AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 30591–30609.
DOI: 10.3934/math.20231564
Received: 07 August 2023
Revised: 14 September 2023
Accepted: 26 October 2023
Published: 10 November 2023

Research article

Some properties for 2-variable modified partially degenerate Hermite (MPDH) polynomials derived from differential equations and their zeros distributions

Gyung Won Hwang[1], **Cheon Seoung Ryoo**[2] **and Jung Yoog Kang**[3,*]

[1] Department of Mathematics, Dong-A University, Busan 604-714, Republic of Korea
[2] Department of Mathematics, Hannam University, Daejeon 34430, Republic of Korea
[3] Department of Mathematics Education, Silla University, Busan 46958, Republic of Korea

* **Correspondence:** Email: jykang@silla.ac.kr; Tel: +82519995581.

Abstract: The 2-variable modified partially degenerate Hermite (MPDH) polynomials are the subject of our study in this paper. We found basic properties of these polynomials and obtained several types of differential equations related to MPDH polynomials. Based on the MPDH polynomials, we looked at the structures of the approximation roots for a particular polynomial and checked the values of the approximate roots. Further, we presented some conjectures for MPDH polynomials.

Keywords: differential equations; symmetric identities; partially degenerate Hermite polynomials; complex zeros
Mathematics Subject Classification: 11B68, 11B83, 26C10, 34A30, 65D20, 65L99

1. Introduction

A solution of the following second-order differential equation

$$g'' - 2\gamma g' + 2ng = 0$$

is the ordinary Hermite polynomials $H_n(\gamma)$.

Actually, for $n \in \{0, 1, 2, \cdots\}$, the following form

$$\frac{d^2H(\gamma)}{d\gamma^2} - 2\gamma\frac{dH(\gamma)}{d\gamma} + 2nH(\gamma) = 0$$

has a solution which is the generating function of the ordinary Hermite polynomials $H_n(\gamma)$.

Let $H_n(\gamma, \nu)$ be 2-variable Hermite polynomials given by the following generating function:

$$\sum_{n=0}^{\infty} H_n(\gamma, \nu)\frac{\tau^n}{n!} = e^{\tau(\gamma+\nu\tau)} \quad \text{(see [9])}. \tag{1}$$

In other words, this polynomial provides the solution of the following heat equation:

$$\frac{\partial}{\partial \nu} H_n(\gamma, \nu) = \frac{\partial^2}{\partial \gamma^2} H_n(\gamma, \nu). \tag{2}$$

It is clear that $H_n(\gamma, 0) = \gamma^n$ and $H_n(2\gamma, -1) = H_n(\gamma)$.

Mathematicians who study polynomials have introduced and continuously developed new special polynomials that can be applied in fields such as combinatorics, numerical analysis, physics, and so on (see [1–5, 7, 12, 14]). Carlitz pioneered the concept of degenerate polynomials, and since then several mathematicians have worked to extend well-known special polynomials such as Bernoulli ([6, 15]), Euler [6] and tangent [13] to encompass their degenerate counterparts. With the discovery of new polynomials, they studied various properties and identities of polynomials, the structure of approximate roots of polynomials, differential equations and more (see [10, 11]). Young [15] showed the symmetric properties, congruence and identities and the relationship with the Stirling number for degenerate Bernoulli polynomials. Ryoo [13] confirmed the structures of approximate roots, properties of approximated real and imaginary roots and the identity for degenerate tangent polynomials. Carlitz [6] has defined the degenerate Stirling, Bernoulli and Eulerian numbers and he studied some degenerate properties of these numbers. In [8], Hwang and Ryoo introduced the new generating function of the 2-variable degenerate Hermite polynomials $\mathcal{H}_n(\gamma, \nu, \zeta)$ as follows:

$$\sum_{n=0}^{\infty} \mathcal{H}_n(\gamma, \nu, \zeta)\frac{\tau^n}{n!} = (1+\zeta)^{\frac{(\gamma+\nu\tau)\tau}{\zeta}}. \tag{3}$$

We can see that (3) reduces to (1) when $(1+\zeta)^{\frac{\tau}{\zeta}} \to e^{\tau}$ as $\zeta \to 0$. $\mathcal{H}_n(\gamma, \nu, \zeta)$ is the solution of the following partial differential equation:

$$\begin{aligned} &\frac{\partial}{\partial \nu}\mathcal{H}_n(\gamma, \nu, \zeta) = \frac{\zeta}{\log(1+\zeta)}\frac{\partial^2}{\partial \gamma^2}\mathcal{H}_n(\gamma, \nu, \zeta), \\ &\mathcal{H}_n(\gamma, 0, \zeta) = \left(\frac{\zeta}{\log(1+\zeta)}\right)^n \gamma^n. \end{aligned} \tag{4}$$

Since $\dfrac{\zeta}{\log(1+\zeta)} \to 1$ as ζ approaches to 0, it is clear that (4) becomes (2).

The purpose of this paper is to construct new modified partially degenerate Hermite (MPDH) polynomials based on the results mentioned above and find some conjectures by looking at the properties of differential equations and approximate roots related to these polynomials.

The structure of this paper is as follows. We find some properties of the newly defined MPDH polynomials in Section 2. Here, we also obtain the relationship between the Stirling numbers of the first kind and the MPDH polynomials. In Section 3, we derive the symmetric properties, which are important properties of MPDH polynomials, and Section 4 presents several differential equations with

MPDH polynomials as solutions. Section 5 introduces the structures and data of approximation roots of MPDH polynomials. We try to understand higher-order MPDH polynomials by looking at various experimental results by substituting various values of variables into the MPDH polynomials. In Section 6, we present some conjectures as a result of the paper, as well as directions for further research.

2. Basic properties for the MPDH polynomials

In this section, a new class of the MPDH polynomials are considered. Further, some properties of these polynomials are also obtained.

We define the MPDH polynomials $\mathbf{H}_n(\gamma,\nu,\zeta)$ by means of the generating function

$$\sum_{n=0}^{\infty}\mathbf{H}_n(\gamma,\nu,\zeta)\frac{\tau^n}{n!}=e^{\gamma\tau}(1+\zeta)^{\frac{\nu\tau^2}{\zeta}}. \tag{5}$$

Since $(1+\zeta)^{\frac{\nu t^2}{\zeta}}\to e^{\nu t^2}$ as $\zeta\to 0$, it is evident that (5) reduces to (1). Observe that degenerate Hermite polynomials $\mathcal{H}_n(\gamma,\nu,\zeta)$ and MPDH polynomials $\mathbf{H}_n(\gamma,\nu,\zeta)$ are completely different.

Now, we recall that the ζ-analogue of the falling factorial sequences are as follows:

$$(\gamma|\zeta)_0=1,(\gamma|\zeta)_n=\gamma(\gamma-\zeta)(\gamma-2\zeta)\cdots(\gamma-(n-1)\zeta),(n\geq 1).$$

Note that $\lim_{\zeta\to 1}(\gamma|\zeta)_n=\gamma(\gamma-1)(\gamma-2)\cdots(\gamma-(n-1))=(\gamma)_n,(n\geq 1)$. We remember that the classical Stirling numbers of the first kind $S_1(n,k)$ and the second kind $S_2(n,k)$ are defined by the relations (see [5,6,8–11,13,15])

$$(\gamma)_n=\sum_{k=0}^{n}S_1(n,k)\gamma^k \text{ and } \gamma^n=\sum_{k=0}^{n}S_2(n,k)(\gamma)_k,$$

respectively. We also have

$$\sum_{n=m}^{\infty}S_2(n,m)\frac{\tau^n}{n!}=\frac{(e^\tau-1)^m}{m!} \text{ and } \sum_{n=m}^{\infty}S_1(n,m)\frac{\tau^n}{n!}=\frac{(\log(1+\tau))^m}{m!}, \tag{6}$$

and we need the binomial theorem: For a variable ν,

$$\begin{aligned}(1+\zeta)^{\frac{\nu\tau^2}{\zeta}}&=\sum_{m=0}^{\infty}\left(\frac{\nu\tau^2}{\zeta}\right)_m\frac{\zeta^m}{m!}\\&=\sum_{l=0}^{\infty}\left(\sum_{m=l}^{\infty}S_1(m,l)\nu^l\zeta^{m-l}\frac{l!}{m!}\right)\frac{\tau^{2l}}{l!}.\end{aligned} \tag{7}$$

