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Research article

Conformal bi-slant submersion from Kenmotsu manifolds

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Abstract: The prospect of conformal bi-slant submersions from a Kenmotsu manifold is discussed in the present article, taking into account that the Reeb vector field ξ is vertical. We looked at the integrability of distributions as well as the geometry of distribution leaves since the concept of bi-slant submersion ensures the presence of slant distributions. Finally, the idea of pluriharmonicity is also described in the paper, along with a supporting example for our research.

Keywords: Kenmotsu manifold; Riemannian submersions; conformal bi-slant submersions
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Abbreviations:

ACM: Almost contact metric; CBSS: Conformal bi-slant submersion; RM: Riemannian manifold; KM: Kenmotsu manifold

1. Introduction

O'Neill [26] and Gray [16] were the ones who first proposed and developed the concept of submersions and immersions. For Riemannian manifolds, they discovered certain Riemannian equations by studying the geometrical characteristics. Submersions theory is an important topic in differential geometry that discusses the properties between differentiable structures. Riemannian submersions is the subject of study throughout both mathematics and physics since it has numerous applications, most notably in the Kaluza-Klein theory and Yang-Mills theory (see [10, 20, 24, 40]). Watson [39] investigated the Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds in the year 1976. Later, Şahin [32] studied geometric characteristics and Riemannian submersions geometry. Using an almost Hermitian manifold, he defined anti-invariant

Riemannian submersions onto Riemannian manifolds. "He demonstrates that, under the almost complex structure of the total manifold, their vertical distribution is anti-invariant." Many authors looked into and expanded on this research by studying anti-invariant submersions [3, 32], semi-invariant submersions [33], slant submersions [13, 34] and semi-slant submersions [19, 27], among other topics. Tastan et al. [37] defined and investigated hemi-slant submersions from almost Hermitian manifolds as a generalization case of semi-invariant and semi-slant submersions.

From almost Hermitian to almost contact metric manifolds, Chinea [11] expanded the notion of Riemannian submersions. He examined base space, total and fibre space from an intrinsic geometric perspective point. Prasad et al. extended the concept of hemi-slant submersions a step further, by defining quasi-bi-slant submersions from an almost contact metric manifold [28, 29]. The results he obtained for submersions were interesting and he also discovered some decomposition theorems.

Fuglede [14] and Ishihara [21] introduced the concept of conformal submersion as a generalization of Riemannian submersions and talked about some of their geometric characteristics. If the positive function $\lambda = 1$, which is dilation, then the conformal submersions become Riemannian submersions. Gudmundsson and Wood [18] investigated conformal holomorphic submersion as a generalization of holomorphic submersion. They were able to obtain the necessary and sufficient conditions for harmonic morphisms of conformal holomorphic submersions. Later on, conformal anti-invariant submersions, [2, 30], conformal semi-invariant submersions [4], conformal slant submersions [6] and conformal semi-slant submersions [5] were studied and defined by Akyol and Şahin. A number of researchers have recently explored the geometry of conformal hemi-slant submersions [1, 23, 35], conformal bi-slant submersions [7] and quasi bi-slant conformal submersions [8] and they have discussed some decomposition theorems. Additionally, they expanded the idea of pluriharmonicity from almost Hermitian manifolds to almost contact metric manifolds. The present paper is a complement of [7]. In [7], ξ is horizontal and in the present paper it is vertical.

In this paper, we investigate conformal bi-slant submersions from a Kenmotsu manifold onto a Riemannian manifold with vertical vector field ξ. The structure of the paper is as follows. Section 2 introduces almost contact metric manifolds, precisely the Kenmotsu manifolds with the properties required for this study. Section 3 includes a definition of conformal bi-slant submersion as well as some noteworthy discoveries. Section 4, details the conditions needed for distribution integrability as well as the total geodesicness of its leaves. This section also discusses how a total space Kenmotsu manifold becomes a locally twisted product manifold. Finally, at the end of the study, the concept of ϕ-pluriharmonicity is addressed.

2. Preliminaries

We start off by providing a few definitions and findings that will be quite helpful for our research and will aid in exploring the central idea of the research paper.

Let $(\bar{O}_1, g_1)$ and $(\bar{O}_2, g_2)$ be Riemannian manifolds, where $\dim(\bar{O}_1) = m$, $\dim(\bar{O}_2) = n$ and $m > n$. A Riemannian submersion $\mathcal{J}\colon \bar{O}_1 \to \bar{O}_2$ is a surjective map of $\bar{O}_1$ onto $\bar{O}_2$ satisfying the following axioms:

(i) $\mathcal{J}$ has maximal rank.

(ii) The differential $\mathcal{J}_*$ preserves the lenghts of horizontal vectors.

For each $q \in \bar{O}_2$, $\mathcal{J}^{-1}(q)$ is an $(m-n)$ dimensional submanifold of $\bar{O}_1$. The submanifolds $\mathcal{J}^{-1}(q), q \in \bar{O}_2$

are called fibers. A vector field on $\bar{O}_1$ is called vertical if it is always tangent to fibers. A vector field on $\bar{O}_1$ is called horizontal if it is always orthogonal to fibers. A vector field X on $\bar{O}_1$ is called basic if X is horizontal and $\mathcal{J}$- related to a vector field X_* on $\bar{O}_2$, i.e., $\mathcal{J}_* X_p = X_{*\mathcal{J}(p)}$, for all $p \in \bar{O}_1$. Note that we denote the projection morphisms on the distributions $\ker\mathcal{J}_*$ and $(\ker\mathcal{J}_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

Definition 2.1. *[9] Let $\mathcal{J}$ be a Riemannian submersion from an ACM manifold $(\bar{O}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{O}_2, g_2)$. Then $\mathcal{J}$ is called a horizontally conformal submersion, if there is a positive function λ such that*

$$g_1(U_1, V_1) = \frac{1}{\lambda^2} g_2(\mathcal{J}_* U_1, \mathcal{J}_* V_1) \tag{2.1}$$

for any $U_1, V_1 \in \Gamma(ker\mathcal{J}_*)^\perp$. It is obvious that every Riemannian submersions is a particularly horizontally conformal submersion with $\lambda = 1$. This λ is usually called the dilation function.

The formulae of $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ are

$$\mathcal{T}(L_1, L_2) = \mathcal{T}_{L_1} L_2 = \mathcal{H}\nabla_{\mathcal{V}L_1} \mathcal{V}L_2 + \mathcal{V}\nabla_{\mathcal{V}L_1} \mathcal{H}L_2, \tag{2.2}$$

$$\mathcal{A}(L_1, L_2) = \mathcal{A}_{L_1} L_2 = \mathcal{V}\nabla_{\mathcal{H}L_1} \mathcal{H}L_2 + \mathcal{H}\nabla_{\mathcal{H}L_1} \mathcal{V}L_2 \tag{2.3}$$

for all vector fields $L_1, L_2 \in \Gamma(T\bar{O}_1)$ [15].

It is obvious that a Riemannian submersion $\mathcal{J}\colon \bar{O}_1 \to \bar{O}_2$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. Taking account the fact from (2.2) and (2.3) we may have

$$\nabla_{\bar{W}_1} \bar{Z}_1 = \mathcal{T}_{\bar{W}_1} \bar{Z}_1 + \bar{\nabla}_{\bar{W}_1} \bar{Z}_1, \tag{2.4}$$

$$\nabla_{\bar{W}_1} \bar{X}_1 = \mathcal{T}_{\bar{W}_1} \bar{X}_1 + \mathcal{H}\nabla_{\bar{W}_1} \bar{X}_1, \tag{2.5}$$

$$\nabla_{\bar{X}_1} \bar{W}_1 = \mathcal{A}_{\bar{X}_1} \bar{W}_1 + \mathcal{V}\nabla_{\bar{X}_1} \bar{W}_1, \tag{2.6}$$

$$\nabla_{\bar{X}_1} \bar{Y}_1 = \mathcal{H}\nabla_{\bar{X}_1} \bar{Y}_1 + \mathcal{A}_{\bar{X}_1} \bar{Y}_1 \tag{2.7}$$

for all $\bar{W}_1, \bar{Z}_1 \in \Gamma(\ker\mathcal{J}_*)$ and $\bar{X}_1, \bar{Y}_1 \in \Gamma(\ker\mathcal{J}_*)^\perp$ where

$$\bar{\nabla}_{\bar{W}_1} \bar{Z}_1 = \mathcal{V}\nabla_{\bar{W}_1} \bar{Z}_1.$$

Then we can easily see that $\mathcal{T}_{\bar{Z}}$ and $\mathcal{A}_{\bar{W}}$ are skew-symmetric, i.e.,

$$g(\mathcal{A}_{\bar{W}} F_1, F_2) = -g(F_1, \mathcal{A}_{\bar{W}} F_2), \quad g(\mathcal{T}_{\bar{Z}} F_1, F_2) = -g(F_1, \mathcal{T}_{\bar{Z}} F_2) \tag{2.8}$$

for all $F_1, F_2 \in \Gamma(T_p \bar{O}_1)$.

Here, we recall the proposition as follows:

Proposition 2.1. *[17] Let $\mathcal{J}\colon \bar{O}_1 \to \bar{O}_2$ be horizontally conformal submersion with dilation λ and $\bar{Z}, \bar{W} \in \Gamma(\ker\mathcal{J}_*)^\perp$, then*

$$\mathcal{A}_{\bar{W}} \bar{Z} = \frac{1}{2}\{\mathcal{V}[\bar{W}, \bar{Z}] - \lambda^2 g_1(\bar{W}, \bar{Z}) \mathrm{grad}_v \frac{1}{\lambda^2}\}. \tag{2.9}$$

Then the second fundamental form of $\mathcal{J}$ is given by

$$(\nabla\mathcal{J}_*)(\bar{W}, \bar{Z}) = \nabla^{\mathcal{J}}_{\bar{W}} \mathcal{J}_* \bar{Z} - \mathcal{J}_* \nabla_{\bar{W}} \bar{Z}. \tag{2.10}$$

A map is said to be totally geodesic if $(\nabla\mathcal{J}_*)(\bar{W}, \bar{Z}) = 0$ for all $\bar{W}, \bar{Z} \in \Gamma(T_p \bar{O}_1)$, where Levi-Civita and pullback connections are ∇ and $\nabla^{\mathcal{J}}$ [38].

Lemma 2.1. *Let $\mathcal{J}$: $\bar{O}_1 \to \bar{O}_2$ be a horizontal conformal submersion. Then, we have*

(i) $(\nabla\mathcal{J}_*)(\bar{W}_1, \bar{Z}_1) = \bar{W}_1(\ln\lambda)\mathcal{J}_*(\bar{Z}_1) + \bar{Z}_1(\ln\lambda)\mathcal{J}_*(\bar{W}_1) - g_1(\bar{W}_1, \bar{Z}_1)\mathcal{J}_*(\mathrm{grad}\ln\lambda)$,

(ii) $(\nabla\mathcal{J}_*)(\bar{U}_1, \bar{V}_1) = -\mathcal{J}_*(\mathcal{T}_{\bar{U}_1}\bar{V}_1)$,

(iii) $(\nabla\mathcal{J}_*)(\bar{W}_1, \bar{U}_1) = -\mathcal{J}_*(\nabla^{\mathcal{J}}_{\bar{W}_1}\bar{U}_1) = -\mathcal{J}_*(\mathcal{A}_{\bar{W}_1}\bar{U}_1)$

for any $\bar{W}_1, \bar{Z}_1 \in \Gamma(ker\mathcal{J}_*)^\perp$ and $\bar{U}_1, \bar{V}_1 \in \Gamma(ker\mathcal{J}_*)$ [9].

