Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(12): 28450–28464.
DOI: 10.3934/math.20231456
Received: 02 September 2023
Revised: 25 September 2023
Accepted: 10 October 2023
Published: 18 October 2023

Research article

Exponential stability of a system of coupled wave equations by second order terms with a past history

Zayd Hajjej[1,*] **and Menglan Liao**[2]

[1] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

[2] School of Mathematics, Hohai University Nanjing, Jiangsu 210098, China

* **Correspondence:** Email: zhajjej@ksu.edu.sa.

Abstract: In this manuscript we consider a coupled, by second order terms, system of two wave equations with a past history acting on the first equation as a stabilizer. We show that the solution of this system decays exponentially by constructing an appropriate Lyapunov function.

Keywords: wave equation; exponential stability; past history
Mathematics Subject Classification: 35L05, 93D23, 74D05

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$ be a bounded open set with regular boundary $\Gamma = \partial\Omega$. A coupled wave equation through second order terms with just one viscoelastic infinite memory term is considered:

$$\begin{cases} y_{tt}(x,t) - a\Delta y(x,t) + c\Delta z(x,t) + \displaystyle\int_0^\infty g(s)\Delta y(x,t-s)ds = 0, \quad \text{in } \Omega \times (0,\infty), \\ z_{tt}(x,t) - \Delta z(x,t) + c\Delta y(x,t) = 0, \quad \text{in } \Omega \times (0,\infty), \\ y = z = 0, \quad \text{on } \Gamma \times (0,\infty), \\ y(x,0) = y_0(x),\ z(x,0) = z_0(x),\ y_t(x,0) = y_1(x),\ z_t(x,0) = z_1(x), \quad \text{in } \Omega, \\ y(x,-t) = f(x,t), \quad \text{in } \Omega \times (0,\infty), \end{cases} \tag{1.1}$$

where y_0, y_1, f, z_0 and z_1 are known functions belonging to appropriate space, $a > 0$ and $c \in \mathbb{R}^*$ such that $a > c^2$ and

$$a = b + c^2, \tag{1.2}$$

where b is a positive constant satisfying

$$l = b - \int_0^\infty g(s)ds > 0. \tag{1.3}$$

The function g verifies some assumptions that will be given in the next section.

The aforementioned model can be used to describe the motion of two elastic membranes subject to an elastic force that pulls one membrane toward the other. This model takes the memory effect into account, which may exist in some materials particularly in low temperature [12].

Note here that (1.1) is stabilized by the infinite memory term $\int_0^\infty g(s)\Delta y(x, t-s)ds$, which appears in only one equation. It is the concept of indirect stabilization that was first introduced by Russell [24] and later on was developed in [2]. Many researchers have been interested in this topic. We start off by reviewing some works related to wave equation with an infinite memory term. Dafermos, in his pioneer paper [9], studied the equation

$$\rho u_{tt} = cu_{xx} - \int_{-\infty}^t g(t-\tau)u_{xx}d\tau,\ x \in [0, 1],\ t \geq 0, \tag{1.4}$$

where ρ and c are positive constants. Under the assumption that g is non-negative, monotonically nonnegative and satisfies a condition likewise (1.3), the author proved that the solutions of (1.4) are asymptotically stable. In [13], Giorgi et al. analyzed the longtime behavior of solutions and proved the existence of a global attractor for solutions (in the autonomous case) in a bounded domain of $\mathbb{R}^3$ of the following semi-linear hyperbolic equation

$$u_{tt} - k(0)\Delta u - \int_0^\infty k'(s)\Delta u(s)ds + g(u) = f, \tag{1.5}$$

where k, g and f are assumed to satisfy certain conditions. By adding a frictional dissipation in (1.5) of the form αu_t, Conti and Pata [8] proved the existence of a regular global attractor. In [6,21], the authors gave necessary and sufficient conditions (on the relaxation function) for the exponential stability of an abstract equation of the form