Note that

$$\mathcal{G}(\tau,\gamma,\nu,\zeta)=e^{\gamma\tau}(1+\zeta)^{\frac{\nu\tau^2}{\zeta}}$$

satisfies

$$\frac{\partial\mathcal{G}(\tau,\gamma,\nu,\zeta)}{\partial\tau}-\left(\gamma+2\nu\frac{\log(1+\zeta)}{\zeta}\right)\mathcal{G}(\tau,\gamma,\nu,\zeta)=0.$$

Substitute the series in (5) for $\mathcal{G}(\tau,\gamma,\nu,\zeta)$ to get

$$\mathbf{H}_{n+1}(\gamma,\nu,\zeta)-\gamma\mathbf{H}_n(\gamma,\nu,\zeta)-\frac{2\nu n\log(1+\zeta)}{\zeta}\mathbf{H}_{n-1}(\gamma,\nu,\zeta)=0, n=1,2,\ldots. \tag{8}$$

This is the recurrence relation for MPDH polynomials. Another recurrence relation comes from

$$\frac{\partial\mathcal{G}(\tau,\gamma,\nu,\zeta)}{\partial\gamma}-\tau\mathcal{G}(\tau,\gamma,\nu,\zeta)=0.$$

This implies

$$\frac{\partial\mathbf{H}_n(\gamma,\nu,\zeta)}{\partial\gamma}-n\mathbf{H}_{n-1}(\gamma,\nu,\zeta)=0, n=1,2,\ldots. \tag{9}$$

Eliminate $\mathbf{H}_{n-1}(\gamma,\nu,\zeta)$ from (8) and (9) to obtain

$$\mathbf{H}_{n+1}(\gamma,\nu,\zeta)-\gamma\mathbf{H}_n(\gamma,\nu,\zeta)-\frac{2\nu\log(1+\zeta)}{\zeta}\frac{\partial\mathbf{H}_n(\gamma,\nu,\zeta)}{\partial\gamma}=0.$$

Differentiate this equation and use (9) again to get

$$n\mathbf{H}_n(\gamma,\nu,\zeta)-\gamma\frac{\partial\mathbf{H}_n(\gamma,\nu,\zeta)}{\partial\gamma}-\frac{2\nu\log(1+\zeta)}{\zeta}\frac{\partial^2\mathbf{H}_n(\gamma,\nu,\zeta)}{\partial\gamma^2}=0, n=0,1,2,\ldots.$$

Thus the MPDH polynomials $\mathbf{H}_n(\gamma,\nu,\zeta)$ in generating function (5) are the solution of differential equation

$$\frac{2\nu\log(1+\zeta)}{\zeta}\frac{\partial^2\mathbf{H}_n(\gamma,\nu,\zeta)}{\partial\gamma^2}+\gamma\frac{\partial\mathbf{H}_n(\gamma,\nu,\zeta)}{\partial\gamma}-n\mathbf{H}_n(\gamma,\nu,\zeta)=0,$$
$$\mathbf{H}_n(\gamma,0,\zeta)=\gamma^n.$$

As another application of the differential equation for $\mathbf{H}_n(\gamma,\nu,\zeta)$, we derive

$$\frac{\log(1+\zeta)}{\zeta}\frac{\partial^2\mathbf{H}_n(\gamma,\nu,\zeta)}{\partial\gamma^2}-\frac{\partial\mathbf{H}_n(\gamma,\nu,\zeta)}{\partial\nu}=0,$$
$$\mathbf{H}_n(\gamma,0,\zeta)=\gamma^n.$$

The generating function (5) is useful for deriving several properties of the MPDH polynomials $\mathbf{H}_n(\gamma,\nu,\zeta)$.

The following basic properties of the MPDH polynomials $\mathbf{H}_n(\gamma,\nu,\zeta)$ are derived form (5). We, therefore, choose to omit the details involved.

Since (8), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbf{H}_n(\gamma,\nu,\zeta)\frac{\tau^n}{n!}&=e^{\gamma\tau}(1+\zeta)^{\frac{\nu\tau^2}{\zeta}}\\&=\sum_{k=0}^{\infty}\gamma^k\frac{\tau^k}{k!}\sum_{m=0}^{\infty}\left(\frac{\nu\tau^2}{\zeta}\right)_m\frac{\zeta^m}{m!}\\&=\sum_{k=0}^{\infty}\gamma^k\frac{\tau^k}{k!}\sum_{l=0}^{\infty}\sum_{m=l}^{\infty}S_1(m,l)\nu^l\zeta^{m-l}\frac{l!}{m!}\frac{\tau^{2l}}{l!}\\&=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}A(l,k),\end{aligned}$$

where

$$A(l,k) := \sum_{m=k}^{\infty} S_1(m,l)\nu^l \zeta^{m-l}\gamma^k \frac{\tau^{2l+k}}{m!k!}.$$

Using the well-known identity

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A(l,k) = \sum_{k=0}^{\infty}\sum_{l=0}^{[\frac{k}{2}]} A(l,k-2l),$$

we obtain

$$\sum_{n=0}^{\infty} \mathbf{H}_n(\gamma,\nu,\zeta)\frac{\tau^n}{n!} = \sum_{k=0}^{\infty}\frac{\tau^k}{k!}\sum_{l=0}^{[\frac{k}{2}]}\left(\sum_{m=l}^{\infty} S_1(m,l)\nu^l\zeta^{m-l}\gamma^{k-2l}\frac{k!}{m!(k-2l)!}\right).$$

On comparing the coefficients of $\frac{\tau^n}{n!}$, we have the following theorem.

Theorem 1. For any positive integer n, we have

$$\mathbf{H}_n(\gamma,\nu,\zeta) = \sum_{k=0}^{[\frac{n}{2}]}\left(\sum_{m=k}^{\infty} S_1(m,k)\nu^k\zeta^{m-k}\gamma^{n-2k}\frac{n!}{m!(n-2k)!}\right).$$

By (5) and Theorem 1, we have the following corollary.

Corollary 1. For any positive integer n, we have

$$\mathbf{H}_n(\gamma,\nu,\zeta) = \sum_{k=0}^{[\frac{n}{2}]}\left(\frac{\log(1+\zeta)}{\zeta}\right)^k \nu^k\gamma^{n-2k}\frac{n!}{k!(n-2k)!}.$$

Since $\lim_{\zeta\to 0}\dfrac{\log(1+\zeta)}{\zeta} = 1$, we get

$$H_n(\gamma,\nu) = n!\sum_{k=0}^{[\frac{n}{2}]}\frac{\nu^k\gamma^{n-2k}}{k!(n-2k)!}.$$

The following basic properties of the MPDH polynomials $\mathbf{H}_n(\gamma,\nu,\zeta)$ are derived from (5). We, therefore, choose to omit the details involved.

Theorem 2. For any positive integer n, we have

(1) $\mathbf{H}_n(\gamma_1+\gamma_2,\nu,\zeta) = \sum_{l=0}^{n}\binom{n}{l}\gamma_2^l\mathbf{H}_{n-l}(\gamma_1,\nu,\zeta).$

(2) $\mathbf{H}_n(\gamma,\nu_1+\nu_2,\zeta) = \sum_{k=0}^{[\frac{n}{2}]}\sum_{m=k}^{\infty}\mathbf{H}_{n-2k}(\gamma,\nu_1,\zeta)S_1(m,k)\frac{\nu_2^k\zeta^{m-k}n!}{m!(n-2k)!}.$

(3) $\mathbf{H}_n(\gamma_1+\gamma_2,\nu_1+\nu_2,\zeta) = \sum_{l=0}^{n}\binom{n}{l}\mathbf{H}_l(\gamma_1,\nu_1,\zeta)\mathbf{H}_{n-l}(\gamma_2,\nu_2,\zeta).$

3. Symmetric identities for the MPDH polynomials

In this section, we give some new symmetric identities for the MPDH polynomials. We also get some explicit formulas and properties for the MPDH polynomials.

