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Research article

Stability analysis of Abel's equation of the first kind

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Abstract: This paper established sufficient conditions for the stability of Abel's differential equation of the first kind. These conditions explicate the impact of the asymptotic behaviors exhibited by the time-varying coefficients on the overall stability of the system. More precisely, we studied the positivity, the continuation and the boundedness of solutions. Additionally, we investigated the attractivity, the asymptotic stability, the uniform stability and the instability of the system. The results were scrutinized by numerical simulations.

Keywords: Abel's equation; attractivity; asymptotic stability; uniform stability; boundedness of solutions
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1. Introduction

In the presence of an additional cubic nonlinearity, the Abel's first-order differential equation bears resemblance to Riccati equations and, consequently, it has the form

$$\dot{y}(t) = a(t)\,y^3(t) + b(t)\,y^2(t) + c(t)\,y(t) + d(t), \tag{1.1}$$

where $t \geq t_0 \in \mathbb{R}$, the solution is $y(t) \in \mathbb{R}$, and the continuous real functions are $a, b, c, d \in C^0(\mathbb{R}, \mathbb{R})$. It is noteworthy that when d is uniformly zero and exactly one of the functions a and b is uniformly zero as well, (1.1) transforms into a Bernoulli equation.

The system's widespread importance originates from its significance in numerous applications in physics [13, 17], fluids [19], control theory [28], finance [33], cosmology [8], cancer therapy [10], biology [9] and the M-theory [37]. Owing to this relevance, numerous mathematical characteristics of the system's states have been studied in the literature. For instance, various analytical solutions have been obtained in [16, 18, 29–31] under restricted conditions. Due to the nonlinearity of the system, it is arduous to generalize these restricted forms. Therefore, many researchers have chosen to numerically solve the system as in [3–5, 32]. The presence of limit cycles in certain classes of the system has also been conducted in [14, 27].

Stability analysis is an essential approach across a wide spectrum of fields within control theory [2, 6, 24, 35] and the qualitative theory of differential equations [11, 26, 34]. This analysis opens the routes for additional knowledge on the behavior patterns of many real-world nonlinear systems [12, 15, 20, 21, 36]. Despite the aforementioned importance of stability analysis in the various fields, there is no existing study in the literature; according to our knowledge, that considers the stability of the celebrated equation (1.1). Therefore, we have dedicated this study to explore that topic. To delve further into the specifics, we focus on the positivity and boundedness of solutions and study some asymptotic behaviors, including uniform stability, attractivity, asymptotic stability and instability of the system in both its homogeneous and nonhomogeneous forms. For the prior notions, we have obtained precise conditions that are related to the signs and asymptotic behaviors of the time-varying coefficients. To demonstrate the proposed results, numerical simulations have been conducted.

The paper is organized as follows. A compilation of mathematical results obtained from the literature has been presented in section two. Conditions for the positivity of solutions have been introduced in section three. The case when d is uniformly zero has been considered in section four in which the instability and the asymptotic stability are investigated. Section five provides sufficient conditions for the origin attractivity and the state convergence of the system. The conclusion section is included at the end of the paper.

2. Background results

For a Lebesgue measurable function $q : \mathbb{R}_+ \to \mathbb{R}^m$, let $\|q\|_\infty$ be the essential supremum of $|q|$ on $\mathbb{R}_+$ where $|\cdot|$ is the Euclidean distance [1]. A strictly increasing function $\gamma \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ is of class $\mathcal{K}$ if $\gamma(0) = 0$. It belongs to class $\mathcal{K}_\infty$ when we have $\lim_{s\to\infty} \gamma(s) = \infty$. Consider the n-dimensional differential equation $\dot{u}(t) = f(t, u(t))$, $t \geq t_0$. We assume the continuity of the function f. Furthermore, we assume that $f(t, 0) = 0$ for every $t \geq t_0$ (hence, the origin is an equilibrium point). The origin is uniformly stable if there is some γ of class $\mathcal{K}$ and a positive number c; that is independent of t_0, such that for every initial value with $|u_0| < c$, each solution is continuable on $[t_0, \infty)$ and $|u(t)| \leq \gamma(|u_0|)$ for all $t \geq t_0$. The origin is locally attractive if for every $t_0 \in \mathbb{R}$, there is some $c > 0$; that may depend on t_0, such that for every $|u_0| < c$, each solution is continuable on $[t_0, \infty)$ with $\lim_{t\to\infty} u(t) = 0$. If the prior conditions are satisfied for every $u_0 \in \mathbb{R}^m$, the origin is globally attractive. On the other hand, the zero solution is asymptotically stable if it is stable and attractive [2].

In the presence of unbounded perturbations, the asymptotic stability of differential equations-based systems is studied in the next theorem.

Theorem 2.1. *[22, Theorem 6.2] Consider the first order class of differential equations $\dot{u}(t) = f(t, u(t))$, where $t \geq t_0$, $u(t) \in \mathbb{R}^m$ is the state and $f : [t_0, \infty) \times \mathbb{R}^m \to \mathbb{R}^m$ is a well-defined function that satisfies $f(\cdot, 0) = 0$. We suppose that all of the classical Carathéodory conditions are satisfied (thus, there exists a solution that is locally absolutely continuous [2, Section 1.1]). Furthermore, we assume that*

(1). Two constants α, β exist such that $\beta > \alpha > 0$ and $(-1)^\alpha = -1$. Moreover, there exist continuous

real functions $Q \in C^0(\mathbb{R},\mathbb{R})$, $E \in C^0(\mathbb{R},\mathbb{R})$ *such that* $Q(\cdot) > 0$, $E(\cdot) > 0$, $\lim_{t\to\infty} \frac{E(t)}{Q(t)} = \infty$ *and a Lebesgue measurable function* $\mu : \mathbb{R} \to \mathbb{R}$ *that satisfies* $\int_{t_0}^{\infty} Q(t)\mu(t)\,dt = \infty$.

(2). $\lim_{t\to\infty} \Lambda(t) = 0$, *where*

$$\Lambda(t) = \frac{Q(t)\dot{E}(t) - \dot{Q}(t)E(t)}{\mu(t)(Q(t))^{\frac{2\beta-\alpha-1}{\beta-\alpha}}(E(t))^{\frac{\beta-2\alpha+1}{\beta-\alpha}}}, \text{ for almost all } t \in (t_0,\infty). \tag{2.1}$$

(3). For each solution $u : [t_0,\omega) \to \mathbb{R}^m$ *(*$[t_0,\omega)$ *is maximal interval of existence), there exist positive constants* σ, c_1, c_2, δ *and a continuously differential function* $V \in C^1(\mathbb{R}\times\mathbb{R}^m,\mathbb{R}_+)$, *satisfying*

$$c_1|\alpha|^\sigma \le V(t,\alpha) \le c_2|\alpha|^\sigma, \text{ for all } t \in \mathbb{R} \text{ and all } \alpha \in \mathbb{R}^m, \tag{2.2}$$

$$\begin{aligned} \left.\frac{\partial V(t,\eta)}{\partial \eta}\right|_{\eta=u(t)} \cdot f(t,u(t)) \quad &+ \quad \left.\frac{\partial V(t,\eta)}{\partial t}\right|_{\eta=u(t)} \\ &\le \left(-Q(t)V^\alpha(t,u(t)) + E(t)V^\beta(t,u(t))\right)\mu(t), \end{aligned} \tag{2.3}$$

for almost all $t \in (t_0,\omega)$ *with* $V(t,u(t)) < \delta$.