Let M be a differentiable manifold of dimension n, is said to be having an almost contact structure (ϕ, ξ, η) if, it carries a tensor field ϕ, vector field ξ and 1-form η on M satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1, \tag{2.11}$$

where, I is identity tensor. The almost contact structure (ϕ, ξ, η) is said to be normal if $N + d\eta \otimes \xi = 0$, where N is the Nijenhuis tensor of ϕ. Suppose that a Riemannian metric tensor g is given in M and satisfies the condition

$$g(\phi\bar{W}, \phi\bar{Z}) = g(\bar{W}, \bar{Z}) - \eta(\bar{W})\eta(\bar{V}Z), \quad \eta(\bar{W}) = g(\bar{W}, \xi) \tag{2.12}$$

for all $\bar{Z}, \bar{W} \in \Gamma(TM)$. Then (ϕ, ξ, η, g)-structure is called an ACM structure, Tanno [36] determined connected ACM manifolds with the largest automorphism groups. The sectional curvature of a plane section containing ξ for such a manifold is constant c. The characterizing equations of these manifolds are

$$(\nabla_{\bar{W}}\phi)\bar{Z} = g(\phi\bar{W}, \bar{Z})\xi - \eta(\bar{Z})\phi\bar{W}. \tag{2.13}$$

These spaces are referred to as Kenmotsu manifolds, since Kenmotsu investigated some of these manifolds' basic differential geometric features [22]. On a KM, we can deduce that

$$\nabla_{\bar{W}}\xi = -\phi^2\bar{W} = \bar{W} - \eta(\bar{W})\xi \tag{2.14}$$

and the covariant derivative of ϕ is defined by

$$(\nabla_{\bar{W}}\phi)\bar{Z} = \nabla_{\bar{W}}\phi\bar{Z} - \phi\nabla_{\bar{W}}\bar{Z} \tag{2.15}$$

for any $\bar{W}, \bar{Z} \in \Gamma(TM)$.

Definition 2.2. *Suppose $\mathfrak{D}$ is a k-dimensional smooth distribution on $\bar{O}_1$. Then an immersed submanifold i: $\bar{O}_2 \hookrightarrow \bar{O}_2$ is called an integral manifold for $\mathfrak{D}$ if for every $x \in \bar{O}_2$, the image of $d_i\bar{O}_2$: $T_p\bar{O}_2 \to T_p\bar{O}_1$ is $\mathfrak{D}_\mathfrak{p}$. We say the distribution $\mathfrak{D}_\mathfrak{p}$ is integrable if through each point of $\bar{O}_1$ there exists an integral manifold of $\mathfrak{D}$.*

Further, a distribution $\mathfrak{D}$ is involutive if it satisfies the Frobenius condition such that if $X, Y \in \Gamma(T\bar{O}_1)$ belongs to $\mathfrak{D}$, so $[X, Y] \in \mathfrak{D}$. Frobenius theorem states that an involutive distribution is integrable.

Definition 2.3. *Let $\bar{O}_1$ be n-dimensional smooth manifold. A foliation $\mathfrak{F}$ of $\bar{O}_1$ is a decomposition of $\bar{O}_1$ into a union of disjoint connected submanifolds $\bar{O}_1 = \cup_{L\in\mathfrak{F}}L$ called the leaves of the foliation, such that for each $m \in \bar{O}_1$, there is a neighborhood U of $\bar{O}_1$ and a smooth submersion f_U: $U \to R^k$ with $f_U^{-1}(x)$ a leaf of $\mathfrak{F}|_U$ the restriction of the foliation to U, for each $x \in R^k$.*

Definition 2.4. *Let $\bar{O}_1$ be a Riemannian manifold, and let $\mathfrak{F}$ be a foliation on $\bar{O}_1$. $\mathfrak{F}$ is totally geodesic if each leaf L is a totally geodesic submanifold of $\bar{O}_1$; that is, any geodesic tangent to L at some point must lie within L.*

3. Conformal bi-slant submersions

Definition 3.1. *Let $(\bar{O}_1, \phi, \xi, \eta, g_1)$ be an ACM manifold and $(\bar{O}_2, g_2)$ be a RM. A conformal submersion $\mathcal{J}$ is said to be a CBSS with vertical ξ if D_{θ_1} and D_{θ_2} are slant distributions with slant angel θ_1 and θ_2 such that* $\ker\mathcal{J}_* = D_{\theta_1} \oplus D_{\theta_2} \oplus < \xi >$*, where $< \xi >$ is a* 1*-dimensional distribution spanned by ξ and $\mathcal{J}$ is called proper if $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.*

If $n_1,\ n_2 \neq 0,\ 0 < \theta_1 < \frac{\pi}{2}$ and $0 < \theta_2 < \frac{\pi}{2}$, then, $\mathcal{J}$ is said to be a proper CBSS with vertical ξ, where n_1, n_2 are the dimensions of D_{θ_1} and D_{θ_2} respectively.

In this part, we provide a non-trivial example to support our research.

Example 3.1. *Let (x_i, y_i, z) be Cartesian coordinates on $\mathbb{R}^{2n+1}$ for $i = 1, 2, 3, \cdots, n$. An ACM structure (ϕ, ξ, η, g) is defined as follows:*

$$\phi\left(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + \ldots\ldots + a_n\frac{\partial}{\partial x_n} + b_1\frac{\partial}{\partial y_1} + b_2\frac{\partial}{\partial y_2} + \ldots\ldots. + b_n\frac{\partial}{\partial y_n} + c\frac{\partial}{\partial z}\right)$$
$$= \left(-b_1\frac{\partial}{\partial x_1} - b_2\frac{\partial}{\partial x_2} - \ldots\ldots - b_n\frac{\partial}{\partial x_n} + a_1\frac{\partial}{\partial y_1} + a_2\frac{\partial}{\partial y_2} + \ldots\ldots + a_n\frac{\partial}{\partial y_n}\right),$$

where $\xi = \frac{\partial}{\partial z}$ and a_i, b_i, c are C^∞- real valued functions in R^{2n+1}. Let $\eta = dz, g$ is Euclidean metric and

$$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots\ldots, \frac{\partial}{\partial y_n}, \frac{\partial}{\partial z}\right\}$$

is orthonormal base field of vectors on $\mathbb{R}^{2n+1}$. Then, it can be easily seen that $(\phi, \xi, \eta, g_{\mathbb{R}^{2n+1}})$ is a Kenmotsu structure on $\mathbb{R}^{2n+1}$.

Define a conformal submersion $\mathcal{J}$: $\mathbb{R}^9 \to \mathbb{R}^4$ such that

$$(x_1, \ldots, x_4, y_1, \ldots, y_4, z) \longrightarrow (x_1, (\cos\theta_1)x_2 + (\sin\theta_1)x_4, (-\cos\theta_2)y_1 + (\sin\theta_2)y_3, y_2),$$

where $(x_1, \ldots, x_4, y_1, \ldots, y_4, z)$ are natural coordinates of $\mathbb{R}^9$ and $(\mathbb{R}^9, g_{\mathbb{R}^9})$ is a KM with above defined structure and $\lambda = \pi^5$. Then it follows that

$$(\ker\mathcal{J}_*)^\perp = \left\{Y = \frac{\partial}{\partial x_1}, \cos\theta_1\frac{\partial}{\partial x_2} + \sin\theta_1\frac{\partial}{\partial x_4}, -\cos\theta_2\frac{\partial}{\partial y_1} + \sin\theta_2\frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_2}\right\},$$

$$(\ker\mathcal{J}_*) = \left\{\bar{W}_1 = \frac{\partial}{\partial x_3}, \bar{W}_2 = \sin\theta_2\frac{\partial}{\partial y_1} + \cos\theta_2\frac{\partial}{\partial y_3}, \bar{W}_3 = \frac{\partial}{\partial y_4}, \bar{W}_4 = \sin\theta_1\frac{\partial}{\partial x_2} - \cos\theta_1\frac{\partial}{\partial x_4}, \bar{W}_5 = \frac{\partial}{\partial z}\right\},$$

$$D_{\theta_1} = \left\{\bar{W}_1 = \frac{\partial}{\partial x_3}, \bar{W}_2 = \sin\theta_2\frac{\partial}{\partial y_1} + \cos\theta_2\frac{\partial}{\partial y_3}, \bar{W}_5 = \frac{\partial}{\partial z}\right\}$$

and

$$D_{\theta_2} = \left\{\bar{W}_3 = \frac{\partial}{\partial y_4}, \bar{W}_4 = \sin\theta_1\frac{\partial}{\partial x_2} - \cos\theta_1\frac{\partial}{\partial x_4}\right\}.$$

Thus, $\mathcal{J}$ is a CBSS with vertical ξ and slant angle θ_1 and θ_2 with $\lambda = \pi^5$.

Suppose that $\mathcal{J}$ is a CBSS with vertical ξ from KM $(\bar{O}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{O}_2, g_2)$, then for any $\bar{U} \in \ker\mathcal{J}_*$

$$\bar{U} = \Re\bar{U} + \mathfrak{L}\bar{U} + \eta(\bar{U})\xi, \tag{3.1}$$

where $\Re\bar{U} \in \Gamma(D_{\theta_1})$ and $\mathfrak{L}\bar{U} \in \Gamma(D_{\theta_2})$.

Also, for $\bar{U} \in \Gamma(\ker\mathcal{J}_*)$

$$\phi\bar{U} = \psi\bar{U} + \zeta\bar{U}, \tag{3.2}$$

where $\psi\bar{U} \in \Gamma(\ker\mathcal{J}_*)$ and $\zeta\bar{U} \in \Gamma(\ker\mathcal{J}_*)^\perp$. For any $\bar{X} \in \Gamma(\ker\mathcal{J}_*)^\perp$, we have

$$\phi\bar{X} = t\bar{X} + f\bar{X}, \tag{3.3}$$

where $t\bar{X} \in \Gamma(\ker\mathcal{J}_*)$ and $f\bar{X} \in \Gamma(\ker\mathcal{J}_*)^\perp$.

The horizontal distribution $(\ker\mathcal{J}_*)^\perp$ is decomposed as

$$(\ker\mathcal{J}_*)^\perp = \zeta D_{\theta_1} \oplus \zeta D_{\theta_2} \oplus \mu, \tag{3.4}$$

such that μ is the complementary distribution to $\zeta D_{\theta_1} \oplus \zeta D_{\theta_2}$ in $\Gamma(\ker\mathcal{J}_*)^\perp$.

Given that $\mathcal{J}\colon \bar{O}_1 \to \bar{O}_2$ is a CBSS with vertical ξ, let's present some insightful findings that will be used throughout the work.

Theorem 3.1. *Let $\mathcal{J}\colon (\bar{O}_1, \phi, \xi, \eta, g_1) \to (\bar{O}_2, g_2)$ be a CBSS with vertical ξ from ACM manifold onto a RM with slant angles θ_1 and θ_2. Then we have*

(i) $\psi^2\bar{U} = -(\cos^2\theta_i)\bar{U}$,

(ii) $g_1(\psi\bar{U}, \psi\bar{V}) = cos^2\theta_i g_1(\bar{U}, \bar{V})$,

(iii) $g_1(\zeta\bar{U}, \zeta\bar{V}) = sin^2\theta_i g_1(\bar{U}, \bar{V})$

for any vector fields $\bar{U}, \bar{V} \in \Gamma(D_{\theta_i})$, where $i = 1, 2$.

Due to similarities with the proof of [12, Theorem 3.4], we omit the proof of the aforementioned result.

Lemma 3.1. *Let $(\bar{O}_1, \phi, \xi, \eta, g_1)$ be a KM and $(\bar{O}_2, g_2)$ be a RM. If $\mathcal{J}\colon \bar{O}_1 \to \bar{O}_2$ is a CBSS with vertical ξ, then we have*

$$\zeta t\bar{X} + f^2\bar{X} = -\bar{X}, \quad \psi t\bar{X} + tf\bar{X} = 0, \quad -\bar{U} + \eta(\bar{U})\xi = \psi^2\bar{U} + t\zeta\bar{U}, \quad \zeta\psi\bar{U} + f\zeta\bar{U} = 0$$

for any $\bar{U} \in \Gamma(\ker\mathcal{J}_*)$ and $\bar{X} \in \Gamma(ker\mathcal{J}_*)^\perp$.