Theorem 3. Let $w_1, w_2 > 0$ and $w_1 \neq w_2$. The following identity holds true:

$$w_1^m \mathbf{H}_m\left(w_2\gamma, w_2^2\nu, \zeta\right) = w_2^m \mathbf{H}_m\left(w_1\gamma, w_1^2\nu, \zeta\right).$$

Proof. Let $w_1, w_2 > 0$ and $w_1 \neq w_2$. We start with

$$G(\tau,\zeta) = e^{w_1 w_2 \gamma\tau}(1+\zeta)^{\frac{w_1^2 w_2^2 \nu\tau^2}{\zeta}},$$

then the expression for $G(\tau,\mu)$ is symmetric in w_1 and w_2:

$$G(\tau,\zeta) = \sum_{n=0}^{\infty} \mathbf{H}_n\left(w_1\gamma, w_1^2\nu, \zeta\right)\frac{(w_2\tau)^n}{n!} = \sum_{n=0}^{\infty} w_2^n \mathbf{H}_n\left(w_1\gamma, w_1^2\nu, \zeta\right)\frac{\tau^n}{n!}.$$

On the similar lines we can obtain that

$$G(\tau,\zeta) = \sum_{n=0}^{\infty} \mathbf{H}_n\left(w_2\gamma, w_2^2\nu, \zeta\right)\frac{(w_1\tau)^n}{n!} = \sum_{n=0}^{\infty} w_1^n \mathbf{H}_n\left(w_2\gamma, w_2^2\nu, \zeta\right)\frac{\tau^n}{n!}.$$

Comparing the coefficients of $\frac{\tau^n}{n!}$ in the last two equations, the expected result of Theorem 1 is achieved. □

For each integer $k \geq 0$, $\mathbf{S}_k(n) = 0^k + 1^k + 2^k + \cdots + (n-1)^k$ is called the sums of powers of consecutive integers. A generalized falling factorial sum $\sigma_k(n,\mu)$ can be defined by the generating function (see [5, 6, 13])

$$\sum_{k=0}^{\infty} \sigma_k(n,\zeta)\frac{\tau^k}{k!} = \frac{(1+\zeta)^{\frac{(n+1)\tau}{\zeta}} - 1}{(1+\zeta)^{\frac{\tau}{\zeta}} - 1}.$$

Note that $\lim_{\zeta\to 0} \sigma_k(n,\zeta) = \mathbf{S}_k(n)$. For $\mu \in \mathbb{C}$, we defined the degenerate Bernoulli polynomials given by the generating function

$$\sum_{n=0}^{\infty} \beta_n(\gamma,\zeta)\frac{\tau^n}{n!} = \frac{\tau}{(1+\zeta)^{\frac{\tau}{\zeta}} - 1}(1+\zeta)^{\frac{\gamma\tau}{\zeta}}.$$

When $\gamma = 0$ and $\beta_n(\zeta) = \beta_n(0,\zeta)$ are called the modified degenerate Bernoulli numbers, note that

$$\lim_{\zeta\to 0} \beta_n(\zeta) = \mathbf{B}_n,$$

where $\mathbf{B}_n$ are called the Bernoulli numbers (see [6, 7]).

The first few of them are

$$\beta_0(\zeta) = \frac{\zeta}{\log(1+\zeta)}, \quad \beta_1(\zeta) = -\frac{1}{2}, \quad \beta_2(\zeta) = \frac{\log(1+\zeta)}{6\zeta}, \quad \beta_3(\zeta) = 0,$$
$$\beta_4(\zeta) = -\frac{1}{30}\left(\frac{\log(1+\zeta)}{\zeta}\right)^3, \quad \beta_5(\zeta) = 0, \quad \beta_6(\zeta) = \frac{1}{42}\left(\frac{\log(1+\zeta)}{\zeta}\right)^5.$$

Again, we now use

$$F(\tau,\mu) = \frac{w_1 w_2 \tau e^{w_1 w_2 \gamma \tau}(1+\zeta)^{\frac{w_1^2 w_2^2 \nu \tau^2}{\zeta}}\left((1+\zeta)^{\frac{w_1 w_2 \tau}{\zeta}} - 1\right)}{\left((1+\zeta)^{\frac{w_1 \tau}{\zeta}} - 1\right)\left((1+\zeta)^{\frac{w_2 \tau}{\zeta}} - 1\right)}.$$

From $F(\tau,\mu)$, we get the following result:

$$\begin{aligned}
F(\tau,\mu) &= \frac{w_1 w_2 \tau e^{w_1 w_2 \gamma \tau}(1+\zeta)^{\frac{w_1^2 w_2^2 \nu \tau^2}{\zeta}}\left((1+\zeta)^{\frac{w_1 w_2 \tau}{\zeta}} - 1\right)}{\left((1+\zeta)^{\frac{w_1 \tau}{\zeta}} - 1\right)\left((1+\zeta)^{\frac{w_2 \tau}{\zeta}} - 1\right)} \\
&= \frac{w_1 w_2 \tau}{\left((1+\zeta)^{\frac{w_1 \tau}{\zeta}} - 1\right)} e^{w_1 w_2 \gamma \tau}(1+\zeta)^{\frac{w_1^2 w_2^2 \nu \tau^2}{\zeta}} \frac{\left((1+\zeta)^{\frac{w_1 w_2 \tau}{\zeta}} - 1\right)}{\left((1+\zeta)^{\frac{w_2 \tau}{\zeta}} - 1\right)} \\
&= w_2 \sum_{n=0}^{\infty} \beta_n(\zeta) \frac{(w_1\tau)^n}{n!} \sum_{n=0}^{\infty} \mathbf{H}_n\left(w_2\gamma, w_2^2\nu, \zeta\right) \frac{(w_1\tau)^n}{n!} \sum_{n=0}^{\infty} \sigma_k(w_1 - 1, \zeta) \frac{(w_2\tau)^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \sum_{m=0}^{i} \binom{n}{i}\binom{i}{m} w_1^i w_2^{n+1-i} \beta_m(\zeta) \mathbf{H}_{i-m}\left(w_2\gamma, w_2^2\nu, \zeta\right) \sigma_{n-i}(w_1 - 1, \zeta) \right) \frac{\tau^n}{n!}.
\end{aligned}$$

In a similar fashion, we have

$$F(\tau,\zeta)=\frac{w_1w_2\tau}{\left((1+\zeta)^{\frac{w_2\tau}{\zeta}}-1\right)}e^{w_1w_2\gamma\tau}(1+\zeta)^{\frac{w_1^2w_2^2\nu\tau^2}{\zeta}}\frac{\left((1+\zeta)^{\frac{w_1w_2\tau}{\zeta}}-1\right)}{\left((1+\zeta)^{\frac{w_1\tau}{\zeta}}-1\right)}$$
$$=w_1\sum_{n=0}^{\infty}\beta_n(\zeta)\frac{(w_2\tau)^n}{n!}\sum_{n=0}^{\infty}\mathbf{H}_n\left(w_1\gamma,w_1^2\nu,\zeta\right)\frac{(w_2\tau)^n}{n!}\sum_{n=0}^{\infty}\sigma_k(w_2-1,\zeta)\frac{(w_1\tau)^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}w_2^iw_1^{n+1-i}\beta_m(\zeta)\mathbf{H}_{i-m}\left(w_1\gamma,w_1^2\nu,\zeta\right)\sigma_{n-i}(w_2-1,\zeta)\right)\frac{\tau^n}{n!}.$$

By comparing the coefficients of $\frac{\tau^n}{n!}$ on the right-hand sides of the last two equations, we have the below theorem.

Theorem 4. Let $w_1,w_2>0$ and $w_1\neq w_2$, then the following identity holds true:

$$\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}w_1^iw_2^{n+1-i}\beta_m\left(\frac{\mu}{w_1}\right)\mathbf{H}_{i-m}\left(w_2\gamma,w_2^2\nu,\zeta\right)\sigma_{n-i}(w_1-1,\zeta)$$
$$=\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}w_2^iw_1^{n+1-i}\beta_m(\zeta)\mathbf{H}_{i-m}\left(w_1\gamma,w_1^2\nu,\zeta\right)\sigma_{n-i}(w_2-1,\zeta).$$

By taking the limit as $\mu\to 0$, we have the following corollary.

