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Research article

Fixed point results for a new α-θ-Geraghty type contraction mapping in metric-like space via $C_{\mathcal{G}}$-simulation functions

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Abstract: This paper aims to introduce the new concept of an α-θ-Geraghty type contraction mapping using $C_{\mathcal{G}}$-simulation in a metric-like space. Additionally, through this type of contraction, we establish fixed point results that generalize several known fixed point results in the literature. We provide some examples as an application that proves the credibility of our results.

Keywords: metric-like space; fixed point; Geraghty contraction; (α, θ)-admissible mapping; $C_{\mathcal{G}}$-simulation function
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1. Introduction

Fixed point theory is an active and evolving area of mathematics, which has many applications in various disciplines. Concepts and techniques of fixed point theory are valuable tools for solving problems in different areas of mathematics (see, for example, [13, 14, 16, 22]). The references that will be mentioned in this paper highlight some specific developments in fixed point theory. The notion of $\mathcal{Z}$-contraction was introduced by Khojasteh et al. [8] using simulation functions, in which a generalized version of the Banach contraction principle was presented. Olgun et al. [9] derived fixed point results for a generalized $\mathcal{Z}$-contraction. Chandok et al. [11] combined simulation functions and C-class functions, leading to the existence and uniqueness of the point of coincidence, which generalized the results in [8, 9]. Geraghty [6] introduced a generalization of this principle using an auxiliary function. There are other results concerning the fixed point theory, for example [7, 19–21, 23, 25].

Samet et al. [17] introduced the concept of α-admissibility and extended the Banach contraction principle. Karapinar [12] further generalized the results of Samet et al. [17] and Khojasteh et al. [8] by introducing the notion of an α-admissible $\mathcal{Z}$-contraction. Chandok [2] introduced the notion of (α, θ)-admissible mappings and obtained fixed point theorems. Alsamir et al. [15] have recently proved fixed point theorems for an (α, θ)-admissible $\mathcal{Z}$-contraction mapping in complete metric-like spaces.

In 2012, Harandi [3] reintroduced the concept of metric-like spaces, which is a general relaxation of some properties of a metric space, while still capturing the essential ideas.

Definition 1. *[3] Let Υ be a nonempty set. A function $d_l : \Upsilon \times \Upsilon \to [0, \infty)$ is defined as a metric-like space on Υ if it satisfies the following conditions for any $\eta, \mu, \nu \in \Upsilon$:*

(d_1) if $d_l(\eta, \mu) = 0$, then $\eta = \mu$,
(d_2) $d_l(\eta, \mu) = d_l(\mu, \eta)$,
(d_3) $d_l(\eta, \mu) \leq d_l(\eta, \nu) + d_l(\nu, \mu)$.

The pair (Υ, d_l) is said to be a metric-like space.

Let us notice that while every partial metric space and metric space could be considered as metric-like spaces, the converse is not necessarily true.

According to reference [3], we have the following topological notions. For each metric-like function d_l, defined on Υ, there exists a topology τ_{d_l} on Υ, introduced by d_l, where the family of open balls, determined by d_l, forms the following basis:

$$B_{d_l}(\eta, \gamma) = \Big\{\mu \in \Upsilon : \ |d_l(\eta, \mu) - d_l(\eta, \eta)| < \gamma\Big\}, \text{ for all } \eta \in \Upsilon \text{ and } \gamma > 0.$$

Let us consider (Υ, d_l) as a metric-like space. The function $Q : \Upsilon \to \Upsilon$ is said to be d_l-continuous at $\eta \in \Upsilon$ if for every $\gamma > 0$ there is $\rho > 0$ such that $Q(B_{d_l}(\eta, \rho)) \subseteq B_{d_l}(Q\eta, \gamma)$. Consequently, if $Q : \Upsilon \to \Upsilon$ is d_l-continuous and if $\lim\limits_{n\to\infty} \eta_n = \eta$, we obtain that $\lim\limits_{n\to+\infty} Q\eta_n = Q\eta$. A sequence $\{\eta_n\}_{n=0}^{+\infty}$, consisting of elements of Υ, is said to be a d_l-Cauchy sequence if the limit $\lim\limits_{n,m\to+\infty} d_l(\eta_n, \eta_m)$ exists and is a finite value. We say that the metric-like space (Υ, d_l) is complete if, for every d_l-Cauchy sequence $\{\eta_n\}_{n=0}^{+\infty}$, there exists an element $\eta \in \Upsilon$ such that

$$\lim_{n\to+\infty} d_l(\eta_n, \eta) = d_l(\eta, \eta) = \lim_{n,m\to+\infty} d_l(\eta_n, \eta_m).$$

In a metric-like space (Υ, d_l), a subset Σ is bounded if there exists a point $\eta \in \Upsilon$ and $N \geq 0$ such that $d_l(\mu, \eta) \leq N$ for every $\mu \in \Sigma$.

Remark 1. *The uniqueness of the limit for a convergent sequence cannot be guaranteed in metric-like spaces.*

The following lemma is required, which is known and useful for the sequel.

Lemma 1. *[1, 3] Let (Υ, d_l) be a metric-like space. Let $\{\eta_n\}$ be a sequence in Υ such that $\eta_n \to \eta$, where $\eta \in \Upsilon$ and $d_l(\eta, \mu) = 0$. Then, for all $\mu \in \Upsilon$, we have $\lim\limits_{n\to+\infty} d_l(\eta_n, \mu) = d_l(\eta, \mu)$.*

In 2015, Khojasteh et al. [8] provided a class Θ of functions $\mathcal{Z} : (\mathbb{R}_+ \cup \{0\})^2 \to \mathbb{R}$ which satisfy the following assumptions:

($\mathcal{Z}_1$) $\mathcal{Z}(0,0) = 0$,
($\mathcal{Z}_2$) $\mathcal{Z}(\varrho,\sigma) < \sigma - \varrho$ for every $\varrho,\sigma > 0$,
($\mathcal{Z}_3$) if $\{\varrho_n\}$ and $\{\sigma_n\}$ are sequences in $\mathbb{R}^+$ such that $\lim_{n\to+\infty} \varrho_n = \lim_{n\to+\infty} \sigma_n > 0$, then

$$\lim_{n\to+\infty} \sup \mathcal{Z}(\varrho_n,\sigma_n) < 0.$$

The functions $\mathcal{Z}$ are called the simulation functions.

Rold'an-L'opez-de-Hierro et al. [10] modified the notion of simulation function by replacing ($\mathcal{Z}_3$) with ($\mathcal{Z}_3'$), where

($\mathcal{Z}_3'$): If $\{\varrho_n\}$ and $\{\sigma_n\}$ are sequences in $\mathbb{R}_+$ such that $\lim_{n\to+\infty} \varrho_n = \lim_{n\to+\infty} \sigma_n > 0$ and $\varrho_n < \sigma_n$, then

$$\lim_{n\to+\infty} \sup \mathcal{Z}(\varrho_n,\sigma_n) < 0.$$

The simulation function satisfying conditions ($\mathcal{Z}_1$), ($\mathcal{Z}_2$) and ($\mathcal{Z}_3'$) is called simulation function in the sense of Roldán-López-de-Hierro.

Definition 2. *[4] The map $\mathcal{G} : (\mathbb{R}_+ \cup \{0\})^2 \to \mathbb{R}$ is said to be a C-class function if it is continuous and satisfies the following assumptions:*

(1) $\mathcal{G}(\sigma,\varrho) \leq \sigma$,
(2) $\mathcal{G}(\sigma,\varrho) = \sigma$ *means that either* $\sigma = 0$ *or* $\varrho = 0$*, for every* $\sigma,\varrho \in \mathbb{R}_+ \cup \{0\}$.

Definition 3. *[21] A map $\mathcal{G} : (\mathbb{R}_+ \cup \{0\})^2 \to \mathbb{R}$ is said to satisfy condition ($C_\mathcal{G}$) if there exists $C_\mathcal{G} \geq 0$ such that:*

($\mathcal{G}1$) $\mathcal{G}(\sigma,\varrho) > C_\mathcal{G}$ *implies that* $\sigma > \varrho$,
($\mathcal{G}2$) $\mathcal{G}(\varrho,\varrho) \leq C_\mathcal{G}$*, for each* $\varrho \in \mathbb{R}_+ \cup \{0\}$.

Definition 4. *[21] A map $\Psi : (\mathbb{R}_+ \cup \{0\})^2 \to \mathbb{R}$ is said to be a $C_\mathcal{G}$-simulation function if it satisfies the following assumptions:*

(1) $\Psi(\varrho,\sigma) < \mathcal{G}(\sigma,\varrho)$ *for each* $\varrho,\sigma > 0$ *such that* $\mathcal{G} : (\mathbb{R}_+ \cup \{0\})^2 \to \mathbb{R}$ *is a C-class function, which satisfies the condition ($C_\mathcal{G}$),*
(2) *if* $\{\varrho_n\}$ *and* $\{\sigma_n\}$ *are two sequences in* $\mathbb{R}_+$ *such that* $\lim_{n\to+\infty} \varrho_n = \lim_{n\to+\infty} \sigma_n > 0$ *and* $\varrho_n < \sigma_n$*, then* $\lim_{n\to+\infty} \sup \Psi(\varrho_n,\sigma_n) < C_\mathcal{G}$.