$$u_{tt} + Au - \int_0^\infty k(s)Au(t-s)ds = 0, \tag{1.6}$$

where A is a self-adjoint strictly positive linear operator with compact inverse. Later on, Guesmia [14] examined (1.6) by considering another self-adjoint and strictly positive operator B (in the integral term) instead of A and by assuming that $D(A) \subset D(B)$, such that the embedding is dense and compact. He proved the stability of the system for a wide class of the relaxation function. For more results about stability of the wave equation with past history, we refer the reader to the references [3, 4, 7, 11, 16, 17, 20, 22, 23]. For a coupled system with infinite memory, we mention the work of Almeida and Santos [5] in which the authors studied a coupled system of wave equations and proved a polynomial decay estimate. Guesmia [15] considered a coupled system of two linear abstract evolution equations of second-order with one infinite memory acting only on the first equation:

$$\begin{cases} u_{tt} + Au - \int_0^\infty g(s)Bu(t-s)ds + \tilde{B}v = 0, & \forall\ t > 0, \\ v_{tt} + \tilde{A}v + \tilde{B}u = 0, & \forall\ t > 0, \end{cases} \tag{1.7}$$

where $A, \tilde{A}$ and B are unbounded self-adjoint linear positive definite operators in a Hilbert space H, with domains $D(A) \subset D(B) \subset H$ and $D(\tilde{A}) \subset H$, such that the embeddings are dense and compact, and $\tilde{B}$ is a self-adjoint linear bounded operator in H. The author proved a stability result of (1.7) for a wide range of integrable kernels that can decay slower than exponential one. Later, Jin et al. [19] improved the result obtained in [15] by assuming much weaker conditions on the convolution kernel. We also cite the works [18, 25] in which the authors studied an abstract system like (1.7) and considered additional terms of the form αu and external forces. We note here that our system does not fit in the framework of the works mentioned above, and their general result does not apply because the condition on the coupling operator fails (in our case, it is an unbounded operator in the state space whereas in the other works it is bounded). We finish this part by citing the recent work of Akil and Hajjej [1] in which the authors studied a similar problem to (1.1), but with only one localized frictional damping instead of a memory term and proved the exponential stability of the system.

The main innovation points of the paper are:

(1) Extending exponential decay outcomes, which have previously been established for the coupling of two viscoelastic wave equations through zero-order or first-order terms, to the realm of coupling by second-order terms.

(2) Removing the assumption of equal wave propagation speeds, a common feature in numerous prior studies.

It's worth noting that (1.1) carries a real-world physical interpretation. For instance, in one dimension, (1.1) describes the behavior of a piezoelectric material exhibiting magnetic effects.

The paper is subdivided as follows: In Section 2, we establish the existence and uniqueness of a solution to (1.1) in an appropriate Hilbert space. By using the perturbed energy method, we prove the exponential decay of the energy associated with (1.1) in the last section.

2. Well-posedness

We use the standard Lebesgue space $L^2(\Omega)$ with its usual norm $\|\cdot\|$. We denote by C_p the embedding constant of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, i.e.

$$\|y\| \leq C_p\|\nabla y\|, \quad \forall\ y \in H_0^1(\Omega).$$

In this paper, we take into account the following conditions:

(H1): $g \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ satisfies $l_0 = \displaystyle\int_0^\infty g(s)ds > 0$ and $g(s) > 0,\ \forall\ s \in \mathbb{R}_+$.

(H2): For any $s \in \mathbb{R}_+,\ g'(s) < 0$ and there exists two positive constants, b_0 and b_1, such that

$$-b_0 g(s) \leq g'(s) \leq -b_1 g(s). \tag{2.1}$$

As in [9], we define

$$\begin{cases} \eta(x,s,t) = y(x,t) - y(x,t-s), \quad \forall\ (x,s,t) \in \Omega \times (0,+\infty) \times (0,+\infty), \\ \eta_0(x,s) = \eta(x,s,0) = f(x,0) - f(x,s), \quad \forall\ (x,s) \in \Omega \times (0,+\infty). \end{cases} \tag{2.2}$$

