AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 28582–28600.
DOI:10.3934/math.20231463
Received: 23 August 2023
Revised: 21 September 2023
Accepted: 09 October 2023
Published: 20 October 2023

Research article

Approximating fixed points of demicontractive mappings in metric spaces by geodesic averaged perturbation techniques

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Abstract: In this article, we introduce the fundamentals of the theory of demicontractive mappings in metric spaces and expose the key concepts and tools for building a constructive approach to approximating the fixed points of demicontractive mappings in this setting. By using an appropriate geodesic averaged perturbation technique, we obtained strong convergence and Δ-convergence theorems for a Krasnoselskij-Mann type iterative algorithm to approximate the fixed points of quasi-nonexpansive mappings within the framework of CAT(0) spaces. The main results obtained in this nonlinear setting are natural extensions of the classical results from linear settings (Hilbert and Banach spaces) for both quasi-nonexpansive mappings and demicontractive mappings. We applied our results to solving an equilibrium problem in CAT(0) spaces and showed how we can approximate the equilibrium points by using our fixed point results. In this context we also provided a numerical example in the case of a demicontractive mapping that is not a quasi-nonexpansive mapping and highlighted the convergence pattern of the algorithm in Table 1. It is important to note that the numerical example is set in non-Hilbert CAT(0) spaces.

Keywords: CAT(0) space; demiclosedness-type property; demicontractive mapping; Δ-convergence theorem; fixed point; Krasnoselskij-Mann iteration; metric space; quasi-nonexpansive mapping; strong convergence theorem
Mathematics Subject Classification: 47H05, 47H10, 47H25

1. Introduction

For a metric space $(\mathcal{H}, d)$ with a nonempty subset $\mathcal{D}$ and a mapping $T : \mathcal{D} \to \mathcal{H}$, a point $p \in \mathcal{D}$ is called a fixed point of T if $Tp = p$. Throughout this paper, the set of such fixed points is denoted as $\mathrm{Fix}(T)$. Fixed points have significant applications across various fields. While contraction self-mappings on complete metric spaces have unique fixed points and are approachable through the Picard scheme, this principle does not hold for more general mappings, like the class of nonexpansive type mappings. A notable instance of these general mappings is the class of demicontractive mappings extensively studied by researchers since its introduction by Măruşter [38,39] and Hicks and Kubicek [24]. The class of demicontractive mappings, which includes nonexpansive and quasi-nonexpansive mappings, appears to be the largest class of nonexpansive type mappings whose fixed points can be approximated by means of iterative schemes.

Recent advancements in the approximation of fixed points for demicontractive mappings within linear spaces can be found in Berinde [8] and references therein.

Moreover, the theoretical study of demicontractive mappings in Hilbert and Banach spaces have important applications in solving various practical problems: Split problem [28, 49, 50, 52, 55], split common fixed point problem [21,29,30,43,56], split feasibility problem with multiple output sets [53], split variational inequality problem and split common null point problem [20], equilibrium problem and split generalized equilibrium problem [22, 23, 42, 57] and more. Furthermore, fixed points of certain demicontractive mappings address real-world problems, including intensity-modulated radiation therapy, dynamic emission tomographic and image reconstruction.

Furthermore, investigations into demicontractive mappings have been very recently extended to nonlinear geodesic spaces, as detailed in references [3, 36, 47, 48] and related literature.

In [34], Krasnoselskij suggested a modification of Picard iteration by using the so-called average mapping $\frac{1}{2}I + \frac{1}{2}T$ as the operator for the iterations. This approach efficiently approximates fixed points of nonexpansive mappings and some of their generalizations. The average mapping is generally considered for $\sigma \in (0, 1)$ by the following definition:

$$T_\sigma : u \mapsto (1 - \sigma)u + \sigma T u.$$

Through this technique of average mapping, several modifications of the Picard iteration have been formulated for generalized mappings, including well-known algorithms like the Krasnoselskij-Mann iteration [35], Ishikawa iteration [26], Noor iteration [41] and Agarwal iteration [2]. The technique of average mapping provides deeper insights into the inherent properties of the underlying operator. One noteworthy property is that T_σ can exhibit asymptotic regularity even when T lacks this property. This geometric characteristic, along with further analysis in this direction, has been explored [12,13,27,34], leading to the concepts of enriched contractions and enriched nonexpansive mappings [6, 7].

It is important to emphasize that convexity and linearity structures significantly influence the average mapping techniques. Moreover, numerous results in fixed point theory, such as extensions from one linear mapping to another or approximation schemes, revolve around the concept of average mapping. CAT(0) spaces, with their flexible linearity and convexity structures, are particularly well-suited for the concept of averaged mappings.

Starting from these facts, the aim of this paper is to introduce the fundamentals of the theory of demicontractive mappings in metric spaces and expose the key concepts and tools for building a constructive approach to approximating the fixed points of demicontractive mappings in this setting. By using an appropriate geodesic averaged perturbation technique, we establish strong convergence and Δ-convergence theorems for a Krasnoselskij-Mann type iterative algorithm to approximate the fixed points of quasi-nonexpansive mappings within the framework of CAT(0) spaces. Next, by using the fact similar to the case of Hilbert spaces, demicontractive mappings in CAT(0) spaces are enriched quasi-nonexpansive mappings, we derive strong and Δ-convergence theorems for approximating fixed points of demicontractive mappings in CAT(0) spaces. The main results obtained in this nonlinear setting are natural extensions of the classical results from linear settings (Hilbert and Banach spaces) for both quasi-nonexpansive mappings and demicontractive mappings. We illustrate the relevance and effectiveness of the main theoretical results by providing appropriate supportive examples.

2. Preliminaries

Let $(\mathcal{H}, d)$ be a metric space and $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. A mapping $T : \mathcal{D} \to \mathcal{H}$ is said to be

(1) *Lipschitz* if there exists $\ell > 0$, such that

$$d(Tu, Tw) \leq \ell d(u, w), \text{ for all } u, w \in \mathcal{D}. \tag{2.1}$$

(2) *Contraction* if (2.1) holds with $\ell < 1$ and *nonexpansive* if (2.1) holds with $\ell = 1$.

(3) *Quasi-nonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$ and

$$d(Tu, p) \leq \ell d(u, p), \text{ for all } u \in \mathcal{D} \text{ and } p \in \mathrm{Fix}(T). \tag{2.2}$$

(4) *κ-demicontractive* [24] if $\mathrm{Fix}(T) \neq \emptyset$ and there exists $\kappa < 1$, such that

$$d^2(Tu, p) \leq d^2(u, p) + \kappa d^2(u, Tu), \text{ for all } u \in \mathcal{D} \text{ and } p \in \mathrm{Fix}(T). \tag{2.3}$$

Remark 2.1. *1) The definition of a demicontractive mapping as given by inequality* (2.3) *is the metric space correspondent of the original definition introduced by Hicks and Kubicsek [24] in Hilbert spaces.*

2) It is well known that contraction mappings are nonexpansive. Also, every nonexpansive mapping with a fixed point is quasi-nonexpansive and every quasi-nonexpansive mapping is demicontractive. However, the converses are not true in general. For supportive examples, see Examples 1.1–1.3 in [8].

A metric space $(\mathcal{H}, d)$ is called a *geodesic space* if, for every two points $u, w \in \mathcal{H}$, there exists a mapping $\phi_u^w : [0, 1] \subset \mathbb{R} \to \mathcal{H}$ satisfying the following:

(a) $\phi_u^w(0) = u$,

(b) $\phi_u^w(1) = w$,

(c) $d\left(\phi_u^w(t_1), \phi_u^w(t_2)\right) = |t_1 - t_2| d(u, w)$ for every $t_1, t_2 \in [0, 1]$.

