AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 29332–29351.
DOI:10.3934/math.20231501
Received: 12 September 2023
Revised: 12 October 2023
Accepted: 15 October 2023
Published: 30 October 2023

Research article

Asynchronously switching control of discrete-time switched systems with a Φ-dependent integrated dwell time approach

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Abstract: In this paper, the asynchronous control problem is investigated and a multiple convex Lyapunov functions (MCLF) approach is introduced for a class of discrete-time switched linear systems under the Φ-dependent integrated dwell time (ΦDIDT) switching strategy. For the problem of asynchronous switching, this paper considers that Lyapunov functions may jump when the subsystem switches or the controller changes. Thus, the constructed MCLF is dependent on both the asynchronous interval and the synchronous interval, and the synchronous interval is divided into the convex interval and non-convex interval parts. Some sufficient conditions of stability with Linear matrix inequality (LMI) forms are obtained, and the asynchronous controller is designed to guarantee the globally uniform exponential stability of the system under study. In addition, the proposed method can degenerate to the existing methods to deal with the asynchronous control problem. Finally, a numerical example illustrates the superiority of the proposed method.

Keywords: switched systems; asynchronous control; globally uniformly exponentially stability; Φ-dependent integrated dwell time; multiple convex Lyapunov function
Mathematics Subject Classification: 34D20, 93D15

1. Introduction

A switched system usually consists of a set of state-space models and a switching signal. The problem of stability/stabilization for switched systems has drawn a great deal of attention and interest in the field of automation [1–6]. Without a doubt, the stability analysis of switched systems is very important and is closely related to various switching strategies, such as average dwell time (ADT) switching [7–15]. The authors point out that there is incorrect thinking about the relationship between mode-dependent ADT (MDADT) and ADT in many existing related studies, where ADT is seen as a special case of MDADT. In fact, the ADT strategy mainly focuses on the compensation effect among subsystems without considering the subsystems' differences. Instead, the MDADT strategy takes these subsystems' differences into account but misses the compensation among subsystems. In the stability study of switched systems, as we know, both the switching strategies and other supported methods are quite important. Therefore, in recent decades, various stability analysis tools have been proposed, mainly including the common Lyapunov function [16–18], the multiple Lyapunov functions [19–22], the multiple discontinuous Lyapunov function [23,24], and the multiple convex Lyapunov function [25,26].

The switching strategies and supported methods above are mainly applied to the system under synchronous switching. However, a class of asynchronously switched systems has become a research hotspot. The literature [27], in both continuous-time and discrete-time contexts, studies the problem of asynchronous switching control for a class of switched linear systems under the ADT strategy by further relaxing the demand of the Lyapunov-like function decreasing during the whole running time of each active subsystem. The literature [28] studies the problem of asynchronous switching control for discrete-time switched systems with MDADT strategy and considers that the lag time of the controllers of different subsystems may be different. The paper [29] investigates the stability of a class of asynchronously switched linear systems by using a mode-dependent integrated dwell time (MDIDT) strategy. It is worth noting that all the above-mentioned works of literature assume that the Lyapunov function may be discontinuous when the controller changes, but it is still continuous when the subsystem switches. Thus, the designed Lyapunov function may be deduced as greatly conservative due to neglecting the jump of the Lyapunov function caused by the subsystem switching. Then, seeking a less conservative result has become an important problem in the stability analysis and asynchronous control of switched systems.

Inspired by the aforementioned works and issues, this article investigates more general stability and stabilization criteria for a class of asynchronously switched linear systems. The main contributions are as follows: (1) A novel MCLF is constructed that considers the jump of the Lyapunov function caused by the subsystem switching. (2) A switching strategy named ΦDIDT is proposed that covers the IDT and MDIDT strategies. (3) Based on the proposed ΦDIDT strategy with the constructed MCLF, some new stability criteria and controller designs of the system under study are obtained, which are more flexible than the existing results [29–31].

The remaining structure is organized as follows: In Section 2, the problem statement and necessary definitions for stability analysis of discrete-time switched linear systems are provided. In Section 3, the stability analysis for the asynchronous switching control of the considered system with ΦDIDT switching is deduced by the MCLF approach. Moreover, the design of the asynchronous controller for the system is obtained. In addition, the proposed method can degenerate into that of Vu and Liberzon [30]. A numerical example illustrates the superiority of the asynchronous control strategy in Section 4. Lastly, it is summarized in Section 5.

For the convenience of reviewing the meanings of abbreviations, the following Table 1 is provided.

Table 1. List of abbreviations.

DT	Dwell Time	MCLF	Multiple Convex Lyapunov Functions
ADT	Average Dwell Time	IDT	Integrated Dwell Time
MDADT	Mode Dependent ADT	MDIDT	Mode Dependent IDT
ΦDADT	Φ Dependent ADT	ΦDIDT	Φ Dependent IDT
LMI	Linear Matrix Inequalities	GUES	Globally Uniformly Exponentially Stable

2. Problem formulation and preliminaries

Some fairly standard notations are used in this paper. $\mathbb{Z}^*$ ($\mathbb{R}$) stands for the set of positive integers (real numbers). $\mathbb{R}^n$ represents the space of n-dimensional real Euclidean and $\mathbb{R}^{n\times n}$ refers to the space of $n\times n$ matrix with all entries being real. $P > 0$ ($P \geq 0$) implies that P is positive definite (semi-definite). Meanwhile, A^T stands for the transpose of a matrix A, and A^{-1} stands for the inverse of a matrix A. For $x \in \mathbb{R}^n$, $\|x\|$ stands for the Euclidean vector norm of x. The notation $\forall$ ($\in, \notin$) denotes "for all" ("in", "not in"). The "$\star$" notation denotes the elements above the main diagonal of a symmetric matrix.

Consider the following discrete-time switched linear system

$$x(k+1) = A_{\varsigma(k)}x(k) + B_{\varsigma(k)}u(k), x(k_0) = x_0, k \geq k_0, \tag{2.1}$$

where $x(k)$ is the system state, $x(k_0) \in \mathbb{R}^n$ stands for initial state, $u(k) \in \mathbb{R}^n$ is control input, $\varsigma(k)$: $[k_0, +\infty) \mapsto \mathfrak{F}_m = \{1, 2, \cdots, m\}$, is a piecewise constant function from the right, called the switching law. Let $k_1 < k_2 < \cdots < k_l < \cdots$, $l \in \mathbb{Z}^*$ be the switching instants of $\varsigma(k)$. A_ν, B_ν, $\forall\nu \in \mathfrak{F}_m$ are constant matrices of appropriate dimensions. Letting $\mathfrak{O} = \{1, 2, \cdots, s\}$, $s \in \mathbb{Z}^*$, $s \leq m$. Define the mapping $\Phi_I : \mathfrak{F}_m \mapsto \mathfrak{O}$ as an epimorphism operator. Set $\Phi_{I\gamma} = \{\nu \in \mathfrak{F}_m \mid \Phi_I(\nu) = \gamma\}$.

Definition 2.1. ([10]) The equilibrium $x = 0$ of system (2.1) with $u(k) \equiv 0$ is globally uniformly exponentially stable (GUES), if, for a given switching signals ς, there exist constants $\epsilon > 0$ and $0 < \lambda < 1$ such that the system satisfies $\|x(k)\| \leq \epsilon\lambda^{(k-k_0)}\|x(k_0)\|$, $\forall k \geq k_0$ with initial condition $x(k_0)$.

Definition 2.2. For $k \in [k_l, k_{l+1})$, $l \in \mathbb{Z}^*$, and $\Phi_I(\nu) = \gamma \in \mathfrak{O}$, if there are a Φ_I-dependent dwell time $\tau_{d\Phi_{I\gamma}} > 0$ and a Φ_I-dependent average dwell time $\tau_{a\Phi_{I\gamma}} > \tau_{d\Phi_{I\gamma}}$ with some scalar $N_{0\Phi_{I\gamma}} > 0$, such that

$$k_{l+1} - k_l \geq \tau_{d\Phi_{I\gamma}}, \tag{2.2}$$

$$N_{\varsigma\Phi_{I\gamma}}(k_0, k) \leq N_{0\Phi_{I\gamma}} + \frac{K_{\Phi_{I\gamma}}(k_0, k)}{\tau_{a\Phi_{I\gamma}}}, \forall k \geq k_0 \geq 0, \tag{2.3}$$

hold, then we say the switching signal $\varsigma(k)$ has a Φ-dependent integrated dwell time (ΦDIDT) $\tau_{a\Phi_{I\gamma}}$ with the minimum dwell time $\tau_{d\Phi_{I\gamma}}$. When there is no ambiguity, it is briefly described as $\varsigma(k)$ having a ΦDIDT $\tau_{a\Phi_{I\gamma}}$. Here, $N_{0\Phi_{I\gamma}}$ stands for the chatter bound, $N_{\varsigma\Phi_{I\gamma}}(k, k_0)$ is the sum of switching numbers of subsystems $\Phi_{I\gamma}$ being activated over $[k_0, k]$, and $K_{\Phi_{I\gamma}}(k, k_0)$ represents the total running time of subsystems $\Phi_{I\gamma}$ over $[k_0, k]$.