Corollary 2. Let $w_1,w_2>0$ and $w_1\neq w_2$, then the following identity holds true:

$$\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}w_1^iw_2^{n+1-i}\mathbf{B}_mH_{i-m}(w_2\gamma,w_2^2\nu)\mathbf{S}_{n-i}(w_1-1)$$
$$=\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}w_2^iw_1^{n+1-i}\mathbf{B}_mH_{i-m}(w_1\gamma,w_1^2\nu)\mathbf{S}_{n-i}(w_2-1).$$

4. Differential equations associated with MPDH polynomials

In this section, we construct the differential equations with coefficients $a_i(N,\gamma,\nu,\zeta)$ arising from the generating functions of the MPDH polynomials:

$$\left(\frac{\partial}{\partial\tau}\right)^N\mathcal{G}(t,\gamma,\nu,\zeta)-a_0(N,\gamma,\nu,\zeta)\mathcal{G}(\tau,\gamma,\nu,\zeta)-\cdots-a_N(N,\gamma,\nu,\zeta)\tau^N\mathcal{G}(\tau,\gamma,\nu,\zeta)=0.$$

By using the coefficients of this differential equation, we can derive explicit identities for the 2-variable MPDH polynomials $\mathbf{H}_n(\gamma,\nu,\zeta)$. Recall that

$$\mathcal{G}=\mathcal{G}(\tau,\gamma,\nu,\zeta)=e^{\gamma\tau}(1+\zeta)^{\frac{\nu\tau^2}{\zeta}}=\sum_{n=0}^{\infty}\mathbf{H}_n(\gamma,\nu,\zeta)\frac{\tau^n}{n!},\quad \zeta,\gamma,\nu,\tau\in\mathbb{C}.\tag{10}$$

Then, by (10) we have

$$\mathcal{G}^{(1)} = \frac{\partial}{\partial\tau}\mathcal{G}(\tau,\gamma,\nu,\zeta) = \frac{\partial}{\partial\tau}\left(e^{\gamma\tau}(1+\zeta)^{\dfrac{\nu\tau^2}{\zeta}}\right) = \left(\gamma + \frac{\log(1+\zeta)}{\zeta}2\nu\tau\right)e^{\gamma\tau}(1+\zeta)^{\dfrac{\nu\tau^2}{\zeta}}$$
$$= (\gamma)\mathcal{G}(\tau,\gamma,\nu,\zeta) + \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)\tau\mathcal{G}(\tau,\gamma,\nu,\zeta) \tag{11}$$

$$\mathcal{G}^{(2)} = \frac{\partial}{\partial\tau}\mathcal{G}^{(1)}(\tau,\gamma,\nu,\zeta) = \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)\mathcal{G}(\tau,\gamma,\nu,\zeta) + \left((\gamma+2\nu\tau)\frac{\log(1+\zeta)}{\zeta}\right)\mathcal{G}^{(1)}(\tau,\gamma,\nu,\zeta)$$
$$= \left(\gamma^2 + 2\nu\frac{\log(1+\zeta)}{\zeta}\right)\mathcal{G}(\tau,\gamma,\nu,\zeta) + \left(4\gamma\nu\frac{\log(1+\zeta)}{\zeta}\right)\tau\mathcal{G}(\tau,\gamma,\nu,\zeta) \tag{12}$$
$$+ \left(4\nu^2\left(\frac{\log(1+\zeta)}{\zeta}\right)^2\right)\tau^2\mathcal{G}(t,\gamma,\nu,\zeta).$$

Continuing this process as shown in (12), we can guess that

$$\mathcal{G}^{(N)} = \left(\frac{\partial}{\partial\tau}\right)^N \mathcal{G}(\tau,\gamma,\nu,\zeta) = \sum_{i=0}^{N} a_i(N,\gamma,\nu,\zeta)\tau^i\mathcal{G}(\tau,\gamma,\nu,\zeta), (N = 0,1,2,\ldots). \tag{13}$$

Differentiating (13) with respect to τ, we have

$$\mathcal{G}^{(N+1)} = \frac{\partial\mathcal{G}^{(N)}}{\partial\tau} = \sum_{i=0}^{N} a_i(N,\gamma,\nu,\zeta)(i)\tau^{i-1}\mathcal{G}(\tau,\gamma,\nu,\zeta) + \sum_{i=0}^{N} a_i(N,\gamma,\nu,\zeta)\tau^i\mathcal{G}^{(1)}(\tau,\gamma,\nu,\zeta)$$
$$= \sum_{i=0}^{N}(i)a_i(N,\gamma,\nu,\zeta)\tau^{i-1}\mathcal{G}(\tau,\gamma,\nu,\zeta) + \sum_{i=0}^{N}(\gamma)a_i(N,\gamma,\nu,\zeta)\tau^i\mathcal{G}(\tau,\gamma,\nu,\zeta)$$
$$+ \sum_{i=0}^{N}\left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)a_i(N,\gamma,\nu,\zeta)\tau^{i+1}\mathcal{G}(\tau,\gamma,\nu,\zeta) \tag{14}$$
$$= \sum_{i=0}^{N-1}(i+1)a_{i+1}(N,\gamma,\nu,\zeta)\tau^i\mathcal{G}(\tau,\gamma,\nu,\zeta) + \sum_{i=0}^{N}(\gamma)a_i(N,\gamma,\nu,\zeta)\tau^i\mathcal{G}(\tau,\gamma,\nu,\zeta)$$
$$+ \sum_{i=1}^{N+1}\left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)a_{i-1}(N,\gamma,\nu,\zeta)\tau^i\mathcal{G}(\tau,\gamma,\nu,\zeta).$$

Now, replacing N by $N+1$ in (13), we find

$$\mathcal{G}^{(N+1)} = \sum_{i=0}^{N+1} a_i(N+1,\gamma,\nu,\zeta)\tau^i\mathcal{G}(\tau,\gamma,\nu,\zeta). \tag{15}$$

Comparing the coefficients on both sides of (14) and (15), we obtain

$$a_0(N+1,\gamma,\nu,\zeta) = a_1(N,\gamma,\nu,\zeta) + (\gamma)a_0(N,\gamma,\nu,\zeta). \tag{16}$$

For $1 \le i \le N-1$, we obtain

$$a_i(N+1,\gamma,\nu,\zeta) = (i+1)a_{i+1}(N,\gamma,\nu,\zeta) + (\gamma)a_i(N,\gamma,\nu,\zeta) + \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)a_{i-1}(N,\gamma,\nu,\zeta). \tag{17}$$

For $i = N$, we obtain

$$a_N(N+1,\gamma,\nu,\zeta) = (\gamma)a_N(N,\gamma,\nu,\zeta) + \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)a_{N-1}(N,\gamma,\nu,\zeta). \tag{18}$$

For $i = N+1$, we obtain

$$a_{N+1}(N+1,\gamma,\nu,\zeta) = \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)a_N(N,\gamma,\nu,\zeta). \tag{19}$$

In addition, by (13) we have

$$\mathcal{G}(\tau,\gamma,\nu,\zeta) = \mathcal{G}^{(0)}(\tau,\gamma,\nu,\zeta) = a_0(0,\gamma,\nu,\zeta)\mathcal{G}(\tau,\gamma,\nu,\zeta). \tag{20}$$

By (20), we get

$$a_0(0,\gamma,\nu,\zeta) = 1. \tag{21}$$

It is not difficult to show that

$$\begin{aligned}(\gamma)\mathcal{G}(\tau,\gamma,\nu,\zeta) + \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)\tau\mathcal{G}(\tau,\gamma,\nu,\zeta) = \mathcal{G}^{(1)}(\tau,\gamma,\nu,\zeta) &= \sum_{i=0}^{2} a_i(1,\gamma,\nu,\zeta)\tau^i\mathcal{G}(\tau,\gamma,\nu,\zeta)\\ &= a_0(1,\gamma,\nu,\zeta)\mathcal{G}(\tau,\gamma,\nu,\zeta) + a_1(1,\gamma,\nu,\zeta)\tau\mathcal{G}(\tau,\gamma,\nu,\zeta).\end{aligned} \tag{22}$$