Proof. Equations (2.14), (3.2) and (3.3) are used to obtain outcomes from simple calculations. □

Let $(\bar{O}_2, g_2)$ is a RM and that $(\bar{O}_1, \phi, \xi, \eta, g_1)$ is a KM. Now, let us check how the Kenmotsu structure on $\bar{O}_1$ affects the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of a $\mathcal{BSCS}$ $\mathcal{J}\colon (\bar{O}_1, \phi, \xi, \eta, g_1) \to (\bar{O}_2, g_2)$

Lemma 3.2. *If $\mathcal{J}\colon \bar{O}_1 \to \bar{O}_2$ is a CBSS with vertical ξ where $(\bar{O}_1, \phi, \xi, \eta, g_1)$ be a KM and $(\bar{O}_2, g_2)$ be a RM, then we have*

(i) $\mathcal{A}_{\bar{X}} t\bar{Y} + \mathcal{H}\nabla_{\bar{X}} f\bar{Y} = f\mathcal{H}\nabla_{\bar{X}}\bar{Y} + \zeta\mathcal{A}_{\bar{X}}\bar{Y}$,

(ii) $\mathcal{V}\nabla_{\bar{X}} t\bar{Y} + \mathcal{A}_{\bar{X}} f\bar{Y} = t\mathcal{H}\nabla_{\bar{X}}\bar{Y} + \psi\mathcal{A}_{\bar{X}}\bar{Y} - g_1(\phi\bar{X}, \bar{Y})\xi$,

(iii) $\mathcal{V}\nabla_{\bar{X}}\psi\bar{V} + \mathcal{A}_{\bar{X}}\zeta\bar{V} = t\mathcal{A}_{\bar{X}}\bar{V} + \psi\mathcal{V}\nabla_{\bar{X}}\bar{V} + g_1(\phi\bar{X}, \bar{V})\xi - \eta(\bar{V})t\bar{X}$,

(iv) $\mathcal{A}_{\bar{X}}\psi\bar{V} + \mathcal{H}\nabla_{\bar{X}}\zeta\bar{V} = f\mathcal{A}_{\bar{X}}\bar{V} + \zeta\mathcal{V}\nabla_{\bar{X}}\bar{V} - \eta(\bar{V})f\bar{X}$,

(v) $\mathcal{V}\nabla_{\bar{V}}t\bar{X} + \mathcal{T}_{\bar{V}}f\bar{X} = \psi\mathcal{T}_{\bar{V}}f\bar{X} + t\mathcal{H}\nabla_{\bar{V}}\bar{X} + g_1(\phi\bar{X}, \bar{V})\xi$,

(vi) $\mathcal{T}_{\bar{V}}t\bar{X} + \mathcal{H}\nabla_{\bar{V}}f\bar{X} = \zeta\mathcal{T}_{\bar{V}}\bar{X} + f\mathcal{H}\nabla_{\bar{V}}\bar{X}$,

(vii) $\mathcal{V}\nabla_{\bar{U}}\psi\bar{V} + \mathcal{T}_{\bar{U}}\zeta\bar{V} + \eta(\bar{V})\psi\bar{U} = t\mathcal{T}_{\bar{U}}\bar{V} + \psi\mathcal{V}\nabla_{\bar{U}}\bar{V} + g_1(\phi\bar{U}, \bar{V})\xi$,

(viii) $\mathcal{T}_{\bar{U}}\psi\bar{V} + \mathcal{H}\nabla_{\bar{U}}\zeta\bar{V} + \eta(\bar{V})\zeta\bar{U} = f\mathcal{T}_{\bar{U}}\bar{V} + \zeta\mathcal{V}\nabla_{\bar{U}}\bar{V}$

for any $\bar{U}, \bar{V} \in \Gamma(\ker \mathcal{J}_*)$ *and* $\bar{X}, \bar{Y} \in \Gamma(\ker\mathcal{J}_*)^\perp$.

Proof. By some simple steps of calculation with using (2.7), (2.15) and (3.3), (*i*) and (*ii*) are easily obtained. In the same manner, from Eqs (2.4)–(2.6), (2.15), (3.2) and (3.3), we will get the desired results. □

We will now go through some fundamental findings that can be used to investigate the conformal bi-slant submersions $\mathcal{J}\colon \bar{O}_1 \to \bar{O}_2$ geometry. Define the following in this regard:

(a) $(\nabla_{\bar{U}}\psi)\bar{V} = \mathcal{V}\nabla_{\bar{U}}\psi\bar{V} - \psi\mathcal{V}\nabla_{\bar{U}}\bar{V}$,

(b) $(\nabla_{\bar{U}}\zeta)\bar{V} = \mathcal{H}\nabla_{\bar{U}}\zeta\bar{V} - \zeta\mathcal{V}\nabla_{\bar{U}}\bar{V}$,

(c) $(\nabla_{\bar{X}}t)\bar{Y} = \mathcal{V}\nabla_{\bar{X}}t\bar{Y} - t\mathcal{H}\nabla_{\bar{X}}\bar{Y}$,

(d) $(\nabla_{\bar{X}}f)\bar{Y} = \mathcal{H}\nabla_{\bar{X}}f\bar{Y} - f\mathcal{H}\nabla_{\bar{X}}\bar{Y}$

for all $\bar{U}, \bar{V} \in \Gamma(\ker \mathcal{J}_*)$ and $\bar{X}, \bar{Y} \in \Gamma(\ker \mathcal{J}_*)^\perp$.

Lemma 3.3. *Let* $(\bar{O}_1, \phi, \xi, \eta, g_1)$ *be a KM and* $(\bar{O}_2, g_2)$ *be a RM. If* $\mathcal{J}\colon \bar{O}_1 \to \bar{O}_2$ *is a CBSS with vertical* ξ*, then we have*

(i) $(\nabla_{\bar{U}}\psi)\bar{V} = t\mathcal{T}_{\bar{U}}\bar{V} - \mathcal{T}_{\bar{U}}\zeta\bar{V} + g_1(\phi\bar{U}, \bar{V})\xi - \eta(\bar{V})\psi\bar{U}$,

(ii) $(\nabla_{\bar{U}}\zeta)\bar{V} = f\mathcal{T}_{\bar{U}}\bar{V} - \mathcal{T}_{\bar{U}}\psi\bar{V} - \eta(\bar{V})\zeta\bar{U}$,

(iii) $(\nabla_{\bar{X}}t)\bar{Y} = \psi\mathcal{A}_{\bar{X}}\bar{Y} - \mathcal{A}_{\bar{X}}f\bar{Y}$,

(iv) $(\nabla_{\bar{X}}f)\bar{Y} = \zeta\mathcal{A}_{\bar{X}}\bar{Y} - \mathcal{A}_{\bar{X}}t\bar{Y}$

for any $\bar{U}, \bar{V} \in \Gamma(\ker\mathcal{J}_*)$ *and* $\bar{X}, \bar{Y} \in \Gamma(\ker\mathcal{J}_*)^\perp$.

Proof. By taking account the fact from (2.4)–(2.7), (2.13), (*i*), (*ii*) part from Lemma 3.2 and from part (a)–(d), it is easy to get the proof of the lemma. □

It is given that ∇ is the Levi-Civita connection of Kenmotsu manifolds $\bar{O}_1$. Let us suppose that the tenors ψ and ζ are parallel, we can write

$$t\mathcal{T}_{\bar{U}}\bar{V} = \mathcal{T}_{\bar{U}}\zeta\bar{V} - g_1(\phi\bar{U}, \bar{V})\xi + \eta(\bar{V})\psi\bar{U}, \quad f\mathcal{T}_{\bar{U}}\bar{V} = \mathcal{T}_{\bar{U}}\psi\bar{V} + \eta(\bar{V})\zeta\bar{U}$$

for any $\bar{X}, \bar{Y} \in \Gamma(T\bar{O}_1)$.

4. Geometry of leaves of distributions

We will talk about the geometry of distribution leaves and integrability conditions in this section. We begin with the prerequisites that must be met in order for the slant distributions to be integrable.

Theorem 4.1. *Let $\mathcal{J}$: $(\bar{O}_1, \phi, \xi, \eta, g_1) \to (\bar{O}_2, g_2)$ be a proper CBSS with vertical ξ with slant angles θ_1 and θ_2, where $(\bar{O}_1, \phi, \xi, \eta, g_1)$ is a KM and $(\bar{O}_2, g_2)$ is a RM. Then the distribution D_{θ_1} is integrable if and only if*

$$\begin{aligned}
&\lambda^{-2} g_2((\nabla \mathcal{J}_*)(\bar{U}_1, \zeta\bar{V}_1), \mathcal{J}_*\zeta\bar{U}_2) - g_1(\phi\bar{U}_1 - \phi\bar{V}_1, \bar{U}_2)\eta(\psi\bar{U}_1) - g_1(\phi\bar{U}_1 - \phi\bar{V}_1, \psi\bar{V}_1 - \psi\bar{U}_1)\eta(\bar{U}_2) \\
&= \lambda^{-2}\{(g_2(\nabla^{\mathcal{J}}_{\bar{U}_1} \mathcal{J}_*\zeta\bar{V}_1 - \nabla^{\mathcal{J}}_{\bar{V}_1} \mathcal{J}_*\zeta\bar{U}_1), \mathcal{J}_*\zeta\bar{U}_2)\} + g_1(T_{\bar{V}_1}\zeta\psi\bar{U}_1 - T_{\bar{U}_1}\zeta\psi\bar{V}_1, \bar{U}_2) \\
&\quad + g_1(T_{\bar{U}_1}\zeta\bar{V}_1 - T_{\bar{V}_1}\zeta\bar{U}_1, \psi\bar{U}_2) + \lambda^{-2} g_2((\nabla \mathcal{J}_*)(\bar{V}_1, \zeta\bar{U}_1), \mathcal{J}_*\zeta\bar{U}_2)
\end{aligned}$$

for any $\bar{U}_1, \bar{V}_1 \in \Gamma(D_{\theta_1})$ and $\bar{U}_2 \in \Gamma(D_{\theta_2})$.

Proof. For any $\bar{U}_1, \bar{V}_1 \in \Gamma(D_{\theta_1})$ and $\bar{U}_2 \in \Gamma(D_{\theta_2})$ and on using (2.12), (2.13) and from (3.2), we have

$$\begin{aligned}
g_1([\bar{U}_1, \bar{V}_1], \bar{U}_2) = {}& g_1(\nabla_{\bar{V}_1}\psi^2\bar{U}_1, \bar{U}_2) - g_1(\nabla_{\bar{U}_1}\psi^2\bar{V}_1, \bar{U}_2) - g_1(\nabla_{\bar{U}_1}\zeta\psi\bar{V}_1, \bar{U}_2) \\
&+ g_1(\nabla_{\bar{V}_1}\zeta\psi\bar{U}_1, \bar{U}_2) + g_1(\nabla_{\bar{U}_1}\zeta\bar{V}_1, \phi\bar{U}_2) - g_1(\nabla_{\bar{V}_1}\zeta\bar{U}_1, \phi\bar{U}_2).
\end{aligned}$$

Considering Theorem 3.1, we have

$$\begin{aligned}
\sin^2\theta_1 g_1([\bar{U}_1, \bar{V}_1], \bar{U}_2) = {}& - g_1(\nabla_{\bar{U}_1}\zeta\psi\bar{V}_1, \bar{U}_2) + g_1(\nabla_{\bar{V}_1}\zeta\psi\bar{U}_1, \bar{U}_2) + g_1(\nabla_{\bar{U}_1}\zeta\bar{V}_1, \phi\bar{U}_2) - g_1(\nabla_{\bar{V}_1}\zeta\bar{U}_1, \phi\bar{U}_2) \\
&+ g_1(\phi\bar{U} - \phi\bar{V}, \psi\bar{V} - \psi\bar{U})\eta(\bar{U}_2) + g_1(\phi\bar{U} - \phi\bar{V}, \bar{U}_2)\eta(\psi\bar{U}).
\end{aligned}$$

By using (2.5), we have

$$\begin{aligned}
\sin^2\theta_1 g_1([\bar{U}_1, \bar{V}_1], \bar{U}_2) = {}& g_1(\mathcal{T}_{\bar{V}_1}\zeta\psi\bar{U}_1 - \mathcal{T}_{\bar{U}_1}\zeta\psi\bar{V}_1, \bar{U}_2) - g_1(\mathcal{T}_{\bar{U}_1}\zeta\bar{V}_1 - \mathcal{T}_{\bar{V}_1}\zeta\bar{U}_1, \psi\bar{U}_2) \\
&+ g_1(\mathcal{H}\nabla_{\bar{U}_1}\zeta\bar{V}_1 - \mathcal{H}\nabla_{\bar{V}_1}\zeta\bar{U}_1, \zeta\bar{U}_2) + g_1(\phi\bar{U} - \phi\bar{V}, \psi\bar{V} - \psi\bar{U})\eta(\bar{U}_2) \\
&+ g_1(\phi\bar{U} - \phi\bar{V}, \bar{U}_2)\eta(\psi\bar{U}).
\end{aligned}$$

Now considering Lemma 2.1 and (2.10), we have

$$\begin{aligned}
\sin^2\theta_1 g_1([\bar{U}_1, \bar{V}_1], \bar{U}_2) = {}& \lambda^{-2} g_2((\nabla^{\mathcal{J}}_{\bar{U}_1} \mathcal{J}_*\zeta\bar{V}_1 - \nabla^{\mathcal{J}}_{\bar{V}_1} \mathcal{J}_*\zeta\bar{U}_1), \mathcal{J}_*\zeta\bar{U}_2)\} + g_1(\mathcal{T}_{\bar{V}_1}\zeta\psi\bar{U}_1 - \mathcal{T}_{\bar{U}_1}\zeta\psi\bar{V}_1, \bar{U}_2) \\
&- g_1(\mathcal{T}_{\bar{U}_1}\zeta\bar{V}_1 - \mathcal{T}_{\bar{V}_1}\zeta\bar{U}_1, \psi\bar{U}_2) - \lambda^{-2} g_2((\nabla \mathcal{J}_*)(\bar{U}_1, \zeta\bar{V}_1), \mathcal{J}_*\zeta\bar{U}_2) \\
&+ \lambda^{-2} g_2((\nabla \mathcal{J}_*)(\bar{V}_1, \zeta\bar{U}_1), \mathcal{J}_*\zeta\bar{U}_2) + g_1(\phi\bar{U} - \phi\bar{V}, \bar{U}_2)\eta(\psi\bar{U}) \\
&+ g_1(\phi\bar{U} - \phi\bar{V}, \psi\bar{V} - \psi\bar{U})\eta(\bar{U}_2).
\end{aligned}$$

□

Studying distribution leaves will be significant since they are crucial to the geometry of conformal bi-slant submersions from the Kenmotsu manifold. In order to do this, we are determining the circumstances in which distributions define total geodesic foliation on M.