Definition 5. *[17] Let $\alpha : \Upsilon^2 \to \mathbb{R}^+$ be a function and $Q : \Upsilon \to \Upsilon$ be a self-mapping. We say that Q is an α-admissible if*

$$\alpha(\eta,\mu) \geq 1 \text{ implies that } \alpha(Q\eta,Q\mu) \geq 1 \text{ for every } \eta,\mu \in \Upsilon. \tag{1.1}$$

Definition 6. *[18] An α-admissible map Q is called triangular α-admissible if $\alpha(\eta,\nu) \geq 1$ and $\alpha(\nu,\mu) \geq 1$, which implies that $\alpha(\eta,\mu) \geq 1$ for all $\eta,\mu,\nu \in \Upsilon$.*

Chandok [2] introduced the notion of (α,θ)-admissible mappings.

Definition 7. *[2] Let Υ be a nonempty set, $Q : \Upsilon \to \Upsilon$ and $\theta, \alpha : \Upsilon \times \Upsilon \to \mathbb{R}^+$. We say that Q is an (α, θ)-admissible mapping if*

$$\begin{cases} \theta(\eta,\mu) \geq 1 \\ and \\ \alpha(\eta,\mu) \geq 1 \end{cases} \quad imply\ that \quad \begin{cases} \theta(Q\eta,Q\mu) \geq 1 \\ and \\ \alpha(Q\eta,Q\mu) \geq 1, \end{cases}$$

for all $\eta, \mu \in \Upsilon$.

Definition 8. *Let Υ be a nonempty set, $Q : \Upsilon \to \Upsilon$ and $\theta, \alpha : \Upsilon \times \Upsilon \to \mathbb{R}^+$. We say that Q is a triangular (α, θ)-admissible mapping if*

(1) Q is an (α, θ)-admissible mapping,
(2) $\theta(\eta,\mu) \geq 1$, $\theta(\mu,\nu) \geq 1$, $\alpha(\eta,\mu) \geq 1$ and $\alpha(\mu,\nu) \geq 1$ imply that $\theta(\eta,\nu) \geq 1$ and $\alpha(\eta,\nu) \geq 1$,

for all $\eta, \mu, \nu \in \Upsilon$.

Example 1. *Let $\Upsilon = [0, \infty)$ and let $Q : \Upsilon \to \Upsilon$ be defined as follows*

$$Q\mu = \begin{cases} \frac{1-\mu^2}{8}, & if\ \mu \in [0,1] \\ 9\mu, & otherwise. \end{cases}$$

Moreover, let $\theta, \alpha : \Upsilon \times \Upsilon \to \mathbb{R}^+$ be defined as follows

$$\theta(\eta,\mu) = \begin{cases} 1, & if\ \eta,\mu \in [0,1] \\ 0, & otherwise \end{cases} \quad and \quad \alpha(\eta,\mu) = \begin{cases} 2, & if\ \eta,\mu \in [0,1] \\ 0, & otherwise. \end{cases}$$

If $\eta, \mu \in \Upsilon$ such that $\theta(\eta,\mu) \geq 1$ and $\alpha(\eta,\mu) \geq 1$, then $\eta, \mu \in [0,1]$. On the other hand, for all $\mu \in [0,1]$ it holds that $Q\mu \in [0,1]$. Hence, it follows that $\theta(Q\eta,Q\mu) \geq 1$ and $\alpha(Q\eta,Q\mu) \geq 1$, which implies that Q is an (α,θ)-admissible mapping. Moreover, for all $\eta,\mu,\nu \in \Upsilon$ such that $\theta(\eta,\mu) \geq 1$, $\theta(\mu,\nu) \geq 1$, $\alpha(\eta,\mu) \geq 1$ and $\alpha(\mu,\nu) \geq 1$, it holds that $\eta,\mu,\nu \in [0,1]$. Thus, it follows that $\theta(\eta,\nu) \geq 1$ and $\alpha(\eta,\nu) \geq 1$, which finally implies that Q is a triangular (α,θ)-admissible mapping.

H. Alsamir et al. [15] established fixed point theorems in the complete metric-like spaces by introducing a new contraction condition using an (α, θ)-admissible mapping and a simulation function.

Definition 9. *[15] Let (Υ, d_l) be a metric-like space where $Q : \Upsilon \to \Upsilon$ and $\alpha, \theta : \Upsilon \times \Upsilon \to \mathbb{R}^+$ are given. Such Q is called an (α, θ)-admissible $\mathcal{Z}$-contraction with respect to $\mathcal{Z}$ if*

$$\mathcal{Z}\Big(\alpha(\eta,\mu)\theta(\eta,\mu)d_l(Q\eta,Q\mu), d_l(\eta,\mu)\Big) \geq 0, \tag{1.2}$$

for all $\eta, \mu \in \Upsilon$, where $\mathcal{Z} \in \Theta$.

Remark 2. *From the definition of simulation function, it is clear that $\mathcal{Z}(\eta,\mu) < 0$ for all $\eta \geq \mu > 0$. Therefore, Q is an (α,θ)-admissible $\mathcal{Z}$-contraction with respect to $\mathcal{Z}$ and then it holds that*

$$\alpha(\eta,\mu)\theta(\eta,\mu)d_l(Q\eta,Q\mu) < d_l(\eta,\mu),$$

for all $\eta, \mu \in \Upsilon$, where $\eta \neq \mu$.

Theorem 1. *[15] Let (Υ, d_l) be a complete metric-like space and $Q : \Upsilon \to \Upsilon$ be an (α, θ)-admissible $\mathcal{Z}$-contraction mapping such that*

i) Q is an (α, θ)-admissible,
ii) there is $\eta_0 \in \Upsilon$ such that $\alpha(\eta_0, Q\eta_0) \geq 1$ and $\theta(\eta_0, Q\eta_0) \geq 1$,
iii) Q is d_l-continuous.

Then, Q possesses a unique fixed point $\eta \in \Upsilon$ such that $d_l(\eta, \eta) = 0$.

Definition 10. *[5] Let $\beta : \mathbb{R}_+ \cup \{0\} \to (0, 1)$, which satisfies the following condition:*

$$\text{for any } \{b_m\} \subset \mathbb{R}^+ \text{ and } \lim_{m\to+\infty} \beta(b_m) = 1 \ \text{ it holds } \lim_{m\to+\infty} b_m = 0^+.$$

Such a function is called a Geraghty function.

We denote the set of Geraghty functions by $\mathcal{F}_G$.

Definition 11. *[6] Let $Q : \Upsilon \to \Upsilon$ be a self-mapping over a metric space (Υ, d). We say that Q is a Geraghty contraction, if there is $\beta \in \mathcal{F}_G$ such that*

$$d(Q\eta, Q\mu) \leq \beta(d(\eta, \mu))d(\eta, \mu), \text{ for every } \eta, \mu \in \Upsilon.$$

Theorem 2. *[6] Let (Υ, d) be a complete metric space and $Q : \Upsilon \to \Upsilon$ be a Geraghty contraction. Then, Q possesses a unique fixed point $\eta \in \Upsilon$ and the sequence $\{Q^n\eta\}$ converges to η.*

This paper aims to demonstrate certain fixed point outcomes for a novel form of α-θ-Geraghty contraction mapping using $C_{\mathcal{G}}$-simulation functions in metric-like spaces. Consequently, we demonstrate that several known fixed point theorems can be easily shown by these main results.

2. Main results

To start the presentation of our main results, we introduce the following definition.

Definition 12. *Let (Υ, d_l) be a metric-like space, $Q : \Upsilon \to \Upsilon$ be a map and $\alpha, \theta : \Upsilon^2 \to \mathbb{R} \cup \{0\}$ be functions. We say that Q is a $\Psi_{C_{\mathcal{G}}}$-α-θ-Geraghty contraction with respect to a $C_{\mathcal{G}}$-simulation function Ψ if there exists $\beta \in \mathcal{F}_G$ such that*

$$\Psi\Big(\alpha(\eta, \mu)\theta(\eta, \mu)d_l(Q\eta, Q\mu), \beta(M(\eta, \mu))M(\eta, \mu)\Big) \geq C_{\mathcal{G}} \tag{2.1}$$

for every $\eta, \mu \in \Upsilon$, where

$$M(\eta, \mu) = \max\Big\{d_l(\eta, \mu), d_l(\eta, Q\eta), d_l(\mu, Q\mu)\Big\}.$$

Remark 3. *(1) From (1) in Definition 4, it is clear that a $C_{\mathcal{G}}$-simulation function must satisfy $\Psi(r, r) < C_{\mathcal{G}}$ for all $r > 0$.*

(2) If Q is a $\Psi_{C_{\mathcal{G}}}$-α-θ-Geraghty contraction with respect to a $C_{\mathcal{G}}$-simulation function Ψ, then

$$\alpha(\eta, \mu)\theta(\eta, \mu)d_l(Q\eta, Q\mu) < M(\eta, \mu),$$

for every $\eta, \mu \in \Upsilon$.

Now, we are in a position to state our first main result.