Then, there exists $r > 0$ *such that for any* $|y(t_0)| < r$, *each solution* $y(t)$ *is continuable on* $[t_0,\infty)$ *and is bounded with* $|y(t)| \le \sqrt[\sigma]{\frac{c_2}{c_1}}\,|y(t_0)|$ *for every* $t \in [t_0,\infty)$ *(so that the origin is uniformly stable). Additionally, the origin exhibits asymptotic stability.*

Lemma 2.1. *[22, Theorem 6.1] In addition to the results of the previous theorem, the mapping* $t \to V(t,u(t))$ *is monotonically decreasing.*

The perturbation term $E(\cdot)$ in the inequality (2.3) is considered unbounded because it is assumed that $\lim_{t\to\infty} \frac{E(t)}{Q(t)} = \infty$. In the next theorem, we consider bounded perturbations.

Theorem 2.2. *[23, Theorem 2.1] For the differential equation* $\dot{u}(t) = E(t) - Q(t)\beta(u(t))$, *where* $t \ge t_0$, $u \in \mathbb{R}$ *is the output,* $\beta \in C^0(\mathbb{R},\mathbb{R})$ *exhibits strict increase with* $\beta(0) = 0$, *and* e, $q \in C^0(\mathbb{R},\mathbb{R}_+)$ *are continuous real-valued functions, if* $Q(\cdot) > 0$, $\int_{t_0}^{\infty} Q(t)\,dt = \infty$ *and* $\lim_{t\to\infty} \frac{E(t)}{Q(t)} = l \in$ *Range* $\{\beta\}$. *Then for every* $u(t_0) \ge 0$, *each solution* $u(t)$ *is global with* $\lim_{t\to\infty} u(t) = \beta^{-1}(l)$.

An invariance principle derived from reference [25] is now being presented in the next proposition.

Proposition 2.1. *[25, Proposition 3.2] For some* $\omega \in (t_0,\infty]$, *let* $v : [t_0,\omega) \to \mathbb{R}_+$ *be a nonnegative locally absolutely continuous function and let* $g_1 \in C^0([t_0,\infty),\mathbb{R}_+)$, $g_2 \in C^0([t_0,\infty),\mathbb{R}_+)$ *be nonnegative continuous functions that satisfy the following*

(1). $v(t_0) < g_2(t_0)$.

(2). $g_1(t) < g_2(t)$, *for every* $t \ge t_0$.

(3). One has

$$\dot{v}(t) \le \dot{g}_2(t) \text{ for almost all } t \in (t_0,\omega) \text{ with } g_1(t) < v(t) < g_2(t). \tag{2.4}$$

Then $v(t) < g_2(t)$, *for all* $t \in [t_0,\omega)$.

3. Positivity of solutions

Since all of the functions a, b, c and d are continuous, the existence of a continuously differentiable solution of (1.1) is guaranteed. This solution has a maximal interval of existence of the form $[t_0, \omega)$ where ω can be infinite [7].

Now, we introduce several conditions for the positiveness of solutions.

Proposition 3.1. *If $d(t) > 0$ for every $t > t_0$, then for each $y(t_0) \geq 0$, the positivity of solutions is guaranteed; i.e. $y(t) \geq 0$ for every $t \in [t_0, \omega)$.*

Proof. By the Eq (1.1), we deduce that

$$\dot{y}(t) = d(t) \text{ for every } t \in (t_0, \omega) \text{ that satisfies } y(t) = 0,$$

and, thus, the fact that $d(\cdot) > 0$ gives

$$\dot{y}(t) > 0 \text{ for every } t \in (t_0, \omega) \text{ that satisfies } y(t) = 0. \tag{3.1}$$

Let $y(t_0) \geq 0$. Suppose that there exists $t_1 \in [t_0, \omega)$ such that $y(t_1) < 0$, then $t_0 < t_1$ because $y(t_0) \geq 0$. Consider the set $A = \{\gamma \in [t_0, t_1)/y(t) \geq 0, \forall t \in [t_0, \gamma]\}$. A is nonempty because $t_0 \in A$. Let $t_2 = \sup(A)$. We have $t_0 \leq t_2 < t_1$ because $y(t_1) < 0$. The definition of t_2 ensures the existence of two sequences $\{t_m < t_2\}_{m=1}^{\infty}$ and $\{t'_m > t_2\}_{m=1}^{\infty}$, satisfying $\lim_{m\to\infty} t'_m = \lim_{m\to\infty} t_m = t_2$, $y(t_m) \geq 0$ and $y(t'_m) < 0$ for each positive integer m. Thus, the continuity of the state y gives $y(t_2) = \lim_{n\to\infty} y(t_n) \geq 0 \geq \lim_{n\to\infty} y(t'_n) = y(t_2)$ so that $y(t_2) = 0$. Let $B = \{t \in [t_2, t_1)/y(t) = 0\}$. The set B is nonempty because $t_2 \in B$. $t_3 := \sup B$. One can easily verify that $y(t_3) = 0$ and $t_2 \leq t_3 < t_1$. We claim that $y(t) < 0$ for all $t \in (t_3, t_1)$. To prove this claim, let us assume that there exists some $\tau \in (t_3, t_1)$ such that $y(\tau) \geq 0$. By the definition of t_3, we have $y(\tau) > 0$. Since $y(\tau) > 0 > y(t_1)$, one can see by the intermediate value theorem, the continuous function y has a root in the interval (τ, t_1) so that there is some constant $\tau' \in (\tau, t_1)$ with $y(\tau') = 0$. This contradicts the facts that $t_3 = \sup B$, $y(t_3) = 0$ and ends the proof of our claim that states that $y(t) < 0$ for all $t \in (t_3, t_1)$. Therefore, we have $y(t_3 + h) < 0$ for all $h \in (0, t_1 - t_3)$ so that the result $y(t_3) = 0$ gives $\frac{y(t_3+h)-y(t_3)}{h} < 0$ for all $h \in (0, t_1 - t_3)$. This implies that $\dot{y}_+(t_3) = \dot{y}(t_3) \leq 0$, which contradicts the inequality (3.1) and ends the proof of the proposition. □

Simulation 1. *Pick $t_0 = 0$. For every $t \geq 0$, let $a(t) = \cos(t)$, $b(t) = \sin(t)$, $c(t) = -t$ and $d(t) = \sin(t) + 2$. Since $d(t) > 0$ for all $t > t_0$, for each $y(t_0) \geq 0$ the positivity of solutions is guaranteed by Proposition 3.1. These solutions can be global or nonglobal. Figure 1 illustrates the solutions $y(t)$ for the initial values $y(0) = 1$, $y(0) = 2$, $y(0) = 3$, $y(0) = 4$ and $y(0) = 5$ where the positiveness of solutions can be easily observed.*

Proposition 3.2. *Suppose that $d(t) \geq 0$ for every $t \geq t_0$. If one of the following sets of conditions is satisfied*

(i) $a(t) \leq 0$, $b(t) \geq 0$, $c(t) \leq 0$ *for all* $t \in [t_0, \infty)$ *and* $y(t_0) \geq 0$.
(ii) $a(t) \geq 0$, $b(t) < 0$, $c(t) > 0$ *for all* $t \in [t_0, \infty)$ *and* $y(t_0) > 0$.
(iii) $a(t) < 0$, $b(t) \geq 0$, $c(t) \geq 0$ *for all* $t \in [t_0, \infty)$ *and* $y(t_0) > 0$. *Furthermore, suppose that at least one of the functions b and c is nonzero for each $t \in [t_0, \infty)$.*

Then, the positivity of solutions is guaranteed.