Theorem 4.2. *Let $\mathcal{J}$: $(\bar{O}_1, \phi, \xi, \eta, g_1) \to (\bar{O}_2, g_2)$ be a CBSS with vertical ξ from a KM onto a RM $\bar{O}_2$. Then D_{θ_1} is not totally geodesic on $\bar{O}_1$.*

Proof. For any vector field $\bar{U}, \bar{V} \in \Gamma(D_\theta)$ with the fact that $\bar{V}$ and ξ are orthogonal, we have

$$g_1(\nabla_{\bar{U}}\bar{V}, \xi) = -g_1(\bar{V}, \nabla_{\bar{U}}\xi).$$

By considering the (2.14), we get

$$g_1(\nabla_{\bar{U}}\bar{V}, \xi) = -g_1(\bar{U}, \bar{V}).$$

Since,

$$\bar{U}, \bar{V} \in \Gamma(D_{\theta_1}), -g_1(\bar{U}, \bar{V}) \neq 0,$$

that is $g_1(\nabla_{\bar{U}}\bar{V}, \xi) \neq 0$. Hence, the distribution is not defines totally geodesic foliation on $\bar{O}_1$. □

Since the Reeb vector field ξ is assumed to be vertical, the slant distribution D_{θ_1} does not define total geodesic foliation. In order to deal with this issue, the geometry of the leaves of the slant distribution $D_{\theta_1}\oplus <\xi>$ is being examined here.

Theorem 4.3. *Let $(\bar{O}_1, \phi, \xi, \eta, g_1)$ be a KM and $(\bar{O}_2, g_2)$ be a RM such that $\mathcal{J}$ is a CBSS with vertical ξ from M onto $\bar{O}_2$. Then the distribution $D_{\theta_1}\oplus <\xi>$ defines totally geodesic foliation on $\bar{O}_1$ if and only if*

$$\begin{aligned}\lambda^{-2}g_2((\nabla\mathcal{J}_*)(\bar{U}_1,\zeta\bar{V}_1),\mathcal{J}_*\zeta\bar{U}_2) &= g_1(\mathcal{T}_{\bar{U}_1}\zeta\bar{V}_1,\psi\bar{U}_2) - g_1(\mathcal{T}_{\bar{U}_1}\zeta\psi\bar{V}_1,\bar{U}_2)\\ &\quad + \lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{U}_1}\mathcal{J}_*\zeta\bar{V}_1,\mathcal{J}_*\zeta\bar{U}_2)\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}&\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{X}}\mathcal{J}_*\zeta\bar{U}_1,\mathcal{J}_*\zeta\bar{V}_1) + g_1(\mathcal{A}_{\bar{X}}\zeta\bar{U}_1,\psi\bar{V}_1)\\ &= \sin^2\theta g_1([\bar{U}_1,\bar{X}],\bar{V}_1) + g_1(\mathcal{A}_{\bar{X}}\zeta\psi\bar{U}_1,\bar{V}_1) + g_1(\operatorname{grad}\ln\lambda,\bar{X})g_1(\zeta\bar{U}_1,\zeta\bar{V}_1)\\ &\quad + g_1(\operatorname{grad}\ln\lambda,\zeta\bar{U}_1)g_1(\bar{X},\zeta\bar{V}_1) - g_1(\operatorname{grad}\ln\lambda,\zeta\bar{V}_1)g_1(\bar{X},\zeta\bar{U}_1)\end{aligned} \tag{4.2}$$

for any $\bar{U}_1, \bar{V}_1 \in \Gamma(D_{\theta_1}\oplus <\xi>)$, $\bar{U}_2 \in \Gamma(D_{\theta_2})$ and $\bar{X} \in \Gamma(\ker\mathcal{J}_*)^\perp$.

Proof. For any $\bar{U}_1, \bar{V}_1 \in \Gamma(D_{\theta_1}\oplus <\xi>)$ and $\bar{U}_2 \in \Gamma(D_{\theta_2})$ with using (2.12), (2.13) and (3.2), we have

$$g_1(\nabla_{\bar{U}_1}\bar{V}_1,\bar{U}_2) = g_1(\nabla_{\bar{U}_1}\zeta\bar{V}_1,\phi\bar{U}_2) - g_1(\nabla_{\bar{U}_1}\zeta\psi\bar{V}_1,\bar{U}_2) - g_1(\nabla_{\bar{U}_1}\psi^2\bar{V}_1,\bar{U}_2).$$

From Theorem 3.1, we can write

$$\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1,\bar{U}_2) = -g_1(\nabla_{\bar{U}_1}\zeta\psi\bar{V}_1,\bar{U}_2) + g_1(\nabla_{\bar{U}_1}\zeta\bar{V}_1,\phi\bar{U}_2).$$

On using (2.5), we have

$$\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1,\bar{U}_2) = g_1(\mathcal{T}_{\bar{U}_1}\zeta\bar{V}_1,\psi\bar{U}_2) - g_1(\mathcal{T}_{\bar{U}_1}\zeta\psi\bar{V}_1,\bar{U}_2) + g_1(\mathcal{H}\nabla_{\bar{U}_1}\zeta\bar{V}_1,\zeta\bar{U}_2).$$

Considering (2.10) and Lemma 2.1, we obtain

$$\begin{aligned}\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1,\bar{U}_2) &= g_1(\mathcal{T}_{\bar{U}_1}\zeta\bar{V}_1,\psi\bar{U}_2) - g_1(\mathcal{T}_{\bar{U}_1}\zeta\psi\bar{V}_1,\bar{U}_2) - \lambda^{-2}g_2((\nabla\mathcal{J}_*)(\bar{U}_1,\zeta\bar{V}_1),\mathcal{J}_*\zeta\bar{U}_2)\\ &\quad + \lambda^{-2}g_2(\nabla_{\bar{U}_1}\mathcal{J}_*\zeta\bar{V}_1,\mathcal{J}_*\zeta\bar{U}_2),\end{aligned}$$

this proves first part of theorem.

On the other hand, $\bar{U}_1, \bar{V}_1 \in \Gamma(D_{\theta_1})$ and $\bar{X} \in \Gamma(\ker\mathcal{J}_*)^\perp$ with using (2.12), (2.13) and (3.2), we can write

$$g_1(\nabla_{\bar{U}_1}\bar{V}_1,\bar{X}) = -g_1([\bar{U}_1,\bar{X}],\bar{V}_1) + g_1(\phi\nabla_{\bar{X}}\psi\bar{U}_1,\bar{V}_1) - g_1(\nabla_{\bar{X}}\zeta\bar{U}_1,\phi\bar{V}_1).$$

Considering Theorem 3.1, we obtained

$$\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1,\bar{X}) = -g_1([\bar{U}_1,\bar{V}_1],\bar{X}) + g_1(\nabla_{\bar{X}}\zeta\psi\bar{U}_1,\bar{V}_1) - g_1(\nabla_{\bar{X}}\zeta\bar{U}_1,\phi\bar{V}_1).$$

On using (2.7), we have

$$\begin{aligned}\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1,\bar{X}) &= \sin^2\theta_1 g_1([\bar{U}_1,\bar{X}],\bar{V}_1) + g_1(\mathcal{A}_{\bar{X}}\zeta\psi\bar{U}_1,\bar{V}_1) - g_1(\mathcal{A}_{\bar{X}}\zeta\bar{U}_1,\psi\bar{V}_1)\\ &\quad - \lambda^{-2}g_2(\mathcal{J}_*\nabla_{\bar{X}}\zeta\bar{U}_1,\mathcal{J}_*\zeta\bar{U}_1).\end{aligned}$$

Using Lemma 2.1, we yields

$$\begin{aligned}\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1,\bar{X}) &= \sin^2\theta_1 g_1([\bar{U}_1,\bar{X}],\bar{V}_1) + g_1(\mathcal{A}_{\bar{X}}\zeta\psi\bar{U}_1,\bar{V}_1) - \lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{X}}\mathcal{J}_*\zeta\bar{U}_1,\mathcal{J}_*\zeta\bar{V}_1)\\ &+ g_1(\mathrm{grad}\ln\lambda,\bar{X})g_1(\zeta\bar{U}_1,\zeta\bar{V}_1) + g_1(\mathrm{grad}\ln\lambda,\zeta\bar{U}_1)g_1(\bar{X},\zeta\bar{V}_1)\\ &- g_1(\mathrm{grad}\ln\lambda,\zeta\bar{V}_1)g_1(\bar{X},\zeta\bar{U}_1) - g_1(\mathcal{A}_{\bar{X}}\zeta\bar{U}_1,\psi\bar{V}_1).\end{aligned}$$

This completes the proof of the Theorem. □

It is obvious that the Theorems 4.1–4.3 hold for distribution D_{θ_2}. Now, we look at certain circumstances that allow horizontal and vertical distributions to be totally geodesic. We commence by giving the findings for vertical distribution.

Theorem 4.4. *Let $(\bar{O}_1,\phi,\xi,\eta,g_1)$ be a KM and $(\bar{O}_2,g_2)$ be a RM such that $\mathcal{J}$ is a CBSS with vertical ξ from $\bar{O}_1$ onto $\bar{O}_2$. Then vertical distribution $(\ker\mathcal{J}_*)$ defines totally geodesic foliation on $\bar{O}_1$ if and only if*

$$\begin{aligned}\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{X}_1}\mathcal{J}_*\zeta\bar{U}_1,\mathcal{J}_*\zeta\bar{V}_1) &= (\cos^2\theta_2-\cos^2\theta_1)g_1(\nabla_{\bar{X}_1}\mathfrak{L}\bar{U}_1,\bar{V}_1) + g_1(\mathcal{A}_{\psi}\bar{V}_1,\zeta\bar{U}_1) - g_1(\mathcal{A}_{\bar{X}_1}\bar{V}_1,\zeta\psi\bar{U}_1)\\ &+ g_1(\mathrm{grad}\ln\lambda,\bar{X}_1)g_1(\zeta\bar{U}_1,\zeta\bar{V}_1) + g_1(\mathrm{grad}\ln\lambda,\zeta\bar{U}_1)g_1(\bar{X}_1,\zeta\bar{V}_1)\\ &- g_1(\mathrm{grad}\ln\lambda,\zeta\bar{V}_1)g_1(\bar{X}_1,\zeta\bar{U}_1) - \sin^2\theta_1 g_1([\bar{U}_1,\bar{X}_1],\bar{V}_1)\\ &- g_1(\phi\bar{X},\psi\bar{U})\eta(\bar{V}) + g_1(\phi\bar{X},\bar{V})\eta(\psi\bar{U})\end{aligned}\tag{4.3}$$

for $\bar{U}_1,\bar{V}_1\in\Gamma(\ker\mathcal{J}_*)$ and $\bar{X}_1\in\Gamma(\ker\mathcal{J}_*)^\perp$.