The image of ϕ_u^w is often called a *geodesic segment* connecting u and w. For $u, w \in \mathcal{H}$ having a unique geodesic segment and for any $t \in [0, 1]$, there exists a unique point x on the segment connecting u and w, denoted by $(1-t)u \oplus tw$, with the following properties:

$$d(u, x) = td(u, w) \quad \text{and} \quad d(x, w) = (1-t)d(u, w). \tag{2.4}$$

It is known that CAT(0) spaces are geodesic spaces with the property that every pair of points is connected by a unique geodesic segment. A set is *convex* if it contains the geodesic segment connecting any pair of its points. Moreover, we have the following inequalities [18] for $u, v, w \in \mathcal{H}$ and $t \in [0, 1]$:

$$d((1-t)u \oplus tv, w) \leq (1-t)d(u, w) + td(v, w), \tag{2.5}$$

$$d^2((1-t)u \oplus tv, w) \leq (1-t)d^2(u, w) + td^2(v, w) - t(1-t)d^2(u, v), \tag{2.6}$$

where $d^2(x, y) = [d(x, y)]^2$ for all $x, y \in \mathcal{H}$. When $t = \frac{1}{2}$, inequality (2.6) reduces to the CN-inequality of Bruhat and Tits [14]. A complete CAT(0) space is called a Hadamard space. For further details on CAT(0) spaces, see [10, 33].

Let $(\mathcal{H}, d)$ be a CAT(0) space and $\{u_n\}$ be a bounded sequence in $\mathcal{H}$. The *asymptotic center* of $\{u_n\}$ is defined by

$$A(\{u_n\}) := \{u \in \mathcal{H} : \limsup_{n\to\infty} d(u, u_n) = \inf_{v\in\mathcal{H}} \limsup_{n\to\infty} d(v, u_n)\}.$$

Furthermore, the sequence $\{u_n\}$ Δ-*converges* to a point w in $\mathcal{H}$ if $\{w\}$ is the asymptotic center of every subsequence of $\{u_n\}$ and it *strongly converges* to w if $\lim_{n\to\infty} d(u_n, w) = 0$. We write $\Delta\text{-}\lim_{n\to\infty} u_n = u$ to mean $\{u_n\}$ is Δ-convergent to u.

In the sequel, we shall need the following concept.

Definition 2.1. *A map $T : \mathcal{D} \to \mathcal{H}$ is said to have a demiclosedness-type property if for any sequence $\{u_n\} \subseteq \mathcal{H}$,*

$$\left.\begin{array}{l} \Delta\text{-}\lim_{n\to\infty} u_n = u \\ \lim_{n\to\infty} d(u_n, Tu_n) = 0 \end{array}\right\} \implies u = Tu. \tag{2.7}$$

Remark 2.2. *The notion of an operator having the demiclosedness-type property introduced in Definition 2.1 corresponds to the notion of demiclosedness introduced by Browder [11] in the case of Banach spaces. This property ensures the weak convergence of Krasnoselskij iteration to the fixed point of a demicontractive operator [24, 39].*

The following theorem is crucial for our approximation results.

Theorem 2.1. *[45] Let $\mathcal{D}$ be a nonempty closed convex subset of a Hadamard space $(\mathcal{H}, d)$. Let $T : \mathcal{D} \to \mathcal{H}$ be a mapping with nonempty fixed point set, which satisfies the demiclosedness-type property* (2.7). *Suppose that $\{u_n\}$ is a sequence in $\mathcal{D}$ such that*

(P1) $d(u_n, Tu_n) \to 0$,

(P2) $\{d(u_n, u^*)\}$ *converges in $\mathbb{R}$ for every* $u^* \in \mathrm{Fix}(T)$.

Then $\{u_n\}$ Δ-converges to a fixed point of T.

3. Demicontractive mappings

In this section, we consider the general setting of metric spaces and discuss significant inequalities associated with the class of demicontractive mappings. We show that demicontractive mappings and enriched quasi-nonexpansive mappings define the same set of mappings.

In the sequel, we take

$$Q(x, y, u, w) := \frac{1}{2}\Big[d^2(x, w) + d^2(y, u) - d^2(x, u) - d^2(y, w)\Big], \tag{3.1}$$

which is a metric version of the following known identity from inner-product spaces:

$$\langle x - y,\ u - w\rangle = \frac{1}{2}\Big[\|x - w\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - w\|^2\Big],$$

where x, y, u, w are arbitrary points in the space. The term in (3.1) is reported as quasilinearization in [5].

Proposition 3.1. *Let $(\mathcal{H}, d)$ be a metric space and $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. A mapping $T : \mathcal{D} \to \mathcal{H}$ is κ-demicontractive if and only if*

$$Q(u, p, u, Tu) \geq \frac{1-\kappa}{2} d^2(u, Tu), \text{ for all } u \in \mathcal{D} \text{ and } p \in \operatorname{Fix}(T). \tag{3.2}$$

Proof. Let $u \in \mathcal{D}$ and $p \in \operatorname{Fix}(T)$. Then,

$$\begin{aligned}
& d^2(Tu, p) \leq d^2(u, p) + \kappa d^2(u, Tu) \\
\iff\ & d^2(u, p) + d^2(u, Tu) + (\kappa - 1)d^2(u, Tu) - d^2(Tu, p) \geq 0. \\
\iff\ & \frac{1}{2}\Big[d^2(u, p) + d^2(u, Tu) - d^2(Tu, p)\Big] + \frac{\kappa - 1}{2} d^2(u, Tu) \geq 0 \\
\iff\ & Q(u, p, u, Tu) + \frac{\kappa - 1}{2} d^2(u, Tu) \geq 0 \\
\iff\ & Q(u, p, u, Tu) \geq \frac{1 - \kappa}{2} d^2(u, Tu)
\end{aligned}$$

as desired. □

Remark 3.1. *It follows from Proposition 3.1 that any mapping $T : \mathcal{D} \to \mathcal{H}$ satisfying*

$$\alpha d^2(u, Tu) \leq Q(u, p, u, Tu), \text{ for all } u \in \mathcal{D} \text{ and } p \in \operatorname{Fix}(T),$$

for some $\alpha > 0$ is κ^*-demicontractive where $\max\{0, 1 - 2\alpha\} < \kappa^* < 1$.

Moreover, when we consider the inequality (3.2) in an inner-product setting, it reduces to the following inequality:

$$\alpha\|u - Tu\|^2 \leq \langle u - p,\ u - Tu\rangle, \text{ for all } u \in \mathcal{D} \text{ and } p \in \operatorname{Fix}(T), \tag{3.3}$$

which corresponds to the original definition of a demicontractive mapping as given by Măruşter in the case of Hilbert spaces [38, 39].

The class of mappings satisfying (3.3) was introduced in 1973 [38] for the case of $\mathbb{R}^n$ and in 1977 [39] in the setting of Hilbert spaces, while the class of demicontractive mappings in the sense of (2.3) was introduced in [24]. Although the two classes were introduced independently, it was later discovered that they coincide in the setting of real Hilbert spaces (see [37, pp. 2]).

Thus, Proposition 3.1 guarantees that this result is true in general metric spaces, too. Hence, there is a need to further investigate the relationship between this class of mappings and other known classes of nonexpansive type mappings.