Let $\mathfrak{O} = \{1\}$ and $\mathfrak{O} = \mathfrak{F}_m$, we can get the following integrated dwell time (IDT) and mode-dependent integrated dwell time (MDIDT) from Definition 2.2.

Definition 2.3. ([30]) For $k \in [k_l, k_{l+1})$, $l \in \mathbb{Z}^*$, if there exist a dwell time $\tau_d > 0$ and ADT $\tau_a > \tau_d$ with some scalar $N_{0\varsigma} > 0$, such that

$$k_{l+1} - k_l \geq \tau_d, \tag{2.4}$$

$$N_\varsigma(k_0, k) \leq N_{0\varsigma} + \frac{K(k_0, k)}{\tau_a}, \forall k \geq k_0 \geq 0, \tag{2.5}$$

hold, then the switching signal $\varsigma(k)$ is called to have an integrated dwell time (IDT) τ_a with the minimum dwell time τ_d (briefly described as IDT τ_a with no ambiguity). Here, $N_{0\varsigma}$ stands for the chatter bound, $N_\varsigma(k, k_0)$ is the sum of switching numbers of all subsystems being activated over $[k_0, k]$, and $K_\varsigma(k, k_0)$ represents the total running time of all subsystems over $[k_0, k]$.

Definition 2.4. ([31]) For $k \in [k_l, k_{l+1})$, $l \in \mathbb{Z}^*$ and $\varsigma(k) = \nu \in \mathfrak{F}_m$, if there exist a mode-dependent dwell time $\tau_{d\nu} > 0$ and an MDADT $\tau_{a\nu} > \tau_{d\nu}$ with some scalar $N_{0\varsigma\nu} > 0$, such that

$$k_{l+1} - k_l \geq \tau_{d\nu}, \tag{2.6}$$

$$N_{\varsigma\nu}(k_0, k) \leq N_{0\varsigma\nu} + \frac{K_\nu(k_0, k)}{\tau_{a\nu}}, \forall k \geq k_0 \geq 0, \tag{2.7}$$

hold, then we say $\varsigma(k)$ has an mode-dependent integrated dwell time (MDIDT) $\tau_{a\nu}$ with the minimum dwell time $\tau_{d\nu}$ (briefly described as MDIDT $\tau_{a\nu}$ with no ambiguity). Here, $N_{0\varsigma\nu}$ stands for the chatter bound, $N_{\varsigma\nu}(k, k_0)$ is the sum of switching numbers of the ν^{th} subsystem being activated over $[k_0, k]$, and $K_\nu(k, k_0)$ represents the total running time of the ν^{th} subsystem over $[k_0, k]$.

Remark 2.1. In essence, ΦDIDT (resp., IDT/MDIDT) is the hybrid between DT and ΦDADT (resp., ADT/MDADT).

Lemma 2.1. ([32]) Given $\mathfrak{X} \in \mathbb{R}^n$ and $\mathcal{Z}^T = \mathcal{Z} \in \mathbb{R}^{n\times n}$ and $\mathcal{D} \in \mathbb{R}^{m\times n}$ meeting rank$(\mathcal{D}) < n$. The following two expressions are equivalent:

1) $\mathfrak{X}\mathcal{Z}\mathfrak{X}^T < 0, \forall \mathfrak{X} \in \{\mathfrak{X} \in \mathbb{R}^n | \mathfrak{X} \neq 0, \mathcal{D}\mathfrak{X} = 0\}$;

2) $\exists \mathfrak{Y} \in \mathbb{R}^{n\times m}$, $\mathcal{Z} + \mathfrak{Y}\mathcal{D} + \mathcal{D}^T\mathfrak{Y}^T < 0$.

For asynchronous switching, we generally assume that the time lags of switching controllers to their corresponding subsystems are $\Delta l \leq k_{l+1} - k_l$. As a matter of convenience, it is assumed that maximal delay of asynchronous switching, $\Delta L = \max_{l\in\mathbb{Z}^*}\{\Delta l\}$, is known a prior without loss of generality. Let $\varsigma(k_{l-1}) = \omega$, $\varsigma(k_l) = \nu$, $\forall \nu, \omega \in \mathfrak{F}_m$. From the notation of above these symbols, the closed-loop system can be described as:

(a) when k is on the asynchronous interval $[k_l, k_l + \Delta l)$, $\forall \nu, \omega \in \mathfrak{F}_m$,

$$x(k+1) = A''_{\nu,\omega}x(k), (A''_{\nu,\omega} = A_\nu + B_\nu K_\omega), \tag{2.8}$$

(b) when k is on the synchronous interval $[k_l + \Delta l, k_{l+1})$, $\forall \nu \in \mathfrak{F}_m$,

$$x(k+1) = A'_\nu x(k), (A'_\nu = A_\nu + B_\nu K_\nu). \tag{2.9}$$

3. Main results

In this section, an MCLF is firstly improved, which is expressed in the form of a convex combination of positive definite matrices. For the study of the system (2.8)–(2.9) under ΦDIDT switching, consider that Lyapunov functions may jump when the subsystem switches or the controller changes. Thus, the constructed MCLF is dependent on both the asynchronous interval and the synchronous interval. As a matter of fact, we can not find accurately the moment of subsystems switching because of the influence of the asynchronous problem. Therefore, it's hard to construct a convex function over the entire synchronous interval. To solve this problem, [29] came up with a new idea that the synchronous interval $[k_{la}, k_{l+1})$ is divided into convex interval $[k_{la}, k_{lb})$ and non-convex interval $[k_{lb}, k_{l+1})$ by $\tau_{d\nu}$, where k_{la} and k_{lb} ($k_{la} = k_l + \Delta L$ and $k_{lb} = k_l + \tau_{d\nu}$) are the starting and ending points of the synchronous convex interval, respectively. In the research of asynchronous switching, it is often required that the asynchronous delay should not exceed a certain dwell time. Moreover, the existence of convex interval $[k_{la}, k_{lb})$ plays a crucial role in the paper. Therefore, it is both natural and necessary to require $k_{la} < k_{lb}$.

Without causing ambiguity, we use k_{lb} to denote $k_l + \tau_{d\Phi_{l\gamma}}$ in the paper. Then it is assumed that $\Delta L < \tau_{d\Phi_{l\gamma}}$.

Similar to the literature [24] and [26], the multiple convex Lyapunov function approach is employed as follows: $\forall n \in \mathcal{N} \triangleq \{1, 2, \cdots, N\}$ where the positive integer N refers to the number of matrices $U_{\nu\omega n} > 0$ ($U_{\nu n} > 0$); nonlinear continuous functions $\hbar_{\nu\omega n}[k_{la} - (k-1)] = \hbar_{\nu\omega n}(k_{la} - k + 1)$ ($\hbar_{\nu n}(k - k_{la})$) are satisfying

$$\hbar_{\nu\omega n}(k_{la} - k + 1) \geq 0, \sum_{n=1}^{N} \hbar_{\nu\omega n}(k_{la} - k + 1) = 1, \tag{3.1}$$

$$\hbar_{\nu\omega n}(0) = a_{\nu\omega n}, \sum_{n=1}^{N} \hbar_{\nu\omega n}(k_{la} - k + 1) = b_{\nu\omega n}, \tag{3.2}$$

$$\hbar_{\nu\omega n}(k_{la} - k + 1) = \frac{b_{\nu\omega n} - a_{\nu\omega n}}{\Delta L}(k_{la} - k + 1) + a_{\nu\omega n}.$$

Then, we have

$$\hbar_{\nu\omega n}(k_{la} + 1 - k + 1) - \hbar_{\nu\omega n}(k_{la} - k + 1) = \frac{b_{\nu\omega n} - a_{\nu\omega n}}{\Delta L}. \tag{3.3}$$

Next, the constructed Lyapunov functions are dependent on both the subsystem and controller. Namely, the Lyapunov function on the asynchronous interval $[k_l, k_{la})$ takes the different one on the convex interval $[k_{(l-1)a}, k_{(l-1)b})$ and the Lyapunov function on the non-convex interval $[k_{lb}, k_{l+1})$ uses the same one on the convex interval $[k_{la}, k_{lb})$ with $k = k_{lb}$, which is more consistent with the engineering reality.