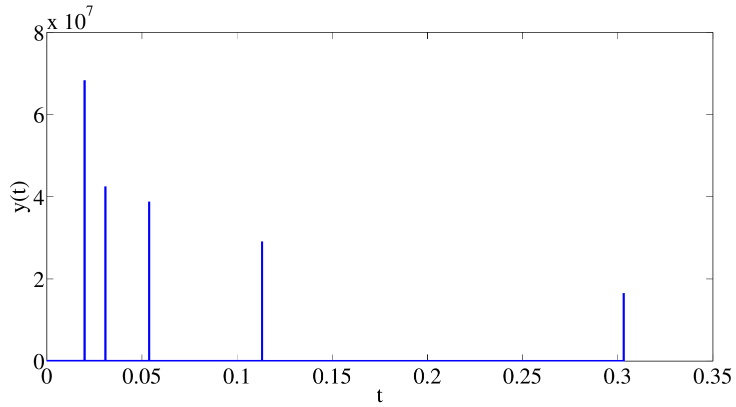

Figure 1. $y(t)$ versus t for the different initial values $y(0) = 1$, $y(0) = 2$, $y(0) = 3$, $y(0) = 4$ and $y(0) = 5$. Observe that all solutions are nonglobal and positive.

Proof. We show each case individually.

Proof for the set of conditions (i). Let $y(t_0) \geq 0$. Let us suppose that for some $t_* \in [t_0, \omega)$, we have $y(t_*) < 0$. Thus, $t_* \in (t_0, \omega)$ because $y(t_0) > 0$. Let $A = \{t \in [t_0, t_*) / y(t) = 0\}$. A is nonempty because y is continuous, $t_0 < t_*$, $y(t_0) \geq 0$ and $y(t_*) < 0$. Let $t_{**} = \sup(A)$. We have $t_{**} \in A$ so that $y(t_{**}) = 0$ because y is continuous. Furthermore, one can show that $y(t) < 0$ for every $t \in (t_{**}, t_*)$. Thus, we obtain by (1.1) and the facts that $a(\cdot) \leq 0$, $b(\cdot) \geq 0$ and $c(\cdot) \leq 0$ that $\dot{y}(t) \geq 0$ for every $t \in (t_{**}, t_*)$. Hence, $y(t_{**}) \leq y(t_*)$. This contradicts the results $y(t_*) < 0$ and $y(t_{**}) = 0$. □

Proof for the set of conditions (ii). Let $y(t_0) > 0$. Since $a(\cdot) \geq 0$, $b(\cdot) < 0$, $c(\cdot) > 0$ and $d(\cdot) \geq 0$, we obtain by (1.1) that

$$\dot{y}(t) \geq b(t)\, y^2(t) + c(t)\, y(t), \text{ for every } t \in (t_0, \omega) \text{ that satisfies } y(t) > 0,$$

so that

$$\dot{y}(t) \geq 0, \text{ for all } t \in (t_0, \omega) \text{ that satisfies } 0 < y(t) < \frac{c(t)}{|b(t)|}. \tag{3.2}$$

Suppose that there exists some $t_* \in [t_0, \omega)$ with $y(t_*) < 0$. We get $t_* \in (t_0, \omega)$ because $y(t_0) > 0$. One can use the facts that $y(t_*) < 0$, $y(t_0) > 0$ along with the intermediate value theorem to show that the set $S = \{t \in (t_0, t_*) : y(t) = 0\}$ is nonempty. The continuity of the solution y gives $t_{**} := \inf S \in S$ that is $y(t_{**}) = 0$ and $t_0 < t_{**}$. Additionally, it is easy to show that $y(t) > 0$ for every $t \in (t_0, t_{**})$. Thus, since $\frac{c(\cdot)}{|b(\cdot)|} > 0$, $y(t_{**}) = 0$ and the mapping $t \to \frac{c(t)}{|b(t)|}$ is continuous, one can prove the existence of some $t^* \in [t_0, t_{**})$ such that $0 < y(t) < \frac{c(t)}{|b(t)|}$ for all $t \in (t^*, t_{**})$. We get $y(t^*) > 0$ by the definition of t_{**}. Hence, we get by the inequality (3.2) that $\dot{y}(t) \geq 0$ for all $t \in (t^*, t_{**})$ so that $y(t_{**}) \geq y(t^*)$, which is a contradiction because $y(t_{**}) = 0$ and $y(t^*) > 0$. □

Proof for the set of conditions (iii). Pick $y(t_0) > 0$. Assume, without loss of generality, that $b(\cdot) > 0$. We deduce by (1.1) that (noting that it is assumed that $a(\cdot) < 0$):

$$\dot{y}(t) \geq a(t)\, y^3(t) + b(t)\, y^2(t), \text{ for every } t \in (t_0, \omega) \text{ that satisfies } y(t) > 0,$$

so that

$$\dot{y}(t) \geq 0, \text{ for all } t \in (t_0, \omega) \text{ that satisfies } 0 < y(t) \leq \frac{b(t)}{|a(t)|}. \tag{3.3}$$

As in the proof of Case (*ii*), we can show the existence of some instants t^* and t_{**} such that $t_0 \le t^* < t_{**}$, $y(t_{**}) = 0$, $y(t^*) > 0$ and $0 < y(t) < \frac{b(t)}{|a(t)|}$, for all $t \in (t^*, t_{**})$. Thus, a contradiction can be concluded by (3.3). □

□

4. Asymptotic behavior of the system associated with unbounded perturbations: Homogeneous form analysis

This section focuses on studying the case when the function $d(t)$ is uniformly zero. In this case the origin $y = 0$ is a rest point for (1.1).

Applying the Lyapunov stability technique on the system under study in its homogeneous form may lead to a differential Lyapunov inequality with an unbounded perturbation. The next theorem proves that; even with the existence of unbounded perturbations, the system is still able to exhibit notions like uniform stability, asymptotic stability and instability depending on the asymptotic properties of the coefficient functions.

Theorem 4.1. *We consider the following two distinct results:*

(i) Suppose that $d(t) = 0$, $c(t) < 0$ for each $t \ge t_0$ and $\int_{t_0}^{\infty} c(t)\,dt = -\infty$. Suppose that for every $t > t_0$, at least one of $a(t)$ and $b(t)$ is nonzero. Define the function $\lambda : [t_0, \infty) \to \mathbb{R}_+$ as $\lambda(t) = \max(|a(t)|, |b(t)|)$ for each $t \ge t_0$. Assume that $\lim_{t\to\infty} \frac{\lambda(t)}{|c(t)|} = \infty$ and

$$\lim_{t\to\infty} \frac{c(t)\,\dot{\lambda}(t) - \dot{c}(t)\,\lambda(t)}{c^2(t)\,\lambda(t)} = 0. \tag{4.1}$$

Then, there is some $r > 0$ such that when $|y(t_0)| < r$, each solution $y(t)$ is continuable on $[t_0, \infty)$. Besides, the mapping $t \to |y(t)|$ is monotonically decreasing with $|y(t)| < |y(t_0)|$ for every $t > t_0$ (and, thus, $y = 0$ is uniformly stable). Furthermore, the equilibrium point $y = 0$ exhibits asymptotic stability.

(ii) If $d(t) = b(t) = 0$, $c(t) > 0$, $a(t) < 0$ for each $t > t_0$, $\int_{t_0}^{\infty} a(t)\,dt = -\infty$ and $\lim_{t\to\infty} \frac{c(t)}{a(t)} = -\infty$, then for each $y(t_0) \in \mathbb{R}$, each nontrivial solution $y(t)$ is global with $\lim_{t\to\infty} |y(t)| = \infty$ (observe that in this case, (1.1) reduces to a Bernoulli equation).

Proof. Since for every $t > t_0$ at least one of $a(t)$ and $b(t)$ is nonzero, we have $\lambda(t) > 0$ for each $t > t_0$. Let $V(t) = y^2(t)$ for each $t \in [t_0, \omega)$. We show each case individually.