Thus, by (10) and (22), we also get

$$a_0(1,\gamma,\nu,\zeta) = \gamma, \quad a_1(1,\gamma,\nu,\zeta) = 2\nu\frac{\log(1+\zeta)}{\zeta}. \tag{23}$$

From (16), we note that

$$\begin{aligned}&a_0(N+1,\gamma,\nu,\zeta) = a_1(N,\gamma,\nu,\zeta) + (\gamma)a_0(N,\gamma,\nu,\zeta),\\ &a_0(N,\gamma,\nu,\zeta) = a_1(N-1,\gamma,\nu,\zeta) + (\gamma)a_0(N-1,\gamma,\nu,\zeta),\\ &\ldots,\\ &a_0(N+1,\gamma,\nu,\zeta) = \sum_{j=0}^{N-1}\gamma^j a_1(N-j,\gamma,\nu,\zeta) + \gamma^{N+1}.\end{aligned} \tag{24}$$

For $1 \le i \le N-1$, from (17) we note that

$$\begin{aligned}&a_i(N+1,\gamma,\nu,\zeta) = (i+1)a_{i+1}(N,\gamma,\nu,\zeta) + (\gamma)a_i(N,\gamma,\nu,\zeta) + \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)a_{i-1}(N,\gamma,\nu,\zeta),\\ &a_i(N,\gamma,\nu,\zeta) = (i+1)a_{i+1}(N-1,\gamma,\nu,\zeta) + (\gamma)a_i(N-1,\gamma,\nu,\zeta)\\ &\qquad + \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)a_{i-1}(N-1,\gamma,\nu,\zeta),\\ &\ldots,\\ &a_i(N+1,\gamma,\nu,\zeta) = (i+1)\sum_{j=0}^{N}\gamma^j a_{i+1}(N-j,\gamma,\nu,\zeta) + \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)\sum_{j=0}^{N}\gamma^j a_{i-1}(N-j,\gamma,\nu,\zeta).\end{aligned} \tag{25}$$

From (18), we have

$$\begin{aligned} a_N(N+1,\gamma,\nu,\zeta) &= (\gamma)a_N(N,\gamma,\nu,\zeta) + \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)a_{N-1}(N,\gamma,\nu,\zeta),\\ a_{N-1}(N,\gamma,\nu,\zeta) &= (\gamma)a_{N-1}(N-1,\gamma,\nu,\zeta) + \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)a_{N-2}(N-1,\gamma,\nu,\zeta),\dots,\\ a_N(N+1,\gamma,\nu,\zeta) &= \gamma\sum_{j=0}^{N-1}\left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)^j a_{N-i}(N-i,\gamma,\nu,\zeta) + \gamma\left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)^N. \end{aligned} \tag{26}$$

Again, by (19) we have

$$\begin{aligned} a_{N+1}(N+1,\gamma,\nu,\zeta) &= \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)a_N(N,\gamma,\nu,\zeta),\\ a_N(N,\gamma,\nu,\zeta) &= \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)a_{N-1}(N-1,\gamma,\nu,\zeta),\dots,\\ a_{N+1}(N+1,\gamma,\nu,\zeta) &= \left(2\nu\frac{\log(1+\zeta)}{\zeta}\right)^{N+1}. \end{aligned} \tag{27}$$

Note that here, the matrix $a_i(j,\gamma,\nu,\zeta)_{0\le i\le N+1,0\le j\le N+1}$ is given by

$$\begin{pmatrix} 1 & \gamma & \gamma^2+\dfrac{2\nu\log(1+\zeta)}{\zeta} & \cdots & \cdot\\ 0 & \dfrac{2\nu\log(1+\zeta)}{\zeta} & \dfrac{4\gamma\nu\log(1+\zeta)}{\zeta} & \cdot\ \cdots & \cdot\\ 0 & 0 & \left(2\nu\dfrac{\log(1+\zeta)}{\zeta}\right)^2 & \cdot\ \cdots & \cdot\\ 0 & 0 & 0 & \cdot\ \cdots & \cdot\\ 0 & 0 & 0 & \cdot\ \cdots & \cdot\\ 0 & 0 & 0 & 0\ \cdots & \cdot\\ \vdots & \vdots & \vdots & \vdots\ \ddots & \cdot\\ 0 & 0 & 0 & 0\ \cdots & \left(2\nu\dfrac{\log(1+\zeta)}{\zeta}\right)^{N+1} \end{pmatrix}$$

Therefore, by (24)–(27), we obtain the following theorem.

Theorem 5. For $N=0,1,2,\dots$, the differential equation

$$\left(\frac{\partial}{\partial\tau}\right)^N\mathcal{G}(\tau,\gamma,\nu,\zeta) - \sum_{i=0}^{N}a_i(N,\gamma,\nu,\zeta)\tau^i\mathcal{G}(\tau,\gamma,\nu,\zeta) = 0$$

has a solution of

$$\mathcal{G} = \mathcal{G}(\tau, \gamma, \nu, \zeta) = e^{\gamma\tau}(1+\zeta)^{\frac{\nu\tau^2}{\zeta}},$$

where

$$a_0(N+1, \gamma, \nu, \zeta) = \sum_{j=0}^{N-1} \gamma^j a_1(N-j, \gamma, \nu, \zeta) + \gamma^{N+1},$$

$$a_N(N+1, \gamma, \nu, \zeta) = \gamma \sum_{j=0}^{N-1} \left(2\nu \frac{\log(1+\zeta)}{\zeta}\right)^j a_{N-i}(N-i, \gamma, \nu, \zeta) + \gamma \left(2\nu \frac{\log(1+\zeta)}{\zeta}\right)^N,$$

$$a_{N+1}(N+1, \gamma, \nu, \zeta) = \left(2\nu \frac{\log(1+\zeta)}{\zeta}\right)^{N+1},$$

$$a_i(N+1, \gamma, \nu, \zeta) = (i+1) \sum_{j=0}^{N} \gamma^j a_{i+1}(N-j, \gamma, \nu, \zeta)$$

$$+ \left(2\nu \frac{\log(1+\zeta)}{\zeta}\right) \sum_{j=0}^{N} \gamma^j a_{i-1}(N-j, \gamma, \nu, \zeta), (1 \le i \le N-1).$$

Here is a plot of the surface for this solution.

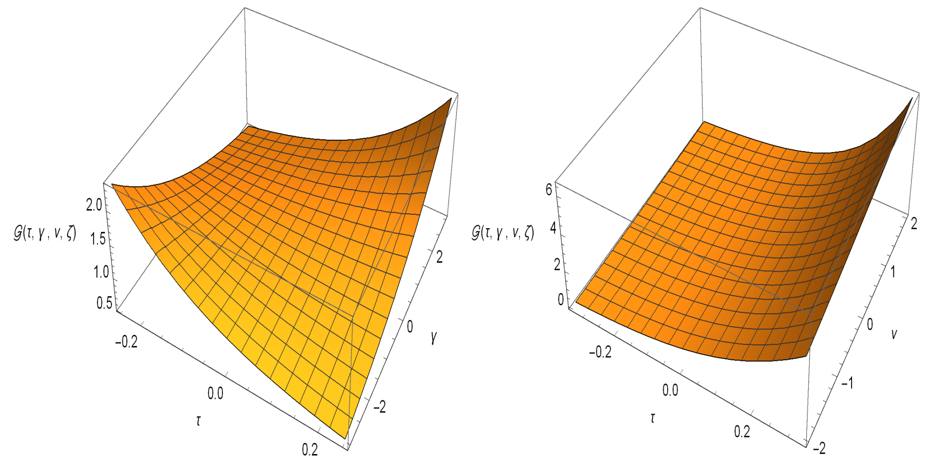


Figure 1. The surface for the solution $\mathcal{G}(\tau, \gamma, \nu, \zeta)$.