Further, for $\forall \nu, \omega \in \mathfrak{F}_m$, we construct an MCLF candidate as follows:
(a) when $k \in [k_l, k_{la})$,

$$\begin{aligned} V_{\nu\omega} &= x^T(k) U_{\nu\omega}(k_{la} - k + 1) x(k) \\ &= x^T(k) \sum_{n=1}^{N} \hbar_{\nu\omega n}(k_{la} - k + 1) U_{\nu\omega n} x(k), \end{aligned} \tag{3.4}$$

(b) when $k \in [k_{la}, k_{lb})$,

$$\begin{aligned} V_{\nu} &= x^T(k) U_{\nu}(k) x(k) \\ &= x^T(k) \sum_{n=1}^{N} \hbar_{\nu n}(k - k_{la}) U_{\nu n} x(k), \end{aligned} \tag{3.5}$$

(c) when $k \in [k_{lb}, k_{l+1})$,

$$\begin{aligned} V_{\nu} &= x^T(k) U_{\nu}(k) x(k) = x^T(k) U_{\nu}(k_{lb} - k_{la}) x(k) \\ &= x^T(k) \sum_{n=1}^{N} \hbar_{\nu n}(k_{lb} - k_{la}) U_{\nu n} x(k). \end{aligned} \tag{3.6}$$

It can be seen from (3.4) that the taken Lyapunov function on the synchronous interval $[k_l + \Delta l, k_{la})$ is the one on the asynchronous interval $[k_l, k_l + \Delta l)$, which is inconsistent with the one on the synchronous interval $[k_{la}, k_{lb})$. As we know, it is unrealistic and unreasonable to predict the asynchronous duration

Δl after each switching in advance. To solve this problem, this paper uses the fixed asynchronous duration ΔL instead of the actual asynchronous duration Δl. Although this brings some conservatism, it provides us with solutions to difficult problems.

Now, we are in a position to deduce the condition of the exponential stability of the system (2.8)–(2.9).

Theorem 3.1. For given scalar $0 < \alpha_\gamma < 1, \beta_\gamma > 1, \mu_{1\gamma} > 0, \mu_{2\gamma} > 1$ with $\alpha_\gamma^{-\Delta L}\beta_\gamma^{\Delta L}\mu_{1\gamma}\mu_{2\gamma} > 1$, $\forall\gamma \in \mathfrak{D}$, $\forall\nu, \omega \in \mathfrak{F}_m$, $\nu \neq \omega$ and $\Phi_I(\nu) = \gamma$, suppose there exist positive matrices $U_{\nu n}$ and matrices Q, $\forall n, r \in \mathcal{N}$, such that

$$\begin{bmatrix} -\alpha_\gamma U_{\nu n} & \star \\ QA'_\nu & U_{\nu n} + \Sigma_{r=1}^{N}\pi_{\nu r}U_{\nu r} - Q - Q^T \end{bmatrix} < 0, \tag{3.7}$$

$$\begin{bmatrix} -\alpha_\gamma \sum_{n=1}^{N} b_{\nu n}U_{\nu n} & \star \\ QA'_\nu & \sum_{n=1}^{N}(b_{\nu n} + \pi_{\nu n})U_{\nu n} - Q - Q^T \end{bmatrix} < 0, \tag{3.8}$$

$$\begin{bmatrix} -\beta_\gamma \sum_{n=1}^{N} b_{\omega n}U_{\nu\omega n} & \star \\ QA''_{\nu\omega} & \sum_{n=1}^{N}(b_{\omega n} + \pi_{\nu\omega n})U_{\nu\omega n} - Q - Q^T \end{bmatrix} < 0, \tag{3.9}$$

$$\sum_{n=1}^{N} a_{\nu\omega n}U_{\nu\omega n} < \mu_{1\gamma}\sum_{n=1}^{N} b_{\omega n}U_{\omega n}, \tag{3.10}$$

$$\sum_{n=1}^{N} a_{\nu n}U_{\nu n} < \mu_{2\gamma}\sum_{n=1}^{N} a_{\nu\omega n}U_{\nu\omega n}, \tag{3.11}$$

hold, where $\pi_{\nu n} = \frac{b_{\nu n} - a_{\nu n}}{\tau_{d\Phi_{I\gamma}} - \Delta L}$. Then, the system (2.8)–(2.9) is GUES for any $\varsigma(k)$ having ΦDIDT

$$\tau_{a\Phi_{I\gamma}} > \tau^*_{a\Phi_{I\gamma}} \geq \max\{\tau_{d\Phi_{I\gamma}}, \frac{\ln(\alpha_\gamma^{-\Delta L}\beta_\gamma^{\Delta L}\mu_{1\gamma}\mu_{2\gamma})}{-\ln\alpha_\gamma}\}, \forall\gamma \in \mathfrak{D}, \forall\nu \in \mathfrak{F}_m. \tag{3.12}$$

Proof: From (3.4), we have

$$\begin{aligned} &V_{\nu\omega}(k+1) - \beta_\gamma V_{\nu\omega}(k) \\ &= \begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix}^T \begin{bmatrix} -\beta_\gamma U_{\nu\omega}(k_{la} - k + 1) & 0 \\ 0 & U_{\nu\omega}(k_{la} - k + 2) \end{bmatrix} \begin{bmatrix} x(k) \\ x(k+1) \end{bmatrix} \\ &= \mathfrak{X}^T \mathcal{Z} \mathfrak{X} < 0, \end{aligned} \tag{3.13}$$

where $k \in [k_l, k_{la})$, $\forall\nu, \omega \in \mathfrak{F}_m$, $\forall\gamma \in \mathfrak{D}$. Let $\mathfrak{Y} = [0 \quad Q^T]^T$, $\mathcal{D}_{\nu\omega} = [\mathcal{A}''_{\nu\omega} \quad -I]$. By (3.1), (3.3), and Lemma 2.1, (3.9) indicates that

$$\begin{aligned} &\mathcal{Z} + \mathfrak{Y}\mathcal{D}_{\nu\omega} + \mathcal{D}_{\nu\omega}^T\mathfrak{Y}^T \\ &= \begin{bmatrix} -\beta_\gamma U_{\nu\omega}(k_{la} - k + 1) & 0 \\ 0 & U_{\nu\omega}(k_{la} - k + 2) \end{bmatrix} + \begin{bmatrix} 0 \\ Q \end{bmatrix} \begin{bmatrix} A''_{\nu\omega} & -I \end{bmatrix} \\ &+ \begin{bmatrix} A''^T_{\nu\omega} \\ -I \end{bmatrix} \begin{bmatrix} 0 & Q^T \end{bmatrix} \\ &= \begin{bmatrix} -\beta_\gamma U_{\nu\omega}(k_{la} - k + 1) & \star \\ QA''_{\nu\omega} & U_{\nu\omega}(k_{la} - k + 2) - Q - Q^T \end{bmatrix} < 0. \end{aligned} \tag{3.14}$$

Further, it follows from (3.13) and (3.14) that

$$V_{\nu\omega}(k+1) \le \beta_\gamma V_{\nu\omega}(k), \forall k \in [k_l, k_{la}). \tag{3.15}$$

In a similar way, if (3.7) and (3.8) hold, we immediately get

$$V_\nu(k+1) \le \alpha_\gamma V_\nu(k), \forall k \in [k_{la}, k_{lb}), \tag{3.16}$$

$$V_\nu(k+1) \le \alpha_\gamma V_\nu(k), \forall k \in [k_{lb}, k_{l+1}). \tag{3.17}$$

At the switching point k_l, $l \in \mathbb{Z}^*$, suppose $\varsigma(k_{l-1}) = \omega$, $\varsigma(k_l) = \nu$, $\forall \nu$, $\omega \in \mathfrak{F}_m$, $\nu \ne \omega$, we have

$$V_{\nu\omega}(k_l) - \mu_{1\gamma} V_\omega(k_l) = x^T(k_l)[U_{\nu\omega}(k_l) - \mu_{1\gamma} U_\omega(k_{(l-1)a})]x(k_l). \tag{3.18}$$

Similariy, at point k_{la}, it is clear that

$$V_\nu(k_{la}) - \mu_{2\gamma} V_{\nu\omega}(k_{la}) = x^T(k_{la})[U_\nu(k_{la}) - \mu_{2\gamma} U_\omega(k_{(l)b})]x(k_{la}). \tag{3.19}$$

According to (3.10) and (3.11), $\forall \gamma \in \mathfrak{O}$, $\forall \nu$, $\omega \in \mathfrak{F}_m$, $\nu \ne \omega$, one can obtain

$$V_{\nu\omega}(k_l) \le \mu_{1\gamma} V_\omega(k_l), \tag{3.20}$$

$$V_\nu(k_{la}) \le \mu_{2\gamma} V_{\nu\omega}(k_{la}). \tag{3.21}$$

From (3.15)–(3.21), one has

$$\begin{aligned} V_{\varsigma(k)}(k) &\le \alpha_{\varsigma(k_l)}^{k-k_{lb}} V_{\varsigma(k_l)}(k_{lb}) \\ &\le \alpha_{\varsigma(k_l)}^{k-k_{la}} V_{\varsigma(k_l)}(k_{la}) \\ &\le \alpha_{\varsigma(k_l)}^{k-k_{la}} \mu_{\varsigma(k_l)_{2\gamma}} V_{\varsigma(k_l)}(k_{la}) \\ &\le \alpha_{\varsigma(k_l)}^{k-k_{la}} \beta_{\varsigma(k_l)}^{\Delta L} \mu_{\varsigma(k_l)_{2\gamma}} V_{\varsigma(k_l)}(k_l) \\ &\le \alpha_{\varsigma(k_l)}^{k-k_{la}} \beta_{\varsigma(k_l)}^{\Delta L} \mu_{\varsigma(k_l)_{1\gamma}} \mu_{\varsigma(k_l)_{2\gamma}} V_{\varsigma(k_{l-1})}(k_l). \end{aligned} \tag{3.22}$$

