p^2)\right\}\int_\Omega |\nabla y_t|^2 dx \\
-\ & \frac{N_1}{4}\int_\Omega |c\nabla y - \nabla z|^2 dx. \quad (3.4)
\end{aligned}
$$

At this point, we choose N_1 for any positive real number and we pick up N_3 and N_2, respectively, such that

$$N_3 > \frac{4N_1}{3c},$$

$$N_2 g_0 > N_1 - N_3\left(c + \frac{3cC_s^2 N_3\left((\rho+2)E(0)\right)^{\frac{2}{\rho+2}}}{4N_1(\rho+1)}\right)$$

and

$$N_2 g_0 > N_1 c + \frac{N_1 l}{b^2+4(b-l)^2}\left(1 + \frac{C_s^{2(\rho+1)}}{\rho+1}\left(\frac{2}{c}E(0)\right)^\rho\right) + N_3(c^2 + c^3 C_p^2).$$

After this, we choose N sufficiently large so that (3.1) holds true and

$$N > \frac{N_2^2 g(0)(b^2+4(b-l)^2)}{2lN_1}(1 + \frac{C_p^2}{\rho+1}).$$

Therefore, it follows for some constants $m, C > 0$ and all $t \ge t_1$ that

$$\mathcal{F}'(t) \le -mE(t) + C(g\circ\nabla y)(t). \quad (3.5)$$

Denote $\mathcal{L}(t) = \mathcal{F}(t) + CE(t)$. Clearly, $\mathcal{L}(t)$ is equivalent to $E(t)$. It follows from (3.5) that

$$\mathcal{L}'(t) \leq -mE(t) + C\int_0^t g(s)\int_\Omega |\nabla y(t) - \nabla y(t-s)|^2 dxds. \tag{3.6}$$

Next, we multiply (3.6) by $\xi(t)$ and use Assumption **(H2)** and (2.3) to obtain

$$\begin{aligned}\xi(t)\mathcal{L}'(t) &\leq -m\xi(t)E(t) + C\xi(t)\int_0^t g(s)\int_\Omega |\nabla y(t) - \nabla y(t-s)|^2 dxds\\ &\leq -m\xi(t)E(t) + C\int_0^t \xi(s)g(s)\int_\Omega |\nabla y(t) - \nabla y(t-s)|^2 dxds\\ &\leq -m\xi(t)E(t) - C\int_0^t g'(s)\int_\Omega |\nabla y(t) - \nabla y(t-s)|^2 dxds\\ &\leq -m\xi(t)E(t) - CE'(t), \quad \forall\ t \geq t_1. \end{aligned}\tag{3.7}$$

Denote $R(t) = \xi(t)\mathcal{L}(t) + CE(t) \sim E(t)$, then we have from (3.7) and the fact that ξ is nonincreasing that, for any $t \geq t_1$,

$$R'(t) \leq -m\xi(t)E(t).$$

Using the fact that $R \sim E$, we obtain

$$R'(t) \leq -\beta_1 R(t)$$

for some positive constant β_1. By applying Gronwall's Lemma, we obtain the existence of a constant $C_1 > 0$ such that

$$R(t) \leq C_1 e^{-\beta_1 \int_{t_1}^t \xi(s)\, ds},$$

which yields to

$$E(t) \leq \beta_2 e^{-\beta_1 \int_{t_1}^t \xi(s)\, ds},$$

for some constant $\beta_2 > 0$. □

Remark 3.2. *By replacing in* (1.1) *the memory term by a past history term of the form* $\int_0^\infty g(s)\Delta y(x, t-s)ds$*, and by defining the new variable* η *(as in [8]) by*

$$\begin{cases} \eta(x, s, t) = y(x, t) - y(x, t-s), \quad \forall\ (x, s, t) \in \Omega \times (0, +\infty) \times (0, +\infty),\\ \eta_0(x, s) = \eta(x, s, 0) = f(x, 0) - f(x, s), \quad \forall\ (x, s) \in \Omega \times (0, +\infty), \end{cases}$$