It is clear that

$$\eta_t(x,s,t) + \eta_s(x,s,t) = y_t(x,t), \ \ x \in \Omega, \ \ s,t > 0.$$

Moreover, we have $\eta(x,0,t)=0,\ x\in\Omega$ and $t>0$. Then, (1.1) is equivalent to

$$\begin{cases} y_{tt}(x,t)-l_1\Delta y(x,t)+c\Delta z(x,t)-\displaystyle\int_0^\infty g(s)\Delta\eta(x,s,t)ds=0, \quad \text{in } \Omega\times(0,\infty),\\ z_{tt}(x,t)-\Delta z(x,t)+c\Delta y(x,t)=0, \quad \text{in } \Omega\times(0,\infty),\\ \eta_t(x,s,t)+\eta_s(x,s,t)=y_t(x,t) \quad \text{in } \Omega\times(0,\infty)\times(0,\infty),\\ y=z=0, \quad \text{on } \Gamma\times(0,\infty),\\ y(x,0)=y_0(x),\ z(x,0)=z_0(x),\ y_t(x,0)=y_1(x),\ z_t(x,0)=z_1(x), \quad \text{in } \Omega,\\ y(x,-t)=f(x,t), \quad \text{in } \Omega\times(0,\infty), \end{cases} \tag{2.3}$$

where

$$l_1=l+c^2. \tag{2.4}$$

We define the space Σ by

$$\Sigma=L_g^2(\mathbb{R}_+;H_0^1(\Omega))=\Big\{\eta:\mathbb{R}_+\to H_0^1(\Omega);\int_0^\infty g(s)\|\nabla\eta(s)\|^2ds<\infty\Big\},$$

equipped with the inner product

$$\langle\eta,\tilde\eta\rangle_\Sigma=\int_0^\infty\int_\Omega g(s)\nabla\eta(s)\cdot\nabla\tilde\eta(s)ds.$$

The state space is given by

$$\mathcal{H}=H_0^1(\Omega)\times L^2(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)\times\Sigma, \tag{2.5}$$

which is a Hilbert space under the scalar product

$$\Big\langle Z,\tilde Z\Big\rangle_{\mathcal{H}}=\int_\Omega(l\nabla u\cdot\nabla\tilde u+v\tilde v+(c\nabla u-\nabla p)\cdot(c\nabla\tilde u-\nabla\tilde p)+q\tilde q)dx+\langle\eta,\tilde\eta\rangle_\Sigma, \tag{2.6}$$

for all $Z=(u,v,p,q,\eta)^\top$ and $\tilde Z=(\tilde u,\tilde v,\tilde p,\tilde q,\tilde\eta)^\top$ in $\mathcal{H}$. The norm in $\mathcal{H}$ is denoted by $\|\cdot\|_{\mathcal{H}}$ and given by

$$\|(u,v,p,q,\eta)\|^2{}_{\mathcal{H}}=\int_\Omega\Big(|v|^2+l|\nabla u|^2+|q|^2+|c\nabla u-\nabla p|^2\Big)dx+\int_0^\infty g(s)\|\nabla\eta(s)\|^2ds.$$

We define the unbounded operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{A}(u,v,p,q,\eta)^\top=\left(v,l_1\Delta u-c\Delta p+\int_0^\infty g(s)\Delta\eta(s)ds,q,\Delta p-c\Delta u,v-\eta_s\right)^\top, \tag{2.7}$$

with domain

$$D(\mathcal{A}):=\Big\{Z:=(u,v,p,q,\eta)^\top\in\Big(H^2(\Omega)\cap H_0^1(\Omega)\Big)\times H_0^1(\Omega)\times\Big(H^2(\Omega)\cap H_0^1(\Omega)\Big)\times H_0^1(\Omega)\times\Sigma;$$
$$\eta_s\in\Sigma,\ \ l_1\Delta u+\int_0^\infty g(s)\Delta\eta(s)ds\in L^2(\Omega)\Big\}.$$

If we set $Z=(y,y_t,z,z_t,\eta)^\top$, then (2.3) may be written as:

$$Z_t=\mathcal{A}Z, \quad Z(0)=Z_0, \tag{2.8}$$

where $Z_0=(y_0,y_1,z_0,z_1,\eta_0)^\top$.