According to [46], a mapping $T : \mathcal{D} \to \mathcal{H}$ is called *α-enriched nonexpansive* if there exists $\alpha \in [0, +\infty)$ such that

$$d^2(Tu, Tw) + \alpha^2 d^2(u, w) + 2\alpha Q(u, w, Tu, Tw) \leq (\alpha + 1)^2 d^2(u, w), \ \forall u, w \in \mathcal{H}. \tag{3.4}$$

Remark 3.2. *It should be noted that the definition of α-enriched nonexpansive by inequality* (3.4) *corresponds to the original definition of enriched nonexpansive mappings given in [6] in the case of a Hilbert space. Thus, it is natural to define an enriched quasi-nonexpansive mapping in metric spaces as follows.*

Definition 3.1. *Let $(\mathcal{H}, d)$ be a metric space and let $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. A mapping $T : \mathcal{D} \to \mathcal{H}$ is called α-enriched quasi-nonexpansive if* $\mathrm{Fix}(T) \neq \emptyset$ *and there exists $\alpha \in [0, +\infty)$, such that*

$$d^2(Tu, p) + \alpha^2 d^2(u, p) + 2\alpha Q(u, p, Tu, p) \leq (\alpha + 1)^2 d^2(u, p), \ \forall \ u \in \mathcal{D}, \ p \in \mathrm{Fix}(T). \tag{3.5}$$

Example 3.1. *Let $\mathcal{H} = \mathcal{D} = \mathbb{R}^3$ be endowed with the metric d defined by*

$$d(u, w) = \sqrt{\sum_{i=1}^{2} (u_i - w_i)^2 + (u_2^2 + w_3 - u_3 - w_2^2)^2},$$

for all $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $w = (w_1, w_2, w_3) \in \mathbb{R}^3$. Consider $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$Tu = T(u_1, u_2, u_3) = -4(u_1, u_2, -4u_2^2).$$

Clearly $\mathrm{Fix}(T) = \{0\}$. *Also, T is not a quasi-nonexpansive mapping with respect to the both usual metric in $(\mathbb{R}^3, d)$, since for any $u = (u_1, 0, 0)$, $u_1 \neq 0$, we have*

$$d(Tu, 0) = 4|u_1| > |u_1| = d(u, 0).$$

However, T is an α-enriched quasi-nonexpansive mapping with respect to $(\mathbb{R}^3, d)$ for $\alpha \geq \frac{3}{2}$. Indeed for all $u \in \mathbb{R}^3$, we have

$$d^2(Tu, 0) = 16 \sum_{i=1}^{2} u_i^2, 2Q(u, 0, Tu, 0) = d^2(u, 0) + d^2(0, Tu) - d^2(u, Tu)$$

$$= \left[\sum_{i=1}^{2} u_i^2 + (u_2^2 - u_3)^2 \right] + \sum_{i=1}^{2} (4u_i)^2 - \left[\sum_{i=1}^{2} (u_i + 4u_i)^2 + (u_2^2 - u_3)^2 \right]$$

$$= \sum_{i=1}^{2} [u_i^2 + (4u_i)^2 - (u_i + 4u_i)^2] = -8 \sum_{i=1}^{2} u_i^2.$$

Thus, we get

$$\begin{aligned} d^2(Tu, 0) + \alpha^2 d^2(u, 0) + 2\alpha Q(u, 0, Tu, 0) &= 16 \sum_{i=1}^{2} u_i^2 + \alpha^2 \sum_{i=1}^{2} u_i^2 + \alpha^2 (u_2^2 - u_3)^2 - 8\alpha \sum_{i=1}^{2} u_i^2 \\ &= (4-\alpha)^2 \sum_{i=1}^{2} u_i^2 + \alpha^2 (u_2^2 - u_3)^2. \end{aligned}$$

It follows that for any $\alpha \geq 0$ such that $|4-\alpha| \leq \alpha + 1$, we get that

$$d^2(Tu, 0) + \alpha^2 d^2(u, 0) + 2\alpha Q(u, 0, Tu, 0) \leq (\alpha + 1)^2 d^2(u, 0).$$

It suffices to take $\alpha \geq \dfrac{3}{2}$.

We now state the main theorem of this section.

Theorem 3.1. *Let $(\mathcal{H}, d)$ be a metric space and let $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{H}$ is a mapping. Then, T is κ-demicontractive mapping if and only if T is an α-enriched quasi-nonexpansive mapping, where $\alpha = \frac{\kappa}{1-\kappa}$.*

Proof. Let $u \in \mathcal{D}$ and $p \in \mathrm{Fix}(T)$. By (3.1), we obtain

$$2Q(u, p, Tu, p) = d^2(u, p) + d^2(Tu, p) - d^2(u, Tu).$$

This is equivalent to

$$d^2(u, Tu) = d^2(u, p) + d^2(Tu, p) - 2Q(u, p, Tu, p).$$

Thus, T is κ-demicontractive mapping if and only if

$$\begin{aligned} & d^2(Tu, p) \leq d^2(u, p) + \kappa d^2(u, Tu) \\ \iff\ & d^2(Tu, p) \leq d^2(u, p) + \kappa[d^2(u, p) + d^2(Tu, p) - 2Q(u, p, Tu, p)] \\ \iff\ & d^2(Tu, p) \leq \frac{1+\kappa}{1-\kappa} d^2(u, p) - \frac{2\kappa}{1-\kappa} Q(u, p, Tu, p) \\ \iff\ & d^2(Tu, p) + \left(\frac{\kappa}{1-\kappa}\right)^2 d^2(u, p) + 2\frac{\kappa}{1-\kappa} Q(u, p, Tu, p) \leq \frac{1+\kappa}{1-\kappa} d^2(u, p) + \left(\frac{\kappa}{1-\kappa}\right)^2 d^2(u, p) \\ \iff\ & d^2(Tu, p) + \left(\frac{\kappa}{1-\kappa}\right)^2 d^2(u, p) + 2\frac{\kappa}{1-\kappa} Q(u, p, Tu, p) \leq \left(\frac{\kappa}{1-\kappa} + 1\right)^2 d^2(u, p) \\ \iff\ & d^2(Tu, p) + \alpha^2 d^2(u, p) + 2\alpha Q(u, p, Tu, p) \leq (\alpha + 1)^2 d^2(u, p), \end{aligned}$$

where $\alpha = \frac{\kappa}{1-\kappa}$. □

Remark 3.3. *Theorem 3.1 signifies that the class of demicontractive mappings coincides with the class of enriched quasi-nonexpansive mappings in the following sense:*

(1) *every* α*-enriched quasi-nonexpansive mapping is* $\frac{\alpha}{1+\alpha}$*-demicontractive;*

(2) *every* κ*-demicontractive mapping is* $\frac{\kappa}{1-\kappa}$*-enriched quasi-nonexpansive.*

The above properties correspond to the similar ones existing in the case of Hilbert spaces (see Theorem 9 in [9]).

Corollary 3.1. *Let* $\mathcal{H}$ *be a real Hilbert space endowed with the usual metric* d *and* $\mathcal{D}$ *be a nonempty subset of* $\mathcal{H}$*. Suppose that* $T : \mathcal{D} \to \mathcal{H}$ *is a mapping. Then* T *is* κ*-demicontractive mapping if and only if*

$$\|\alpha u + Tu - (\alpha + 1)p\| \leq (\alpha + 1)\|u - p\|, \text{ for all } u \in \mathcal{D} \text{ and } p \in \mathrm{Fix}(T), \tag{3.6}$$

where $\alpha = \frac{\kappa}{1-\kappa}$*.*

The immediate result holds because (3.5) reduces to (3.6) in real Hilbert spaces with the usual distance d.

4. Average perturbation technique and fixed point approximation

In this section, we consider a geodesic space $(\mathcal{H}, d)$ in which every pair of points is connected by a unique geodesic segment. Suppose that $\mathcal{D}$ is a nonempty subset of $\mathcal{H}$ and $T : \mathcal{D} \to \mathcal{H}$ is a mapping. For $\sigma \in (0, 1]$, the average perturbation T_σ of T is defined by

$$T_\sigma u = (1 - \sigma)u \oplus \sigma Tu, \ \forall u \in \mathcal{D}. \tag{4.1}$$

This is a well-defined mapping by the properties of the space. Moreover, it follows from Lemma 3.5 of [46] that

A1) $\mathrm{Fix}(T) = \mathrm{Fix}(T_\sigma)$;

A2) $d^2(T_\sigma u, T_\sigma w) \leq (1 - \sigma)^2 d^2(u, w) + \sigma^2 d^2(Tu, Tw) + 2\sigma(1 - \sigma)Q(u, w, Tu, Tw)$

for all $u, w \in \mathcal{D}$. The last inequality guarantees existence of σ, upon which T_σ is nonexpansive mapping in CAT(0) spaces whenever T is enriched nonexpansive mapping. This is one of the impacts of the geodesic average perturbation technique on the class of enriched contractions. In a similar fashion, we investigate the effect of geodesic average perturbation techniques for the class of demicontractive mappings.