Proof of (i). For all $t \in (t_0, \omega)$, system (1.1) gives

$$\begin{aligned} \dot{V}(t) &= 2c(t)\,y^2(t) + 2b(t)\,y^3(t) + 2a(t)\,y^4(t) \\ &\le 2c(t)\,V(t) + 2\,|b(t)|\,V^{\frac{3}{2}}(t) + 2\,|a(t)|\,V^2(t) \\ &\le 2c(t)\,V(t) + 2\max(|b(t)|, |a(t)|)\left(V^{\frac{3}{2}}(t) + V^2(t)\right). \end{aligned}$$

Thus, we get

$$\dot{V}(t) \le 2c(t)\,V(t) + 4\lambda(t)\,V^{\frac{3}{2}}(t) \text{ for every } t \in (t_0, \omega) \text{ with } V(t) < 1. \tag{4.2}$$

Therefore, inequality (2.3) is satisfied with $Q(\cdot) = -2c(\cdot)$, $E(\cdot) = 4\lambda(\cdot)$, $\alpha = 1$, $\beta = \frac{3}{2}$, $\delta = 1$ and μ is the identity function. Observe that $0 < \alpha < \beta$ and that $Q(\cdot) > 0$, $E(\cdot) > 0$. Since $V(\cdot) = y^2(\cdot)$, inequality (2.2) is satisfied with $c_1 = c_2 = 1$ and $\sigma = 2$. The function Λ defined in (2.1) satisfies

$$\Lambda(t) = \frac{\dot{c}(t)\lambda(t) - c(t)\dot{\lambda}(t)}{2\lambda(t)c^2(t)}, \text{ for almost all } t > t_0.$$

Thus, we conclude by (4.1) that $\lim_{t\to\infty}\Lambda(t) = 0$. On the other hand, $\lim_{t\to\infty}\frac{E(t)}{Q(t)} = \infty$ because it is assumed that $\lim_{t\to\infty}\frac{\lambda(t)}{|c(t)|} = \infty$ and $\int_{t_0}^{\infty} Q(t)\mu(t)\,dt = \infty$ (note that $\int_{t_0}^{\infty} c(t)\,dt = -\infty$). Therefore, Theorem 2.1 ensures that when the initial value is small enough, each solution $y(t)$ is continuable on $[t_0,\infty)$. Moreover, $y = 0$ is uniformly stable and is asymptotically stable. In addition, Lemma 2.1 implies that the mapping $t \to |V(t)|$ is strictly decreasing and, hence, $t \to |y(t)|$ exhibits the same behavior because $V = y^2$. □

Proof of (ii). Consider an initial condition $y(t_0) \in \mathbb{R}$. Let $y(t)$ be a nontrivial solution of (1.1) and let $[t_0,\omega)$ be the maximal interval of existence. To prove $\omega = \infty$, assume that $\omega < \infty$ and $\lim_{t\to\omega-}|y(t)| = \infty$. Consider the Lyapunov function $V = y^2$. We have $\dot{V}(t) = 2v(t)\dot{v}(t)$ and (1.1) gives (observe that b and d are uniformly zero)

$$\dot{V}(t) = 2a(t)V^2(t) + 2c(t)V(t), \text{ for all } t \in (t_0,\omega). \tag{4.3}$$

Since $\lim_{t\to\omega-}|y(t)| = \infty$, we have $\lim_{t\to\omega-}V(t) = \infty$. Thus, one can utilize the fact that $\omega < \infty$ to prove that there exists some $t' \in (t_0,\omega)$ with $V(t) > -\frac{c(t)}{a(t)}$ for all $t \in (t',\omega)$ (note that $-\frac{c(\cdot)}{a(\cdot)} > 0$). Therefore, (4.3) yields $\dot{V}(t) < 0$ for every $t \in (t',\omega)$, which is a contradiction because $\lim_{t\to\omega-}V(t) = \infty$. Thus, $\omega = \infty$. Now, we need to prove that $\lim_{t\to\infty}|y(t)| = \infty$. To this end, define $\Psi \in C^0([t_0,\infty),\mathbb{R}_+)$ as $\Psi(t) = -\frac{c(t)}{2a(t)}$ for all $t \geq t_0$. Observe that $\Psi(\cdot) > 0$, and it is assumed that $\lim_{t\to\infty}\Psi(t) = \infty$. We obtain by (4.3) that

$$\dot{V}(t) \geq -2a(t)V^2(t) \text{ for almost all } t > t_0 \text{ with } V(t) \leq \Psi(t). \tag{4.4}$$

Next, let us consider the following three subcases.

Subcase (ii-1): There exists $t_* > t_0$ in a way that $V(t) \geq \Psi(t)$ for each $t \geq t_*$.
Since $\lim_{t\to\infty}\Psi(t) = \infty$ and $V(\cdot) \geq \Psi(\cdot)$, the result $\lim_{t\to\infty}V(t) = \infty$ comes true so that $\lim_{t\to\infty}|y(t)| = \infty$.

Subcase (ii-2): There exists $t_* > t_0$ in a way that $V(t) \leq \Psi(t)$ for every $t \geq t_*$.
We conclude by (4.4) and the fact that $a(\cdot) < 0$ that $\dot{V}(t) \geq 0$ for almost all $t > t_*$ so that V is nondecreasing. Since $y(t)$ is not the trivial solution, the Lyapunov function is not uniformly zero. Thus, there exists $t^* > t_0$ such that $V(t^*) > 0$ and, hence, $V(t) \geq V(t^*)$ for all $t \geq t^*$. Thus, (4.4) gives $\dot{V}(t) \geq -2a(t)V^2(t^*)$ for all $t > t^*$ (noting that $a(\cdot) < 0$). Therefore, the Fundamental Theorem of Calculus leads to

$$V(t) = V(t^*) + \int_{t^*}^{t}\dot{V}(\tau)\,d\tau \geq V(t^*) - 2V^2(t^*)\int_{t^*}^{t} a(\tau)\,d\tau, \forall t \geq t^*.$$

Thus, we get $\lim_{t\to\infty}V(t) = \lim_{t\to\infty}|y(t)| = \infty$ because $\int_{t_0}^{\infty} a(t)\,dt = -\infty$.

Subcase (ii-3): Both Subcases (ii-1) and (ii-2) are incorrect.
Given $\varepsilon > 0$, the fact that $\lim_{t\to\infty} \Psi(t) = \infty$ ensures the existence of $T_0 > t_0$ such that $\Psi(t) > \varepsilon$ for every $t \geq T_0$. In accordance with the present subcase, there must be $\{t_m \geq T_0\}_{m=1}^{\infty}$ and $\{t'_m \geq T_0\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty} t_m = \lim_{m\to\infty} t'_m = \infty$ and for each positive integer m, we have

$$V(t_m) < \Psi(t_m) \text{ and } V(t'_m) > \Psi(t'_m).$$

This guarantees the presence of two numbers $t_1 > T_0$ and $t_2 > T_0$ satisfying $t_1 < t_2$, $\Psi(t_1) < V(t_1)$ and $\Psi(t_2) > V(t_2)$. By applying the intermediate value theorem on the continuous function $\chi := V - \Psi$ and the compact interval $[t_1, t_2]$, we conclude that there exists some $T_1 \in (t_1, t_2)$ such that $\chi(T_1) = 0$ so that $\Psi(T_1) = V(T_1)$.