Theorem 3. *Let (Υ, d_l) be a complete metric-like space and $Q : \Upsilon \to \Upsilon$ be a $\Psi_{C_{\mathcal{G}}}$-α-θ-Geraghty contraction satisfying the following conditions:*

(1) Q is a triangular (α, θ)-admissible,
(2) there is $\eta_0 \in \Upsilon$ such that $\alpha(\eta_0, Q\eta_0) \geq 1$ and $\theta(\eta_0, Q\eta_0) \geq 1$,
(3) either
  (3a) Q is d_l-continuous
  or
  (3b) if $\{\eta_n\} \subset \Upsilon$ such that $\theta(\eta_n, \eta_{n+1}) \geq 1$ and $\alpha(\eta_n, \eta_{n+1}) \geq 1$ for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} d_l(\eta_n, \eta) = d_l(\eta, \eta)$, then we obtain $\theta(\eta_n, \eta) \geq 1$ and $\alpha(\eta_n, \eta) \geq 1$ for every $n \in \mathbb{N}$.

Then, Q possesses a unique fixed point $\eta \in \Upsilon$ with $d_l(\eta, \eta) = 0$.

Proof. From (2), there is $\eta_0 \in \Upsilon$ such that $\theta(\eta_0, Q\eta_0) \geq 1$ and $\alpha(\eta_0, Q\eta_0) \geq 1$. We define a sequence $\{\eta_n\}$ with an initial point η_0 such that $\eta_{n+1} = Q\eta_n$, for all $n \geq 0$. If $\eta_m = \eta_{m+1}$ for some $m \in \mathbb{N}$, then η_m is a fixed point of Q and the proof is completed. Let us assume that $\eta_n \neq Q\eta_n$, for all $n \in \mathbb{N}$. Using the (α, θ)-admissibility of Q and assumptions (2) we obtain the following:

$$\alpha(\eta_1, \eta_2) \geq 1 \text{ and } \theta(\eta_1, \eta_2) \geq 1.$$

Repeating this process, we obtain the following:

$$\alpha(\eta_n, \eta_{n+1}) \geq 1 \text{ and } \theta(\eta_n, \eta_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

Since Q is $\Psi_{C_{\mathcal{G}}}$-θ-α-Geraghty contraction, when taking $\eta = \eta_n$ and $\mu = \eta_{n+1}$ in (2.1), we obtain the following:

$$\Psi\Big(\alpha(\eta_n, \eta_{n+1})\theta(\eta_n, \eta_{n+1})d_l(Q\eta_n, Q\eta_{n+1}), \beta(M(\eta_n, \eta_{n+1}))M(\eta_n, \eta_{n+1})\Big) \geq C_{\mathcal{G}}.$$

Thus,

$$\begin{aligned} C_{\mathcal{G}} &\leq \Psi\Big(\alpha(\eta_n, \eta_{n+1})\theta(\eta_n, \eta_{n+1})d_l(Q\eta_n, Q\eta_{n+1}), \beta(M(\eta_n, \eta_{n+1}))M(\eta_n, \eta_{n+1})\Big) \\ &< \mathcal{G}\Big(\beta(M(\eta_n, \eta_{n+1}))M(\eta_n, \eta_{n+1}), \alpha(\eta_n, \eta_{n+1})\theta(\eta_n, \eta_{n+1})d_l(Q\eta_n, Q\eta_{n+1})\Big). \end{aligned}$$

Since $\mathcal{G}$ is a C-class function which satisfies the condition $(C_{\mathcal{G}})$, we obtain the following:

$$\alpha(\eta_n, \eta_{n+1})\theta(\eta_n, \eta_{n+1})d_l(Q\eta_n, Q\eta_{n+1}) < \beta(M(\eta_n, \eta_{n+1}))M(\eta_n, \eta_{n+1}). \tag{2.3}$$

Thus, from (2.2) and (2.3), we obtain the following:

$$\begin{aligned} d_l(\eta_{n+1}, \eta_{n+2}) &\leq \alpha(\eta_n, \eta_{n+1})\theta(\eta_n, \eta_{n+1})d_l(Q\eta_n, Q\eta_{n+1}) \\ &< \beta(M(\eta_n, \eta_{n+1}))M(\eta_n, \eta_{n+1}) \\ &< M(\eta_n, \eta_{n+1}). \end{aligned} \tag{2.4}$$

On the other hand, we have

$$M(\eta_n, \eta_{n+1}) = \max\Big\{d_l(\eta_n, \eta_{n+1}), d_l(\eta_n, Q\eta_n),\ d_l(\eta_{n+1}, Q\eta_{n+1})\Big\}$$

$$= \max\Big\{d_l(\eta_n,\eta_{n+1}), d_l(\eta_n,\eta_{n+1}),\ d_l(\eta_{n+1},\eta_{n+2})\Big\}$$
$$= \max\Big\{d_l(\eta_n,\eta_{n+1}), d_l(\eta_{n+1},\eta_{n+2})\Big\}.$$

From the inequality (2.4), if $M(\eta_n,\eta_{n+1}) = d_l(\eta_{n+1},\eta_{n+2})$, then we obtain the following contradiction:

$$d_l(\eta_{n+1},\eta_{n+2}) < d_l(\eta_{n+1},\eta_{n+2}).$$

Hence, $M(\eta_n,\eta_{n+1}) = d_l(\eta_n,\eta_{n+1})$ and, consequently, we obtain the following from the inequality (2.4):

$$d_l(\eta_{n+1},\eta_{n+2}) < d_l(\eta_n,\eta_{n+1})\ .$$

Thus, for all $n \in \mathbb{N}\cup\{0\}$, we have $d_l(\eta_n,\eta_{n+1}) > d_l(\eta_{n+1},\eta_{n+2})$. Therefore, $d_l(\eta_n,\eta_{n+1})$ is a strictly decreasing sequence of positive real numbers. Then, we can find a number $\gamma \geq 0$ such that

$$\lim_{n\to+\infty} d_l(\eta_n,\eta_{n+1}) = \lim_{n\to+\infty} M(\eta_n,\eta_{n+1}) = \gamma.$$

Let us assume that $\gamma > 0$. Therefore, using the inequality (2.4), we obtain the following:

$$\lim_{n\to+\infty} \alpha(\eta_n,\eta_{n+1})\theta(\eta_n,\eta_{n+1})d_l(Q\eta_n,Q\eta_{n+1}) = \gamma \tag{2.5}$$

and

$$\lim_{n\to+\infty} \beta(d_l(\eta_n,\eta_{n+1}))d_l(\eta_n,\eta_{n+1}) = \gamma. \tag{2.6}$$

Let $\sigma_n = \beta(d_l(\eta_n,\eta_{n+1}))d_l(\eta_n,\eta_{n+1})$ and $\varrho_n = \alpha(\eta_n,\eta_{n+1})\theta(\eta_n,\eta_{n+1})d_l(Q\eta_n,Q\eta_{n+1})$.
Using (2.1) and (2) from Definition 4, we obtain the following:

$$C_{\mathcal{G}} \leq \lim_{n\to+\infty} \sup \Psi\Big(\varrho_n,\sigma_n\Big)$$
$$< C_{\mathcal{G}},$$

which is impossible, and this implies that $\gamma = 0$.
Hence,

$$\lim_{n\to+\infty} d_l(\eta_n,\eta_{n+1}) = 0. \tag{2.7}$$

Now, we prove that $\{\eta_n\}$ is a d_l-Cauchy sequence. Assuming the opposite that it is not, it follows that there exists ϵ for which we can find subsequences $\{\eta_{n_\lambda}\}$ and $\{\eta_{m_\lambda}\}$ of $\{\eta_n\}$, where $m_\lambda > n_\lambda > \lambda$ such that for all λ,

$$d_l(\eta_{n_\lambda},\eta_{m_\lambda}) \geq \epsilon. \tag{2.8}$$

Furthermore, we can choose a m_k, related to n_k, such that it is the smallest integer for which it holds that $m_\lambda > n_\lambda$ and (2.8). Using (2.8), we obtain the following:

$$d_l(\eta_{n_\lambda},\eta_{m_\lambda-1}) < \epsilon. \tag{2.9}$$

According to (2.8), (2.9) and the triangle inequality, we can obtain the following:

$$\begin{aligned}\epsilon &\leq d_l(\eta_{n_\lambda}, \eta_{m_l})\\ &\leq d_l(\eta_{n_l}, \eta_{m_l-1}) + d_l(\eta_{m_\lambda-1}, \eta_{m_\lambda})\\ &< \epsilon + d_l(\eta_{m_\lambda-1}, \eta_{m_\lambda}).\end{aligned}$$

From (2.7) and letting $n \to +\infty$ in the previous inequality, we can obtain the following:

$$\lim_{n\to+\infty} d_l(\eta_{n_\lambda}, \eta_{m_\lambda}) = \epsilon. \tag{2.10}$$

Additionally, from the triangle inequality, we have the following:

$$\begin{cases} d_l(\eta_{n_\lambda+1}, \eta_{m_\lambda}) \leq d_l(\eta_{n_\lambda}, \eta_{n_\lambda+1}) + d_l(\eta_{n_\lambda}, \eta_{m_\lambda})\\ \text{and}\\ d_l(\eta_{n_\lambda}, \eta_{m_\lambda}) \leq d_l(\eta_{n_\lambda+1}, \eta_{m_\lambda}) + d_l(\eta_{n_\lambda}, \eta_{n_\lambda+1}).\end{cases}$$