Proposition 4.1. *Let* $(\mathcal{H}, d)$ *be a CAT(0) space and* $\mathcal{D}$ *be a nonempty subset of* $\mathcal{H}$*. If* $T : \mathcal{D} \to \mathcal{H}$ *is* κ*-demicontractive mapping, then the average perturbation* T_σ *of* T *is* κ^**-demicontractive where* $\max\left\{0, 1 + \frac{\kappa-1}{\sigma}\right\} \leq \kappa^* < 1$*.*

Proof. Let $u \in \mathcal{D}$ and $p \in \mathrm{Fix}(T)$. By (2.6) and the hypothesis that T is κ-demicontractive, we obtain

$$\begin{aligned} d^2(T_\sigma u, p) &= d^2((1 - \sigma)u \oplus \sigma Tu, p) \\ &\leq (1 - \sigma)d^2(u, p) + \sigma d^2(Tu, p) - \sigma(1 - \sigma)d^2(u, Tu) \\ &\leq (1 - \sigma)d^2(u, p) + \sigma[d^2(u, p) + \kappa d^2(u, Tu)] - \sigma(1 - \sigma)d^2(u, Tu) \end{aligned}$$

$$= d^2(u,p) + \sigma(\kappa + \sigma - 1)d^2(u,Tu).$$

This and (2.4) yield

$$\begin{aligned} d^2\left(T_\sigma u, p\right)| &\le d^2(u,p) + \frac{\kappa+\sigma-1}{\sigma}\sigma^2 d^2(u,Tu) \\ &= d^2(u,p) + \frac{\kappa+\sigma-1}{\sigma} d^2(u,T_\sigma u) \\ &= d^2(u,p) + \left(1 + \frac{\kappa-1}{\sigma}\right) d^2(u,T_\sigma u) \\ &\le d^2(u,p) + \max\left\{0, 1 + \frac{\kappa-1}{\sigma}\right\} d^2(u,T_\sigma u) \qquad (4.2) \\ &\le d^2(u,p) + \kappa^* d^2(u,T_\sigma u) \end{aligned}$$

as desired. □

Proposition 4.2. *Let $(\mathcal{H}, d)$ be a CAT(0) space and $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. If $T : \mathcal{D} \to \mathcal{H}$ is κ-demicontractive mapping, then for any $\sigma \in]0, 1-\kappa]$, the average perturbation T_σ of T is quasi-nonexpansive.*

Proof. Observe that

$$\sigma \in (0, 1-\kappa] \quad \Longrightarrow \quad \max\left\{0, 1 + \frac{\kappa-1}{\sigma}\right\} = 0.$$

Thus, the desired result is achieved using (4.2) of the proof of Proposition 4.1. □

Corollary 4.1. *Let $(\mathcal{H}, d)$ be a CAT(0) space and $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. If $T : \mathcal{D} \to \mathcal{D}$ is an α-enriched quasi-nonexpansive mapping, then for any $\sigma \in \left]0, \frac{1}{1+\alpha}\right]$, the average perturbation T_σ of T is quasi-nonexpansive.*

Proof. Since$T$ is an α-enriched quasi-nonexpansive mapping, it follows from Theorem 3.1 that T is $\frac{\alpha}{1+\alpha}$-demicontractive. Consequently, Proposition 4.2 yields the desired result. □

Next, we state some direct consequences of the results, which are linear versions of Propositions 4.1 and 4.2.

Corollary 4.2. *[8, Lemma 3.1] Let $\mathcal{H}$ be a real Hilbert space endowed with the usual metric d and $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. If $T : \mathcal{D} \to \mathcal{H}$ is κ-demicontractive mapping, then the average perturbation T_σ of T is κ^*-demicontractive, where* $\max\left\{0, 1 + \frac{\kappa-1}{\sigma}\right\} \le \kappa^* < 1$.

Corollary 4.3. *[8, Lemma 3.2] Let $\mathcal{H}$ be a real Hilbert space endowed with the usual metric d and $\mathcal{D}$ be a nonempty subset of $\mathcal{H}$. If $T : \mathcal{D} \to \mathcal{H}$ is κ-demicontractive mapping, then for any $\sigma \in]0, 1-\kappa]$, the average perturbation T_σ of T is quasi-nonexpansive.*

Given a mapping T, the set of fixed points of T is necessarily required to be nonempty for an iterative sequence to approach an element of the set. Under this assumption, various classes of mappings such as Banach contraction mappings, Kannan mappings, Bianchini mappings, nonexpansive mappings, Suzuki mappings and several others belong to the class of quasi-nonexpansive mappings. Furthermore, according to Proposition 4.2, the geodesic average perturbation of a demicontractive mapping is also a quasi-nonexpansive mapping. It is important to note that every strictly pseudocontractive mapping is demicontractive under this assumption.

One of the well-known schemes for finding fixed points of quasi-nonexpansive mappings is the Krasnoselskij-Mann algorithm. This algorithm has been studied and modified in various aspects by different scholars. In the setting of CAT(0) spaces, the algorithm is updated as follows:

$$u_{n+1} = (1-\alpha_n)u_n \oplus \alpha_n T u_n,\ n \geq 1, \tag{4.3}$$

where $\{\alpha_n\} \subseteq [0,1]$. Next, we utilize the scheme in (4.3) to approximate a fixed point of a demicontractive mapping through a quasi-nonexpansive mapping.

Theorem 4.1. *Let $(\mathcal{H}, d)$ be a CAT(0) space and $\mathcal{D}$ be a nonempty convex subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{D}$ is a quasi-nonexpansive mapping and $\{u_n\}$ is a sequence generated by* (4.3) *with $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. Then,*

(P1) $d(u_n, Tu_n) \to 0$,

(P2) $\{d(u_n, u^*)\}$ converges in $\mathbb{R}$ for every $u^* \in \mathrm{Fix}(T)$.

Proof. Let $u^* \in \mathrm{Fix}(T)$. Using (2.6), (4.3) and the hypothesis that T is a quasi-nonexpansive mapping, we have

$$\begin{aligned} d^2(u_{n+1},\ u^*) &= d^2((1-\alpha_n)u_n \oplus \alpha_n T u_n,\ u^*) \\ &\leq (1-\alpha_n)d^2(u_n, u^*) + \alpha_n d^2(Tu_n, u^*) - \alpha_n(1-\alpha_n)d^2(u_n, Tu_n) \\ &\leq d^2(u_n, u^*) - \alpha_n(1-\alpha_n)d^2(u_n, Tu_n). \end{aligned}$$

This implies that

$$d(u_{n+1},\ u^*) \leq d(u_n, u^*) \tag{4.4}$$

and

$$\begin{aligned} d^2(u_n, Tu_n) &\leq \frac{1}{\alpha_n(1-\alpha_n)}\Big[d^2(u_n, u^*) - d^2(u_{n+1},\ u^*)\Big] \\ &\leq \frac{1}{a(1-b)}\Big[d^2(u_n, u^*) - d^2(u_{n+1},\ u^*)\Big]. \end{aligned} \tag{4.5}$$

Thus, (4.4) yields (P2) of Theorem 2.1. Consequently, (4.5) together with (P2) yield (P1). □

Based on the preceding facts, we obtain the following results as direct applications of Theorem 2.1 together with Theorem 4.1.

Corollary 4.4. *Let $(\mathcal{H}, d)$ be a complete CAT(0) space and $\mathcal{D}$ be a nonempty closed convex subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{D}$ is a quasi-nonexpansive mapping that satisfies the demiclosedness-type property* (2.7) *and $\{u_n\}$ is a sequence generated by* (4.3) *with $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ Δ-converges to a fixed point of T.*

Proof. By Theorem 4.1, conditions (P1) and (P2) of Theorem 2.1 hold. Consequently, Theorem 2.1 yields the desired result using the demiclosedness-type property of T. □

In real Hilbert spaces, the Δ-convergence coincides with weak convergence and, thus, we have the next result which corresponds to Theorem 8 in [19] (see also Theorem 4.3 in [8]).