We claim that $V(t) > \varepsilon$ for every $t > T_1$. To prove it, we use the contradiction technique and assume that there is some $T_2 > T_1$ with $V(T_2) \leq \varepsilon$. Since $\Psi(T_1) = V(T_1)$, the set $S = \{t \in [T_1, T_2) / \Psi(t) = V(t)\}$ is nonempty. Thus, the continuity of V and Ψ imply that $T_3 := \sup S \in S$ (note that $T_3 < T_2$ because $V(T_2) \leq \varepsilon$ and $\Psi(\cdot) > \varepsilon$ on $[T_0, \infty)$). Therefore, one can verify that $V(t) < \Psi(t)$ for all $t \in (T_3, T_2)$. Thus, we get by (4.4) and the fact that $a(\cdot) < 0$ that $\dot{V}(t) \geq 0$ for all $t \in (T_3, T_2)$. This means V is nondecreasing on (T_3, T_2) and, hence, $V(T_2) \geq V(T_3)$. Thus, the facts that $\Psi(t) > \varepsilon$, for every $t \geq T_0$, $T_3 \in S$ and $V(T_2) \leq \varepsilon$ give the contradicted statement $\Psi(T_3) > \varepsilon \geq V(T_2) \geq V(T_3) = \Psi(T_3)$. This finishes the proof of our claim, which states that $\lim_{t\to\infty} V(t) = \infty$ so that $\lim_{t\to\infty} |y(t)| = \infty$. □

□

Comment 1. *The inequality (4.2) in the proof of Result (i), indicates the unbounded nature of the perturbation because it is assumed that* $\lim_{t\to\infty} \frac{\lambda(t)}{|c(t)|} = \infty$ *and* $c(\cdot) < 0$. *The same holds true for the inequality (4.3) (in the proof of Result (ii)) based on the assumptions* $\lim_{t\to\infty} \frac{c(t)}{|a(t)|} = \infty$ *with* $a(\cdot) < 0$.

Simulation 2. *Given* $t_0 = 1$, *let b and d be uniformly zero.*

For the case $a(t) = t^2$ *and* $c(t) = -t$ *for every* $t \geq t_0 = 1$, *we have* $a(\cdot) \neq 0$, $c(\cdot) < 0$ *and* $\int_{t_0}^{\infty} c(t)\,dt = -\infty$. *Also, we get* $\lambda(t) = \max(|a(t)|, |b(t)|) = \max\left(t^2, 0\right) = t^2$ *for all* $t \geq t_0 = 1$. *We obtain*

$$\lim_{t\to\infty} \frac{\lambda(t)}{|c(t)|} = \lim_{t\to\infty} \frac{t^2}{t} = \infty$$

and

$$\lim_{t\to\infty} \frac{c(t)\dot{\lambda}(t) - \dot{c}(t)\lambda(t)}{c^2(t)\lambda(t)} = \lim_{t\to\infty} \frac{(-t)(2t) - (-1)\left(t^2\right)}{(t^2)(t^2)} = 0,$$

so that condition (4.1) is satisfied. Therefore, by Result (i) of Theorem 4.1, we conclude that there is some $r > 0$ *such that when* $|y(t_0)| < r$, *every solution is global and bounded with* $|y(\cdot)| < |y(t_0)|$, $y = 0$ *exhibits uniform stability and asymptotic stability and the mapping* $t \to |y(t)|$ *is strictly decreasing. This is shown in Figure 2.*

For the case $a(t) = -t$ *and* $c(t) = t^2$ *for all* $t \geq t_0 = 1$, *note that* $a(\cdot) < 0$, $c(\cdot) > 0$ *and* $b(\cdot) = 0$ *and, thus, the positivity of solutions is guaranteed for all* $y(t_0) \in \mathbb{R}$ *by Item (iii) of Proposition 3.2. We have* $\int_{t_0}^{\infty} a(t)\,dt = -\infty$ *and* $\lim_{t\to\infty} \frac{c(t)}{a(t)} = \lim_{t\to\infty} \frac{t^2}{-t} = -\infty$. *Therefore, we conclude by Result (ii) of Theorem 4.1 that for any* $y(t_0) \in \mathbb{R}$, *each nontrivial solution* $y(t)$ *is globally defined and* $\lim_{t\to\infty} |y(t)| = \infty$. *This is demonstrated in Figure 3.*

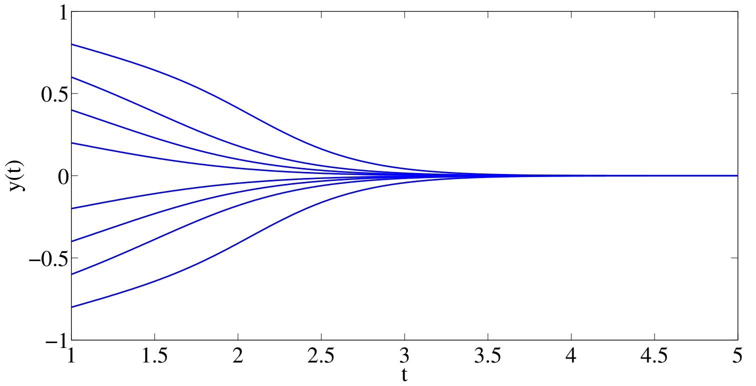

Figure 2. $y(t)$ versus t for the initial values $\pm 0.2, \pm 0.4, \pm 0.6, \pm 0.8$. Resulting from Theorem 4.1, these simulations have been created incorporating the conditions stated in Result (i), which includes the assumption $c(\cdot) < 0$. Take note that each solution $y(t)$ is continuable on $[t_0, \infty)$ and converges to zero as t goes to infinity. Moreover, the mapping $t \to |y(t)|$ is monotonically decreasing so that $|y(t)| < |y(t_0)|$ for every $t \geq t_0$. This empathizes the uniform stability and the asymptotic stability, even though this particular case is associated with a Lyapunov inequality involving an unbounded perturbation.

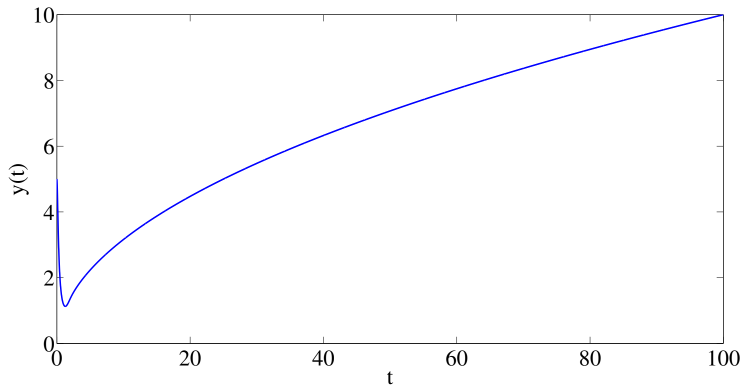

Figure 3. $y(t)$ versus t for the initial value five. This simulation has been created based on the conditions outlined in Result (ii) of Theorem 4.1, including the assumption $a(\cdot) < 0$. Observe that the nontrivial solution $y(t)$ is continuable on $[t_0, \infty)$, nonnegative and diverges to infinity.

5. Sufficient conditions for the attractivity

The next theorem introduces two sets of conditions for the local and global attractivity of the nonhomogeneous differential equation under study where the system has been transformed into linear and nonlinear nonautonomous differential Lyapunov inequalities with vanishing perturbations.