Then, one can further get

$$\begin{aligned} V_{\varsigma(k)} &\le \alpha_{\varsigma(k_l)}^{k-k_{la}} \alpha_{\varsigma(k_{l-1})}^{k_l-k_{(l-1)a}} \cdots \alpha_{\varsigma(k_0)}^{k_1-k_0} \beta_{\varsigma(k_l)}^{\Delta L} \beta_{\varsigma(k_{l-1})}^{\Delta L} \cdots \beta_{\varsigma(k_1)}^{\Delta L} \\ &\quad \times \mu_{\varsigma(k_l)_{1\gamma}} \mu_{\varsigma(k_{l-1})_{1\gamma}} \cdots \mu_{\varsigma(k_1)_{1\gamma}} \mu_{\varsigma(k_l)_{2\gamma}} \mu_{\varsigma(k_{l-1})_{2\gamma}} \cdots \mu_{\varsigma(k_1)_{2\gamma}} V_{\varsigma(k_0)}(k_0) \\ &= \alpha_{\varsigma(k_0)}^{\Delta L} \beta_{\varsigma(k_0)}^{-\Delta L} \mu_{\varsigma(k_0)_{1\gamma}}^{-1} \mu_{\varsigma(k_0)_{2\gamma}}^{-1} \prod_{\gamma=1}^{s} [(\alpha_\gamma^{-\Delta L} \beta_\gamma^{\Delta L} \mu_{1\gamma} \mu_{2\gamma})^{N_{\Phi_\gamma}(k,k_0)} \alpha_\gamma^{K_{\Phi_\gamma}(k,k_0)}] \\ &\quad \times V_{\varsigma(k_0)}(k_0). \end{aligned} \tag{3.23}$$

Moreover, if (2.3) and $\alpha_\gamma^{-\Delta L} \beta_\gamma^{\Delta L} \mu_{1\gamma} \mu_{2\gamma} > 1$ hold, (3.23) can be rewritten as

$$\begin{aligned} V_{\varsigma(k)}(k) &\le \alpha_{\varsigma(k_0)}^{\Delta L} \beta_{\varsigma(k_0)}^{-\Delta L} \mu_{\varsigma(k_0)_{1\gamma}}^{-1} \mu_{\varsigma(k_0)_{2\gamma}}^{-1} \prod_{\gamma=1}^{s} [(\alpha_\gamma^{-\Delta L} \beta_\gamma^{\Delta L} \mu_{\gamma_{1\gamma}} \mu_{\gamma_{2\gamma}})^{N_{0\Phi_{I\gamma}} + \frac{K_{\Phi_{I\gamma}}(k,k_0)}{\tau_{a\Phi_{I\gamma}}}} \\ &\quad \times \alpha_\gamma^{K_{\Phi_{I\gamma}}(k,k_0)}] V_{\varsigma(k_0)}(k_0) \end{aligned}$$

$$= \alpha_{\varsigma(k_0)}^{\Delta L}\beta_{\varsigma(k_0)}^{-\Delta L}\mu_{\varsigma(k_0)_{1\gamma}}^{-1}\mu_{\varsigma(k_0)_{2\gamma}}^{-1}\prod_{\gamma=1}^{s}\{(\alpha_\gamma^{-\Delta L}\beta_\gamma^{\Delta L}\mu_{\gamma_{1\gamma}}\mu_{\gamma_{2\gamma}})^{N_{0\Phi_{I\gamma}}}$$

$$\times [(\alpha_\gamma^{-\Delta L}\beta_\gamma^{\Delta L}\mu_{1\gamma}\mu_{2\gamma})^{\frac{1}{\tau_{a\Phi_{I\gamma}}}}\alpha_\gamma]^{K_{\Phi_{I\gamma}}(k,k_0)}\}V_{\varsigma(k_0)}(k_0).$$

Considering $\tau_{a\Phi_{I\gamma}} > \tau^*_{a\Phi_{I\gamma}} \geq \max\{\tau_{d\Phi_{I\gamma}}, \frac{\ln \alpha_\gamma^{-\Delta L}\beta_\gamma^{\Delta L}\mu_{1\gamma}\mu_{2\gamma}}{-\ln \alpha_\gamma}\}$, $\forall \gamma \in \mathfrak{O}$, $\forall \nu \in \mathfrak{F}_m$, we have $0 < (\alpha_\gamma^{-\Delta L}\beta_\gamma^{\Delta L}\mu_{1\gamma}\mu_{2\gamma})^{\frac{1}{\tau_{a\Phi_{I\gamma}}}}\alpha_\gamma < 1$, and it follows that

$$V_{\varsigma(k)}(k) \leq \max_{\gamma\in\mathfrak{O}}\{\alpha_\gamma^{\Delta L}\}\max_{\gamma\in\mathfrak{O}}\{\beta_\gamma^{-\Delta L}\}\max_{\gamma\in\mathfrak{O}}\{\mu_{1\gamma}^{-1}\}\max_{\gamma\in\mathfrak{O}}\{\mu_{2\gamma}^{-1}\}$$

$$\times \prod_{\gamma=1}^{s}\{(\alpha_\gamma^{-\Delta L}\beta_\gamma^{\Delta L}\mu_{1\gamma}\mu_{2\gamma})^{N_{0\Phi_{I\gamma}}}$$

$$\times \max_{\gamma\in\mathfrak{O}}[(\alpha_\gamma^{-\Delta L}\beta_\gamma^{\Delta L}\mu_{1\gamma}\mu_{2\gamma})^{\frac{1}{\tau_{a\Phi_{I\gamma}}}}\alpha_\gamma]\}^{K_{\Phi_{I\gamma}}(k,k_0)}V_{\varsigma(k_0)}(k_0).$$

Therefore, we conclude that the system (2.8)–(2.9) is GUES. It is proven.

Next, we give the design of the asynchronous controller to guarantee the GUES of the asynchronously switched control system (2.8)–(2.9).

Theorem 3.2. For given scalars $0 < \alpha_\gamma < 1, \beta_\gamma > 1, \mu_{1\gamma} > 0, \mu_{2\gamma} > 1$ with $\alpha_\gamma^{-\Delta L}\beta_\gamma^{\Delta L}\mu_{1\gamma}\mu_{2\gamma} > 1$, $\forall\gamma \in \mathfrak{O}$, $\forall\nu,\ \omega \in \mathfrak{F}_m$, $\nu \neq \omega$, suppose there exist positive matrices $H_{\nu n}$, matrices Y_ν, and symmetric invertible matrix X, $\forall r, n \in \mathcal{N}$, such that

$$\begin{bmatrix} -\alpha_\gamma H_{\nu n} & \star \\ A_\nu X + B_\nu Y_\nu & H_{\nu n} + \sum_{r=1}^{N}\pi_{\nu r}H_{\nu r} - 2X \end{bmatrix} < 0, \tag{3.24}$$

$$\begin{bmatrix} -\alpha_\gamma \sum_{n=1}^{N} b_{\nu n}H_{\nu n} & \star \\ A_\nu X + B_\nu Y_\nu & \sum_{n=1}^{N}(b_{\nu n} + \pi_{\nu n})H_{\nu n} - 2X \end{bmatrix} < 0, \tag{3.25}$$

$$\begin{bmatrix} -\beta_\gamma \sum_{n=1}^{N} b_{\omega n}H_{\nu\omega n} & \star \\ A_\nu X + B_\nu Y_\omega & \sum_{n=1}^{N}(b_{\omega n} + \pi_{\omega n})H_{\nu\omega n} - 2X \end{bmatrix} < 0, \tag{3.26}$$

$$\sum_{n=1}^{N} a_{\nu\omega n}H_{\nu\omega n} < \mu_{1\gamma}\sum_{n=1}^{N} b_{\omega n}H_{\omega n}, \tag{3.27}$$

$$\sum_{n=1}^{N} a_{\nu n}H_{\nu n} < \mu_{2\gamma}\sum_{n=1}^{N} a_{\nu\omega n}H_{\nu\omega n}, \tag{3.28}$$

hold, where $\pi_{\nu n} = \frac{b_{\nu n} - a_{\nu n}}{\tau_{d\Phi_{I\gamma}-\Delta L}}$. Then there is a state feedback controller such that the resulting closed-loop system of (2.8)–(2.9) is GUES for any switching signal satisfying (3.12), and the feedback gain can be given by

$$K_\nu = Y_\nu X^{-1}. \tag{3.29}$$

Proof: For $k \in [k_l, k_{la})$, let

$$H_{\nu\omega n} = X^T U_{\nu\omega n}X, Y_\omega = K_\omega X, X = Q^{-1}.$$

From (3.26), we have

$$\begin{bmatrix} -\beta_\gamma \sum_{n=1}^{N} b_{\omega n} Q^{-1^T} H_{\nu\omega n} Q^{-1} & \star \\ A_\nu Q^{-1} + B_\nu K_\omega Q^{-1} & Q^{-1^T} [\sum_{n=1}^{N} (b_{\omega n} + \pi_{\omega n}) H_{\nu\omega n}] Q^{-1} - 2Q^{-1} \end{bmatrix} < 0. \tag{3.30}$$

Pre- and post-multiplying both sides of the inequality in (3.30) by diag $\{Q, Q\}$ yields

$$\begin{bmatrix} -\beta_\gamma \sum_{n=1}^{N} b_{\omega n} H_{\nu\omega n} & \star \\ Q(A_\nu + B_\nu K_\omega) & \sum_{n=1}^{N} (b_{\omega n} + \pi_{\omega n}) H_{\nu\omega n} - 2Q \end{bmatrix} < 0, \tag{3.31}$$

which it can ensure (3.9). We omit the same part here, the conditions (3.7), (3.8) and (3.11) also can be guaranteed by (3.24), (3.25) and (3.28). According to Theorem 3.1, the switched system (2.8)–(2.9) is GUES.