In the left picture of Figure 1, we choose $-3 \le \gamma \le 3, -\frac{1}{4} \le \tau \le \frac{1}{4}, \zeta = 1/3$, and $\nu = 2$. In the right picture of Figure 1, we choose $-1 \le \nu \le 1, -\frac{1}{3} \le \tau \le \frac{1}{3}, \zeta = 1/3$, and $\gamma = 5$.

Making N-times derivative for (8) with respect to τ, we have

$$\left(\frac{\partial}{\partial\tau}\right)^N \mathcal{G}(\tau, \gamma, \nu, \zeta) = \sum_{m=0}^{\infty} \mathbf{H}_{m+N}(\gamma, \nu, \zeta) \frac{t^m}{m!}. \tag{28}$$

By the Cauchy product and by multiplying the exponential series $e^{\gamma\tau} = \sum_{m=0}^{\infty} \gamma^m \frac{\tau^m}{m!}$ in both sides of (28), we get

$$\begin{aligned} e^{-n\frac{\log(1+\zeta)}{\zeta}\tau} \left(\frac{\partial}{\partial\tau}\right)^N \mathcal{G}(\tau,\gamma,\nu,\zeta) &= \left(\sum_{m=0}^{\infty} \left(-n\frac{\log(1+\zeta)}{\zeta}\right)^m \frac{\tau^m}{m!}\right)\left(\sum_{m=0}^{\infty} \mathbf{H}_{m+N}(\gamma,\nu,\zeta)\frac{\tau^m}{m!}\right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k}\left(-n\frac{\log(1+\zeta)}{\zeta}\right)^{m-k} \mathbf{H}_{N+k}(\gamma,\nu,\zeta)\right)\frac{\tau^m}{m!}. \end{aligned} \tag{29}$$

By the Leibniz rule and inverse relation, we have

$$\begin{aligned} & e^{-n\frac{\log(1+\zeta)}{\zeta}\tau} \left(\frac{\partial}{\partial\tau}\right)^N \mathcal{G}(\tau,\gamma,\nu,\zeta) \\ &= \sum_{k=0}^{N} \binom{N}{k}\left(n\frac{\log(1+\zeta)}{\zeta}\right)^{N-k} \left(\frac{\partial}{\partial\tau}\right)^k \left(e^{-n\frac{\log(1+\zeta)}{\zeta}t} \mathcal{G}(\tau,\gamma,\nu,\zeta)\right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{N} \binom{N}{k}\left(n\frac{\log(1+\zeta)}{\zeta}\right)^{N-k} \mathbf{H}_{m+k}\left(\gamma - n\frac{\log(1+\zeta)}{\zeta},\nu,\zeta\right)\right)\frac{\tau^m}{m!}. \end{aligned} \tag{30}$$

Hence, by (29) and (30) and comparing the coefficients of $\frac{\tau^m}{m!}$, we get the following theorem.

Theorem 6. Let m, n, N be nonnegative integers, then

$$\begin{aligned} & \sum_{k=0}^{m} \binom{m}{k}(-n)^{m-k}\left(\frac{\log(1+\zeta)}{\zeta}\right)^{m-k} \mathbf{H}_{N+k}(\gamma,\nu,\zeta) \\ &= \sum_{k=0}^{N} \binom{N}{k} n^{N-k}\left(\frac{\log(1+\zeta)}{\zeta}\right)^{N-k} \mathbf{H}_{m+k}\left(\gamma - n\frac{\log(1+\zeta)}{\zeta},\nu,\zeta\right). \end{aligned} \tag{31}$$

If we take $m = 0$ in (31), then we have the following corollary.

Corollary 3. For $N = 0, 1, 2, \ldots,$ we have

$$\mathbf{H}_N(\gamma,\nu,\zeta) = \sum_{k=0}^{N} \binom{N}{k} n^{N-k}\left(\frac{\log(1+\zeta)}{\zeta}\right)^{N-k} \mathbf{H}_k\left(\gamma - n\frac{\log(1+\zeta)}{\zeta},\nu,\zeta\right).$$

By (26) and Theorem 6, we have

$$\begin{aligned} & a_0(N,\gamma,\nu,\zeta)\mathcal{G}(\tau,\gamma,\nu,\zeta) + a_1(N,\gamma,\nu,\zeta)\tau\mathcal{G}(\tau,\gamma,\nu,\zeta) \\ & + \cdots \\ & + a_{2N-1}(N,\gamma,\nu,\zeta)\tau^{N-1}\mathcal{G}(\tau,\gamma,\nu,\nu) + a_{2N}(N,\gamma,\nu,\zeta)\tau^N\mathcal{G}(\tau,\gamma,\nu,\zeta) \\ & = \sum_{m=0}^{\infty} \mathbf{H}_{m+N}(\gamma,\nu,\zeta)\frac{\tau^m}{m!}. \end{aligned}$$

Hence, we have the following theorem.

Theorem 7. For $N = 0, 1, 2, \ldots$, we get

$$\mathbf{H}_{N+m}(\gamma, \nu, \mu) = \sum_{i=0}^{m} \frac{\mathbf{H}_{m-i}(\gamma, \nu, \mu) a_i(N, \gamma, \nu, \mu) m!}{(m-i)!}. \tag{32}$$

If we take $m = 0$ in (32), then we have the below corollary.

Corollary 4. For $N = 0, 1, 2, \ldots$, we have

$$\mathbf{H}_N(\gamma, \nu, \mu) = a_0(N, \gamma, \nu, \mu)\mathbf{H}_0(\gamma, \nu, \mu) = a_0(N, \gamma, \nu, \mu),$$

where

$$a_0(0, \gamma, \nu, \mu) = 1,$$
$$a_0(N+1, \gamma, \nu, \mu) = \sum_{i=0}^{N} \gamma^i a_1(N-i, \gamma, \nu, \mu) + \gamma^{N+1}.$$

Hence, we have the following theorem.

Theorem 8. For $N = 0, 1, 2, \ldots$ and $0 \leq m \leq 2N$, we get

$$\sum_{k=0}^{[\frac{m}{2}]} (N)_k \zeta^k \mathbf{H}_{N+m-2k}(\gamma, \nu, \zeta) \frac{m!}{(m-2k)!k!} = \sum_{i=0}^{m} \frac{\mathbf{H}_{m-i}(\gamma, \nu, \zeta) a_i(N, \gamma, \nu, \zeta) m!}{(m-i)!}. \tag{33}$$

If we take $m = 0$ in (33), then we have the below corollary.

Corollary 5. For $N = 0, 1, 2, \ldots$, we have

$$\mathbf{H}_N(\gamma, \nu, \zeta) = a_0(N, \gamma, \nu, \zeta)\mathbf{H}_0(\gamma, \nu, \zeta) = a_0(N, \gamma, \nu, \zeta),$$

where

$$a_0(0, \gamma, \nu, \zeta) = 1,$$
$$a_0(N+1, \gamma, \nu, \zeta) = \sum_{j=0}^{N-1} \gamma^j a_1(N-j, \gamma, \nu, \zeta) + \gamma^{N+1}.$$

The first few of them are

$$\mathbf{H}_0(\gamma, \nu, \zeta) = 1,$$
$$\mathbf{H}_1(\gamma, \nu, \zeta) = \gamma,$$
$$\mathbf{H}_2(\gamma, \nu, \zeta) = \gamma^2 + \frac{2\nu \log(1+\zeta)}{\zeta},$$
$$\mathbf{H}_3(\gamma, \nu, \zeta) = \gamma^3 + \frac{6\gamma\nu \log(1+\zeta)}{\zeta},$$
$$\mathbf{H}_4(\gamma, \nu, \zeta) = \gamma^4 + \frac{12\gamma^2\nu \log(1+\zeta)}{\zeta} + 12\nu^2 \left(\frac{\log(1+\zeta)}{\zeta}\right)^2,$$
$$\mathbf{H}_5(\gamma, \nu, \zeta) = \gamma^5 + \frac{20\gamma^3\nu \log(1+\zeta)}{\zeta} + 60\gamma\nu^2 \left(\frac{\log(1+\zeta)}{\zeta}\right)^2,$$

$$\mathbf{H}_6(\gamma,\nu,\zeta) = \gamma^6 + \frac{30\gamma^4\nu\log(1+\zeta)}{\zeta} + 180\gamma^2\nu^2\left(\frac{\log(1+\zeta)}{\zeta}\right)^2 + 120\nu^3\left(\frac{\log(1+\zeta)}{\zeta}\right)^3.$$