Proof. On taking $\bar{U}_1,\bar{V}_1\in\Gamma(\ker\mathcal{J}_*)$ and $\bar{X}_1\in\Gamma(\ker\mathcal{J}_*)^\perp$ with using (2.12), (2.13) and (3.2), we have

$$g_1(\nabla_{\bar{U}_1}\bar{V}_1,\bar{X}_1) = -g_1([\bar{U}_1,\bar{X}_1],\bar{V}_1) + g_1(\nabla_{\bar{X}_1}\phi\psi\bar{U}_1,\bar{V}_1) - g_1(\nabla_{\bar{X}_1}\zeta\bar{U}_1,\phi\bar{V}_1).$$

On using decomposition (3.1) and Theorem 3.1, we get

$$\begin{aligned}g_1(\nabla_{\bar{U}_1}\bar{V}_1,\bar{X}_1) &= -g_1([\bar{U}_1,\bar{X}_1],\bar{V}_1) - \cos^2\theta_1 g_1(\nabla_{\bar{X}_1}\mathfrak{R}\bar{U}_1,\bar{V}_1) - \cos^2\theta_2 g_1(\nabla_{\bar{X}_1}\mathfrak{L}\bar{U}_1,\bar{V}_1)\\ &+ g_1(\nabla_{\bar{X}_1}\zeta\psi\bar{U}_1,\bar{V}_1) - g_1(\nabla_{\bar{X}_1}\zeta\bar{U}_1,\psi\bar{V}_1) - g_1(\nabla_{\bar{X}_1}\zeta\bar{U}_1,\zeta\bar{V}_1)\\ &- g_1(\phi\bar{X},\psi\bar{U})\eta(\bar{V}) + g_1(\phi\bar{X},\bar{V})\eta(\psi\bar{U}).\end{aligned}$$

On using (2.7), we can write

$$\begin{aligned}\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1,\bar{X}_1) &= (\cos^2\theta_2-\cos^2\theta_1)g_1(\nabla_{\bar{X}_1}\mathfrak{L}\bar{U}_1,\bar{V}_1) - \sin^2\theta_1 g([\bar{U}_1,\bar{X}_1],\bar{V}_1)\\ &+ g_1(\mathcal{A}_{\bar{X}_1}\psi\bar{V}_1,\zeta\bar{U}_1) - g_1(\mathcal{A}_{\bar{X}_1}\bar{V}_1,\zeta\psi\bar{U}_1) - g_1(\mathcal{H}\nabla_{\bar{X}_1}\zeta\bar{U}_1,\zeta\bar{V}_1)\\ &- g_1(\phi\bar{X},\psi\bar{U})\eta(\bar{V}) + g_1(\phi\bar{X},\bar{V})\eta(\psi\bar{U}).\end{aligned}$$

Using (2.10), we get

$$\begin{aligned}\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1,\bar{X}_1) &= (\cos^2\theta_2-\cos^2\theta_1)g_1(\nabla_{\bar{X}_1}\mathfrak{L}\bar{U}_1,\bar{V}_1) + g_1(\mathcal{A}_{\bar{X}_1}\psi\bar{V}_1,\zeta\bar{U}_1)\\ &- g_1(\mathcal{A}_{\bar{X}_1}\bar{V}_1,\zeta\psi\bar{U}_1) + \lambda^{-2}g_2((\nabla\mathcal{J}_*)(\bar{X}_1,\zeta\bar{U}_1),\mathcal{J}_*\zeta\bar{V}_1)\\ &- \lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{X}_1}\mathcal{J}_*\zeta\bar{U}_1,\mathcal{J}_*\zeta\bar{V}_1) - \sin^2\theta_1 g_1([\bar{U}_1,\bar{X}_1],\bar{V}_1)\\ &- g_1(\phi\bar{X},\psi\bar{U})\eta(\bar{V}) + g_1(\phi\bar{X},\bar{V})\eta(\psi\bar{U}).\end{aligned}$$

Considering Lemma 2.1, we have

$$\begin{aligned}\sin^2\theta_1 g_1(\nabla_{\bar U_1}\bar V_1,\bar X_1) &= (\cos^2\theta_2-\cos^2\theta_1)g_1(\nabla_{\bar X_1}\mathfrak{L}\bar U_1,\bar V_1)+g_1(\mathcal{A}_{\bar X_1}\psi\bar V_1,\zeta\bar U_1)-g_1(\mathcal{A}_{\bar X_1}\bar V_1,\zeta\psi\bar U_1)\\&+g_1(\operatorname{grad}\ln\lambda,\bar X_1)g_1(\zeta\bar U_1,\zeta\bar V_1)+g_1(\operatorname{grad}\ln\lambda,\zeta\bar U_1)g_1(\bar X_1,\zeta\bar V_1)\\&-g_1(\operatorname{grad}\ln\lambda,\zeta\bar V_1)g_1(\bar X_1,\zeta\bar U_1)-\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar X_1}\mathcal{J}_*\zeta\bar U_1,\mathcal{J}_*\zeta\bar V_1)\\&-\sin^2\theta_1 g_1([\bar U_1,\bar X_1],\bar V_1)-g_1(\phi\bar X,\psi\bar U)\eta(\bar V)+g_1(\phi\bar X,\bar V)\eta(\psi\bar U).\end{aligned}$$

This completes the proof of the Theorem. □

Similarly, we examined the totally geodesic prerequisite for horizontal distributions.

Theorem 4.5. *Let $(\bar O_1,\phi,\xi,\eta,g_1)$ be a KM and $(\bar O_2,g_2)$ be a RM such that $\mathcal{J}$ is a CBSS with vertical ξ from $\bar O_1$ onto $\bar O_2$. Then horizontal distribution $(\ker\mathcal{J}_*)^\perp$ is totally geodesic if and only if*

$$\begin{aligned}\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar X_1}\mathcal{J}_*\zeta\bar U_1,\mathcal{J}_*f\bar X_2) &= -g_1(\mathcal{A}_{\bar X_1}\zeta\bar U_1,t\bar X_2)-\eta(\bar X_2)g_1(\phi\bar X_1,\bar U_1)+g_1(\operatorname{grad}\ln\lambda,\bar X_1)g_1(\zeta\bar U_1,f\bar X_2)\\&+g_1(\operatorname{grad}\ln\lambda,\zeta\bar U_1)g_1(\bar X_1,f\bar X_2)-g_1(\operatorname{grad}\ln\lambda,f\bar X_2)g_1(\bar X_1,\zeta\bar U_1)\\&-g_1(\operatorname{grad}\ln\lambda,\bar X_1)g_1(\zeta\psi\bar U_1,\bar X_2)-g_1(\operatorname{grad}\ln\lambda,\zeta\psi\bar U_1)g_1(\bar X_1,f\bar X_2)\\&+g_1(\operatorname{grad}\ln\lambda,\bar X_2)g_1(\bar X_1,\zeta\psi\bar U_1)+\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar X_1}\mathcal{J}_*\zeta\psi\bar U_1,\mathcal{J}_*\bar X_2)\end{aligned}\tag{4.4}$$

for any $\bar X_1,\bar X_2\in\Gamma(\ker\mathcal{J}_*)^\perp$, $\bar U_1\in\Gamma(\mathrm{Ker}J_*)$.

Proof. For any $\bar X_1,\bar X_2\in\Gamma(\ker\mathcal{J}_*)^\perp$ and $\bar U_1\in\Gamma(\ker J_*)$ with using (2.12), (2.13) and (3.2), we have

$$g_1(\nabla_{\bar X_1}\bar X_2,\bar U_1)=g_1(\nabla_{\bar X_1}\phi\psi\bar U_1,\bar X_2)-g_1(\nabla_{\bar X_1}\zeta\bar U_1,\phi\bar X_2)-\eta(\bar U_1)g_1(\bar X_1,\bar X_2).$$

By using Theorem 3.1, we can write

$$\sin^2\theta_1 g_1(\nabla_{\bar X_1}\bar X_2,\bar U_1)=g_1(\nabla_{\bar X_1}\zeta\psi\bar U_1,\bar X_2)-g_1(\nabla_{\bar X_1}\zeta\bar U_1,\phi\bar X_2)-\eta(\bar U_1)g_1(\bar X_1,\bar X_2)+\eta(\psi\bar U_1)g_1(\phi\bar X_1,\bar X_2).$$

From (2.7), we get

$$\begin{aligned}\sin^2\theta_1 g_1(\nabla_{\bar X_1}\bar X_2,\bar U_1) &= -g_1(\mathcal{A}_{\bar X_1}\zeta\bar U_1,t\bar X_2)-\lambda^{-2}g_2(\mathcal{J}_*\nabla_{\bar X_1}\zeta\bar U_1,\mathcal{J}_*f\bar X_2)\\&+\lambda^{-2}g_2(\mathcal{J}_*\nabla_{\bar X_1}\zeta\psi\bar U_1,\mathcal{J}_*\bar X_2)-\eta(\bar U_1)g_1(\bar X_1,\bar X_2)+\eta(\psi\bar U_1)g_1(\phi\bar X_1,\bar X_2).\end{aligned}$$

Considering Lemma 2.1, we have

$$\begin{aligned}\sin^2\theta_1 g_1(\nabla_{\bar X_1}\bar X_2,\bar U_1) &= -g_1(\mathcal{A}_{\bar X_1}\zeta\bar U_1,t\bar X_2)-\eta(\bar U_1)g_1(\bar X_1,\bar X_2)+\eta(\psi\bar U_1)g_1(\phi\bar X_1,\bar X_2)\\&-\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar X_1}\mathcal{J}_*\zeta\bar U_1,\mathcal{J}_*f\bar X_2)+g_1(\operatorname{grad}\ln\lambda,\bar X_1)g_1(\zeta\bar U_1,f\bar X_2)\\&+g_1(\operatorname{grad}\ln\lambda,\zeta\bar U_1)g_1(\bar X_1,f\bar X_2)-g_1(\operatorname{grad}\ln\lambda,f\bar X_2)g_1(\bar X_1,\zeta\bar U_1)\\&-g_1(\operatorname{grad}\ln\lambda,\bar X_1)g_1(\zeta\psi\bar U_1,\bar X_2)-g_1(\operatorname{grad}\ln\lambda,\zeta\psi\bar U_1)g_1(\bar X_1,f\bar X_2)\\&+g_1(\operatorname{grad}\ln\lambda,\bar X_2)g_1(\bar X_1,\zeta\psi\bar U_1)+\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar X_1}\mathcal{J}_*\zeta\psi\bar U_1,\mathcal{J}_*\bar X_2).\end{aligned}$$

□

It is now fascinating to investigate if the whole space $\bar O_1$ can become a locally twisted product manifold under specific circumstances. We find some criteria that make total space $\bar O_1$ a locally twisted product manifold in the following result. Here, we give the definition of the twisted product manifold defined by Ponge [31]. Let g_B be a Riemannian metric tensor on the manifold $B=\bar O_1\times\bar O_2$ and assume that the canonical foliations $D_{\bar O_1}$ and $D_{\bar O_2}$ intersect perpendicularly everywhere. The g_B is a metric tensor of

(i) a twisted product if and only if $D_{\bar{O}_1}$ is totally geodesic foliation and $D_{\bar{O}_2}$ is totally umbilical foliation,

(ii) a usually product of Riemannian manifolds if and only if $D_{\bar{O}_1}$ and $D_{\bar{O}_2}$ are totally geodesic foliations,

(iii) a warped product if and only if $D_{\bar{O}_1}$ is totally geodesic foliation and $D_{\bar{O}_2}$ is a spheric foliation, i.e., it is umbilical and its mean curvature vector field is parallel.