Remark 3.1. The ΦDIDT strategy covers the IDT and MDIDT ones. On the one hand, let $\mathfrak{O} = \{1\}$, and replace α_γ, β_γ, $\mu_{1\gamma}$ and $\mu_{2\gamma}$ in Theorems 3.1 and 3.2 with α, β, μ_1 and μ_2, and we can obtain the corresponding results of stability based on the IDT strategy. On the other hand, let $\mathfrak{O} = \mathfrak{F}_m$ and $\Phi_I(\nu) = \nu$ ($\forall \nu \in \mathfrak{F}_m$), and replace $\alpha_\gamma, \beta_\gamma, \mu_{1\gamma}$ and $\mu_{2\gamma}$ in Theorems 3.1 and 3.2 with $\alpha_\nu, \beta_\nu, \mu_{1\nu}$ and $\mu_{2\nu}$, and we can obtain the corresponding stability criterion under the MDIDT strategy. We have omitted these easily obtained results because of spatial limitations. So the ΦDIDT strategy can unify the IDT and MDIDT strategies.

Remark 3.2. As we know, the IDT strategy only focuses on the compensation effect between subsystems but does not take into account the difference between subsystems. Contrariwise, the MDIDT strategy mainly concerns the difference between subsystems but does not consider the compensation between subsystems. For some given $(\Phi_I, \mathfrak{O})$, $\mathfrak{O} \neq \{1\}$ and $\mathfrak{O} \neq \mathfrak{F}_m$, it takes into account both the compensation effect between the ν and the ω subsystems ($\nu \neq \omega$, ν, $\omega \in \Phi_{I\gamma}$) and the difference between $\Phi_{I\gamma}$ and $\Phi_{I\iota}$ ($\gamma \neq \iota$). The fact is that some different stability results with their own advantages can be obtained by choosing different $(\Phi_I, \mathfrak{O})$. So we can't decide which one is better. It is easy to know that when the number of subsystems is limited, we can give all the possibilities of $(\Phi_I, \mathfrak{O})$. For instance, take $\mathfrak{F}_m = \{1, 2, 3\}$, theoretically, function Φ has 13 forms, including 1 form for $\mathfrak{O} = \{1\}$, 6 forms for $\mathfrak{O} = \{1, 2\}$ and 6 forms for $\mathfrak{O} = \{1, 2, 3\}$. Nevertheless, some forms can be classified as the same type; for example, $\Phi_{I1} = \{2, 3\}$, $\Phi_{I2} = \{1\}$ and $\Phi_{I1} = \{1\}$, $\Phi_{I2} = \{2, 3\}$. Therefore, the function Φ is finally categorized into 5 types as follows:

(i) for $\mathfrak{O} = \{1\}$, then $\Phi_{I1} = \{1, 2, 3\}$, which corresponds the IDT results.

(ii) if $\mathfrak{O} = \{1, 2\}$, then there are 3 classification forms: ① $\Phi_{I1} = \{1, 2\}$, $\Phi_{I2} = \{3\}$; ② $\Phi_{I1} = \{1, 3\}$, $\Phi_{I2} = \{2\}$; ③ $\Phi_{I1} = \{2, 3\}$, $\Phi_{I2} = \{1\}$.

(iii) when $\mathfrak{O} = \{1, 2, 3\}$, then $\Phi_{I1} = \{1\}$, $\Phi_{I2} = \{2\}$, $\Phi_{I3} = \{3\}$, which corresponds the MDIDT results.

Remark 3.3. For $\mathfrak{O} = \{1, 2, 3\}$ case, if we take the special value $\mu_{1\gamma} = 1$ in Theorems 3.1 and 3.2, then our results in this paper will degenerate to the results of in [29], which implies that [29] is a special case of the new conclusion.

Remark 3.4. There is the problem of how to properly select design parameters $\alpha_\gamma, \beta_\gamma, \mu_{1\gamma}$ and $\mu_{2\gamma}$ in the implementation of Theorems 3.1 and 3.2. These parameters are coupled with the decision matrices $U_{\nu n}$, Q and matrices $H_{\nu n}$, Y_ν, X in Theorems 3.1 and 3.2, respectively, making it difficult to solve them simultaneously. An effective algorithm for selecting appropriate design parameters is proposed here.

Step 1: Select sufficiently small α_γ and sufficiently large β_γ, $\mu_{1\gamma}$ and $\mu_{2\gamma}$ to ensure a large feasible range of decision variables in Theorems 3.1 and 3.2.

Step 2: If there are the solutions of $\alpha_\gamma^@$, $\beta_\gamma^@$, $\mu_{1\gamma}^@$ and $\mu_{2\gamma}^@$ in Step 1, proceed to the next step, otherwise terminate.

Step 3: Fix $\alpha_\gamma^@$, and gradually reduce β_γ, $\mu_{1\gamma}$ and $\mu_{2\gamma}$ in sequence while ensuring the feasibility of Theorems 3.1 and 3.2. Then one can obtain $\beta_\gamma = \beta_\gamma^*$, $\mu_{1\gamma} = \mu_{1\gamma}^*$ and $\mu_{2\gamma} = \mu_{2\gamma}^*$.

Step 4: Fix β_γ^*, $\mu_{1\gamma}^*$ and $\mu_{2\gamma}^*$, and gradually increase α_γ while ensuring the feasibility of Theorems 3.1 and 3.2. Then one can get $\alpha_\gamma = \alpha_\gamma^*$.

Step 5: Obtain a set of relatively ideal design parameters $(\alpha_\gamma^* \beta_\gamma^*, \mu_{1\gamma}^*, \mu_{2\gamma}^*)$.

Remark 3.5. Consider that Lyapunov functions may not jump when the subsystem switches or the controller changes. Then, we look for the relationship of the constructed Lyapunov functions on the interval $[k_l, k_{la}]$.

Let $V(k)$ be a function defined on the interval $[k_l, k_{la}]$, suppose that there are $m-1$ points on the interval $[k_l, k_{la}]$, which are

$$k-1 = k_l^{(0)} < k_l^{(1)} < k_l^{(2)} < \cdots < k_l^{(i-1)} < k_l^{(i)} = k_{la}.$$

They divide $[k_l, k_{la})$ into m cells $\Delta k_l^{(i)} = [k_l^{(i)}, k_l^{(i-1)}]$, $i = 1, 2, \cdots m$. Denote

$$\| K \| = \max_{1\le i\le m}\{\Delta k_l^{(i)}\}.$$

Take any point $\xi_i \in \Delta k_l^{(i)}$, we have

$$V(\xi_1)\Delta k_l^{(1)} + V(\xi_2)\Delta k_l^{(2)} + \cdots + V(\xi_{i-1})\Delta k_l^{(i-1)} + V(\xi_i)\Delta k_l^{(i)} = \sum_{i=1}^{m} V(\xi_i)\Delta k_l^{(i)}.$$

Let

$$J = \lim_{\|K\|\to 0} \sum_{i=1}^{m} V(\xi_i)\Delta k_l^{(i)} = \int_{k-1}^{k_{la}} V(k)dk.$$

Further, one can know

$$V_m(k) = \frac{J}{\Delta L}k + V_\omega(k-1),$$

$$V_\omega(k-1) = x^T(k)U_\omega(k_{(l-1)b} - k_{(l-1)a})x(k).$$

It satisfies at point k_{la} that

$$V_\nu(k_{la}) = x^T(k)U_\nu(k - k_{la})x(k) + V_\omega(k-1) = \frac{J}{\Delta L}k_{la} + V_\omega(k-1). \tag{3.32}$$

From the functional relation of the above, we can obtain the following Corollaries 3.1 and 3.2.