Therefore,

$$\begin{cases} d_l(\eta_{n_\lambda+1}, \eta_{m_\lambda}) - d_l(\eta_{n_\lambda}, \eta_{m_\lambda}) \leq d_l(\eta_{n_\lambda}, \eta_{n_\lambda+1})\\ \text{and}\\ d_l(\eta_{n_\lambda}, \eta_{m_\lambda}) - d_l(\eta_{n_\lambda+1}, \eta_{m_\lambda}) \leq d_l(\eta_{n_\lambda}, \eta_{n_\lambda+1}).\end{cases}$$

Hence,

$$|d_l(\eta_{n_\lambda+1}, \eta_{m_\lambda}) - d_l(\eta_{n_\lambda}, \eta_{m_\lambda})| \leq d_l(\eta_{n_\lambda}, \eta_{n_\lambda+1}).$$

Using (2.7), (2.10) and taking $\lim_{\lambda\to+\infty}$ on both sides of the previous inequality, we can conclude that

$$\lim_{\lambda\to+\infty} d_l(\eta_{n_\lambda+1}, \eta_{m_\lambda}) = \epsilon. \tag{2.11}$$

Similarly, we show that

$$\lim_{\lambda\to+\infty} d_l(\eta_{n_\lambda+1}, \eta_{m_\lambda+1}) = \epsilon. \tag{2.12}$$

Moreover, using (2.2) and the fact that Q is a triangular (α, θ)-admissible, we can conclude that

$$\theta(\eta_{n_\lambda}, \eta_{m_\lambda}) \geq 1 \text{ and } \alpha(\eta_{n_\lambda}, \eta_{m_\lambda}) \geq 1. \tag{2.13}$$

Let $\eta = \eta_{m_k}$ and $\mu = \eta_{n_k}$. Then, using (2.1), we obtain the following:

$$\begin{aligned}C_{\mathcal{G}} &\leq \Psi\Big(\alpha(\eta_{n_\lambda}, \eta_{m_\lambda})\theta(\eta_{n_\lambda}, \eta_{m_\lambda})d_l(Q\eta_{n_\lambda}, Q\eta_{m_\lambda}), \beta(M(\eta_{n_\lambda}, \eta_{m_\lambda}))M(\eta_{n_\lambda}, \eta_{m_\lambda})\Big)\\ &< \mathcal{G}\Big(\beta(M(\eta_{n_\lambda}, \eta_{m_\lambda}))M(\eta_{n_\lambda}, \eta_{m_\lambda}), \alpha(\eta_{n_\lambda}, \eta_{m_\lambda})\theta(\eta_{n_\lambda}, \eta_{m_\lambda})d_l(Q\eta_{n_\lambda}, Q\eta_{m_\lambda})\Big).\end{aligned}$$

Therefore,

$$\alpha(\eta_{n_\lambda}, \eta_{m_\lambda})\theta(\eta_{n_\lambda}, \eta_{m_\lambda})d_l(Q\eta_{n_\lambda}, Q\eta_{m_\lambda}) < \beta(M(\eta_{n_\lambda}, \eta_{m_\lambda}))M(\eta_{n_\lambda}, \eta_{m_\lambda}).$$

Thus,

$$\begin{aligned} d_l(\eta_{n_\lambda+1},\eta_{m_\lambda+1}) &\le \alpha(\eta_{n_\lambda},\eta_{m_\lambda})\theta(\eta_{n_\lambda},\eta_{m_\lambda})d_l(Q\eta_{n_\lambda},Q\eta_{m_\lambda}) \\ &< \beta(M(\eta_{n_\lambda},\eta_{m_\lambda}))M(\eta_{n_\lambda},\eta_{m_\lambda}) \\ &< M(\eta_{n_\lambda},\eta_{m_\lambda}). \end{aligned} \tag{2.14}$$

Since

$$\begin{aligned} M(\eta_{n_\lambda},\eta_{m_\lambda}) &= \max\Big\{d_l(\eta_{n_\lambda},\eta_{m_\lambda}), d_l(\eta_{n_\lambda},Q\eta_{n_\lambda}),\ d_l(\eta_{m_\lambda},Q\eta_{m_\lambda})\Big\} \\ &= \max\Big\{d_l(\eta_{n_\lambda},\eta_{m_\lambda}), d_l(\eta_{n_\lambda},\eta_{n_\lambda+1}),\ d_l(\eta_{m_\lambda},\eta_{m_\lambda+1})\Big\}, \end{aligned}$$

then, from (2.7) and (2.10), we can obtain the following:

$$\lim_{\lambda\to+\infty} M(\eta_{n_\lambda},\eta_{m_\lambda}) = \epsilon. \tag{2.15}$$

Using (2.12), (2.14) and (2.15), we obtain the following:

$$\lim_{\lambda\to+\infty} \alpha(\eta_{n_\lambda},\eta_{m_\lambda})\theta(\eta_{n_\lambda},\eta_{m_\lambda})d_l(Q\eta_{n_\lambda},Q\eta_{m\lambda}) = \epsilon$$

and

$$\lim_{\lambda\to+\infty} \beta(M(\eta_{n_\lambda},\eta_{m_\lambda}))M(\eta_{n_\lambda},\eta_{m_\lambda}) = \epsilon.$$

Let $\sigma_n = \beta(M(\eta_{n_\lambda},\eta_{m_\lambda}))M(\eta_{n_\lambda},\eta_{m_\lambda})$ and $\varrho_n = \alpha(\eta_{n_\lambda},\eta_{m_\lambda})\theta(\eta_{n_\lambda},\eta_{m_\lambda})d_l(Q\eta_{n_\lambda},Q\eta_{m\lambda})$.
Therefore, using (2.1) and (2) from Definition 4, we obtain the following:

$$\begin{aligned} C_{\mathcal{G}} &\le \lim_{\lambda\to+\infty}\sup \Psi\Big(\varrho_n,\sigma_n\Big) \\ &< C_{\mathcal{G}}, \end{aligned}$$

which is impossible. This implies that $\{\eta_n\}$ is a d_l-Cauchy sequence. From (2.7) and since (Υ, d_l) is a complete metric-like space, there is some $\eta \in \Upsilon$ such that

$$\lim_{n\to+\infty} d_l(\eta_n,\eta) = d_l(\eta,\eta) = \lim_{n,m\to+\infty} d_l(\eta_n,\eta_m) = 0, \tag{2.16}$$

which implies that $d_l(\eta,\eta) = 0$.
Case 1: If condition (3a) is satisfied, the d_l-continuity of Q implies that

$$\begin{aligned} \lim_{n\to+\infty} d_l(\eta_{n+1},Q\eta) &= \lim_{n\to+\infty} d_l(Q\eta_n,Q\eta) \\ &= d_l(Q\eta,Q\eta). \end{aligned} \tag{2.17}$$

By Lemma 1 and (2.16), we obtain the following:

$$\lim_{n\to+\infty} d_l(\eta_{n+1},Q\eta) = d_l(\eta,Q\eta). \tag{2.18}$$

Combining (2.17) and (2.18), we obtain the following:

$$d_l(Q\eta, Q\eta) = d_l(\eta, Q\eta),$$

which implies that $Q\eta = \eta$.

Case 2: Let us suppose that $d_l(Q\eta, \eta) > 0$. From the condition (3b), we can conclude that $\theta(\eta_n, \eta) \geq 1$ and $\alpha(\eta_n, \eta) \geq 1$ for every $n \in \mathbb{N}$. According to (2.1), we obtain the following:

$$\begin{aligned} C_{\mathcal{G}} &\leq \Psi\Big(\alpha(\eta_n, \eta)\theta(\eta_n, \eta)d_l(Q\eta_n, Q\eta), \beta(M(\eta_n, \eta))M(\eta_n, \eta)\Big) \\ &< \mathcal{G}\Big(\beta(M(\eta_n, \eta))M(\eta_n, \eta), \alpha(\eta_n, \eta)\theta(\eta_n, \eta)d_l(Q\eta_n, Q\eta)\Big), \end{aligned} \tag{2.19}$$

which implies that

$$\begin{aligned} \alpha(\eta_n, \eta)\theta(\eta_n, \eta)d_l(Q\eta_n, Q\eta) &\leq \beta(M(\eta_n, \eta))M(\eta_n, \eta) \\ &< M(\eta_n, \eta), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} M(\eta_n, \eta) &= \max\left\{d_l(\eta_n, \eta),\ d_l(\eta_n, Q\eta_n), d_l(\eta, Q\eta)\right\} \\ &= \max\left\{d_l(\eta_n, \eta), d_l(\eta_n, \eta_{n+1}), d_l(\eta, Q\eta)\right\}. \end{aligned}$$

Then, from (2.16), we can obtain the following:

$$\begin{aligned} \lim_{n\to+\infty} M(\eta_n, \eta) &= \lim_{n\to+\infty} \max\left\{d_l(\eta_n, \eta), d_l(\eta_n, \eta_{n+1}), d_l(\eta, Q\eta)\right\} \\ &= \max\left\{d_l(\eta, \eta), d_l(\eta, \eta), d_l(\eta, Q\eta)\right\} \\ &= d_l(\eta, Q\eta). \end{aligned} \tag{2.21}$$