Corollary 4.5. *Let $\mathcal{H}$ be a real Hilbert space endowed with the usual metric d and $\mathcal{D}$ be a nonempty closed convex subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{D}$ is a quasi-nonexpansive mapping that satisfies the demiclosedness-type property* (2.7) *and $\{u_n\}$ is a sequence generated by*

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n, \; n \geq 1$$

with $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges weakly to a fixed point of T.

Remark 4.1. *According to Remark 2.2, the fact that the mapping T satisfies the demiclosedness-type property* (2.7) *means that $I - T$ is demiclosed at zero, and so by Corollary 4.5 we recover Theorem 8 [19].*

Corollary 4.6. *Let $\mathcal{H}$, $\mathcal{D}$, T and $\{u_n\}$ be the same as in Corollary 4.4. Suppose $\mathcal{D}$ is compact, then $\{u_n\}$ converges strongly to a common fixed point of T.*

Proof. Since D is compact, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ that strongly converges to $u^* \in D$. Also, by Theorem 4.1, conditions (P1) and (P2) of Theorem 2.1 hold. Following the lines of proof of Theorem 2.1 with $\{u_{n_k}\}$ in place of the subsequence of $\{u_n\}$ that Δ-converges, (P2) yields that $\lim_{n\to\infty} d(u_n, u^*) = 0$, where $u^* \in \mathrm{Fix}(T)$. □

Theorem 4.2. *Let $(\mathcal{H}, d)$ be a complete CAT(0) space and $\mathcal{D}$ be a nonempty closed convex subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{D}$ is a κ-demicontractive mapping that satisfies the demiclosedness-type property* (2.7) *and $\{u_n\}$ is a sequence generated by* (4.3) *with $0 < a \leq \alpha_n \leq (1 - \kappa)b \leq b < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ Δ-converges to a fixed point of T.*

Proof. Let $\sigma \in]0, 1 - \kappa]$. Since T is κ-demicontractive, it follows from Proposition 4.2 that T_σ is quasi-nonexpansive. By (2.4), the updated iterate u_{n+1} in (4.3) is chosen in the geodesic segment connecting u_n and Tu_n in the sense that

$$d(u_{n+1}, u_n) = \alpha_n d(u_n, Tu_n).$$

This is equivalent to u_{n+1} being chosen in the geodesic segment connecting u_n and Tu_n such that

$$d(u_{n+1}, u_n) = \alpha_n d(u_n, Tu_n) = \frac{\alpha_n}{\sigma} d(u_n, T_\sigma u_n).$$

This implies that

$$u_{n+1} = (1 - \beta_n)u_n \oplus \beta_n T_\sigma u_n, \; n \geq 1, \tag{4.6}$$

where $\beta_n = \dfrac{\alpha_n}{\sigma}$. Based on the hypotheses of the theorem, it follows from Corollary 4.4 that $\{u_n\}$ generated by (4.6) Δ-converges to a fixed point of T_σ provided $\beta_n \in [a, b] \subset]0, 1[$, for all $n \in \mathbb{N}$. By A1), the convergence is to a fixed point of T. Observe that since $0 < \sigma \leq 1 - \kappa$, then $\beta_n \in [a, b] \subset]0, 1[$ gives that $a\sigma \leq \alpha_n \leq (1 - \kappa)b$, for all $n \in \mathbb{N}$. □

The next result is the linear version of Theorem 4.2 that corresponds to the weak convergence result established in [39].

Corollary 4.7. *[39, Theorem 1] Let $\mathcal{H}$ be a real Hilbert space endowed with the usual metric d and $\mathcal{D}$ be a nonempty closed convex subset of $\mathcal{H}$. Suppose that $T : \mathcal{D} \to \mathcal{D}$ is a κ-demicontractive mapping that satisfies the demiclosedness-type property* (2.7) *and $\{u_n\}$ is a sequence generated by*

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n,\ n \geq 1,$$

with $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges weakly to a fixed point of T.

Remark 4.2. *In view of Remark 2.2, the fact that the mapping T satisfies the demiclosedness-type property* (2.7) *means that $I - T$ is demiclosed at zero, and, hence, by Corollary 4.7 we recover Theorem 1 [39].*

The next result follows similar lines of proof as in Corollary 4.6.

Corollary 4.8. *Let $\mathcal{H}$, $\mathcal{D}$, T and $\{u_n\}$ be the same as in Theorem 4.2. Suppose $\mathcal{D}$ is compact, then $\{u_n\}$ converges strongly to a common fixed point of T.*

5. Applications

Let $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction with $F(u, u) = 0$ for all $u \in \mathcal{H}$. Consider the problem of finding

$$u^* \in \mathcal{H} \quad \text{such that} \quad F(u^*, w) \geq 0, \quad \forall\, w \in \mathcal{H}. \tag{5.1}$$

This problem is known as the *equilibrium problem* and has been extensively analyzed by many scholars due to its applicability in various nonlinear problems. Detailed discussions on this problem in the context of Hadamard spaces can be found in [25, 31]. As a special case, let $\mathcal{H} = \mathbb{R}^3$ and define the bifunction F as follows:

$$F(u, w) := \sum_{i=1}^{2} \left(3u_i w_i - 5u_i^2\right) - \frac{1}{2}\left(u_2^2 - u_3 + w_3 - w_2^2\right)^2,\ \forall\, u, w \in \mathcal{H}. \tag{5.2}$$

Suppose that T is given as in Example 3.1. Then, for all $u, w \in \mathbb{R}^3$, we have that

$$d^2(Tu, w) = \left(\sum_{i=1}^{2} (-4u_i - w_i)^2\right) \quad \text{and} \quad d^2(u, Tu) = \sum_{i=1}^{2} (u_i - (-4u_i))^2.$$

Consequently, we can express $F(u, w)$ as

$$\begin{aligned}
F(u, w) &= \frac{1}{2}\sum_{i=1}^{2}\left[16u_i^2 + w_i^2 + 8u_i w_i - \left(u_i^2 + w_i^2 - 2u_i w_i\right) - 25u_i^2\right] - \frac{1}{2}\left(u_2^2 - u_3 + w_3 - w_2^2\right)^2 \\
&= \frac{1}{2}\sum_{i=1}^{2}(4u_i + w_i)^2 - \frac{1}{2}\sum_{i=1}^{2}(u_i - w_i)^2 - \frac{1}{2}\left(u_2^2 - u_3 + w_3 - w_2^2\right)^2 - \frac{25}{2}\sum_{i=1}^{2} u_i^2 \\
&= \frac{1}{2}\left[\left(\sum_{i=1}^{2}(-4u_i - w_i)^2\right) - \left(\sum_{i=1}^{2}(u_i - w_i)^2 + \left(u_2^2 - u_3 + w_3 - w_2^2\right)^2\right) - \left(\sum_{i=1}^{2}(u_i - (-4u_i))^2\right)\right] \\
&= \frac{1}{2}\left[d^2(Tu, w) - d^2(u, w) - d^2(u, Tu)\right] \\
&= Q(u, Tu, w, u).
\end{aligned}$$

We now have the following result.

Proposition 5.1. *For $\beta > 0$, consider the bifunction $F_\beta : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ defined by*

$$F_\beta(u, w) = \beta\, Q(u, Tu, w, u)\ \forall\ u, w \in \mathcal{H},$$

where $T : \mathcal{H} \to \mathcal{H}$ is a mapping. Then, the set of the equilibrium points of F_β coincides with the set of fixed points of T.