Corollary 3.1. For given scalars $0 < \alpha_\gamma < 1$, $\mu_\gamma > 1$, $\forall\gamma \in \mathfrak{O}$, with $\alpha_\gamma^{-\Delta L}\mu_\gamma > 1$, suppose there exist matrices $P_{\nu n}^* > 0$ and matrices Q^*, $\forall\nu \in \mathfrak{F}_m$, $\forall r, n \in \mathcal{N}$ such that

$$\begin{bmatrix} -\alpha_\gamma U_{\nu n}^* & \star \\ Q^* A_\nu' & U_{\nu n}^* + \Sigma_{r=1}^N \pi_{\nu r} U_{\nu r}^* - Q^* - Q^{*T} \end{bmatrix} < 0, \tag{3.33}$$

$$\begin{bmatrix} -\alpha_\gamma \sum_{n=1}^N b_{\nu n} U_{\nu n}^* & \star \\ Q^* A_\nu' & \sum_{n=1}^N (b_{\nu n} + \pi_{\nu n}) U_{\nu n}^* - Q^* - Q^{*T} \end{bmatrix} < 0, \tag{3.34}$$

$$\sum_{n=1}^{N} a_{\nu n} U_{\nu n}^{*} < \mu_{\gamma} \sum_{n=1}^{N} b_{\omega n} U_{\omega n}^{*}, \tag{3.35}$$

hold, where $\pi_{\nu n} = \frac{b_{\nu n}-a_{\nu n}}{\tau_{d\Phi_{I\gamma}-\Delta L}}$. Then, the system (2.8)–(2.9) is GUES for any $\varsigma(k)$ having ΦDIDT

$$\tau_{a\Phi_{I\gamma}} > \tau_{a\Phi_{I\gamma}}^{*} \geq \max\{\tau_{d\Phi_{I\gamma}}, \frac{\ln \alpha_{\gamma}^{-\Delta L}\mu_{\gamma}}{-\ln \alpha_{\gamma}}\}, \forall \gamma \in \mathfrak{O}. \tag{3.36}$$

Proof: Integrating the proof of Theorem 3.1 with (3.32), it can be concluded.

Corollary 3.2. For given scalars $0 < \alpha_{\gamma} < 1$, $\mu_{\gamma} > 1$, $\forall \gamma \in \mathfrak{O}$, with $\alpha_{\gamma}^{-\Delta L}\mu_{\gamma} > 1$, suppose there exist matrices $H_{\nu n}^{*} > 0$, matrices Y_{ν}^{*}, and symmetric invertible matrix X^{*}, $\forall \nu \in \mathfrak{F}_{m}$, $\forall r, n \in \mathcal{N}$, such that

$$\begin{bmatrix} -\alpha_{\gamma} H_{\nu n}^{*} & \star \\ A_{\nu}X + B_{\nu}Y_{\nu}^{*} & H_{\nu n}^{*} + \sum_{r=1}^{N} \pi_{\nu r} H_{\nu r}^{*} - 2X^{*} \end{bmatrix} < 0, \tag{3.37}$$

$$\begin{bmatrix} -\alpha_{\gamma} \sum_{n=1}^{N} b_{\nu n} H_{\nu n}^{*} & \star \\ A_{\nu}X + B_{\nu}Y_{\nu}^{*} & \sum_{n=1}^{N} (b_{\nu n} + \pi_{\nu n}) H_{\nu n}^{*} - 2X^{*} \end{bmatrix} < 0, \tag{3.38}$$

$$\sum_{n=1}^{N} a_{\nu n} H_{\nu n}^{*} < \mu_{\gamma} \sum_{n=1}^{N} b_{\omega n} H_{\omega n}^{*}, \tag{3.39}$$

hold, where $\pi_{\nu n} = \frac{b_{\nu n}-a_{\nu n}}{\tau_{d\Phi_{I\gamma}-\Delta L}}$, then there is the state feedback controller such that the resulting closed-loop system of (2.8)–(2.9) is GUES for any switching signal satisfying

$$\tau_{a\Phi_{I\gamma}} > \tau_{a\Phi_{I\gamma}}^{*} \geq \max\{\tau_{d\Phi_{I\gamma}}, \frac{\ln \alpha_{\gamma}^{-\Delta L}\mu_{\gamma}}{-\ln \alpha_{\gamma}}\}, \forall \gamma \in \mathfrak{O}. \tag{3.40}$$

Moreover, the feedback gain is given by

$$K_{\nu}^{*} = Y_{\nu}^{*} X_{\nu}^{*-1}.$$

Proof: Integrating the proof of Theorem 3.2 with (3.32), it can be concluded.

4. Numerical example

In this section, a simple numerical example in the discrete-time domain will be provided to verify the effectiveness of the theoretical results.

Consider the switched linear system (2.8)–(2.9) with subsystem matrices

$$A_1 = \begin{bmatrix} -0.40 & 0.23 \\ 0.16 & -6.71 \end{bmatrix}, A_2 = \begin{bmatrix} -0.60 & 0 \\ 0.42 & -48.12 \end{bmatrix}, A_3 = \begin{bmatrix} -3.16 & -0.44 \\ -3.43 & -1.19 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.2 \\ -0.3 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1.0 \end{bmatrix}, B_3 = \begin{bmatrix} -0.1 \\ 0.5 \end{bmatrix}.$$

By using the Matlab LMI Toolbox to solve the conditions in Theorem 3.2 with $\Phi_{I1} = \{1, 2\}$, $\Phi_{I2} = \{3\}$ (no loss of generality) and other parameters referring to the corresponding columns in Table 2, the feasible solutions are obtained

$$X = \begin{bmatrix} -0.2020 & * \\ 0.3483 & 0.3390 \end{bmatrix}, X^{-1} = \begin{bmatrix} -1.7862 & * \\ 1.8352 & 0.3390 \end{bmatrix},$$

and

$$Y_1 = \begin{bmatrix} 3.9980 & 2.6600 \end{bmatrix}, Y_2 = \begin{bmatrix} 0.4018 & 2.2180 \end{bmatrix}, Y_3 = \begin{bmatrix} 5.178 & 3.885 \end{bmatrix},$$

and switching signals satisfy $\tau^*_{a\Phi_{I1}} = 3.0200$ and $\tau^*_{a\Phi_{I2}} = 2.4393$, then the controller-gain matrices can be given as follows:

$$K_1 = Y_1X^{-1} = \begin{bmatrix} -2.2418 & 10.1498 \end{bmatrix}, K_2 = Y_2X^{-1} = \begin{bmatrix} 3.3528 & 2.3606 \end{bmatrix},$$

$$K_3 = Y_3X^{-1} = \begin{bmatrix} -2.1191 & 13.6375 \end{bmatrix}.$$

Thus, the closed-loop systems are obtained with matrices $A'_\nu = A_\nu + B_\nu K_\nu$

$$A'_1 = \begin{bmatrix} 0.0484 & -1.7999 \\ 0.8328 & -3.7049 \end{bmatrix}, A'_2 = \begin{bmatrix} -0.6000 & 0 \\ 33.9480 & 24.6060 \end{bmatrix},$$

$$A'_3 = \begin{bmatrix} -3.3629 & -1.8037 \\ -4.4896 & 5.6288 \end{bmatrix}.$$

It is worth noting that the larger parameter N will incur an additional computational burden. To reduce the complexity of the calculation, we take $N = 2$ in the example. When applying Theorem 3.2, different controller designs are generally obtained for different Φ_I, resulting in different closed-loop subsystems. In this situation, it is difficult to compare IDT, MDIDT and ΦDIDT switching strategies. To verify the comprehensiveness and comparison of the presented results, the switching strategies in the following tables are all based on the same controllers mentioned above.

The following facts can be obtained from Tables 2 and 3:

Table 2. Comparison of the results under three switching strategies ($\Delta L = 2$).