It holds that

$$d_l(Q\eta_n, Q\eta) \leq \alpha(\eta_n, \eta)\theta(\eta_n, \eta)d_l(Q\eta_n, Q\eta). \tag{2.22}$$

Then, from (2.20) and (2.22), we conclude that

$$\begin{aligned} d_l(Q\eta_n, Q\eta) &\leq \alpha(\eta_n, \eta)\theta(\eta_n, \eta)d_l(Q\eta_n, Q\eta) \\ &\leq \beta(M(\eta_n, \eta))M(\eta_n, \eta) \\ &< M(\eta_n, \eta). \end{aligned} \tag{2.23}$$

Hence,

$$\frac{d_l(\eta_{n+1}, Q\eta)}{d_l(\eta, Q\eta)} \leq \beta(M(\eta_n, \eta))\frac{M(\eta_n, \eta)}{d_l(\eta, Q\eta)} < \frac{M(\eta_n, \eta)}{d_l(\eta, Q\eta)}. \tag{2.24}$$

Using (2.21), (2.24) and taking $n \to +\infty$, we obtain the following:

$$\lim_{n\to+\infty} \beta(M(\eta_n, \eta)) = 1.$$

Since β is a Geraghty function, it follows that

$$\lim_{n\to+\infty} M(\eta_n, \eta) = 0,$$

which is a contradiction. Therefore, $d_l(\eta, Q\eta) = 0$, which implies that $\eta = Q\eta$. The proof is finished. □

In order to establish the uniqueness, an additional condition is necessary, which is as follows:
(C) $\alpha(\eta,\mu) \geq 1$ and $\theta(\eta,\mu) \geq 1$ for every η,μ in Fix (Q), where Fix(Q) denotes the collection of all fixed points of Q.

Theorem 4. *If the conditions of the Theorem 3 are satisfied, particularly the condition (C), then the operator Q possesses a unique fixed point.*

Proof. In order to prove the uniqueness of the fixed point, let us suppose that there exist $\mu,\eta \in \Upsilon$ such that $Q\mu = \mu$, $Q\eta = \eta$ and $\mu \neq \eta$. From Theorem 3, we have the following:

$$d_l(\eta,\eta) = d_l(\mu,\mu) = 0. \tag{2.25}$$

On the other hand, according to (2.1), we obtain the following:

$$\begin{aligned} C_{\mathcal{G}} &\leq \Psi\Big(\alpha(\eta,\mu)\theta(\eta,\mu)d_l(Q\eta,Q\mu), \beta(M(d_l(\eta,\mu))M(d_l(\eta,\mu))\Big) \\ &< \mathcal{G}\Big(\beta(M(d_l(\eta,\mu))M(d_l(\eta,\mu)), \alpha(\eta,\mu)\theta(\eta,\mu)d_l(Q\eta,Q\mu)\Big) \\ &= \mathcal{G}\Big(\beta(M(d_l(\eta,\mu))M(d_l(\eta,\mu)), \alpha(\eta,\mu)\theta(\eta,\mu)d_l(\eta,\mu)\Big). \end{aligned}$$

Thus,

$$\begin{aligned} \alpha(\eta,\mu)\theta(\eta,\mu)d_l(\eta,\mu) &< \beta(M(d_l(\eta,\mu))M(d_l(\eta,\mu)) \\ &< M(d_l(\eta,\mu)). \end{aligned} \tag{2.26}$$

From (2.25), we conclude that

$$\begin{aligned} M(d_l(\eta,\mu)) &= \max\Big\{d_l(\eta,\mu), d_l(\eta,Q\eta), d_l(\mu,Q\mu)\Big\} \\ &= \max\Big\{d_l(\eta,\mu), d_l(\eta,\eta), d_l(\mu,\mu)\Big\} \\ &= d_l(\eta,\mu). \end{aligned}$$

Hence, from (2.26) and the condition (C), we conclude that

$$\begin{aligned} d_l(\eta,\mu) &\leq \alpha(\eta,\mu)\theta(\eta,\mu)d_l(\eta,\mu) \\ &< d_l(\eta,\mu), \end{aligned}$$

which is a contradiction, thus $\eta = \mu$. □

Theorem 5. *Let (Υ, d_l) be a complete metric-like space and $Q : \Upsilon \to \mathcal{K}(\Upsilon)$ be a map, such that*

$$\theta(\eta,\mu) \geq 1 \text{ and } \alpha(\eta,\mu) \geq 1 \text{ implies that } \Psi\Big(d_l(Q\eta,Q\mu),\ \beta(E(\eta,\mu))E(\eta,\mu)\Big) \geq C_{\mathcal{G}} \tag{2.27}$$

for every $\eta,\mu \in \Upsilon$, $\eta \neq \mu$, where

$$E(\eta,\mu) = d_l(\eta,\mu) + |d_l(\eta,Q\eta) - d_l(\mu,Q\mu)|.$$

Additionally, assume the following:

(1) Q is a triangular (α,θ)-admissible,

(2) there is $\eta_0 \in \Upsilon$ *such that* $\theta(\eta_0, Q\eta) \geq 1$ *and* $\alpha(\eta_0, Q\eta) \geq 1$,

(3) either

(3a) Q *is* d_l*-continuous*

or

(3b) if $\{\eta_n\} \subset \Upsilon$ *such that* $\theta(\eta_n, \eta_{n+1}) \geq 1$ *and* $\alpha(\eta_n, \eta_{n+1}) \geq 1$, *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to+\infty} d_l(\eta_n, \eta) = d_l(\eta, \eta)$, *then we obtain* $\theta(\eta_n, \eta) \geq 1$ *and* $\alpha(\eta_n, \eta) \geq 1$ *for every* $n \in \mathbb{N}$.

Then, Q *possesses a unique fixed point* $\eta \in \Upsilon$ *with* $d_l(\eta, \eta) = 0$.

Proof. From (2), there is $\eta_0 \in \Upsilon$ such that $\theta(\eta_0, Q\eta_0) \geq 1$ and $\alpha(\eta_0, Q\eta_0) \geq 1$. We define a sequence $\{\eta_n\}$ with an initial point η_0 such that $\eta_{n+1} = Q\eta_n$, for all $n \geq 0$. If $\eta_m = \eta_{m+1}$ for some $m \in \mathbb{N}$, then η_m is a fixed point of Q and the proof is completed. Let us assume that $\eta_n \neq Q\eta_n$, for all $n \in \mathbb{N}$. Using the (α, θ)-admissibility of Q and assumptions (2), we obtain the following:

$$\alpha(\eta_1, \eta_2) \geq 1 \text{ and } \theta(\eta_1, \eta_2) \geq 1.$$

Repeating this process, we obtain the following:

$$\alpha(\eta_n, \eta_{n+1}) \geq 1 \text{ and } \theta(\eta_n, \eta_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.28}$$

We conclude that $d_l(\eta_n, \eta_{n+1})$ is decreasing. Let us assume that

$$d_l(\eta_n, \eta_{n+1}) < d_l(\eta_{n+1}, \eta_{n+2}). \tag{2.29}$$

From (2.27) and (2.28), we find that

$$\begin{aligned} C_{\mathcal{G}} &\leq \Psi\Big(d_l(Q\eta_n, Q\eta_{n+1}), \beta(E(\eta_n, \eta_{n+1}))E(\eta_n, \eta_{n+1})\Big) \\ &< \mathcal{G}\Big(\beta(E(\eta_n, \eta_{n+1}))E(\eta_n, \eta_{n+1}), d_l(Q\eta_n, Q\eta_{n+1})\Big), \end{aligned}$$

and, thus, we obtain the following:

$$d_l(Q\eta_n, Q\eta_{n+1}) < \beta(E(\eta_n, \eta_{n+1}))E(\eta_n, \eta_{n+1}). \tag{2.30}$$

Hence, from inequality (2.30) we obtain the following:

$$\begin{aligned} d_l(\eta_{n+1}, \eta_{n+2}) &= d_l(Q\eta_n, Q\eta_{n+1}) \\ &< \beta(E(\eta_n, \eta_{n+1}))E(\eta_n, \eta_{n+1}) \\ &< E(\eta_n, \eta_{n+1}). \end{aligned} \tag{2.31}$$

On the other hand, we have

$$E(\eta_n, \eta_{n+1}) = d_l(\eta_n, \eta_{n+1}) + |d_l(\eta_n, Q\eta_n) - d_l(\eta_{n+1}, Q\eta_{n+1})|.$$

Using (2.29), we obtain the following:

$$\begin{aligned} E(\eta_n, \eta_{n+1}) &= d_l(\eta_n, \eta_{n+1}) - d_l(\eta_n, Q\eta_n) + d_l(\eta_{n+1}, Q\eta_{n+1}) \\ &\leq d_l(\eta_{n+1}, \eta_{n+2}). \end{aligned} \tag{2.32}$$