Proof. For $u^* \in \mathrm{Fix}(T)$, it is clear that

$$F_\beta(u^*, w) = \beta\, Q\,(u^*, u^*, w, u^*) = 0,\ \forall\ w \in \mathcal{H}.$$

For the converse, suppose that $F_\beta(u^*, w) \geq 0$ for all $w \in \mathcal{H}$. Then, we get

$$\begin{aligned} 0 &\leq d^2\,(Tu^*, w) - d^2\,(u^*, w) - d^2\,(u^*, Tu^*) \\ &\leq d^2\,(Tu^*, w) - d^2\,(u^*, Tu^*)\,, \end{aligned}$$

for all $w \in \mathcal{H}$. This implies that

$$d\,(u^*, Tu^*) \leq d\,(Tu^*, w)$$

for all $w \in \mathcal{H}$. Since w can be Tu^*, we conclude that $u^* \in \mathrm{Fix}(T)$. □

Next, we show that $(\mathcal{H}, d)$ is a CAT(0) space and we can approximate the equilibrium of F_β using our fixed-point results. Indeed, for $u, w \in \mathbb{R}^3$, let $\phi : [0, 1] \to \mathbb{R}^3$ be defined by

$$\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t)),$$

where $\phi_i(t) = (1 - t)u_i + tw_i$ for $i = 1, 2$, and

$$\phi_3(t) = (\Phi_2(t))^2 - (1 - t)(u_2^2 - u_3) - t(w_2^2 - w_3).$$

It can be easily shown that $(\mathcal{H}, d)$ is a CAT(0) space and ϕ is the geodesic connecting u and w. This follows from a straightforward computation using the identity

$$\Big(tx + (1 - t)y\Big)^2 = tx^2 + (1 - t)y^2 - t(1 - t)(x - y)^2, \quad \forall x, y \in \mathbb{R}.$$

Thus, the inequality (2.6) holds. It is also worth noting that $(\mathcal{H}, d)$ is not a Hilbert space.

Based on Example 3.1, we observe that T is not a quasi-nonexpansive mapping with respect to $(\mathcal{H}, d)$, but it is a $\frac{3}{2}$-enriched quasi-nonexpansive mapping. Therefore, according to Theorem 3.1, T is $\frac{3}{5}$-demicontractive with respect to $(\mathcal{H}, d)$. Additionally, T satisfies a demiclosedness-type property. To demonstrate this, let $\{u_n\}$ be a sequence such that $\Delta\text{-}\lim_{n\to\infty} u_n = u^*$ and $\lim_{n\to\infty} d(u_n, Tu_n) = 0$. Then, for $\sigma = \frac{2}{5}$, we have

$$T_\sigma u = (1 - \sigma)u \oplus \sigma Tu = \left(-u_1, -u_2, \frac{2}{5}u_2^2 + \frac{3}{5}u_3\right) \text{ for } u = (u_1, u_2, u_3),$$

and, consequently,

$$d(u_n, T_\sigma u^*) \leq d(u_n, T_\sigma u_n) + d(T_\sigma u_n, T_\sigma u^*) = \sigma d(u_n, Tu_n) + d(T_\sigma u_n, T_\sigma u^*)$$

$$= \sigma d(u_n, Tu_n) + \sqrt{\sum_{i=1}^{2}(u_{n,i} - u_i^*)^2 + \left(u_{n,2}^2 + \left(\frac{2}{5}(u_2^*)^2 + \frac{3}{5}(u_3^*)\right) - \left(\frac{2}{5}u_{n,2}^2 + \frac{3}{5}u_{n,3}\right) - (u_2^*)^2\right)^2}$$

$$= \sigma d(u_n, Tu_n) + \sqrt{\sum_{i=1}^{2}(u_{n,i} - u_i^*)^2 + \left(\frac{3}{5}\right)^2 \left(u_{n,2}^2 + u_3^* - u_{n,3} - (u_2^*)^2\right)^2}$$

$$\leq d(u_n, Tu_n) + \sqrt{\sum_{i=1}^{2}(u_{n,i} - u_i^*)^2 + \left(u_{n,2}^2 + u_3^* - u_{n,3} - (u_2^*)^2\right)^2} = d(u_n, Tu_n) + d(u_n, u^*).$$

Thus, $\limsup_{n\to\infty} d(u_n, T_\sigma u^*) \leq \limsup_{n\to\infty} d(u_n, u^*)$. Utilizing the uniqueness of the asymptotic center (see [17, Proposition 7]), we conclude that $x^* = T_\sigma x^*$. Hence, $x^* \in \mathrm{Fix}(T_\sigma) = \mathrm{Fix}(T)$, leading to the conclusion that $x^* = Tx^*$.

To gain insight into the behavior of the Krasnoselskij-Mann algorithm (4.3) with respect to the fixed point $(0, 0, 0)$, we set $\alpha_n = \dfrac{n}{7n+1}$ and obtain the results displayed in Table 1.

Table 1. Few values of the sequence terms for three distinct initial points.

n	u_n	u_n	u_n
1	(13, 11, 10)	(18, -20, 5)	(-20, 17, -8)
2	(4.875, 4.125, -80.1094)	(6.75, -7.5, -289.375)	(-7.5, 6.375, -219.2344)
3	(1.625, 1.375, -82.2844)	(2.25, -2.5, -293.2917)	(-2.5, 2.125, -220.7094)
4	(0.51705, 0.4375, -72.5052)	(0.71591, -0.79545, -258.0623)	(-0.79545, 0.67614, -194.0553)
5	(0.16046, 0.13578, -62.651)	(0.22218, -0.24687, -222.9521)	(-0.24687, 0.20984, -167.6392)
6	(0.04903, 0.041487, -53.9637)	(0.067888, -0.075431, -192.0333)	(-0.075431, 0.064116, -144.3897)
7	(0.014823, 0.012543, -46.4352)	(0.020524, -0.022805, -165.2423)	(-0.022805, 0.019384, -124.2455)
8	(0.0044469, 0.0037628, -39.9344)	(0.0061573, -0.0068414, -142.1088)	(-0.0068414, 0.0058152, -106.8514)
9	(0.0013263, 0.0011222, -34.3296)	(0.0018364, -0.0020404, -122.1637)	(-0.0020404, 0.0017344, -91.8548)
10	(0.00039374, 0.00033316, -29.502)	(0.00054518, -0.00060575, -104.9845)	(-0.00060575, 0.00051489, -78.9377)
⋮	⋮	⋮	⋮
90	(2.5882e-47, 2.19e-47, -0.00013713)	(3.5836e-47, -3.9818e-47, -0.000488)	(-3.9818e-47, 3.3845e-47, -0.00036693)
91	(7.4241e-48, 6.2819e-48, -0.00011757)	(1.028e-47, -1.1422e-47, -0.0004184)	(-1.1422e-47, 9.7084e-48, -0.00031459)
92	(2.1295e-48, 1.8019e-48, -0.0001008)	(2.9485e-48, -3.2761e-48, -0.00035872)	(-3.2761e-48, 2.7847e-48, -0.00026972)
93	(6.1078e-49, 5.1681e-49, -8.6426e-05)	(8.457e-49, -9.3966e-49, -0.00030755)	(-9.3966e-49, 7.9871e-49, -0.00023125)
94	(1.7518e-49, 1.4823e-49, -7.4099e-05)	(2.4255e-49, -2.695e-49, -0.00026368)	(-2.695e-49, 2.2908e-49, -0.00019826)
95	(5.0241e-50, 4.2511e-50, -6.3529e-05)	(6.9564e-50, -7.7293e-50, -0.00022607)	(-7.7293e-50, 6.5699e-50, -0.00016998)
96	(1.4408e-50, 1.2192e-50, -5.4467e-05)	(1.995e-50, -2.2167e-50, -0.00019383)	(-2.2167e-50, 1.8842e-50, -0.00014574)
97	(4.132e-51, 3.4963e-51, -4.6698e-05)	(5.7212e-51, -6.3569e-51, -0.00016618)	(-6.3569e-51, 5.4033e-51, -0.00012495)
98	(1.1849e-51, 1.0026e-51, -4.0037e-05)	(1.6406e-51, -1.8229e-51, -0.00014247)	(-1.8229e-51, 1.5495e-51, -0.00010712)
99	(3.3978e-52, 2.875e-52, -3.4325e-05)	(4.7046e-52, -5.2273e-52, -0.00012215)	(-5.2273e-52, 4.4432e-52, -9.1843e-05)
100	(9.7428e-53, 8.2439e-53, -2.9429e-05)	(1.349e-52, -1.4989e-52, -0.00010472)	(-1.4989e-52, 1.2741e-52, -7.8742e-05)