Theorem 5.1. *We introduce the following separate results:*

(i) Suppose that $a(t) < 0$ *for all* $t \geq t_0$, $\int_{t_0}^{\infty} a(t)\, dt = -\infty$ *and*

$$\lim_{t\to\infty} \frac{b(t)}{a(t)} = \lim_{t\to\infty} \frac{d(t)}{a(t)} = 0. \tag{5.1}$$

Furthermore, assume that either $\lim_{t\to\infty}\frac{c(t)}{a(t)}=0$ *or* $c(t)<0$ *for all* $t\geq t_0$, *then for every initial value* $y(t_0)\in\mathbb{R}$, *each solution* $y(t)$ *of (1.1) is continuable on* $[t_0,\infty)$ *with* $\lim_{t\to\infty}y(t)=0$ *(so that the origin is globally attractive).*

(ii) Assume that $c(t)<0$ *for all* $t\geq t_0$, $\int_{t_0}^{\infty}c(t)\,dt=-\infty$,

$$\lim_{t\to\infty}\frac{a(t)}{c(t)}=\lim_{t\to\infty}\frac{b(t)}{c(t)}=\lim_{t\to\infty}\frac{d(t)}{c(t)}=0, \tag{5.2}$$

and the initial time t_0 *is sufficiently large to satisfy*

$$\left\|\frac{\max(|a|,|b|,|d|)}{|c|}\right\|_\infty<\frac{1}{3}. \tag{5.3}$$

Then there is some $r>0$ *such that for every* $|y(t_0)|<r$, *each solution* $y(t)$ *of (1.1) is continuable on* $[t_0,\infty)$ *with* $|y(\cdot)|<r$ *and* $\lim_{t\to\infty}y(t)=0$ *(so that the origin is locally attractive). Additionally, it is worth mentioning that if* $a(t)<0$ *for every* $t\geq t_0$, *the conditions (5.2) and (5.3) can be relaxed to be* $\lim_{t\to\infty}\frac{b(t)}{c(t)}=\lim_{t\to\infty}\frac{d(t)}{c(t)}=0$ *and* $\left\|\frac{\max(|b|,|d|)}{|c|}\right\|_\infty<\frac{1}{2}$, *respectively.*

Proof. Let $V(t)=y^2(t)$ for each $t\in[t_0,\omega)$.

Proof of Result (i). Consider the case $\lim_{t\to\infty}\frac{c(t)}{a(t)}=0$. We get by (1.1) that for all $t\in(t_0,\omega)$:

$$\begin{aligned}\dot V(t)&=2y(t)\dot y(t)\\&=2a(t)y^4(t)+2b(t)y^3(t)+2c(t)y^2(t)+2d(t)y(t)\\&\leq 2a(t)V^2(t)+2|b(t)|V^{\frac32}(t)+2|c(t)|V(t)+2|d(t)|\sqrt{V(t)}\\&\leq 2a(t)V^2(t)+e(t)\left(V^{\frac32}(t)+V(t)+\sqrt{V(t)}\right),\end{aligned} \tag{5.4}$$

where $e(t)=2\max(|b(t)|,|c(t)|,|d(t)|)$ for each $t\geq t_0$. Given an initial value $y(t_0)\in\mathbb{R}$, set $\delta>\max\left(1,V(t_0),\left\|\frac{3e}{2|a|}\right\|_\infty^4\right)>0$ so that $V(t_0)<\delta$ and $\left(\frac{3e(t)}{2|a(t)|}\right)^4<\delta$ for every $t\geq t_0$ and, hence,

$$\sqrt{\frac{3e(t)\sqrt{\delta^3}}{2|a(t)|}}<\delta,\ \text{ for every } t\geq t_0. \tag{5.5}$$

We conclude by (5.4) that

$$\dot V(t)\leq 2a(t)V^2(t)+e(t)\left(\delta^{\frac32}+\delta+\sqrt{\delta}\right),\ \text{ for every } t\in(t_0,\infty) \text{ with } V(t)<\delta,$$

and, thus, using the fact $\delta>1$ gives

$$\dot V(t)\leq 2a(t)V^2(t)+3\delta^{\frac32}e(t),\ \text{ for every } t\in(t_0,\infty) \text{ that satisfies } V(t)<\delta, \tag{5.6}$$

so that (see (5.5))

$$\dot V(t)\leq 0,\ \text{ for all } t\in(t_0,\omega)\ \text{ that satisfies } \sqrt{\frac{3e(t)\sqrt{\delta^3}}{2|a(t)|}}<V(t)<\delta.$$

Therefore, since $V(t_0) < \delta$, we conclude that all assumptions of Proposition 2.1 are fulfilled with $g_1(t) = \sqrt{\frac{3e(t)\sqrt{\delta^3}}{2|a(t)|}}$ and $g_2(t) = \delta$ for every $t \geq t_0$. Therefore, $V(t) < \delta$ for all $t \in [t_0, \omega)$ and thus $|y(\cdot)| < r := \sqrt{\delta}$. Furthermore, (5.6) implies

$$\dot{V}(t) \leq 2a(t)V^2(t) + 3\delta^{\frac{3}{2}}e(t), \text{ for all } t \in (t_0, \omega).$$

Consider the differential equation $\dot{v}(t) = 2a(t)v^2(t) + 3\delta^{\frac{3}{2}}e(t)$. We have $\lim_{t\to\infty}\frac{e(t)}{a(t)} = 0$ by (5.1) and the definition of the function e. Since we have $a(t) < 0$ for all $t \geq t_0$ and $\int_{t_0}^{\infty} a(t)\,dt = -\infty$, all assumptions of Theorem 2.2 are met with $l = 0$, $Q(\cdot) = -2a(\cdot)$, $E(\cdot) = 3\delta^{\frac{3}{2}}e(\cdot)$ and $\beta(\alpha) = \alpha^2$ for all $\alpha \in \mathbb{R}$ (observe that $\lim_{t\to\infty}\frac{E(t)}{Q(t)} = 0$ because $\lim_{t\to\infty}\frac{e(t)}{a(t)} = 0$ and that $\int_{t_0}^{\infty} Q(t)\,dt = \infty$ because $\int_{t_0}^{\infty} a(t)\,dt = -\infty$). As a result, the classical comparison principle along with Theorem 2.2 guarantee that for any $y(t_0) \in \mathbb{R}$, each solution $y(t)$ is global and the origin is globally attractive.

Consider the case $c(t) < 0$ for all $t \geq t_0$. Since $c(t) < 0$ for all $t \in (t_0, \omega)$ we obtain by (1.1) that

$$\begin{aligned}\dot{V}(t) &= 2a(t)V^2(t) + 2|b(t)|V^{\frac{3}{2}}(t) + 2|d(t)|\sqrt{V(t)} \\ &\leq 2a(t)V^2(t) + 2\max(|b(t)|, |d(t)|)\left(V^{\frac{3}{2}}(t) + \sqrt{V(t)}\right).\end{aligned} \tag{5.7}$$

Pick $\delta > \max\left(1, V(t_0), \left\|\frac{2\max(|b|,|d|)}{|a|}\right\|_\infty^4\right) > 0$ and let $|y(t_0)| < r := \sqrt{\delta}$. One can use Proposition 2.1 to show that $V(t) < \delta$ for all $t \geq t_0$ and, hence, (5.7) leads to

$$\dot{V}(t) \leq 2a(t)V^2(t) + 4\delta^{\frac{3}{2}}\max(|b(t)|, |d(t)|), \text{ for all } t \in (t_0, \omega).$$

As we have done in the proof of the case $\lim_{t\to\infty}\frac{c(t)}{a(t)} = 0$, Theorem 2.2 and the comparison principle can end the proof of this case. □