Hence, from (2.32), the inequality (2.31) turns into

$$\begin{aligned} d_l(\eta_{n+1}, \eta_{n+2}) &< E(\eta_n, \eta_{n+1}) \\ &< d_l(\eta_{n+1}, \eta_{n+2}), \end{aligned} \tag{2.33}$$

which is a contradiction. Consequently, we conclude that $\{d_l(\eta_n, \eta_{n+1})\}$ is a decreasing sequence. Therefore, $d_l(\eta_n, \eta_{n+1})$ is a decreasing sequence of positive real numbers. Hence, there is $\gamma \geq 0$ such that

$$\lim_{n\to+\infty} d_l(\eta_n, \eta_{n+1}) = \gamma.$$

Thus,

$$\lim_{n\to+\infty} E(\eta_n, \eta_{n+1}) = \gamma.$$

Let us assume that $\gamma > 0$. Then, using the inequality (2.31), we obtain the following:

$$\lim_{n\to+\infty} \beta(E(\eta_n, \eta_{n+1})) = 1. \tag{2.34}$$

Since β is a Geraghty function, then

$$\lim_{n\to+\infty} E(\eta_n, \eta_{n+1}) = 0,$$

is a contradiction and hence $\gamma = 0$. Therefore,

$$\lim_{n\to+\infty} d_l(\eta_n, \eta_{n+1}) = 0. \tag{2.35}$$

Now, we prove that $\{\eta_n\}$ is a d_l-Cauchy sequence. Assuming the opposite that it is not, it follows that there exists ϵ for which we can find subsequences $\{\eta_{n_\lambda}\}$ and $\{\eta_{m_\lambda}\}$ of $\{\eta_n\}$, where $m_\lambda > n_\lambda > \lambda$ such that for every λ,

$$d_l(\eta_{n_\lambda}, \eta_{m_\lambda}) \geq \epsilon. \tag{2.36}$$

With the same reasoning as in the Theorem 3, we show that

$$\lim_{\lambda\to+\infty} d_l(\eta_{n_\lambda}, \eta_{m_\lambda}) = \lim_{\lambda\to+\infty} d_l(\eta_{n_\lambda+1}, \eta_{m_\lambda+1}) = \epsilon \tag{2.37}$$

and, consequently, it follows that

$$\lim_{\lambda\to+\infty} E(\eta_{n_\lambda}, \eta_{m_\lambda}) = \epsilon. \tag{2.38}$$

On the other hand, let $\eta = \eta_{n_\lambda}$ and $\mu = \eta_{m_\lambda}$. Since Q is a triangular (α, θ)-admissible, then, using (2.27) and (2.28) we obtain the following:

$$\begin{aligned} C_{\mathcal{G}} &\leq \Psi\Big(d_l(Q\eta_{n_\lambda}, Q\eta_{m_\lambda}), \beta(E(\eta_{n_\lambda}, \eta_{m_\lambda}))E(\eta_{n_\lambda}, \eta_{m_\lambda})\Big) \\ &< \mathcal{G}\Big(\beta(E(\eta_{n_\lambda}, \eta_{m_\lambda}))E(\eta_{n_\lambda}, \eta_{m_\lambda}), d_l(Q\eta_{n_\lambda}, Q\eta_{m_\lambda})\Big). \end{aligned}$$

Hence,

$$\begin{aligned} d_l(Q\eta_{n_\lambda}, Q\eta_{m_\lambda}) &< \beta(E(\eta_{n_\lambda}, \eta_{m_\lambda}))E(\eta_{n_\lambda}, \eta_{m_\lambda}) \\ &< E(\eta_{n_\lambda}, \eta_{m_\lambda}). \end{aligned}$$

Thus,

$$\begin{aligned} d_l(Q\eta_{n_\lambda}, Q\eta_{m_\lambda}) &< \beta(E(\eta_{n_\lambda}, \eta_{m_\lambda}))E(\eta_{n_\lambda}, \eta_{m_\lambda}) \\ &< E(\eta_{n_\lambda}, \eta_{m_\lambda}). \end{aligned}$$

Using (2.37) and (2.38), we obtain the following:

$$\lim_{\lambda\to+\infty} d_l(Q\eta_{n_\lambda}, Q\eta_{m_\lambda}) = \epsilon$$

and

$$\lim_{\lambda\to+\infty} \beta(E(\eta_{n_\lambda}, \eta_{m_\lambda}))E(\eta_{n_\lambda}, \eta_{m_\lambda}) = \epsilon.$$

Then, using (2.27) and (2) from Definition 4, we obtain the following:

$$\begin{aligned} C_{\mathcal{G}} &\leq \lim_{\lambda\to+\infty} \sup \Psi\Big(d_l(Q\eta_{n_\lambda}, Q\eta_{m_\lambda}), \beta(E(\eta_{n_\lambda}, \eta_{m_\lambda}))E(\eta_{n_\lambda}, \eta_{m_\lambda})\Big) \\ &< C_{\mathcal{G}}, \end{aligned}$$

which is a contradiction. This implies that $\{\eta_n\}$ is a d_l-Cauchy sequence. Using (2.35) and the completeness of (Υ, d_l), there is some $\eta \in \Upsilon$ such that

$$\lim_{n\to+\infty} d_l(\eta_n, \eta) = d_l(\eta, \eta) = \lim_{n,m\to+\infty} d_l(\eta_n, \eta_m) = 0, \tag{2.39}$$

which implies that $d_l(\eta, \eta) = 0$.

Case 1: From (3a), if Q is a d_l-continuous mapping, with the same reasoning as in Case 1 of the Theorem 3, we show that $Q\eta = \eta$.

Case 2: From (3b), if Q is not a d_l-continuous mapping, we obtain $\alpha(\eta_n, \eta) \geq 1$ for every $n \in \mathbb{N}$. According to (2.27), we have

$$\begin{aligned} C_{\mathcal{G}} &\leq \Psi\Big(d_l(Q\eta_n, Q\eta), \beta(E(\eta_n, \eta))E(\eta_n, \eta)\Big) \\ &< \mathcal{G}\Big(\beta(E(\eta_n, \eta))E(\eta_n, \eta), d_l(Q\eta_n, Q\eta)\Big), \end{aligned} \tag{2.40}$$

and, thus, we obtain the following:

$$\begin{aligned} d_l(Q\eta_n, Q\eta) &\leq \beta(E(\eta_n, \eta))E(\eta_n, \eta) \\ &< E(\eta_n, \eta), \end{aligned} \tag{2.41}$$

where

$$E(\eta_n, \eta) = d_l(\eta_n, \eta) + |d_l(\eta_n, Q\eta_n) - d_l(\eta, Q\eta)|$$

$$= d_l(\eta_n, \eta) + |d_l(\eta_n, \eta_{n+1}) - d_l(\eta, Q\eta)|.$$

Then, from (2.39), we obtain the following:

$$\lim_{n\to+\infty} E(\eta_n, \eta) = d_l(\eta, Q\eta). \tag{2.42}$$

Since we know that $\alpha(\eta_n, \eta) \geq 1$, from (2.41), we conclude that

$$\begin{aligned} d_l(\eta_{n+1}, Q\eta) &\leq d_l(Q\eta_n, Q\eta) \\ &\leq \beta(E(\eta_n, \eta))E(\eta_n, \eta) \\ &< E(\eta_n, \eta). \end{aligned} \tag{2.43}$$

Let us suppose that $d_l(\eta, Q\eta) > 0$. Then, from (2.42), (2.43) and letting $n \to +\infty$, we obtain the following:

$$\lim_{n\to+\infty} \beta(E(\eta_n, \eta)) = 1.$$

Since β is a Gerahty function, then

$$\lim_{n\to+\infty} E(\eta_n, \eta) = 0,$$

which is a contradiction. This implies that $d_l(\eta, Q\eta) = 0$, and thus, $\eta = Q\eta$. The proof is finished. □

In order to prove the uniqueness, it is necessary to add the condition (C).

Theorem 6. *If conditions of Theorem 5 are satisfied, particularly the condition (C), then Q possesses a unique fixed point.*

Proof. To prove the uniqueness of the fixed point, suppose that there exist $\mu, \eta \in \Upsilon$ such that $Q\mu = \mu$, $Q\eta = \eta$ and $\mu \neq \eta$. According to Theorem 5, we have

$$d_l(\eta, \eta) = d_l(\mu, \mu) = 0. \tag{2.44}$$

On the other hand, according to (2.27), we obtain the following:

$$\begin{aligned} C_{\mathcal{G}} &\leq \Psi\Big(\alpha(\eta, \mu)\theta(\eta, \mu)d_l(Q\eta, Q\mu), \beta(E(\eta, \mu)E(\eta, \mu)\Big) \\ &< \mathcal{G}\Big(\beta(E(\eta, \mu)E(\eta, \mu), \alpha(\eta, \mu)\theta(\eta, \mu)d_l(Q\eta, Q\mu)\Big) \\ &= \mathcal{G}\Big(\beta(E(\eta, \mu)E(\eta, \mu), \alpha(\eta, \mu)\theta(\eta, \mu)d_l(\eta, \mu)\Big). \end{aligned}$$

Thus,

$$\begin{aligned} \alpha(\eta, \mu)\theta(\eta, \mu)d_l(\eta, \mu) &< \beta(E(\eta, \mu)E(\eta, \mu) \\ &< E(\eta, \mu). \end{aligned} \tag{2.45}$$

From (2.44), we conclude that

$$E(d_l(\eta, \mu) = d_l(\eta, \mu) + |d_l(\eta, Q\eta) - d_l(\mu, Q\mu)|$$

$$
\begin{aligned}
&= d_l(\eta,\mu) + |d_l(\eta,\eta) - d_l(\mu,\mu)| \\
&= d_l(\eta,\mu).
\end{aligned}
$$