5. Zeros of the MPDH polynomials

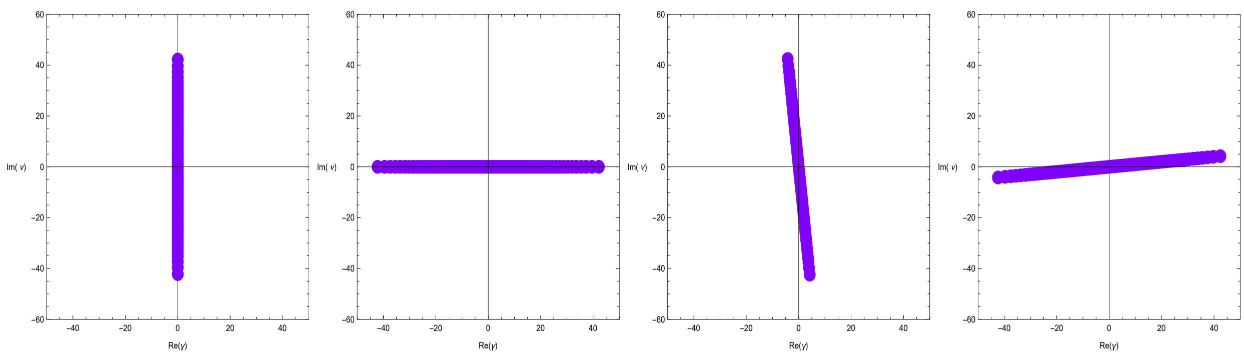
This section shows the benefits of supporting theoretical prediction through numerical experiments and finding new interesting pattern of the zeros of the MPDH equations $\mathbf{H}_n(\gamma,\nu,\zeta) = 0$. By using a computer, the MPDH polynomials $\mathbf{H}_n(\gamma,\nu,\zeta)$ can be determined explicitly. We investigate the zeros of the MPDH equations $\mathbf{H}_n(\gamma,\nu,\zeta) = 0$. The zeros of the $\mathbf{H}_n(\gamma,\nu,\zeta) = 0$ for $n = 60, \nu = 5, -5, 5+i, -5-i, \zeta = 1/3$ and $\gamma \in \mathbb{C}$ are displayed in Figure 2.

In Figure 2(a), we choose $n = 50, \zeta = 1/3$ and $\nu = 5$. In Figure 2(b), we choose $n = 50, \zeta = 1/3$ and $\nu = -5$. In Figure 2(c), we choose $n = 50, \zeta = 1/3$ and $\nu = 5+i$. In Figure 2(d), we choose $n = 50, \zeta = 1/3$ and $\nu = -5-i$.

Stacks of zeros of the MPDH equations $\mathbf{H}_n(\gamma,\nu,\zeta) = 0$ for $1 \le n \le 50, \zeta = 1/3$ from a 3D structure are presented in Figure 3.

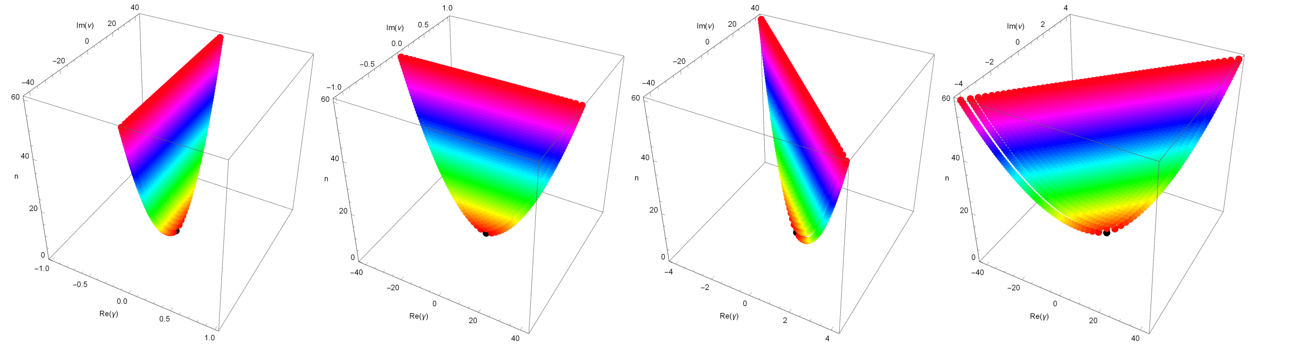
In Figure 3(a), we choose $\nu = 5$. In Figure 3(b), we choose $\nu = -5$. In Figure 3(c), we choose $\nu = 5+i$. In Figure 3(d), we choose $\nu = -5-i$.

Our numerical results for the approximate solutions of real zeros of the MPDH equations $\mathbf{H}_n(\gamma,\nu,\zeta) = 0$ are displayed as Tables 1 and 2.



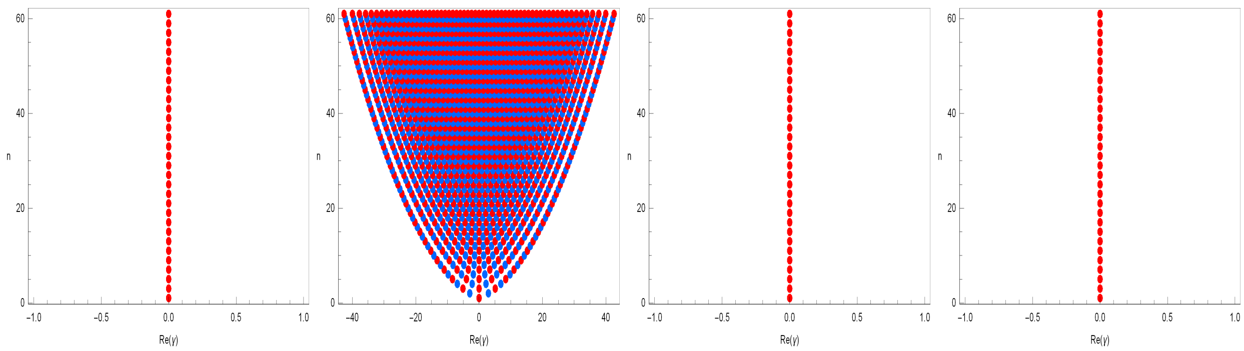
(a) (b) (c) (d)

Figure 2. Zeros of $\mathbf{H}_n(\gamma,\nu,\zeta) = 0$.



(**a**) (**b**) (**c**) (**d**)

Figure 3. Stacks of zeros of $\mathbf{H}_n(\gamma, \nu, \zeta) = 0, \quad 1 \le n \le 60$.



(**a**) (**b**) (**c**) (**d**)

Figure 4. Real zeros of $\mathbf{H}_n(\gamma, \nu, \zeta) = 0$ for $1 \le n \le 60$.

Table 1. Numbers of real and complex zeros of $\mathbf{H}_n(\gamma, \nu, \zeta) = 0$.

	$\nu = 5, \zeta = 1/3$		$\nu = -5, \zeta = 1/3$	
degree n	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	0	2	2	0
3	1	2	3	0
4	0	4	4	0
5	1	4	5	0
6	0	6	6	0
7	1	6	7	0
8	0	8	8	0
9	1	8	9	0
10	0	10	10	0
11	1	10	11	0
12	0	12	12	0

We observed a remarkable regular structure of the complex roots of the MPDH equations

$\mathbf{H}_n(\gamma, \nu, \zeta) = 0$, and also hope to verify the same kind of regular structure of the complex roots of the MPDH equations $\mathbf{H}_n(\gamma, \nu, \zeta) = 0$ (Table 1).

Plot of real zeros of the MPDH equations $\mathbf{H}_n(\gamma, \nu, \zeta) = 0$ for $1 \leq n \leq 50, \zeta = 1/3$ structure are presented in Figure 4.