Proof Result (ii). For each $t \in (t_0, \omega)$, the equation (1.1) leads to

$$\begin{aligned}\dot{V}(t) &= 2c(t)y^2(t) + 2b(t)y^3(t) + 2a(t)y^4(t) + 2d(t)y(t) \\ &\leq 2c(t)V(t) + 2|b(t)|V^{\frac{3}{2}}(t) + 2|a(t)|V^2(t) + 2|d(t)|\sqrt{V(t)} \\ &\leq 2c(t)V(t) + e(t)\left(V^{\frac{3}{2}}(t) + V^2(t) + \sqrt{V(t)}\right),\end{aligned} \tag{5.8}$$

where $e(t) = 2\max(|a(t)|, |b(t)|, |d(t)|)$ for all $t \geq t_0$. (5.3) ensures the existence of some $\delta > 0$ such that

$$\left\|\frac{3e}{2|c|}\right\|_\infty^2 < \delta < 1,$$

and, hence,

$$\frac{3e(t)\sqrt{\delta}}{2|c(t)|} < \delta, \text{ for all } t \geq t_0. \tag{5.9}$$

We get by (5.8) that

$$\dot{V}(t) \leq 2c(t)V(t) + e(t)\left(\delta^{\frac{3}{2}} + \delta^2 + \sqrt{\delta}\right), \text{ for every } t \in (t_0, \omega) \text{ with } V(t) < \delta,$$

and, thus, since $\delta < 1$, one has

$$\dot{V}(t) \leq 2c(t)V(t) + 3\sqrt{\delta}\,e(t), \text{ for every } t \in (t_0, \omega) \text{ with } V(t) < \delta, \tag{5.10}$$

which leads to (see (5.9))

$$\dot{V}(t) \leq 0, \text{ for all } t \in (t_0, \omega) \text{ with } \frac{3e(t)\sqrt{\delta}}{2|c(t)|} < V(t) < \delta. \tag{5.11}$$

Consider an initial value $|y(t_0)| < r := \sqrt{\delta}$ so that $V(x_0) < \delta$. We observe by (5.11) that all assumptions of Proposition 2.1 are satisfied with $g_1(t) = \frac{3e(t)\sqrt{\delta}}{2|c(t)|}$ and $g_2(t) = \delta$ for every $t \geq t_0$ and, thus, $V(t) < \delta$ for all $t \in [t_0, \omega)$ so that $|y(\cdot)| < r$. Moreover, (5.10) leads to

$$\dot{V}(t) \leq 2c(t)V(t) + 3\delta^{\frac{1}{2}}e(t), \text{ for all } t \in (t_0, \omega).$$

The system $\dot{v}(t) = 2c(t)v(t) + 3\delta^{\frac{1}{2}}e(t)$ has the form of the system mentioned in Theorem 2.2 with $Q(\cdot) = -2c(\cdot)$, $E(\cdot) = 3\delta^{\frac{1}{2}}e(\cdot)$, and β is the identity mapping. We have $Q(t) > 0$ because $c(t) < 0$ for all $t \geq t_0$. Note that by (5.2) and the definition of the function e, one has $\lim_{t\to\infty}\frac{E(t)}{Q(t)} = 0$ so that $l = 0$. Furthermore, $\int_{t_0}^{\infty} Q(t)\,dt = \infty$ because $\int_{t_0}^{\infty} c(t)\,dt = -\infty$. All premises of Theorem 2.2 are fulfilled. Therefore, a comparison principle can show that $V(\cdot) \leq v(\cdot)$ so that $y(t)$ is global and the zero solution is locally attractive.

Finally, if $a(t) < 0$ for every $t \geq t_0$, we replace the conditions (5.2) and (5.3) by $\lim_{t\to\infty}\frac{b(t)}{c(t)} = \lim_{t\to\infty}\frac{d(t)}{c(t)} = 0$ and $\left\|\frac{\max(|b|,|d|)}{|c|}\right\|_\infty < \frac{1}{2}$, respectively. For all $t \geq t_0$, equation (1.1) leads for all $t \in (t_0, \omega)$ to

$$\begin{aligned}\dot{V}(t) &= 2c(t)y^2(t) + 2b(t)y^3(t) + 2d(t)y(t)\\ &\leq 2c(t)V(t) + 2|b(t)|V^{\frac{3}{2}}(t) + 2|d(t)|\sqrt{V(t)}\\ &\leq 2c(t)V(t) + 2\max(|b(t)|, |d(t)|)\left(V^{\frac{3}{2}}(t) + \sqrt{V(t)}\right).\end{aligned} \tag{5.12}$$

Let δ be a positive number such that $\left\|\frac{2\max(|b|,|d|)}{|c|}\right\|_\infty^2 < \delta < 1$. The existence of δ can be seen by the assumption $\left\|\frac{\max(|b|,|d|)}{|c|}\right\|_\infty < \frac{1}{2}$. Let $|y(t_0)| < r := \sqrt{\delta}$. Proposition 2.1 ensures that $V(t) < \delta$ for all $t \in (t_0, \omega)$ and, hence, (5.12) gives

$$\dot{V}(t) \leq 2c(t)V(t) + 4\sqrt{\delta}(\max(|b(t)|, |d(t)|)) \text{ for all } t \in (t_0, \omega).$$

As in the prior analysis, one can use Theorem 2.2 to end the proof of the present case. □

□

Simulations. Let $t_0 = 0$, $y(t_0) = 1$, $a(t) = -1 - t^3$, $b(t) = \sin(t)$, $c(t) = \cos(t)$ and $d(t) = \sin(t) + 2$ for every $t \geq t_0 = 0$. Since $d(\cdot) > 0$, we deduce by Proposition 3.1 that $y(t)$ is nonnegative. In addition, observe that $a(\cdot) < 0$ and $\int_{t_0}^{\infty} a(t)\,dt = -\infty$, and we get

$$\lim_{t\to\infty}\frac{c(t)}{a(t)} = \lim_{t\to\infty}\frac{\cos(t)}{-1-t^3} = 0,$$

$$\lim_{t\to\infty}\frac{b(t)}{a(t)} = \lim_{t\to\infty}\frac{\sin(t)}{-1-t^3} = 0,$$

$$\lim_{t\to\infty}\frac{d(t)}{a(t)} = \lim_{t\to\infty}\frac{\sin(t)+2}{-1-t^3} = 0.$$

Thus, assumption (5.1) is satisfied. Therefore, we have by Result (i) of Theorem 5.1 that for every $y(t_0) \in \mathbb{R}$, each solution $y(t)$ of (1.1) is global and the origin is globally attractive. This is illustrated in Figure 4.

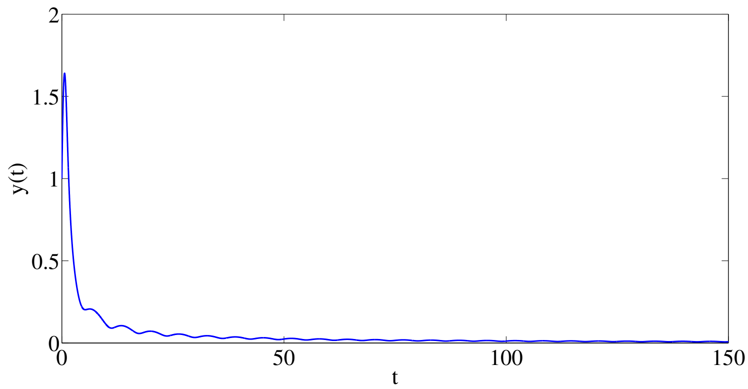

Figure 4. The positivity of the global solution and the attractivity of the origin can be readily observed.

In the subsequent lemma, we utilize Theorem 5.1 to deduce conditions for the convergence of the system's state.