Hence, from (2.45) and the condition (C), we conclude that

$$
\begin{aligned}
d_l(\eta,\mu) &\le \alpha(\eta,\mu)\theta(\eta,\mu)d_l(\eta,\mu) \\
&< d_l(\eta,\mu),
\end{aligned}
$$

which is a contradiction, so $\eta = \mu$. □

Example 2. *Let us consider $\Upsilon = \{0,1,2\}$ and let $d_l : \Upsilon^2 \to \mathbb{R}^+$ be defined as follows:*

$$
\begin{aligned}
&d_l(0,0) = 0, \\
&d_l(1,0) = d_l(0,1) = 7, \\
&d_l(0,2) = d_l(2,0) = 3, \\
&d_l(1,2) = d_l(2,1) = 4, \\
&d_l(1,1) = 3, d_l(2,2) = 1.
\end{aligned}
$$

Clearly, (Υ, d_l) is metric-like space. Let $\theta, \alpha : \Upsilon^2 \to \mathbb{R}^+$ be defined as follows:

$$
\theta(\eta,\mu) = \begin{cases} \frac{3}{2}, & \text{if} \quad \eta \in \{0,1,2\} \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \alpha(\eta,\mu) = \begin{cases} 2, & \text{if} \quad \eta \in \{0,1,2\} \\ 0, & \text{otherwise.} \end{cases}
$$

Let $Q : \Upsilon \to \Upsilon$ be defined as follows:

$$
Q\eta = \begin{cases} 2, & \eta \in \{1,2\} \\ 0, & \text{otherwise.} \end{cases}
$$

Then, for all $\eta,\mu \in \Upsilon$ such that $\theta(\eta,\mu) \ge 1$ and $\alpha(\eta,\mu) \ge 1$, we obtain $\theta(Q\eta, Q\mu) \ge 1$ and $\alpha(Q\eta, Q\mu) \ge 1$, which implies that Q is an (α,θ)-admissible. Moreover, for all $\eta,\mu,\nu \in \Upsilon$ such that $\theta(\eta,\mu) \ge 1$, $\theta(\mu,\nu) \ge 1$, $\alpha(\eta,\mu) \ge 1$ and $\alpha(\mu,\nu) \ge 1$, it holds that $\eta,\mu,\nu \in \{0,1,2\}$. Thus, it follows that $\theta(\eta,\nu) \ge 1$ and $\alpha(\eta,\nu) \ge 1$, which finally implies that Q is a triangular (α,θ)-admissible mapping.

On the other hand, if we take $\eta = 1$, then the condition (2) of the Theorem 3 is satisfied. Next, let us consider the following mappings

$$
\begin{aligned}
\mathcal{G}(\sigma,\varrho) &= \sigma - \varrho, \\
\Psi(\varrho,\sigma) &= \frac{3}{4}\varrho - \sigma, \\
\beta(t) &= \frac{1}{t+1},
\end{aligned}
$$

for every $t,\sigma,\varrho \in \mathbb{R}_+ \cup \{0\}$. It is clear that β, $\mathcal{G}$ and Ψ are the Geraghty function, the C-class function and the $C_{\mathcal{G}}$-simulation function, respectively.

Now, we consider the following cases.

***Case 1.** $(\eta,\mu) = (0,0)$: It holds that $M(0,0) = \max\{d_l(0,0), d_l(0,Q0), d_l(0,Q0)\} = 0$. Then,*

$$
\Psi\Big(\alpha(0,0)\theta(0,0)d_l(Q0,Q0),\ \beta(M(0,0))M(0,0)\Big) = \Psi\left(2\cdot\frac{3}{2}\cdot 0,\ 0\right) = \Psi(0,0) = 0.
$$

Case 2. $(\eta,\mu)=(0,1)$*: It holds that* $M(0,1)=\max\{d_l(0,1),d_l(0,Q0),d_l(1,Q1)\}=7$. *Then,*

$$\Psi\Big(\alpha(0,1)\theta(0,1)d_l(Q0,Q1),\ \beta(M(0,1))M(0,1)\Big)=\Psi\left(2\cdot\frac{3}{2}\cdot 3,\ \beta(7)\cdot 7\right)=\Psi\left(9,\frac{7}{8}\right)=\frac{27}{4}-\frac{7}{8}.$$

Case 3. $(\eta,\mu)=(0,2)$*: It holds that* $M(0,2)=\max\{d_l(0,2),d_l(0,Q0),d_l(2,Q2)\}=3$. *Then,*

$$\Psi\Big(\alpha(0,2)\theta(0,2)d_l(Q0,Q2),\ \beta(M(0,2))M(0,2)\Big)=\Psi\left(2\cdot\frac{3}{2}\cdot 3,\ \beta(3)\cdot 3\right)=\Psi\left(9,\frac{3}{4}\right)=\frac{27}{4}-\frac{3}{4}.$$

Case 4. $(\eta,\mu)=(1,1)$*: It holds that* $M(1,1)=\max\{d_l(1,1),d_l(1,Q1),d_l(1,Q1)\}=4$. *Then,*

$$\Psi\Big(\alpha(1,1)\theta(1,1)d_l(Q1,Q1),\ \beta(M(1,1))M(1,1)\Big)=\Psi\left(2\cdot\frac{3}{2}\cdot 1,\ \beta(4)\cdot 4\right)=\Psi\left(3,\frac{4}{5}\right)=\frac{9}{4}-\frac{4}{5}.$$

Case 5. $(\eta,\mu)=(1,2)$*: It holds that* $M(1,2)=\max\{d_l(1,2),d_l(1,Q1),d_l(2,Q2)\}=4$. *Then,*

$$\Psi\Big(\alpha(1,2)\theta(1,2)d_l(Q1,Q2),\ \beta(M(1,2))M(1,2)\Big)=\Psi\left(2\cdot\frac{3}{2}\cdot 1,\ \beta(4)\cdot 4\right)=\Psi\left(3,\frac{4}{5}\right)=\frac{9}{4}-\frac{4}{5}.$$

Case 6. $(\eta,\mu)=(2,2)$*: It holds that* $M(2,2)=\max\{d_l(2,2),d_l(2,Q2),d_l(2,Q2)\}=1$ *Then,*

$$\Psi\Big(\alpha(2,2)\theta(2,2)d_l(Q2,Q2),\ \beta(M(2,2))M(2,2)\Big)=\Psi\left(2\cdot\frac{3}{2}\cdot 1,\ \beta(1)\cdot 1\right)=\Psi\left(3,\frac{1}{2}\right)=\frac{9}{4}-\frac{1}{2}.$$

Finally, for all $\eta,\mu\in\Upsilon$ *we have the following:*

$$\begin{aligned}0&\le\Psi\Big(\alpha(\eta,\mu)\theta(\eta,\mu)d_l(Q\eta,Q\mu),\beta(M(\eta,\mu))M(\eta,\mu)\Big)\\&<\mathcal{G}\Big(\alpha(\eta,\mu)\theta(\eta,\mu)d_l(Q\eta,Q\mu),\beta(M(\eta,\mu))M(\eta,\mu)\Big).\end{aligned}\tag{2.46}$$

Then, using inequality (2.46) and Definition 12, it is clear that the mapping Q *is* $\Psi_{C_{\mathcal{G}}}$*-*α*-*θ*-Geraghty contraction where* $C_{\mathcal{G}}=0$*. Thus, assumptions of Theorem 3 are satisfied. Hence,* Q *possesses fixed points in* Υ.

Example 3. *Let* $\Upsilon=[0,\infty)$, $d_l(\eta,\mu)=(\eta+\mu)$ *for all* $\eta,\ \mu\in\Upsilon$ *and* $Q:\Upsilon\to\Upsilon$ *be defined as follows:*

$$Q\mu=\begin{cases}\frac{\mu}{16}, & \text{if } 0\le\mu\le 1\\ 4\mu, & \text{otherwise.}\end{cases}$$

Let us consider $\Psi(\eta,\mu)=\eta\gamma-\mu$*, where* $0\le\dfrac{1}{4}<\gamma<1$ *and define* $\alpha,\theta:\Upsilon\times\Upsilon\to\mathbb{R}_+$ *as*

$$\theta(\eta,\mu)=\begin{cases}\frac{5}{3}, & \text{if } 0\le\eta,\mu\le 1\\ 0, & \text{otherwise}\end{cases}$$

and

$$\alpha(\eta,\mu)=\begin{cases}\frac{6}{5}, & \text{if } 0\le\eta,\mu\le 1\\ 0, & \text{otherwise.}\end{cases}$$