(1.1) *becomes*

$$\begin{cases} |y_t|^\rho y_{tt} - \kappa\Delta y - c\Delta y_{tt} + c\Delta z - \displaystyle\int_0^\infty g(s)\Delta\eta(x, s, t)ds = 0, \quad in\ \Omega \times (0, \infty) \times (0, \infty),\\ z_{tt} - \Delta z - \frac{1}{c}\Delta z_{tt} + c\Delta y = 0, \quad in\ \Omega \times (0, \infty),\\ \eta_t(x, s, t) + \eta_s(x, s, t) = y_t(x, t) \quad in\ \Omega \times (0, \infty) \times (0, \infty),\\ y = z = 0, \quad on\ \Gamma \times (0, \infty),\\ y(x, 0) = y_0(x),\ z(x, 0) = z_0(x),\ y_t(x, 0) = y_1(x),\ z_t(x, 0) = z_1(x), \quad in\ \Omega,\\ y(x, -t) = f(x, t), \quad in\ \Omega \times (0, \infty), \end{cases}\tag{3.8}$$

where $\kappa = l + c^2$. The energy of solutions of (3.8) *is defined by*

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{\rho+2}\int_\Omega |y_t|^{\rho+2}dx + \frac{l}{2}\int_\Omega |\nabla y|^2 dx + \frac{c}{2}\int_\Omega |\nabla y_t|^2 dx + \frac{1}{2}\int_\Omega |z_t|^2 dx \\ &+ \frac{1}{2c}\int_\Omega |\nabla z_t|^2 dx + \frac{1}{2}\int_\Omega |c\nabla y - \nabla z|^2 dx + \int_0^\infty \int_\Omega g(s)|\nabla \eta(s)|^2 dxds.\end{aligned}$$

Define

$$\mathcal{G}(t) = M\mathcal{E}(t) + M_1 A(t) + M_2 B_1(t) + M_3 D(t),$$

where

$$B_1(t) = \int_\Omega \left(c\Delta y_t - \frac{1}{\rho+1}|y_t|^\rho y_t\right)\int_0^\infty g(s)\eta(s)dsdx.$$

Now, we suppose that g satisfies

(H3): $g \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ *satisfies* $\int_0^\infty g(s)ds > 0$ *and* $g(s) > 0, \ \forall \ s \in \mathbb{R}_+$.

(H4): *For any* $s \in \mathbb{R}_+, \ g'(s) < 0$ *and there exists two positive constants* b_0 *and* b_1 *such that*

$$-b_0 g(s) \le g'(s) \le -b_1 g(s).$$

By proceeding as in the last section, we can prove for suitable choices of M, M_1, M_2 *and* M_3 *that*

$$\mathcal{G}'(t) \le -C_2\mathcal{E}(t), \ \forall \ t \ge 0,$$

for some positive constant C_2. *Therefore, we have the following result:*

Theorem 3.3. *Assume (***H3***) and (***H4***), then the energy of solutions of* (3.8) *decays exponentially, i.e., there exists positive constants* μ *and* ζ *such that*

$$\mathcal{E}(t) \le \mu\mathcal{E}(0)e^{-\zeta t}, \ \ \forall \ t \ge 0. \tag{3.9}$$

4. Examples

In this section, we give two examples that illustrate explicit formulas for the decay rates of the energy.

(1) Let $g(t) = pe^{-k(1+t)^q}, \ t \ge 0$, where $p > 0$, $0 < q \le 1$ and $p > 0$ are chosen so that g satisfies (1.3). It holds that

$$g'(t) = -pqk(1+t)^{q-1}e^{-k(1+t)^q} = -\xi(t)g(t),$$

where $\xi(t) = qk(1+t)^{q-1}$. From (3.2), we obtain that

$$E(t) \le \beta_2 e^{-\beta_1 k(1+t)^q}, \ \forall \ t \ge 0.$$

(2) Let $g(t) = \frac{a}{(1+t)^p}$, where $p > 1$ and $a > 0$ are chosen such that (1.3) holds true. One has

$$g'(t) = \frac{-ap}{(1+t)^{p+1}} = -\xi(t)g(t),$$

where $\xi(t) = \frac{p}{1+t}$.
Therefore, it follows from (3.2) that

$$E(t) \le \frac{C}{(1+t)^p}, \ \forall \ t \ge 0.$$

5. Conclusions

This paper focused on the stability of solutions for a system of two coupled quasi-linear and linear wave equations in a bounded domain of $\mathbb{R}^n$, subject to viscoelasticity dissipative term existing only in the first equation. This system modeled the motion of two elastic membranes subject to an elastic force that pulls one membrane toward the other. As a future work, we can change the type of damping by considering, for example, structural damping (of the form Δy_t), Balakrishnan-Taylor damping (of the form $(\nabla y, \nabla y_t)\Delta y$) or strong damping (of the form $\Delta^2 y_t$).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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