Lemma 5.1. *Suppose that $a(t) < 0$ for all $t \geq t_0$, $\int_{t_0}^{\infty} a(t)\,dt = -\infty$ and*

$$-\frac{1}{3}\lim_{t\to\infty}\frac{b(t)}{a(t)} = -\sqrt[3]{\lim_{t\to\infty}\frac{d(t)}{a(t)}} = L \in \mathbb{R}. \tag{5.13}$$

Furthermore, we assume that either $\lim_{t\to\infty}\frac{c(t)}{a(t)} = 3L^2$ or $3L^2 a(\cdot) + 2L\,b(\cdot) + c(\cdot) < 0$. Then, for every initial value $y(t_0) \in \mathbb{R}$, each solution $y(t)$ of (1.1) is globally defined and $\lim_{t\to\infty} y(t) = L$.

Proof. First. we consider the case $\lim_{t\to\infty}\frac{c(t)}{a(t)} = 3L^2$. Let $z(\cdot) = y(\cdot) - L$. We get by (1.1); for all $t \in (t_0, \omega)$, that

$$\begin{aligned} \dot{z}(t) &= \dot{y}(t) \\ &= a(t)(z(t)+L)^3 + b(t)(z(t)+L)^2 + c(t)(z(t)+L) + d(t) \\ &= a_*(t)z^3(t) + b_*(t)z^2(t) + c_*(t)z(t) + d_*(t), \end{aligned} \tag{5.14}$$

where (for every $t \in [t_0, \infty)$)

$$\begin{aligned} a_*(t) &= a(t) \\ b_*(t) &= 3L\,a(t) + b(t) \\ c_*(t) &= 3L^2 a(t) + 2L\,b(t) + c(t) \\ d_*(t) &= L^3 a(t) + L^2 b(t) + Lc(t) + d(t). \end{aligned}$$

We deduce by (5.13) that $\lim_{t\to\infty}\frac{b(t)}{a(t)} = -3L$ and, hence,

$$\lim_{t\to\infty}\frac{b_*(t)}{a_*(t)} = \lim_{t\to\infty}\frac{3L\,a(t)+b(t)}{a(t)} = \lim_{t\to\infty}\left(3L + \frac{b(t)}{a(t)}\right) = 3L - 3L = 0. \tag{5.15}$$

Since $\lim_{t\to\infty}\frac{c(t)}{a(t)}=3L^2$ and $\lim_{t\to\infty}\frac{b(t)}{a(t)}=-3L$, we get

$$\begin{aligned}\lim_{t\to\infty}\frac{c_*(t)}{a_*(t)} &= \lim_{t\to\infty}\frac{3L^2 a(t)+2Lb(t)+c(t)}{a(t)}\\ &= \lim_{t\to\infty}\left(3L^2+2L\frac{b(t)}{a(t)}+\frac{c(t)}{a(t)}\right)\\ &= 3L^2+2L(-3L)+3L^2\\ &= 0. \end{aligned}\tag{5.16}$$

We get by (5.13) that $\lim_{t\to\infty}\frac{d(t)}{a(t)}=-L^3$ so that

$$\begin{aligned}\lim_{t\to\infty}\frac{d_*(t)}{a_*(t)} &= \lim_{t\to\infty}\frac{L^3 a(t)+L^2 b(t)+Lc(t)+d(t)}{a(t)}\\ &= \lim_{t\to\infty}\left(L^3+L^2\frac{b(t)}{a(t)}+L\frac{c(t)}{a(t)}+\frac{d(t)}{a(t)}\right)\\ &= L^3+L^2(-3L)+L\left(3L^2\right)-L^3\\ &= 0. \end{aligned}\tag{5.17}$$

On the other hand, since $a_*(\cdot)=a(\cdot)$, we have $a_*(t)<0$ for all $t\geq t_0$ and $\int_{t_0}^{\infty}a_*(t)\,dt=-\infty$ because it is assumed that $a(t)<0$ for all $t\geq t_0$ and $\int_{t_0}^{\infty}a(t)\,dt=-\infty$. These facts along with (5.15)-(5.17) ensure that; when considering the differential equation (5.14), all conditions of Item (i) in Theorem 5.1 are satisfied. Therefore, for any initial condition $z(t_0)\in\mathbb{R}$, each solution $z(t)$ of (5.14) is globally defined and $\lim_{t\to\infty}z(t)=0$; thus, the solution $y(t)$ of (1.1) is global and $\lim_{t\to\infty}y(t)=L$.

Second; for the case $3L^2 a(\cdot)+2Lb(\cdot)+c(\cdot)<0$, we have $c_*(\cdot)<0$. One can derive analogous arguments to the first case to prove that Item (i) of Theorem 5.1 ends the proof. □

Comment 2. *Observe that when $L=0$, Lemma 5.1 reduces to Item (i) of Theorem 5.1.*

Simulation 3. *Set $t_0=0$, $y(t_0)=1$, $a(t)=-1-t$, $b(t)=6t+1$, $c(t)=-12t$ and $d(t)=8t$ for every $t\geq t_0=0$. We have $a(\cdot)<0$, $b(\cdot)\geq 0$, $c(\cdot)\leq 0$ and $y(t_0)\geq 0$. Thus, the positivity of the solution is guaranteed by Item (i) of Proposition 3.2. Moreover, we get $\int_{t_0}^{\infty}a(t)\,dt=-\infty$ and*

$$\lim_{t\to\infty}\frac{c(t)}{a(t)}=\lim_{t\to\infty}\frac{-12t}{-1-t}=12,$$

$$\lim_{t\to\infty}\frac{b(t)}{a(t)}=\lim_{t\to\infty}\frac{6t+1}{-1-t}=-6,$$

$$\lim_{t\to\infty}\frac{d(t)}{a(t)}=\lim_{t\to\infty}\frac{8t}{-1-t}=-8.$$

Thus, assumption (5.13) is satisfied with $L=2$ (note that $\lim_{t\to\infty}\frac{c(t)}{a(t)}=3L^2$) and, hence, Lemma 5.1 guarantees that for any initial value $y(t_0)\in\mathbb{R}$, the solution $y(t)$ of (1.1) is global and $\lim_{t\to\infty}y(t)=L=2$, as shown in Figure 5.

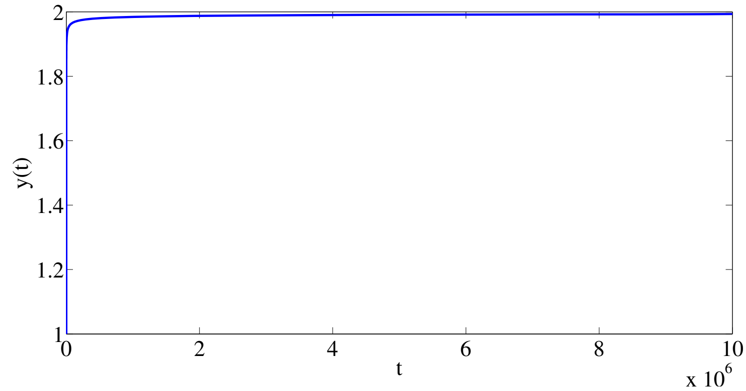

Figure 5. This simulation provides clarity that the solution $y(t)$ is global, nonnegative and converges to $L = 2$ as t goes to infinity.

6. Conclusions

Stability analysis of Abel's differential equation of the first kind has been conducted. More precisely, conditions for the positivity of solutions have been derived in Propositions 3.1 and 3.2. Additionally, it has been clarified in section four that applying the Lyapunov technique to the homogenous form of the equation may give a differential inequality with an unbounded perturbation. For the prior case, Theorem 4.1 has derived sufficient conditions for the continuation and boundedness of solutions, the uniform stability, the asymptotic stability and the instability of the equation. In addition, the local/global origin attractivity has been investigated in Theorem 5.1. Based on the results of the aforementioned theorem, conditions have been provided for the state convergence in Lemma 5.1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The corresponding author states that there are no conflicts of interest.

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