We shall prove that the Theorem 3 can be applied. Clearly, (Υ, d_l) *is a complete metric-like space. Let* $\eta, \mu \in \Upsilon$ *such that* $\alpha(\eta, \mu) \geq 1$ *and* $\theta(\eta, \mu) \geq 1$*. Since* $\eta, \mu \in [0, 1]$*, then* $Q\eta \in [0, 1]$*,* $Q\mu \in [0, 1]$*,* $\alpha(Q\eta, Q\mu) = 1$ *and* $\theta(Q\eta, Q\mu) = 1$*. Hence,* Q *is* (α, θ)*-admissible. Moreover, for all* $\eta, \mu, \nu \in \Upsilon$ *such that* $\theta(\eta, \mu) \geq 1$*,* $\theta(\mu, \nu) \geq 1$*,* $\alpha(\eta, \mu) \geq 1$ *and* $\alpha(\mu, \nu) \geq 1$*, it holds that* $\eta, \mu, \nu \in [0, 1]$*. Thus, it follows that* $\theta(\eta, \nu) \geq 1$ *and* $\alpha(\eta, \nu) \geq 1$*, which finally implies that* Q *is a triangular* (α, θ)*-admissible mapping. Condition (2) is satisfied when* $\eta_0 = 1$ *and condition (3b) is satisfied when* $\eta_n = Q^n\eta_0 = \frac{1}{16^n}$*.*

If $\eta \in [0, 1]$*, then* $\alpha(\eta, \mu) = \frac{6}{5}$ *and* $\theta(\eta, \mu) = \frac{5}{3}$*. Now, let us consider the following mappings*

$$\begin{aligned} \mathcal{G}(\sigma, \varrho) &= \sigma - \varrho, \\ \Psi(\varrho, \sigma) &= \frac{1}{2}\varrho - \sigma, \\ \beta(t) &= \frac{1}{t+1}, \end{aligned}$$

for every $t, \sigma, \varrho \in \mathbb{R}_+ \cup \{0\}$*. It is clear that* β*,* $\mathcal{G}$ *and* Ψ *are the Geraghty function, the C-class function and the* $C_{\mathcal{G}}$*-simulation function, respectively. We have the following:*

$$\begin{aligned} d_l(Q\eta, Q\mu) &= Q\eta + Q\mu \\ &= \frac{1}{16} d_l(\eta, \mu) \\ &\leq \frac{1}{16} M(\eta, \mu). \end{aligned}$$

On the other hand, for any $\eta, \mu \in \Upsilon$*, we obtain* $M(\eta, \mu) \in [0, 2]$*. Hence,*

$$3M(\eta, \mu) - M^2(\eta, \mu) \geq 0.$$

Then,

$$\begin{aligned} \Psi\Big(\alpha(\eta, \mu)\theta(\eta, \mu)d_l(Q\eta, Q\mu), \beta(M(\eta, \mu))M(\eta, \mu)\Big) &= \frac{1}{2}M(\eta, \mu)\beta(M(\eta, \mu)) - \alpha(\eta, \mu)\theta(\eta, \mu)d_l(Q\eta, Q\mu) \\ &= \frac{1}{2}M(\eta, \mu)\beta(M(\eta, \mu)) - 2d_l(Q\eta, Q\mu) \\ &= \frac{M(\eta, \mu)}{2(M(\eta, \mu) + 1)} - 2d_l(Q\eta, Q\mu) \\ &\geq \frac{M(\eta, \mu)}{2(M(\eta, \mu) + 1)} - \frac{1}{8}M(\eta, \mu) \\ &= \frac{3M(\eta, \mu) - M^2(\eta, \mu)}{8(M(\eta, \mu) + 1)} \geq 0. \end{aligned}$$

If $\eta \in [0, 1]$ *and* $\mu \notin [0, 1]$*, then* $\Psi\Big(\alpha(\eta, \mu)\theta(\eta, \mu)d_l(Q\eta, Q\mu), \beta(M(\eta, \mu))M(\eta, \mu)\Big) \geq 0$ *since* $\alpha(\eta, \mu) = \theta(\eta, \mu) = 0$*. Finally, for all* $\eta, \mu \in \Upsilon$*, we obtain the following:*

$$\begin{aligned} C_{\mathcal{G}} = 0 &\leq \Psi\Big(\alpha(\eta, \mu)\theta(\eta, \mu)d_l(Q\eta, Q\mu), \beta(M(\eta, \mu))M(\eta, \mu)\Big) \\ &< \mathcal{G}\Big(\alpha(\eta, \mu)\theta(\eta, \mu)d_l(Q\eta, Q\mu), \beta(M(\eta, \mu))M(\eta, \mu)\Big). \end{aligned}$$

Consequently, all assumptions of Theorem 3 are satisfied and hence Q *has unique fixed point, which is* 0*.*

3. Consequences

Using our main results, we can easily reach several classical fixed point results, which we present in this section.

Corollary 1. *Let* (Υ, d_l) *be a complete metric-like space and* $\mathcal{Q} : \Upsilon \to \mathcal{K}(\Upsilon)$ *be a map such that*

$$\Psi\Big(\alpha(\eta,\mu)\theta(\eta,\mu)d_l(\mathcal{Q}\eta,\mathcal{Q}\mu),\ \beta(\mathcal{D}(\eta,\mu))\mathcal{D}(\eta,\mu)\Big) \geq C_{\mathcal{G}}$$

for every $\eta,\mu \in \Upsilon$*, where* $\eta \neq \mu$ *and*

$$\mathcal{D}(\eta,\mu) = \max\left\{d_l(\eta,\mu), \frac{1}{2}(d_l(\eta,\mathcal{Q}\eta) + d_l(\mu,\mathcal{Q}\mu))\right\}.$$

Moreover, assume the following:

(1) there is $\eta_0 \in \Upsilon$ *such that* $\theta(\eta_0,\ \eta_1) \geq 1$ *and* $\alpha(\eta_0,\ \eta_1) \geq 1$*,*
(2) $\mathcal{Q}$ *is triangular* (α,θ)*-admissible,*
(3) either

(3a) $\mathcal{Q}$ *is* d_l*-continuous*
or
(3b) if $\{\eta_n\} \subset \Upsilon$ *such that* $\theta(\eta_n,\eta_{n+1}) \geq 1$ *and* $\alpha(\eta_n,\eta_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to+\infty} \eta_n = \eta \in \Upsilon$*, then we obtain* $\alpha(\eta_n,\eta) \geq 1$ *for every* $n \in \mathbb{N}$*.*

Then, $\mathcal{Q}$ *possesses a fixed point.*

Proof. We have the following:

$$\begin{aligned}\mathcal{D}(\eta,\mu) &= \max\left\{d_l(\eta,\mu), \frac{1}{2}(d_l(\eta,\mathcal{Q}\eta) + d_l(\mu,\mathcal{Q}\mu))\right\}\\ &\leq \max\left\{d_l(\eta,\mu), d_l(\eta,\mathcal{Q}\eta), d_l(\mu,\mathcal{Q}\mu)\right\}\\ &= M(\eta,\mu).\end{aligned}$$

Then, we can obtain the result using Theorem 3. □

Corollary 2. *Let* (Υ, d_l) *be a complete metric-like space and let* $\mathcal{Q} : \Upsilon \to \Upsilon$ *be a map, such that*

$$\mathcal{Z}\Big(\alpha(\eta,\mu)\theta(\eta,\mu)d_l(\mathcal{Q}\eta,\mathcal{Q}\mu),\ \beta(d_l(\eta,\mu))d_l(\eta,\mu)\Big) \geq 0$$

for every $\eta,\mu \in \Upsilon$*, where* $\eta \neq \mu$ *and*

i) $\mathcal{Q}$ *is triangular* (α,θ)*-admissible*
ii) there exists $\eta_0 \in \Upsilon$ *such that* $\alpha(\eta_0,\mathcal{Q}\eta_0) \geq 1$ *and* $\theta(\eta_0,\mathcal{Q}\eta_0) \geq 1$
iii) $\mathcal{Q}$ *is* d_l*-continuous.*

Then, $\mathcal{Q}$ *possesses a fixed point* $\eta \in \Upsilon$ *such that* $d_l(\eta,\eta) = 0$*.*

Proof. We have the following:

$$
\begin{aligned}
d_l(\eta,\mu) &\leq \max\left\{d_l(\eta,\mu), d_l(\eta,Q\eta), d_l(\mu,Q\mu)\right\}\\
&= M(\eta,\mu).
\end{aligned}
$$

For $C_{\mathcal{G}} = 0$, $\Psi(\varrho,\sigma) = \mathcal{Z}(\varrho,\sigma)$ and $\mathcal{G}(\sigma,\varrho) = \sigma - \varrho$, all assumptions of Theorem 3 are satisfied and Q possesses an unique fixed point. □

In Theorem 3, if we consider that $\theta(\eta,\mu) = \alpha(\eta,\mu) = 1$ for every $\eta,\mu \in \Upsilon$, we obtain the following result.

Corollary 3. *Let (Υ, d_l) be a complete metric-like space and Q be a self-mapping on Υ. Assume that there is $C_{\mathcal{G}}$-simulation function Ψ such that*

$$\Psi\big(d_l(Q\eta,Q\mu),\beta(M(\eta,\mu))M(\eta,\mu)\big) \geq C_{\mathcal{G}}$$

for all $\eta,\mu \in \Upsilon$. Then, Q possesses an unique fixed point $\eta \in \Upsilon$ such that $d_l(\eta,\eta) = 0$.

4. Conclusions

In the context of metric-like space, we developed the idea of the $\Psi_{C_{\mathcal{G}}}$-α-θ-Geraghty contraction mapping and explored some relevant findings about the existence and uniqueness of a fixed point for such mappings via $C_{\mathcal{G}}$-simulation function, which, in return, generalizes, extends and combines findings from the literature.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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