$\mathfrak{O}$	{1}/ IDT [30]	{1,2}			{1,2,3}/ MDIDT [31]
Φ_I	$\Phi_{I1} = \{1,2,3\}$	$\Phi_{I1} = \{1,2\}, \Phi_{I2} = \{3\}$	$\Phi_{I1} = \{1,3\}, \Phi_{I2} = \{2\}$	$\Phi_{I1} = \{1\}, \Phi_{I2} = \{2,3\}$	$\Phi_{I1} = \{1\}, \Phi_{I2} = \{2\}, \Phi_{I3} = \{3\}$
μ	$\mu_{11} = 2$	$\mu_{11} = 2, \mu_{12} = 2$	$\mu_{11} = 2, \mu_{12} = 2$	$\mu_{11} = 2, \mu_{12} = 2$	$\mu_{11} = 2, \mu_{12} = 2, \mu_{13} = 2$
	$\mu_{21} = 0.6$	$\mu_{21} = 0.6, \mu_{22} = 0.6$	$\mu_{21} = 0.6, \mu_{22} = 0.6$	$\mu_{21} = 0.3, \mu_{22} = 0.3$	$\mu_{21} = 0.3, \mu_{22} = 0.6, \mu_{23} = 0.6$
α	$\alpha_1 = 0.64$	$\alpha_1 = 0.5, \alpha_2 = 0.2$	$\alpha_1 = 0.64, \alpha_2 = 0.32$	$\alpha_1 = 0.65, \alpha_2 = 0.2$	$\alpha_1 = 0.5, \alpha_2 = 0.64, \alpha_3 = 0.32$
β	$\beta_1 = 1.25$	$\beta_1 = 1.3, \beta_2 = 1.3$	$\beta_1 = 1.25, \beta_2 = 1.25$	$\beta_1 = 1.66, \beta_2 = 1.66$	$\beta_1 = 1.3, \beta_2 = 1.25, \beta_3 = 1.25$
H_1	$\begin{bmatrix} 0.2949 & \star \\ -0.0598 & 0.3383 \end{bmatrix}$	$\begin{bmatrix} 0.1385 & \star \\ -0.0370 & 0.1779 \end{bmatrix}$	$\begin{bmatrix} 0.1515 & \star \\ -0.0558 & 0.1516 \end{bmatrix}$	$\begin{bmatrix} 0.1289 & \star \\ -0.0345 & -0.1500 \end{bmatrix}$	$\begin{bmatrix} 0.2088 & \star \\ -0.0879 & 0.2651 \end{bmatrix}$
H_2	$\begin{bmatrix} 0.3018 & \star \\ -0.0551 & 0.3390 \end{bmatrix}$	$\begin{bmatrix} 0.3272 & \star \\ -0.0519 & 0.3332 \end{bmatrix}$	$\begin{bmatrix} 0.1236 & \star \\ -0.0410 & 0.1213 \end{bmatrix}$	$\begin{bmatrix} 0.6531 & \star \\ -0.1185 & 0.6968 \end{bmatrix}$	$\begin{bmatrix} 0.2881 & \star \\ -0.0852 & 0.3315 \end{bmatrix}$
H_3	$\begin{bmatrix} 0.2935 & \star \\ -0.0569 & 0.3427 \end{bmatrix}$	$\begin{bmatrix} 0.2131 & \star \\ -0.0518 & 0.2666 \end{bmatrix}$	$\begin{bmatrix} 0.6734 & \star \\ -0.2456 & 0.7296 \end{bmatrix}$	$\begin{bmatrix} 0.6031 & \star \\ -0.1274 & 0.7225 \end{bmatrix}$	$\begin{bmatrix} 0.2444 & \star \\ -0.0972 & 0.3359 \end{bmatrix}$
Signal design	$\tau^*_{a\Phi_{I1}} = 3.4085$	$\tau^*_{a\Phi_{I1}} = 3.0200$ $\tau^*_{a\Phi_{I2}} = 2.4393$	$\tau^*_{a\Phi_{I1}} = 3.4085$ $\tau^*_{a\Phi_{I2}} = 2.5516$	$\tau^*_{a\Phi_{I1}} = 3.1672$ $\tau^*_{a\Phi_{I2}} = 2.3124$	$\tau^*_{a\Phi_{I1}} = 2.0200, \tau^*_{a\Phi_{I2}} = 3.4085$ $\tau^*_{a\Phi_{I3}} = 2.5516$
Signal instance	$\tau_1 = 3.5, \tau_2 = 2.2$ $\tau_3 = 4.8$	$\tau_1 = 1.2, \tau_2 = 5.1$ $\tau_3 = 2.5$	$\tau_1 = 5.5, \tau_2 = 2.6$ $\tau_3 = 1.4$	$\tau_1 = 3.2, \tau_2 = 2.2$ $\tau_3 = 2.4$	$\tau_1 = 2.1, \tau_2 = 3.6$ $\tau_3 = 2.6$
Figure of signal	Figure 1(a)	Figure 2(a)	Figure 3(a)	Figure 4(a)	Figure 5(a)
State response with $x_0 = (7, -1)^T$	Figure 1(b)	Figure 2(b)	Figure 3(b)	Figure 4(b)	Figure 5(b)

(I) For $\mathfrak{O} = \{1\}$ ($\Phi_{I1} = \{1, 2, 3\}$) and $\mathfrak{O} = \{1, 2, 3\}$ ($\Phi_{I1} = \{1\}, \Phi_{I2} = \{2\}, \Phi_{I3} = \{3\}$) cases, we can obtain the IDT and MDIDT strategies, respectively.

(II) For different Φ_I, the ΦDIDT method provides the different results of admissible signals with their own merits. Let $\mathfrak{O} = \{1, 2\}$, for case (i): $\Phi_{I1} = \{1, 2\}$, $\Phi_{I2} = \{3\}$, the 1st and 2nd modes have IDT ≥ 3.0200 and the 3rd mode has IDT ≥ 2.4393; for case (ii): $\Phi_{I1} = \{1, 3\}$, $\Phi_{I2} = \{2\}$, the 1st and 3rd modes have IDT ≥ 3.4085, and the 2nd mode has IDT ≥ 2.5516; for case (iii): $\Phi_{I1} = \{2, 3\}$, $\Phi_{I2} = \{1\}$, the 2nd and 3rd modes have IDT ≥ 3.1672, and the 1st mode has IDT ≥ 2.3124.

(III) A fact can be shown from Table 2, and some different stability results with their own advantages can be obtained by choosing different (Φ_I, $\mathfrak{O}$). So we can't decide which is better.

(IV) The IDT strategy only focuses on the compensation effect between subsystems but does not pay attention to the difference between subsystems. On the contrary, the MDIDT strategy mainly notices the differences between systems but does not consider the compensation between subsystems. For the three cases of $\mathfrak{O} = \{1, 2\}$, we think about not only the differences between the 2nd and 3rd subsystems, the 1st and 3rd subsystems, and the 1st and 2nd subsystems and the rest of the subsystems, but also the compensation effect between them. Thus, the ΦDIDT results cover the IDT and MDIDT ones, which can be shown in Table 2.

(V) It follows from Table 3 that the MDIDT has a smaller value of $\tau^*_{a\Phi_\gamma}$ than the MDIDT value τ^*_{av} in the literature [29]. Let $\mu_{11} = 2$, $\mu_{21} = 0.3$, $\mu_{12} = 2$, $\mu_{22} = 0.6$, $\mu_{13} = 2$, $\mu_{23} = 0.6$, and $\Delta L = 2$. By solving the conditions in our Theorem 3.1, we can obtain $\tau^*_{a\Phi_{I1}} = 2.0200$, $\tau^*_{a\Phi_{I2}} = 3.4085$, and $\tau^*_{a\Phi_{I3}} = 2.5516$. Letting $\mu_{11} = 2$, $\mu_{12} = 2$, $\mu_{13} = 2$, and $\Delta L = 2$, we can obtain $\tau^*_{a1} = 3.7570$, $\tau^*_{a2} = 4.5531$, and $\tau^*_{a3} = 3.0000$ by Theorem 1 in the literature [29]. So the new result has a larger feasible region than the result in the literature [29].

Table 3. Comparison between the results in this paper and the results in Cui et al. (2021) [29] under MDIDT ($\Delta L = 2$).

$\mathfrak{D}$	MDIDT	MDIDT in [29]
Φ_I	$\Phi_{I1} = \{1\}, \Phi_{I2} = \{2\}, \Phi_{I3} = \{3\}$	$\Phi_{I1} = \{1\}, \Phi_{I2} = \{2\}, \Phi_{I3} = \{3\}$
μ	$\mu_{11} = 2, \mu_{12} = 2, \mu_{13} = 2$	$\mu_{11} = 2, \mu_{12} = 2, \mu_{13} = 2$
	$\mu_{21} = 0.3, \mu_{22} = 0.6, \mu_{23} = 0.6$	
α	$\alpha_1 = 0.5, \alpha_2 = 0.64, \alpha_3 = 0.32$	$\alpha_1 = 0.5, \alpha_2 = 0.64, \alpha_3 = 0.32$
	$\beta_1 = 1.3, \beta_2 = 1.25, \beta_3 = 1.25$	$\beta_1 = 1.3, \beta_2 = 1.25, \beta_3 = 1.25$
H_1	$\begin{bmatrix} 0.2088 & \star \\ -0.0879 & 0.2651 \end{bmatrix}$	$\begin{bmatrix} 0.3353 & \star \\ -0.4369 & 0.5743 \end{bmatrix}$
H_2	$\begin{bmatrix} 0.2881 & \star \\ -0.0852 & 0.3315 \end{bmatrix}$	$\begin{bmatrix} 0.3335 & \star \\ -0.2775 & 0.3993 \end{bmatrix}$
H_3	$\begin{bmatrix} 0.2444 & \star \\ -0.0972 & 0.3359 \end{bmatrix}$	$\begin{bmatrix} 1.6078 & \star \\ -3.5011 & 7.6239 \end{bmatrix}$
Signal	$\tau^*_{a\Phi_{I1}} = 2.022$	$\tau^*_{a1} = 3.7570$
design	$\tau^*_{a\Phi_{I2}} = 3.4085$	$\tau^*_{a2} = 4.5531$
	$\tau^*_{a\Phi_{I3}} = 2.5516$	$\tau^*_{a3} = 3.0000$
Signal	$\tau_1 = 2.1$	$\tau_1 = 3.8$
instance	$\tau_2 = 3.6$	$\tau_2 = 4.6$
	$\tau_3 = 2.6$	$\tau_3 = 3.1$

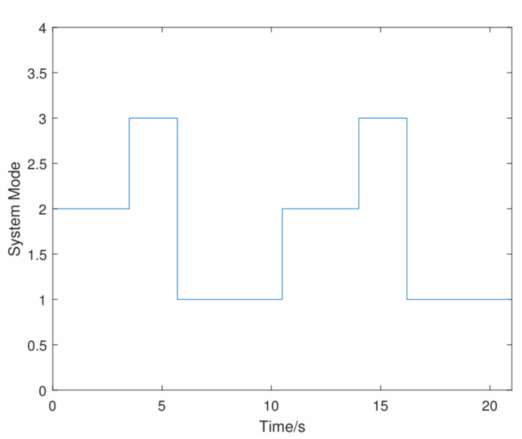

Figure 1(a). The switching signal $\theta^1(k)$.