6. Conclusion remarks

In this article, we have extracted and analyzed the substantial properties of demicontractive mappings in the context of metric fixed point approximation. Based on Theorem 3.1, we have established that demicontractive mappings are enriched quasi-nonexpansive mappings in a general metric space. Furthermore, it is evident that geodesic average perturbation techniques play a significant role in approximating the fixed points of such mappings within the framework of CAT(0) spaces. Moreover, by geodesic average perturbation, we have proven that if T is κ-demicontractive, then T_σ is κ^*-demicontractive for $1 + \kappa/\sigma - 1/\sigma \leq \kappa^* < 1$, and it is quasi-nonexpansive for $\sigma \leq 1 - \kappa$. These properties are established in Propositions 4.1 and 4.2.

We have leveraged the Krasnoselskij-Mann iterative algorithm to approximate the fixed points of quasi-nonexpansive mappings, as shown in Theorem 4.1 and Corollary 4.4. Extending this algorithm through geodesic average perturbation techniques, we have successfully applied it to approximate the fixed points of demicontractive mappings, as expounded in Theorem 4.2. Furthermore, we presented an application of our results to solving an equilibrium problem in CAT(0) spaces and showed how we can approximate the equilibrium of F_β using our fixed point results. Related to this problem, we also provided numerical examples in the case of a demicontractive mapping that is not a quasi-nonexpansive mapping and highlighted the convergence pattern of the algorithm in Table 1. It is important to note that the numerical example is set in non-Hilbert CAT(0) spaces.

This work contributes to a unified understanding of demicontractive mappings and extends and generalizes many existing results found in the literature. Notably, it complements the work presented in [8] by extending the analysis from a linear framework to the broader scope of CAT(0) spaces. As such, the findings presented herein are applicable to all $\mathbb{R}$-trees, Hadamard manifolds and all CAT(κ) spaces where $\kappa \leq 0$.

Our results also open new avenues for applying demicontractive mappings to the solution of various problems in nonlinear analysis, like equilibrium problems, split problems, split feasibility problems, split common fixed point problems, split variational inequality problems, split common null point problems, split generalized equilibrium problems and more ([1, 4, 15, 16, 20–23, 28–32, 36, 40, 42–44, 47–57] and references therein) by extending the existing results from linear settings to nonlinear settings.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. Moreover, this research has received funding support from the NSRF through the Program Management Unit for Human Resources & Institutional Development, Research and Innovation [grant number B39G660025]. The first author was supported by the "Petchra Pra Jom Klao Ph.D. Research Scholarship" from "King Mongkut's University of Technology Thonburi" with the contract number 67/2563. The work was initiated while the first author visited North University Center at Baia Mare, Technical University of Cluj-Napoca, Romania during April-October, 2023.

Conflict of interest

The authors declare they have no conflict of interest.

References

1. A. Adamu, A. Adam, Approximation of solutions of split equality fixed point problems with applications, *Carpathian J. Math.*, **37** (2021), 381–392. https://doi.org/10.37193/CJM.2021.03.02
2. R. Agarwal, D. O'Regan, D. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, *J. Nonlinear Convex Anal.*, **8** (2007), 61–79.
3. K. Aremu, L. Jolaoso, C. Izuchukwu, O. Mewomo, Approximation of common solution of finite family of monotone inclusion and fixed point problems for demicontractive multivalued mappings in CAT(0) spaces, *Ricerche Mat.*, **69** (2020), 13–34. https://doi.org/10.1007/s11587-019-00446-y
4. T. Bantaojai, C. Garodia, I. Uddin, N. Pakkaranang, P. Yimmuang, A novel iterative approach for solving common fixed point problems in geodesic spaces with convergence analysis, *Carpathian J. Math.*, **37** (2021), 145–160. https://doi.org/10.37193/CJM.2021.02.01
5. I. Berg, I. Nikolaev, Quasilinearization and curvature of Aleksandrov spaces, *Geom. Dedicata*, **133** (2008), 195–218. https://doi.org/10.1007/s10711-008-9243-3
6. V. Berinde, Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces, *Carpathian J. Math.*, **35** (2019), 293–304. https://doi.org/10.37193/cjm.2019.03.04
7. V. Berinde, M. Păcurar, Approximating fixed points of enriched contractions in Banach spaces, *J. Fixed Point Theory Appl.*, **22** (2020), 38. https://doi.org/10.1007/s11784-020-0769-9
8. V. Berinde, Approximating fixed points of demicontractive mappings via the quasi-nonexpansive case, *Carpathian J. Math.*, **39** (2023), 73–84. https://doi.org/10.37193/cjm.2023.01.04
9. V. Berinde, M. Păcurar, Fixed points theorems for unsaturated and saturated classes of contractive mappings in Banach spaces, *Symmetry*, **13** (2021), 713. https://doi.org/10.3390/sym13040713
10. M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Berlin: Springer, 1999. https://doi.org/10.1007/978-3-662-12494-9
11. F. Browder, Semicontractive and semiaccretive nonlinear mappings in Banach spaces, *Bull. Amer. Math. Soc.*, **74** (1968), 660–665. https://doi.org/10.1090/s0002-9904-1968-11983-4
12. F. Browder, W. Petryshyn, The solution by iteration of nonlinear functional equations in Banach spaces, *Bull. Amer. Math. Soc.*, **72** (1966), 571–575. https://doi.org/10.1090/s0002-9904-1966-11544-6
13. F. Browder, W. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.*, **20** (1967), 197–228. https://doi.org/10.1016/0022-247x(67)90085-6

14. F. Bruhat, J. Tits, Groupes réductifs sur un corps local, *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, **41** (1972), 5–251. https://doi.org/10.1007/bf02715544

15. K. Calderón, M. Khamsi, J. Martínez-Moreno, Perturbed approximations of fixed points of nonexpansive mappings in $CAT_p(0)$ spaces, *Carpathian J. Math.*, **37** (2021), 65–79. https://doi.org/10.37193/CJM.2021.01.07

16. M. Choban, About convex structures on metric spaces, *Carpathian J. Math.*, **38** (2022), 391–404. https://doi.org/10.37193/cjm.2022.02.10

17. S. Dhompongsa, W. Kirk, B. Sims, Fixed points of uniformly Lipschitzian mappings, *Nonlinear Anal.*, **65** (2006), 762–772. https://doi.org/10.1016/j.na.2005.09.044

18. S. Dhompongsa, B. Panyanak, On Δ-convergence theorems in CAT(0) spaces, *Comput. Math. Appl.*, **56** (2008), 2572–2579. https://doi.org/10.1016/j.camwa.2008.05.036

19. W. Dotson, Fixed points of quasi-nonexpansive mappings, *J. Aust. Math. Soc.*, **13** (1972), 167–170. https://doi.org/10.1017/s144678870001123x

20. M. Eslamian, General algorithms for split common fixed point problem of demicontractive mappings, *Optimization*, **65** (2016), 443–465. https://doi.org/10.1080/02331934.2015.1053883

21. Q. Fan, J. Peng, H. He, Weak and strong convergence theorems for the split common fixed point problem with demicontractive operators, *Optimization*, **70** (2021), 1409–1423. https://doi.org/10.1080/02331934.2021.1890074

22. A. Hanjing, S. Suantai, The split fixed point problem for demicontractive mappings and applications, *Fixed Point Theory*, **21** (2020), 507–524. https://doi.org/10.24193/fpt-ro.2020.2.37

23. A. Hanjing, S. Suantai, Y. Cho, Hybrid inertial accelerated extragradient algorithms for split pseudomonotone equilibrium problems and fixed point problems of demicontractive mappings, *Filomat*, **37** (2023), 1607–1623. https://doi.org/10.2298/FIL2305607H

24. T. Hicks, J. Kubicek, On the Mann iteration process in a Hilbert space, *J. Math. Anal. Appl.*, **59** (1977), 498–504. https://doi.org/10.1016/0022-247x(77)90076-2

25. A. Inuwa, P. Chaipunya, P. Kumam, S. Salisu, Equilibrium problems and proximal algorithm using tangent space products, *Carpathian J. Math.*, in press.