In Figure 4(a), we choose $\nu = 5$. In Figure 4(b), we choose $\nu = -5$. In Figure 4(c), we choose $\nu = 5 + i$. In Figure 4(d), we choose $\nu = -5 - i$.

Next, we calculated an approximate solution satisfying $\mathbf{H}_n(\gamma, \nu, \zeta) = 0, \gamma \in \mathbb{C}$. The results are given in Table 2. In Table 2, we choose $\nu = -5$ and $\zeta = 1/3$.

Table 2. Approximate solutions of $\mathbf{H}_n(\gamma, \nu, \zeta) = 0, \gamma \in \mathbb{R}$.

degree n	γ
1	0
2	−2.9378, 2.9378
3	−5.0884, 0, 5.0884
4	−6.8580, −2.1797, 2.1797, 6.8580
5	−8.3931, −3.9825, 0, 3.9825, 8.3931
6	−9.7659, −5.5500, −1.8117, 1.8117
	5.5500, 9.7659
7	−11.018, −6.9530, −3.3914, 0, 3.3914
	6.9530, 11.018
8	−12.176, −8.2330, −4.8077, −1.5837
	1.5837, 4.8077, 8.2330, 12.176

6. Observations

In this article, we introduced the MPDH polynomials and got new symmetric identities for MPDH polynomials. We derived the symmetric property, one of the important properties of MPDH polynomials. We have shown several types of differential equations with $\mathbf{H}_m(\gamma, \nu, \zeta)$ as their solution. We also observed the symmetric properties of the roots of $\mathbf{H}_m(\gamma, \nu, \zeta) = 0$, which appeared differently as the values of the variables γ and ν changed. As a result, it was found that the distribution of the roots of $\mathbf{H}_m(\gamma, \nu, \zeta) = 0$ had a very regular pattern, and through numerical experiments we found the following conjectures are possible.

Here, we use the notation as follows.

(i) $R_{\mathbf{H}_m(\gamma,\nu,\zeta)}$: the number of real zeros of $\mathbf{H}_m(\gamma, \nu, \zeta) = 0$ lying on the real plane $Im(\gamma) = 0$,

(ii) $C_{\mathbf{H}_m(\gamma,\nu,\zeta)}$: the number of complex zeros of $\mathbf{H}_m(\gamma, \nu, \zeta) = 0$.

(iii) $\mathbb{C}$: the set of complex numbers.

Since m is the degree of the polynomial $\mathbf{H}_m(\gamma, \nu, \zeta)$, we have $R_{\mathbf{H}_m(\gamma,\nu,\zeta)} = m - C_{\mathbf{H}_m(\gamma,\nu,\zeta)}$.

We realized the regular pattern of the complex roots of the MPDH equations $\mathbf{H}_n(\gamma, \nu, \zeta) = 0$ related to ν and ζ. We made these conjectures. Proving or disproving the following conjectures will be our future task.

Conjecture 1. Let m be an odd positive integer and $b > 0$.

$$R_{\mathbf{H}_m(\gamma,b,\zeta)} = 1, \quad C_{\mathbf{H}_m(\gamma,b,\zeta)} = 2\left[\frac{m}{2}\right].$$

Conjecture 2. For $b < 0$,

$$R_{\mathbf{H}_m(\gamma,b,\zeta)} = m, \quad C_{\mathbf{H}_m(\gamma,b,\zeta)} = 0.$$

Conjecture 3. Let m be odd positive integer and $b \in \mathbb{C}$.

$$\mathbf{H}_m(0, b, \zeta) = 0.$$

When we study more ν and ζ variables, it is still unsolved that the Conjectures 1 and 2 are true or false for all variables ν and ζ.

We observe that solutions of the MPDH equations $\mathbf{H}_m(\gamma, b, \zeta) = 0$ have $Re(\gamma) = 0$ reflection symmetry for $b \in \mathbb{R}$. We guess that solutions of the MPDH equations $\mathbf{H}_m(\gamma, b, \zeta) = 0$ do not have $Re(\gamma) = b$ reflection symmetry for $b \in \mathbb{C} \setminus \mathbb{R}$ (see Figures 2–4).

Conjecture 4. The zeros of $\mathbf{H}_m(\gamma, a, \zeta) = 0$, $a \in \mathbb{R}$ have $Im(\gamma) = 0$ reflection symmetry analytic complex functions. The zeros of $\mathbf{H}_m(\gamma, a, \zeta) = 0, a \in \mathbb{C} \setminus \mathbb{R}$ do not have $Im(\gamma) = 0$ reflection symmetry analytic complex functions.

Finally, how many zeros does $\mathbf{H}_m(\gamma, \nu, \zeta) = 0$ have? We cannot determine whether $\mathbf{H}_m(\gamma, \nu, \zeta) = 0$ has m distinct solutions. We want to know the number of complex zeros $C_{\mathbf{H}_m(\gamma,\nu,\zeta)}$ of $\mathbf{H}_m(\gamma, \nu, \zeta) = 0$.

Conjecture 5. For $b \in \mathbb{C}$, $\mathbf{H}_m(\gamma, b, \zeta) = 0$ has m distinct solutions.

When we study more m variables, it is still unsolved whether the conjecture is true or false for all variables m (see Tables 1 and 2).

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Acknowledgments

This work was supported by the Dong-A University research fund.

Conflict of interest

The authors declare no conflict of interest.

References

1. L. C. Andrews, *Special Functions for Engineers and Applied Mathematicians*, New York: Macmillan, 1985.
2. P. Appell, J. K. de Fériet, *Fonctions Hypergéométriques et Hypersphériques: Polynomes d Hermite*, Paris: Gauthier-Villars, 1926.
3. N. Alam, W. A. Khan, S. Araci, H. N. Zaidi, A. Al-Taleb, Evaluation of the Poly-Jindalrae and Poly-Gaenari Polynomials in Terms of Degenerate Functions, *Symmetry*, **15** (2023), 1587. https://doi.org/10.3390/sym15081587
4. G. E. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge: Cambridge University Press, 1999.
5. G. Arfken, *Mathematical Methods for Physicists*, 3 Eds., Orlando: Academic Press, 1985.
6. L. Carlitz, Degenerate Stiling, Bernoulli and Eulerian numbers, *Util. Math.*, **15** (1979), 51–88.
7. A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions Volume 3*, New York: Krieger, 1981.
8. K. W. Hwang, C. S. Ryoo, Differential equations associated with two variable degenerate Hermite polynomials, *Mathematics*, **8** (2020), 228. http://doi.org/10.3390/math8020228
9. K. W. Hwang, C. S. Ryoo, Some Identities Involving Two-Variable Partially Degenerate Hermite Polynomials Induced from Differential Equations and Structure of Their Roots, *Mathematics*, **7** (2020), 632. https://doi.org/10.3390/math8040632
10. C. S. Ryoo, *Some Identities Involving 2-Variable Modified Degenerate Hermite Polynomials Arising from Differential Equations and Distribution of Their Zeros*, London: IntechOpen, 2020. http://doi.org/10.5772/intechopen.92687
11. C. S. Ryoo, J. Y. Kang, Some Identities Involving Degenerate q-Hermite Polynomials Arising from Differential Equations and Distribution of Their Zeros, *Symmetry*, **14** (2022), 706. https://doi.org/10.3390/sym14040706
12. C. S. Ryoo, Some identities involving Hermite Kampé de Fériet polynomials arising from differential equations and location of their zeros, *Mathematics*, **7** (2019), 23. http://doi.org/10.3390/math7010023
13. C. S. Ryoo, Notes on degenerate tangent polynomials, *Global J. Pure Appl. Math.*, **11** (2015), 3631–3637.
14. C. S. Ryoo, R. P. Agarwal, J. Y. Kang, Differential equations arising from Bell-Carlitz polynomials and computation of their zeros, *Neural Parallel Sci. Comput.*, **24** (2016), 453–462.
15. P. T. Young, Degenerate Bernoulli polynomials, generalized factorial sums, and their applications, *J. Number Theory*, **128** (2008), 738–758.

AIMS Press