Theorem 4.6. *Let $\mathcal{J}$ be a CBSS with vertical ξ from a KM $(\bar{O}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{O}_2, g_2)$. Then $\bar{O}_1$ is a locally twisted product manifold of the form $\bar{O}_{1(\ker\mathcal{J}_*)} \times_\lambda \bar{O}_{1(\ker\mathcal{J}_*)^\perp}$ if and only if*

$$\frac{1}{\lambda^2} g_2(\nabla^{\mathcal{J}}_{\phi\bar{W}} \mathcal{J}_*\phi\bar{V}, \mathcal{J}_*\phi f\bar{X}) = g_1(\phi\bar{V}, \phi\bar{W}) g_1(\text{grad} \ln\lambda, \mathcal{J}_*\phi f\bar{V}) - g_1(\nabla_{\bar{V}}\phi\bar{W}, t\bar{X}) \tag{4.5}$$

and

$$g_1(\bar{X}, \bar{Y})H = t\mathcal{A}_{\bar{X}} t\bar{Y} - t\bar{X}(\ln\lambda) f\bar{Y} + t(\text{grad}\ln\lambda) g_1(\bar{X}, f\bar{Y}) + \phi\mathcal{J}_*(\nabla^{\mathcal{J}}_{\bar{X}} \mathcal{J}_* f\bar{Y}) + g_1(\bar{X}, \bar{Y})\xi, \tag{4.6}$$

where H is mean curvature vector and for any $\bar{V}, \bar{W} \in \Gamma(\ker\mathcal{J}_*)$ and $\bar{X}, \bar{Y} \in \Gamma(\ker\mathcal{J}_*)^\perp$.

Proof. For any $\bar{V}, \bar{W} \in \Gamma(\ker\mathcal{J}_*)$ and $\bar{X}, \bar{Y} \in \Gamma(\ker\mathcal{J}_*)^\perp$, we have

$$g_1(\nabla_{\bar{V}}\bar{W}, \bar{X}) = g_1(\mathcal{H}\nabla_{\bar{V}}\phi\bar{W}, f\bar{X}) + g_1(\mathcal{T}_{\bar{V}}\phi\bar{W}, t\bar{X}).$$

Since ∇ is torsion free, $[\bar{V}, \phi\bar{W}] \in \Gamma(\ker\mathcal{J}_*)$, we have

$$g_1(\nabla_{\bar{V}}\bar{W}, \bar{X}) = g_1(\nabla_{\bar{V}}\phi\bar{W}, t\bar{X}) + g_1(\nabla_{\phi\bar{W}}\phi\bar{V}, \phi f\bar{X}).$$

Since $\mathcal{J}$ is CBSS with vertical ξ, by using Lemma 2.1 and from the fact that $g_1(f\bar{X}, \phi\bar{V}) = 0$ for $\bar{X} \in (\ker\mathcal{J}_*)^\perp$ and $\bar{V} \in (\ker\mathcal{J}_*)$, we have

$$\begin{aligned} g_1(\nabla_{\bar{V}}\bar{W}, \bar{X}) = g_1(\nabla_{\bar{V}}\phi\bar{W}, t\bar{X}) &+ \frac{1}{\lambda^2} g_2(\nabla^{\mathcal{J}}_{\phi\bar{W}} \mathcal{J}_*\phi\bar{V}, \mathcal{J}_*(\phi f\bar{X})) \\ &- g_1(\phi\bar{V}, \phi\bar{W}) g_1(\text{grad}\ln\lambda, \mathcal{J}_*(\phi f\bar{V})). \end{aligned}$$

It follows that $\bar{O}_{1(ker\mathcal{J}_*)}$ is totally geodesic if and only if the (4.5) holds good. Now, for $\bar{X}, \bar{Y} \in \Gamma(\ker\mathcal{J}_*)^\perp$, $\bar{V} \in \Gamma(\ker\mathcal{J}_*)$, we have

$$g_1(\nabla_{\bar{X}}\bar{Y}, \bar{V}) = g_1(\mathcal{A}_{\bar{X}} t\bar{Y} + \mathcal{V}\nabla_{\bar{X}} t\bar{Y}, \phi\bar{V}) + g_1(\mathcal{A}_{\bar{X}} f\bar{Y} + \mathcal{H}\nabla_{\bar{X}} f\bar{Y}, \phi\bar{V}) + g_1(\bar{X}, \bar{Y})\eta(\bar{V}).$$

From above equation, we get

$$g_1(\nabla_{\bar{X}}\bar{Y}, \bar{V}) = g_1(\mathcal{A}_{\bar{X}} t\bar{Y}, \phi\bar{V}) + g_1(\mathcal{H}\nabla_{\bar{X}} f\bar{Y}, \phi\bar{V}) + g_1(\bar{X}, \bar{Y})\eta(\bar{V}).$$

Since $\mathcal{J}$ is a CBSS with vertical ξ, from (2.4) and on using Lemma 2.1, we get

$$\begin{aligned} g_1(\nabla_{\bar{X}}\bar{Y}, \bar{V}) = g_1(\mathcal{A}_{\bar{X}} t\bar{Y}, \phi\bar{V}) &- \frac{1}{\lambda^2} g_2(\text{grad}\ln\lambda, \bar{X}) \frac{1}{\lambda^2} g_2(\mathcal{J}_* f\bar{Y}, \mathcal{J}_*\phi\bar{V}) \\ &- \frac{1}{\lambda^2} g_2(\text{grad}\ln\lambda, f\bar{Y}) \frac{1}{\lambda^2} g_2(\mathcal{J}_*\bar{X}, \mathcal{J}_*\phi\bar{V}) \\ &+ \frac{1}{\lambda^2} g_2(\bar{X}, f\bar{Y}) \frac{1}{\lambda^2} g_2(\mathcal{J}_*\text{grad}\ln\lambda, \mathcal{J}_*\phi\bar{V}) \\ &+ \frac{1}{\lambda^2} g_2(\nabla^{\mathcal{J}}_{\bar{X}} \mathcal{J}_* f\bar{Y}, \mathcal{J}_*\phi\bar{V}) + g_1(\bar{X}, \bar{Y})\eta(\bar{V}). \end{aligned}$$

Moreover using the fact that $g_1(f\bar{X}, \phi\bar{V}) = 0$, for $\bar{X} \in \Gamma(\ker \mathcal{J}_*)^\perp$, $\bar{V} \in \Gamma(\ker \mathcal{J}_*)$, we arrived at

$$\begin{aligned} g_1(\nabla_{\bar{X}}\bar{Y}, \bar{V}) =& g_1(\mathcal{A}_{\bar{X}} t\bar{Y}, \phi\bar{V}) + \frac{1}{\lambda^2} g_2(\nabla^{\mathcal{J}}_{\bar{X}} \mathcal{J}_* f\bar{Y}, \mathcal{J}_*\phi\bar{V}) \\ &- \frac{1}{\lambda^2} g_2(\operatorname{grad}\ln\lambda, f\bar{Y}) \frac{1}{\lambda^2} g_2(\mathcal{J}_*\bar{X}, \mathcal{J}_*\phi\bar{V}) \\ &+ \frac{1}{\lambda^2} g_2(\bar{X}, f\bar{Y}) \frac{1}{\lambda^2} g_2(\mathcal{J}_*\operatorname{grad}\ln\lambda, \mathcal{J}_*\phi\bar{V}) + g_1(\bar{X}, \bar{Y})\eta(\bar{V}). \end{aligned}$$

With the last equation, we can say that $\bar{O}_{1(\ker \mathcal{J}_*)^\perp}$ is totally umbilical if and only if the (4.6) satisfied. This proves the theorem completely. □

5. ϕ-Pluriharmonicity of CBSS with vertical ξ

Now, we recall the concept of $\mathcal{J}$-pluriharmonicity which is defined by Ohnita [25] and extend the notion from a almost Hermitian manifold to ACM manifold.

Let $\mathcal{J}$ be a CBSS with vertical ξ from KM $(\bar{O}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{O}_2, g_2)$ with slant angles θ_1 and θ_2. Then conformal bi-slant is ϕ-pluriharmonic, D_{θ_i}-ϕ-pluriharmonic, $ker\mathcal{J}_*$-ϕ-pluriharmonic, $(ker\mathcal{J}_*)^\perp$-ϕ-pluriharmonic and $((ker\mathcal{J}_*)^\perp - ker\mathcal{J}_*)$-$\phi$-pluriharmonic if

$$(\nabla\mathcal{J}_*)(\bar{W}, \bar{Z}) + (\nabla\mathcal{J}_*)(\phi\bar{W}, \phi\bar{Z}) = 0 \tag{5.1}$$

for any $\bar{W}, \bar{Z} \in \Gamma(\mathfrak{D}^{\theta_i})$, for any $\bar{W}, \bar{Z} \in \Gamma(ker\mathcal{J}_*)$, for any $\bar{W}, \bar{Z} \in \Gamma(ker\mathcal{J}_*)^\perp$ and for any $\bar{W} \in \Gamma(ker\mathcal{J}_*)^\perp, \bar{Z} \in \Gamma(ker\mathcal{J}_*)$.

Theorem 5.1. *Let $\mathcal{J}$ be a CBSS with vertical ξ from KM $(\bar{O}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{O}_2, g_2)$ with slant angles θ_1 and θ_2. Suppose that $\mathcal{J}$ is $\mathfrak{D}^{\theta_g}$-ϕ-pluriharmonic. Then D_{θ_1} defines totally geodesic foliation $\bar{O}_1$ if and only if*

$$\begin{aligned} &\mathcal{J}_*(\zeta\mathcal{T}_{\psi\bar{U}}\zeta\psi\bar{V} + f\mathcal{H}\nabla_{\psi\bar{U}}\zeta\psi\bar{V}) - \mathcal{J}_*(\mathcal{A}_{\zeta\bar{U}}\psi\bar{V} + \mathcal{H}\nabla_{\psi\bar{U}}\zeta\bar{V}) \\ &= \cos^2\theta_1 \mathcal{J}_*(f\mathcal{T}_{\psi\bar{U}}\bar{V} + \zeta\mathcal{V}\nabla_{\psi\bar{U}}\bar{V}) + \nabla^{\mathcal{J}}_{\psi\bar{U}}\mathcal{J}_*\phi\bar{V} \\ &\quad - \zeta\bar{U}(\ln\lambda)\mathcal{J}_*\zeta\bar{V} - \zeta\bar{V}(\ln\lambda)\mathcal{J}_*\zeta\bar{U} + g_1(\zeta\bar{U}, \zeta\bar{V})\mathcal{J}_*(\operatorname{grad}\ln\lambda) \end{aligned}$$

for any $\bar{U}, \bar{V} \in \Gamma(D_{\theta_1})$.

Proof. For any $\bar{U}, \bar{V} \in \Gamma(D_{\theta_1})$ and since, $\mathcal{J}$ is D_{θ_1}-ϕ-pluriharmonic, then by using (2.3) and (2.4), we have

$$\begin{aligned} 0 =& (\nabla\mathcal{J}_*)(\bar{U}, \bar{V}) + (\nabla\mathcal{J}_*)(\phi\bar{U}, \phi\bar{V}), \\ \mathcal{J}_*(\nabla_{\bar{U}}\bar{V}) =& -\mathcal{J}_*(\nabla_{\phi\bar{U}}\phi\bar{V}) + \nabla^{\mathcal{J}}_{\phi\bar{U}}\mathcal{J}_*(\phi\bar{V}) \\ =& -\mathcal{J}_*(\mathcal{A}_{\zeta\bar{U}}\psi\bar{V} + \mathcal{V}\nabla_{\zeta\bar{U}}\psi\bar{V} + \mathcal{T}_{\psi\bar{U}}\zeta\bar{V} + \mathcal{H}\nabla_{\psi\bar{U}}\zeta\bar{V}) + \mathcal{J}_*(\phi\nabla_{\psi\bar{U}}\phi\psi\bar{V} \\ &+ (\nabla\mathcal{J}_*)(\zeta\bar{U}, \zeta\bar{V}) - \nabla^{\mathcal{J}}_{\zeta\bar{U}}\mathcal{J}_*\zeta\bar{V} + \nabla^{\mathcal{J}}_{\phi\bar{U}}\mathcal{J}_*\phi\bar{V}. \end{aligned}$$