26. S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44** (1974), 147–150. https://doi.org/10.1090/s0002-9939-1974-0336469-5

27. S. Ishikawa, Fixed points and iteration of a nonexpansive mapping in a Banach space, *Proc. Amer. Math. Soc.*, **59** (1976), 65–71. https://doi.org/10.1090/s0002-9939-1976-0412909-x

28. P. Jailoka, S. Suantai, Split null point problems and fixed point problems for demicontractive multivalued mappings, *Mediterr. J. Math.*, **15** (2018), 204. https://doi.org/10.1007/s00009-018-1251-4

29. P. Jailoka, S. Suantai, Split common fixed point and null point problems for demicontractive operators in Hilbert spaces, *Optim. Method. Softw.*, **34** (2019), 248–263. https://doi.org/10.1080/10556788.2017.1359265

30. P. Jailoka, S. Suantai, The split common fixed point problem for multivalued demicontractive mappings and its applications, *RACSAM*, **113** (2019), 689–706. https://doi.org/10.1007/s13398-018-0496-x

31. K. Juagwon, W. Phuengrattana, Iterative approaches for solving equilibrium problems, zero point problems and fixed point problems in Hadamard spaces, *Comp. Appl. Math.*, **42** (2023), 75. https://doi.org/10.1007/s40314-023-02209-w

32. Y. Kimura, Resolvents of equilibrium problems on a complete geodesic space with curvature bounded above, *Carpathian J. Math.*, **37** (2021), 463–476. https://doi.org/10.37193/CJM.2021.03.09

33. W. Kirk, N. Shahzad, *Fixed point theory in distance spaces*, Cham: Springer, 2014. https://doi.org/10.1007/978-3-319-10927-5

34. M. Krasnosel'skii, Two remarks on the method of successive approximations, *Uspekhi Matematicheskikh Nauk*, **10** (1955), 123–127.

35. W. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506–510. https://doi.org/10.1090/s0002-9939-1953-0054846-3

36. M. Minjibir, S. Salisu, Strong and Δ-convergence theorems for a countable family of multivalued demicontractive maps in Hadamard spaces, *Nonlinear Functional Analysis and Applications*, **27** (2022), 45–58. https://doi.org/10.22771/nfaa.2022.27.01.03

37. C. Moore, Iterative approximation fixed points of demicontractive maps, *Proceedings of The Abdus Salam International Centre for Theoretical Physics*, 1998, 1–12.

38. Ş. Măruşter, Sur le calcul des zéros d'un opérateur discontinu par itération, *Can. Math. Bull.*, **16** (1973), 541–544. https://doi.org/10.4153/cmb-1973-088-7

39. Ş. Măruşter, The solution by iteration of nonlinear equations in Hilbert spaces, *Proc. Amer. Math. Soc.*, **63** (1977), 69–73. https://doi.org/10.1090/s0002-9939-1977-0636944-2

40. M. Nnakwe, J. Ezeora, Strong convergence theorems for variational inequalities and fixed point problems in Banach spaces, *Carpathian J. Math.*, **37** (2021), 477–487. https://doi.org/10.37193/cjm.2021.03.10

41. M. Noor, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.*, **251** (2000), 217–229. https://doi.org/10.1006/jmaa.2000.7042

42. F. Ogbuisi, F. Isiogugu, A new iterative algorithm for pseudomonotone equilibrium problem and a finite family of demicontractive mappings, *Abstr. Appl. Anal.*, **2020** (2020), 3183529. https://doi.org/10.1155/2020/3183529

43. A. Padcharoen, P. Kumam, Y. Cho, Split common fixed point problems for demicontractive operators, *Numer. Algor.*, **82** (2019), 297–320. https://doi.org/10.1007/s11075-018-0605-0

44. S. Panja, M. Saha, R. Bisht, Existence of common fixed points of non-linear semigroups of enriched Kannan contractive mappings, *Carpathian J. Math.*, **38** (2022), 169–178. https://doi.org/10.37193/cjm.2022.01.14

45. S. Salisu, M. Minjibir, P. Kumam, S. Sriwongsa, Convergence theorems for fixed points in CATp(0) spaces, *J. Appl. Math. Comput.*, **69** (2023), 631–650. https://doi.org/10.1007/s12190-022-01763-6

46. S. Salisu, P. Kumam, S. Sriwongsa, On fixed points of enriched contractions and enriched nonexpansive mappings, *Carpathian J. Math.*, **39** (2022), 237–254. https://doi.org/10.37193/cjm.2023.01.16

47. S. Salisu, P. Kumam, S. Sriwongsa, Strong convergence theorems for fixed point of multi-valued mappings in Hadamard spaces, *J. Inequal. Appl.*, **2022** (2022), 143. https://doi.org/10.1186/s13660-022-02870-5

48. S. Salisu, P. Kumam, S. Sriwongsa, Convergence theorems for monotone vector field inclusions and minimization problems in Hadamard spaces, *Anal. Geom. Metr. Space.*, **11** (2023), 20220150. https://doi.org/10.1515/agms-2022-0150

49. S. Suantai, P. Jailoka, A self-adaptive algorithm for split null point problems and fixed point problems for demicontractive multivalued mappings, *Acta Appl. Math.*, **170** (2020), 883–901. https://doi.org/10.1007/s10440-020-00362-6

50. R. Suparatulatorn, P. Cholamjiak, S. Suantai, Self-adaptive algorithms with inertial effects for solving the split problem of the demicontractive operators, *RACSAM*, **114** (2020), 40. https://doi.org/10.1007/s13398-019-00737-x

51. A. Wang, J. Zhao, Self-adaptive iterative algorithms for the split common fixed point problem with demicontractive operators, *J. Nonlinear Var. Anal.*, **5** (2021), 573–587.

52. Y. Wang, X. Fang, J. Guan, T. Kim, On split null point and common fixed point problems for multivalued demicontractive mappings, *Optimization*, **70** (2021), 1121–1140. https://doi.org/10.1080/02331934.2020.1764952

53. F. Wang, The split feasibility problem with multiple output sets for demicontractive mappings, *J. Optim. Theory Appl.*, **195** (2022), 837–853. https://doi.org/10.1007/s10957-022-02096-x

54. J. Xiao, Y. Wang, A viscosity method with inertial effects for split common fixed point problems of demicontractive mappings, *J. Nonlinear Funct. Anal.*, **2022** (2022), 17. https://doi.org/10.23952/jnfa.2022.17

55. Y. Yao, Y. Liou, M. Postolache, Self-adaptive algorithms for the split problem of the demicontractive operators, *Optimization*, **67** (2018), 1309–1319. https://doi.org/10.1080/02331934.2017.1390747

56. Y. Yao, J. Yao, Y. Liou, M. Postolache, Iterative algorithms for split common fixed points of demicontractive operators without priori knowledge of operator norms, *Carpathian J. Math.*, **34** (2018), 459–466. https://doi.org/10.37193/cjm.2018.03.23

57. C. Zhang, Y. Li, Y. Wang, On solving split generalized equilibrium problems with trifunctions and fixed point problems of demicontractive multi-valued mappings, *J. Nonlinear Convex Anal.*, **21** (2020), 2027–2042.