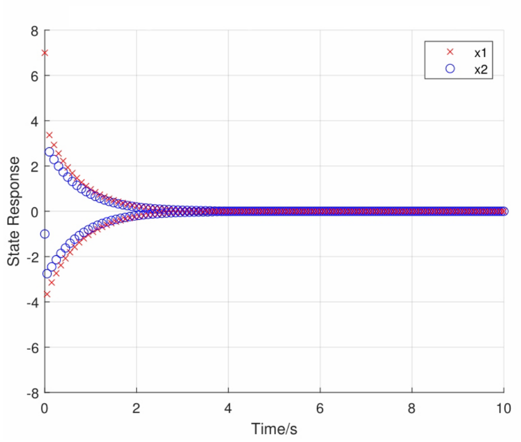

Figure 1(b). The state response of the system under $\theta^1(k)$.

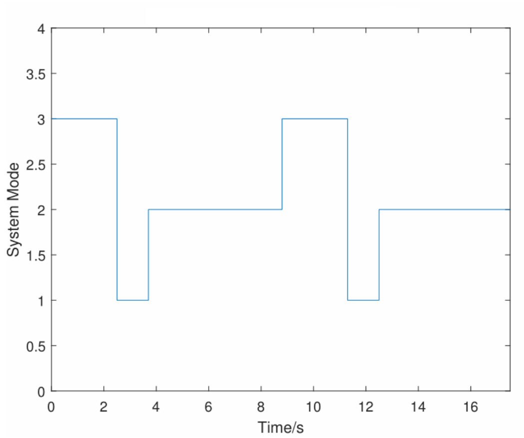

Figure 2(a). The switching signal $\theta^2(k)$.

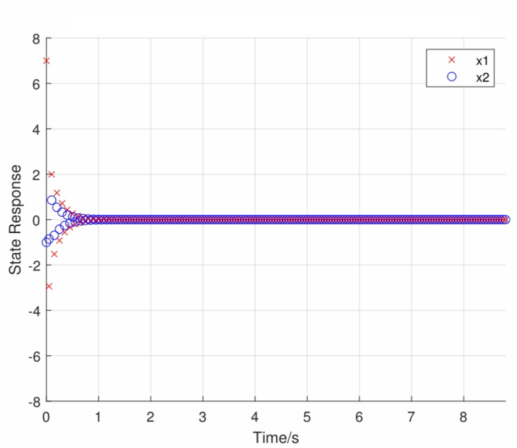

Figure 2(b). The state response of the system under $\theta^2(k)$.

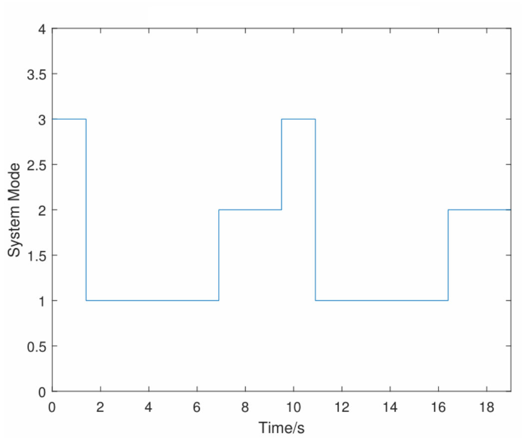

Figure 3(a). The switching signal $\theta^3(k)$.

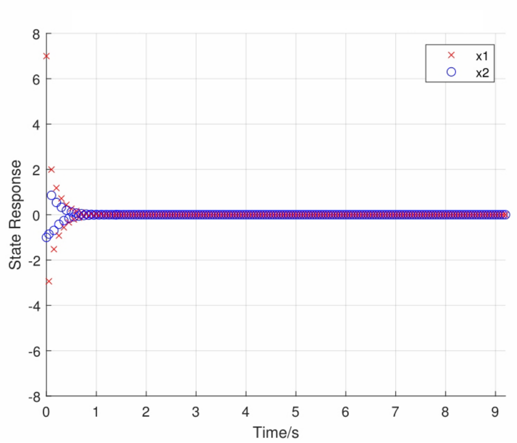

Figure 3(b). The state response of the system under $\theta^3(k)$.

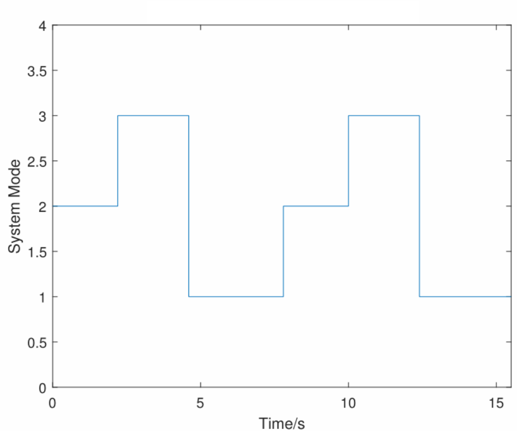

Figure 4(a). The switching signal $\theta^4(k)$.

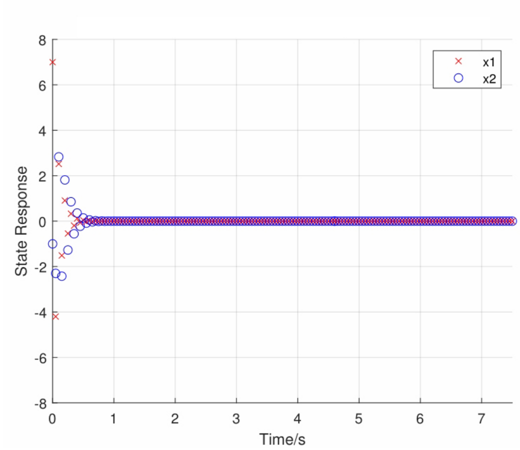

Figure 4(b). The state response of the system under $\theta^4(k)$.

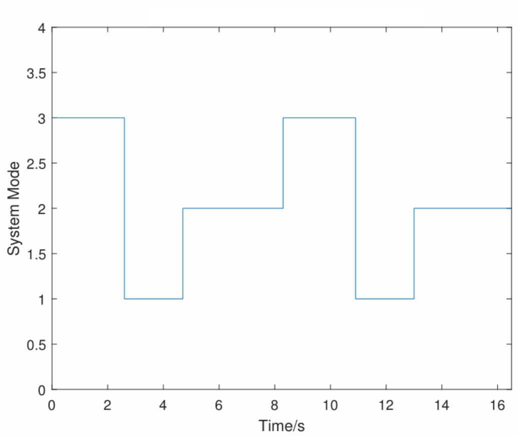

Figure 5(a). The switching signal $\theta^5(k)$.

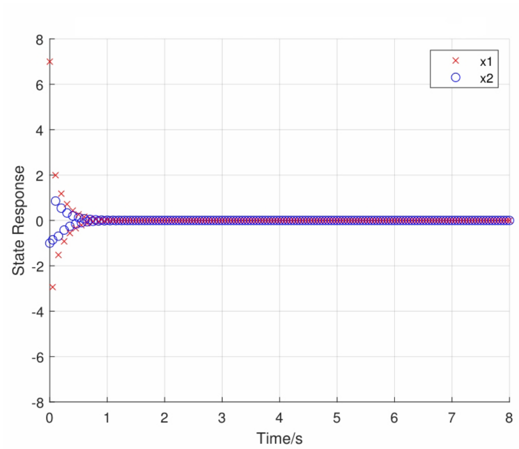

Figure 5(b). The state response of the system under $\theta^5(k)$.

5. Conclusions

In this paper, a new switching strategy ΦDIDT has been proposed and a new MCLF has been introduced for the asynchronous control problem of a class of discrete-time switched linear systems. Different from the existing studies, the paper considers that Lyapunov functions may jump when both the subsystem switches or the controller changes. A numerical example makes some comparisons among different switching strategies to demonstrate the effectiveness of the presented techniques.

Although the methods and techniques presented in this paper are applied to discrete-time switched systems, they are also applicable to continuous-time cases by adjusting the Lyapunov function appropriately, which is our work at hand. In addition, these methods and technologies are expected to be extended to T-S fuzzy systems, Markov jump systems, etc., which are some of the future research directions. On the other hand, some improved forms of ADT/MDADT/ΦDADT, such as persistent DT [2,33], weighted ADT [15] and binary F-dependent ADT [34] have been proposed. Therefore, extending the techniques of this paper to the corresponding persistent IDT/weighted IDT/binary F-dependent IDT forms is another meaningful follow-up work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Supported by Fundamental Research Program of Shanxi Province (202103021224249) and Fund Program for the Scientific Activities of Selected Returned Overseas Professionals in Shanxi Province (20220023).

Conflict of interest

The authors declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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