By using (2.10), (3.1) with Theorem 3.1, the above equation finally takes the form

$$\begin{aligned} \mathcal{J}_*(\nabla_{\bar{U}}V) =& -\cos^2\theta_1\mathcal{J}_*(t\mathcal{T}_{\psi\bar{U}}\bar{V} + f\mathcal{T}_{\psi\bar{U}}\bar{V} + \psi\mathcal{V}\nabla_{\psi\bar{U}}\bar{V} + \zeta\mathcal{V}\nabla_{\psi\bar{U}}\bar{V}) + \nabla^{\mathcal{J}}_{\phi\bar{U}}\mathcal{J}_*\phi\bar{V} \\ &+ \mathcal{J}_*(\psi\mathcal{T}_{\psi\bar{U}}\zeta\psi\bar{V} + \zeta\mathcal{T}_{\psi\bar{U}}\zeta\psi\bar{V} + t\mathcal{H}\nabla_{\psi\bar{U}}\zeta\psi\bar{V} + f\mathcal{H}\nabla_{\psi\bar{U}}\zeta\psi\bar{V}) \\ &- \mathcal{J}_*(\mathcal{A}_{\zeta\bar{U}}\psi\bar{V} + \mathcal{V}\nabla_{\zeta\bar{U}}\psi\bar{V} + \mathcal{T}_{\psi\bar{U}}\zeta\bar{V} + \mathcal{H}\nabla_{\psi\bar{U}}\zeta\bar{V}) - \nabla^{\mathcal{J}}_{\zeta\bar{U}}\mathcal{J}_*\zeta\bar{V} \\ &+ \zeta\bar{U}(\ln\lambda)\mathcal{J}_*\zeta\bar{V} + \zeta\bar{V}(\ln\lambda)\mathcal{J}_*\zeta\bar{U} - g_1(\zeta\bar{U}, \zeta\bar{V})\mathcal{J}_*(\operatorname{grad}\ln\lambda) \end{aligned}$$

from which we get the desired result. □

Theorem 5.2. *Let $\mathcal{J}$ be a CBSS with vertical ξ from KM $(\bar{O}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{O}_2, g_2)$ with slant angles θ_1 and θ_2. Suppose that $\mathcal{J}$ is D_{θ_2}-ϕ-pluriharmonic. Then D_{θ_2} defines totally geodesic foliation $\bar{O}_1$ if and only if*

$$\begin{aligned}&\mathcal{J}_*(\zeta\mathcal{T}_{\psi\bar{Z}}\zeta\psi\bar{W} + f\mathcal{H}\nabla_{\psi\bar{Z}}\zeta\psi\bar{W}) - \mathcal{J}_*(\mathcal{A}_{\zeta\bar{Z}}\psi\bar{W} + \mathcal{H}\nabla_{\psi\bar{Z}}\zeta\bar{W})\\ &= \cos^2\theta_2\mathcal{J}_*(f\mathcal{T}_{\psi\bar{Z}}\bar{W} + \zeta\bar{\mathcal{W}}\nabla_{\psi\bar{Z}}\bar{W}) + \nabla^{\mathcal{J}}_{\psi\bar{Z}}\mathcal{J}_*\phi\bar{W}\\ &\quad - \zeta\bar{Z}(\ln\lambda)\mathcal{J}_*\zeta\bar{W} - \zeta\bar{W}(\ln\lambda)\mathcal{J}_*\zeta\bar{Z} + g_1(\zeta\bar{Z}, \zeta\bar{W})\mathcal{J}_*(\operatorname{grad}\ln\lambda)\end{aligned}$$

for any $\bar{Z}, \bar{W} \in \Gamma(D_{\theta_2})$.

Proof. Due to the similarity of proof of above result to Theorem 5.1, we omit it. □

Theorem 5.3. *Let $\mathcal{J}$ be a CBSS with vertical ξ from KM $(\bar{O}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{O}_2, g_2)$ with slant angles θ_1 and θ_2. Suppose that $\mathcal{J}$ is $((ker\mathcal{J}_*)^\perp - ker\mathcal{J}_*)$-$\phi$-pluriharmonic. Then the horizontal distribution $(ker\mathcal{J}_*)^\perp$ defines totally geodesic foliation on $\bar{O}_1$ if and only if*

$$\begin{aligned}&cos^2\theta_1\mathcal{J}_*\{f\mathcal{T}_{t\bar{X}}\Re\bar{U} + \zeta\mathcal{V}\nabla_{t\bar{X}}\Re\bar{U} + f\mathcal{A}_{C\bar{X}}\Re\bar{U} + \zeta\mathcal{V}\nabla_{C\bar{X}}\Re\bar{U}\}\\ &+ cos^2\theta_2\mathcal{J}_*\{f\mathcal{T}_{t\bar{X}}\mathfrak{L}\bar{U} + \zeta\mathcal{V}\nabla_{t\bar{X}}\mathfrak{L}\bar{U} + f\mathcal{A}_{f\bar{X}}\mathfrak{L}U + \zeta\mathcal{V}\nabla_{f\bar{X}}\mathfrak{L}U + \eta(\psi U)f\bar{X}\}\\ &= \mathcal{J}_*\{\zeta\mathcal{T}_{t\bar{X}}\zeta\psi\Re\bar{U} + f\mathcal{H}\nabla_{t\bar{X}}\zeta\psi\Re\bar{U} + \zeta\mathcal{T}_{t\bar{X}}\zeta\psi\mathfrak{L}\bar{U} + f\mathcal{H}\nabla_{t\bar{X}}\zeta\psi\mathfrak{L}\bar{U}\}\\ &\quad + \mathcal{J}_*\{\zeta\mathcal{A}_{f\bar{X}}\zeta\psi\Re\bar{U} + \zeta\mathcal{A}_{f\bar{X}}\zeta\psi\mathfrak{L}\bar{U} - \mathcal{H}\nabla_{t\bar{X}}\zeta\bar{U}\} + \nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}\\ &\quad - f\bar{X}(\ln\lambda)\mathcal{J}_*\zeta\psi\Re\bar{U} - \zeta\psi\Re\bar{U}(\ln\lambda)\mathcal{J}_*f\bar{X} + g_1(f\bar{X}, \zeta\psi\Re\bar{U})\mathcal{J}_*(\operatorname{grad}\ln\lambda)\\ &\quad - f\bar{X}(\ln\lambda)\mathcal{J}_*\zeta\psi\mathfrak{L}\bar{U} - \zeta\psi\mathfrak{L}\bar{U}(\ln\lambda)\mathcal{J}_*f\bar{X} + g_1(f\bar{X}, \zeta\psi\mathfrak{L}\bar{U})\mathcal{J}_*(\operatorname{grad}\ln\lambda)\\ &\quad + \mathcal{J}_*(\nabla_{\bar{X}}\bar{U}) + \nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U} + \nabla^{\mathcal{J}}_{f\bar{X}}\mathcal{J}_*\zeta\psi\Re\bar{U} + \nabla^{\mathcal{J}}_{f\bar{X}}\mathcal{J}_*\zeta\psi\mathfrak{L}\bar{U}\end{aligned}$$

for any $\bar{X} \in \Gamma(ker\mathcal{J}_*)^\perp$ and $\bar{U} \in \Gamma(ker\mathcal{J}_*)$

Proof. For any $\bar{X} \in \Gamma(ker\mathcal{J}_*)^\perp$ and $\bar{U} \in \Gamma(ker\mathcal{J}_*)$, since $\mathcal{J}$ is $((ker\mathcal{J}_*)^\perp - ker\mathcal{J}_*)$-$\phi$-pluriharmonic, then by using (2.4), (2.10) and (3.1), we get

$$\mathcal{J}_*(\nabla_{f\bar{X}}\zeta\bar{U}) = -\mathcal{J}_*(\nabla_{t\bar{X}}\psi\bar{U} + \nabla_{t\bar{X}}\zeta\bar{U} + \nabla_{f\bar{X}}\psi\bar{U}) + \mathcal{J}_*(\nabla_{\bar{X}}\bar{U}) + \nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}.$$

By using (2.11), we have

$$\begin{aligned}\mathcal{J}_*(\nabla_{f\bar{X}}\zeta\bar{U}) &= -\mathcal{J}_*(\mathcal{T}_{t\bar{X}}\zeta\bar{U} + \mathcal{H}\nabla_{t\bar{X}}\zeta\bar{U}) + \mathcal{J}_*(\nabla_{\bar{X}}\bar{U}) + \nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}\\ &\quad + \mathcal{J}_*\{\phi\nabla_{t\bar{X}}\phi\psi\bar{U} - \eta(\nabla_{t\bar{X}}\psi\bar{U})\xi - \eta(\psi\bar{U})t\bar{X} + \eta(\psi\bar{U})\eta(t\bar{X})\xi\}\\ &\quad + \mathcal{J}_*\{\phi\nabla_{f\bar{X}}\phi\psi\bar{U} - \eta(\nabla_{f\bar{X}}\psi\bar{U})\xi - \eta(\psi\bar{U})f\bar{X} + \eta(\psi\bar{U})\eta(f\bar{X})\xi\}.\end{aligned}$$

Now on using decomposition (2.8), Theorem 3.1 with (2.10), we may yields

$$\begin{aligned}\mathcal{J}_*(\nabla_{f\bar{X}}\zeta\bar{U}) &= \mathcal{J}_*\{-\cos^2\theta_1\phi\nabla_{t\bar{X}}\Re\bar{U} - \cos^2\theta_2\phi\nabla_{t\bar{X}}\mathfrak{L}\bar{U} + \eta(\psi\bar{U})f\bar{X}\}\\ &\quad + \mathcal{J}_*\{\phi\nabla_{t\bar{X}}\zeta\psi\Re\bar{U} + \phi\nabla_{t\bar{X}}\zeta\psi\mathfrak{L}\bar{U} + \phi\nabla_{f\bar{X}}\zeta\psi\Re\bar{U} + \phi\nabla_{f\bar{X}}\zeta\psi\mathfrak{L}\bar{U}\}\\ &\quad + \mathcal{J}_*\{-\cos^2\theta_1\phi\nabla_{f\bar{X}}\Re\bar{U} - \cos^2\theta_2\phi\nabla_{f\bar{X}}\mathfrak{L}\bar{U}\}\\ &\quad - \mathcal{J}_*(\mathcal{H}\nabla_{t\bar{X}}\zeta\bar{U}) + \mathcal{J}_*(\nabla_{\bar{X}}\bar{U}) + \nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}.\end{aligned}$$

From (2.4)–(2.7) and after simple calculation, we may write

$$\begin{aligned}\mathcal{J}_*(\nabla_{f\bar{X}}\zeta\bar{U}) =& -cos^2\theta_1\mathcal{J}_*\{f\mathcal{T}_{t\bar{X}}\mathfrak{R}\bar{U} + \zeta\mathcal{V}\nabla_{t\bar{X}}\mathfrak{R}\bar{U} + f\mathcal{A}_{f\bar{X}}\mathfrak{R}\bar{U} + \zeta\mathcal{V}\nabla_{f\bar{X}}\mathfrak{R}\bar{U}\}\\ &- cos^2\theta_2\mathcal{J}_*\{f\mathcal{T}_{t\bar{X}}\mathfrak{L}\bar{U} + \zeta\mathcal{V}\nabla_{t\bar{X}}\mathfrak{L}\bar{U} + f\mathcal{A}_{f\bar{X}}\mathfrak{L}\bar{U} + \zeta\mathcal{V}\nabla_{f\bar{X}}\mathfrak{L}\bar{U} + \eta(\psi\bar{U})f\bar{X}\}\\ &+ \mathcal{J}_*\{\zeta\mathcal{T}_{t\bar{X}}\zeta\psi\mathfrak{R}\bar{U} + f\mathcal{H}\nabla_{t\bar{X}}\zeta\psi\mathfrak{R}\bar{U} + \zeta\mathcal{T}_{t\bar{X}}\zeta\psi\mathfrak{L}\bar{U} + f\mathcal{H}\nabla_{t\bar{X}}\zeta\psi\mathfrak{L}\bar{U}\}\\ &+ \mathcal{J}_*\{\zeta\mathcal{A}_{f\bar{X}}\zeta\psi\mathfrak{R}\bar{U} + \zeta\mathcal{A}_{f\bar{X}}\zeta\psi\mathfrak{L}\bar{U} - \mathcal{H}\nabla_{t\bar{X}}\zeta\bar{U}\} + \nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}\\ &+ \mathcal{J}_*(f\mathcal{H}\nabla_{f\bar{X}}\zeta\psi\mathfrak{R}\bar{U} + f\mathcal{H}\nabla_{f\bar{X}}\zeta\psi\mathfrak{R}\bar{U}) + \mathcal{J}_*(\nabla_{\bar{X}}\bar{U}).\end{aligned}$$

On using the conformality of $\mathcal{J}$ with (2.4) and from Lemma 2.1, we finally have

$$\begin{aligned}\mathcal{J}_*(\nabla_{f\bar{X}}\zeta\bar{U}) =& -cos^2\theta_1\mathcal{J}_*\{f\mathcal{T}_{t\bar{X}}\mathfrak{R}\bar{U} + \zeta\mathcal{V}\nabla_{t\bar{X}}\mathfrak{R}\bar{U} + f\mathcal{A}_{C\bar{X}}\mathfrak{R}\bar{U} + \zeta\mathcal{V}\nabla_{C\bar{X}}\mathfrak{R}\bar{U}\}\\ &- cos^2\theta_2\mathcal{J}_*\{f\mathcal{T}_{t\bar{X}}\mathfrak{L}\bar{U} + \zeta\mathcal{V}\nabla_{t\bar{X}}\mathfrak{L}\bar{U} + f\mathcal{A}_{f\bar{X}}\mathfrak{L}U + \zeta\mathcal{V}\nabla_{f\bar{X}}\mathfrak{L}U + \eta(\psi U)f\bar{X}\}\\ &+ \mathcal{J}_*\{\zeta\mathcal{T}_{t\bar{X}}\zeta\psi\mathfrak{R}\bar{U} + f\mathcal{H}\nabla_{t\bar{X}}\zeta\psi\mathfrak{R}\bar{U} + \zeta\mathcal{T}_{t\bar{X}}\zeta\psi\mathfrak{L}\bar{U} + f\mathcal{H}\nabla_{t\bar{X}}\zeta\psi\mathfrak{L}\bar{U}\}\\ &+ \mathcal{J}_*\{\zeta\mathcal{A}_{f\bar{X}}\zeta\psi\mathfrak{R}\bar{U} + \zeta\mathcal{A}_{f\bar{X}}\zeta\psi\mathfrak{L}\bar{U} - \mathcal{H}\nabla_{t\bar{X}}\zeta\bar{U}\} + \nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}\\ &- f\bar{X}(\ln\lambda)\mathcal{J}_*\zeta\psi\mathfrak{R}\bar{U} - \zeta\psi\mathfrak{R}\bar{U}(\ln\lambda)\mathcal{J}_*f\bar{X} + g_1(f\bar{X},\zeta\psi\mathfrak{R}\bar{U})\mathcal{J}_*(\operatorname{grad}\ln\lambda)\\ &- f\bar{X}(\ln\lambda)\mathcal{J}_*\zeta\psi\mathfrak{L}\bar{U} - \zeta\psi\mathfrak{L}\bar{U}(\ln\lambda)\mathcal{J}_*f\bar{X} + g_1(f\bar{X},\zeta\psi\mathfrak{L}\bar{U})\mathcal{J}_*(\operatorname{grad}\ln\lambda)\\ &+ \mathcal{J}_*(\nabla_{\bar{X}}\bar{U}) + \nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U} + \nabla^{\mathcal{J}}_{f\bar{X}}\mathcal{J}_*\zeta\psi\mathfrak{R}\bar{U} + \nabla^{\mathcal{J}}_{f\bar{X}}\mathcal{J}_*\zeta\psi\mathfrak{L}\bar{U},\end{aligned}$$

which completes the proof of theorem. □

Now, we are giving some definitions of integrablity and totally geodesic of leaves of distributions.

Definition 5.1. *Suppose D is a k-dimensional smooth distribution on M. Then An immersed submanifold i: N ↪ M is called an integral manifold for D if for every $x \in N$, the image of d_iN: $T_pN \to T_pM$ is D_p. We say the distribution D_p is integrable if through each point of M there exists an integral manifold of D.*

Further, a distribution D is involutive if it satisfies the Frobenius condition such that if $X, Y \in \Gamma(TM)$ belongs to D, so $[X, Y] \in D$. Frobenius theorem states that an involutive distribution is integrable.

Definition 5.2. *Let M be n-dimensional smooth manifold. A foliation $\mathfrak{F}$ of M is a decomposition of M into a union of disjoint connected submanifolds $M = \cup_{L\in\mathfrak{F}}L$ called the leaves of the foliation, such that for each $m \in M$, there is a neighborhood U of M and a smooth submersion f_U: $U \to R^k$ with $f_U^{-1}(x)$ a leaf of $\mathfrak{F}|_U$ the restriction of the foliation to U, for each $x \in R^k$.*

Definition 5.3. *Let M be a Riemannian manifold, and let $\mathfrak{F}$ be a foliation on M. $\mathfrak{F}$ is totally geodesic if each leaf L is a totally geodesic submanifold of M; that is, any geodesic tangent to L at some point must lie within L.*

6. Conclusions

The properties of submersions between Riemannian manifolds have emerged as an interesting area of study in contact as well as complex geometry. The geometry of conformal bi-slant submersion, whose base manifold is a Kenmotsu manifold, was examined in this study. We suppose that the Reeb vector field ξ is vertical and establishes the condition of integrability of slant distributions because the

Reeb vector field ξ plays a crucial role in the geometry of leaves of the distributions. Additionally, the total geodesics of the leaves of distributions were determined. Furthermore, the idea of pluriharmonicity from the Kenmotsu manifold was explored.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflicts of interest.

References

1. M. A. Akyol, Y. Gündüzalp, Hemi-slant submersions from almost product Riemannian manifolds, *Gulf J. Math.*, **4** (2016), 15–27. http://doi.org/10.56947/gjom.v4i3.70
2. M. A. Akyol, B. Şahin, Conformal anti-invariant submersion from almost Hermitian manifolds, *Turk. J. Math.*, **40** (2016), 38–42. https://doi.org/10.3906/mat-1408-20
3. M. A. Akyol, B. Şahin, Conformal anti-invariant submersions from almost Hermitian manifolds, *Turk. J. Math.*, **40** (2016), 43–70. http://doi.org/10.3906/mat-1408-20
4. M. A. Akyol, B. Şahin, Conformal semi-invariant submersions, *Commun. Contemp. Math.*, **19** (2017), 1650011. http://doi.org/10.1142/S0219199716500115
5. M. A. Akyol, Conformal semi-slant submersions, *Int. J. Geom. Methods Mod. Phys.*, **14** (2017), 1750114. http://doi.org/10.1142/S0219887817501146
6. M. A. Akyol, B. Şahin, Conformal slant submersions, *Hacettepe J. Math. Stat.*, **48** (2019), 28–44. http://doi.org/10.15672/HJMS.2017.506
7. I. Al-Dayel, T. Fatima, S. Deshmukh, M. Shuaib, A note on conformal bi-slant submersion from Kenmotsu manifold, *J. Geom. Phys.*, **190** (2023) 104864. http://doi.org/10.1016/j.geomphys.2023.104864
8. I. Al-Dayel, M. Shuaib, S. Deshmukh, T. Fatima, ϕ-pluriharmonicity in quasi bi-slant conformal $\xi^{\perp}$-submersions: a comprehensive study, *AIMS Math.*, **8** (2023), 21746–21768. http://doi.org/10.3934/math.20231109
9. P. Baird, J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, Oxford University Press, 2003.
10. J. P. Bourguignon, H. B. Lawson, Stability and isolation phenomena for Yang-Mills fields, *Commun. Math. Phys.*, **79** (1981), 189–230. http://doi.org/10.1007/BF01942061

11. D. Chinea, Almost contact metric submersions, *Rend. Circ. Mat. Palerm.*, **34** (1985), 89–104. http://doi.org/10.1007/BF02844887
12. J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, M. Fernandez, Slant submanifolds in Sasakian manifolds, *Glasg. Math. J.*, **42** (2000), 125–138. http://doi.org/10.1017/S0017089500010156
13. I. K. Erken, C. Murathan, On slant Riemannian submersions for cosymplectic manifolds, *Bull. Korean Math. Soc.*, **51** (2014), 1749–1771. http://doi.org/10.4134/BKMS.2014.51.6.1749
14. B. Fuglede, Harmonic morphisms between Riemannian manifolds, *Ann. I'institut Fourier*, **28** (1978), 107–144. http://doi.org/10.5802/aif.691
15. M. Falcitelli, S. Ianus, A. M. Pastore, *Riemannian submersions and related topics*, World Scientific, 2004.
16. A. Gray, Pseudo-Riemannian almost product manifolds and submersions, *J. Math. Mech.*, **16** (1967), 715–737.
17. S. Gudmundsson, *The geometry of harmonic morphisms*, Ph. D. thesis, University of Leeds, 1992.
18. S. Gudmundsson, J. C. Wood, Harmonic morphisms between almost Hermitian manifolds, *Boll. Un. Mat. Ital.*, **11** (1997), 185–197.
19. Y. Gunduzalp, Semi-slant submersions from almost product Riemannian manifolds, *Demonstr. Math.*, **49** (2016), 345–356. http://doi.org/10.1515/dema-2016-0029
20. S. Ianus, M. Visinescu, Kaluza-Klein theory with scalar fields and generalised Hopf manifolds, *Class. Quantum Grav.*, **4** (1987), 1317–1325. http://doi.org/10.1088/0264-9381/4/5/026
21. T. Ishihara, A mapping of Riemannian manifolds which preserves harmonic functions, *J. Math. Kyoto Univ.*, **19** (1979), 215–229. http://doi.org/10.1215/kjm/1250522428
22. K. Kenmotsu, A class of almost contact Riemannian manifolds, *Tohoku Math. J.*, **24** (1972), 93–103. http://doi.org/10.2748/tmj/1178241594
23. S. Kumar, S. Kumar, S. Pandey, R. Prasad, Conformal hemi-slant submersions from almost Hermitian manifolds, *Commun. Korean Math. Soc.*, **35** (2020), 999–1018 https://doi.org/10.4134/CKMS.c190448
24. M. T. Mustafa, Applications of harmonic morphisms to gravity, *J. Math. Phys.*, **41** (2000), 6918–6929. http://doi.org/10.1063/1.1290381
25. Y. Ohnita, On pluriharmonicity of stable harmonic maps, *J. London Math. Soc.*, **2** (1987), 563–568. http://doi.org/10.1112/jlms/s2-35.3.563
26. B. O'Neill, The fundamental equations of a submersion, *Michigan Math. J.*, **13** (1966), 459–469. http://doi.org/10.1307/mmj/1028999604
27. K. S. Park, R. Prasad, Semi-slant submersions, *Bull. Korean Math. Soc.*, **50** (2013), 951–962. http://doi.org/10.4134/BKMS.2012.49.2.329
28. R. Prasad, S. S. Shukla, S. Kumar, On quasi bi-slant submersions, *Mediterr. J. Math.*, **16** (2019), 155. https://doi.org/10.1007/s00009-019-1434-7
29. R. Prasad, M. A. Akyol, P. K. Singh, S. Kumar, On quasi bi-slant submersions from Kenmotsu manifolds onto any Riemannian manifolds, *J. Math. Ext.*, **16** (2022), 1–25. https://doi.org/10.30495/JME.2022.1588

30. R. Prasad, S. Kumar, Conformal anti-invariant submersions from nearly Kaehler manifolds, *Palest. J. Math.*, **8** (2019), 237–247.
31. R. Ponge, H. Reckziegel, Twisted products in pseudo-Riemannian geometry, *Geom. Dedicata*, **48** (1993), 15–25. https://doi.org/10.1007/BF01265674
32. B. Şahin, Anti-invariant Riemannian submersions from almost Hermitian manifolds, *Central Eur. J. Math.*, **3** (2010), 437–447. https://doi.org/10.2478/s11533-010-0023-6
33. B. Şahin, Semi-invariant Riemannian submersions from almost Hermitian manifolds, *Canad. Math. Bull.*, **56** (2013), 173–183. https://doi.org/10.4153/CMB-2011-144-8
34. B. Şahin, Slant submersions from almost Hermitian manifolds, *Bull. Math. Soc. Sci. Math. Roumanie*, **1** (2011) 93–105.
35. M. Shuaib, T. Fatima, A note on conformal hemi-slant submersions, *Afr. Mat.*, **34** (2023), 4. https://doi.org/10.1007/s13370-022-01036-2
36. S. Tanno, The automorphism groups of almost contact metric manfolds, *Tohoku Math. J.*, **21** (1969), 21–38.
37. H. M. Taştan, B. Şahin, Ş. Yanan, Hemi-slant submersions, *Mediterr. J. Math.*, **13** (2016), 2171–2184. https://doi.org/10.1007/s00009-015-0602-7
38. H. Urakawa, *Calculus of variations and harmonic maps*, American Mathematical Society, 2013.
39. B. Watson, Almost Hermitian submersions, *J. Differ. Geom.*, **11** (1976), 147–165. https://doi.org/10.4310/jdg/1214433303
40. B. Watson, G, G'-Riemannian submersions and nonlinear gauge field equations of general relativity, *Global Anal.*, **57** (1983), 324–349.

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