Theorem 2.1. *Assume that the assumptions* **(H1)**-**(H2)** *hold. Then, the operator* $\mathcal{A}$ *generates a* C_0 *semigroup of contractions* $e^{\mathcal{A}t}$ *on* $\mathcal{H}$.

Proof. We start by proving that $\mathcal{A}$ is dissipative; that is

$$\langle \mathcal{A}Z, Z\rangle_{\mathcal{H}} \leq 0, \ \forall\, Z = (u, v, p, q, \eta) \in D(\mathcal{A}).$$

In fact, we have for any $Z = (u, v, p, q, \eta) \in D(\mathcal{A})$ that

$$\begin{aligned}\langle \mathcal{A}Z, Z\rangle_{\mathcal{H}} &= l\int_{\Omega} \nabla v \nabla u dx + \int_{\Omega}\left(l_1\Delta u - c\Delta p + \int_0^{\infty} g(s)\Delta\eta(s)ds\right) v dx + \int_{\Omega} (c\nabla v - \nabla q)(c\nabla u - \nabla p)\, dx \\ &+ \int_{\Omega} (-c\Delta u + \Delta p)\, q dx + \int_0^{\infty}\int_{\Omega} g(s)\nabla(v - \eta(s))\nabla\eta(s) dx ds.\end{aligned}$$

Integrating by parts, the righthand side of the last equality yields to

$$\langle \mathcal{A}Z, Z\rangle_{\mathcal{H}} = \frac{1}{2}\int_0^{\infty}\int_{\Omega} g'(s)|\nabla\eta(s)|^2 dx ds \leq 0,$$

which implies that $\mathcal{A}$ is dissipative.

Next, we shall show that $0 \in \rho(\mathcal{A})$ (where $\rho(\mathcal{A})$ represents the resolvent set of $\mathcal{A}$). Given a vector $F = (\xi_1, \xi_2, h, k, \nu) \in \mathcal{H}$, we look for $Z = (u, v, p, q, \eta) \in D(\mathcal{A})$, such that

$$\mathcal{A}Z = F.$$

By the definition of $\mathcal{A}$, we obtain

$$v = \xi_1, \tag{2.9}$$

$$l_1\Delta u - c\Delta p + \int_0^{\infty} g(s)\Delta\eta(s)ds = \xi_2, \tag{2.10}$$

$$q = h, \tag{2.11}$$

$$-c\Delta u + \Delta p = k, \tag{2.12}$$

$$v - \eta_s = \nu. \tag{2.13}$$

Inserting (2.9) in (2.13) and using the fact that $\eta(x, 0, t) = 0$, we have

$$\eta = \int_0^s (\xi_1 - \nu(r))\, dr.$$

It is easy to see that since $\nu \in H_0^1(\Omega)$, then $\eta, \eta_s \in \Sigma$.

Now, using (2.4) and combining (2.10) and (2.12), we infer that

$$\Delta u = \frac{1}{l}\left(ck + \xi_2 - \int_0^{\infty} g(s)\Delta\eta(s)ds\right), \tag{2.14}$$

$$\Delta p = \frac{1}{l}\left(l_1 k + c\xi_2 - c\int_0^{\infty} g(s)\Delta\eta(s)ds\right). \tag{2.15}$$

Let $(\varphi, \psi) \in H_0^1(\Omega)\times H_0^1(\Omega)$. Multiplying (2.14) and (2.15) by φ and ψ, respectively, and then integrating by parts over Ω, one has

$$\begin{aligned}\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega \nabla p \nabla \psi dx = & - \frac{1}{l}\int_\Omega \left(ck + \xi_2 - \int_0^\infty g(s)\Delta\eta(s)ds \right)\varphi dx \\ & - \frac{1}{l}\int_\Omega \left(l_1 k + c\xi_2 - c\int_0^\infty g(s)\Delta\eta(s)ds \right)\psi dx. \end{aligned} \tag{2.16}$$

(2.16) can be rewritten as:

$$\mathbf{a}\left((u, p), (\varphi, \psi)\right) = L(\varphi, \psi),$$

where $\mathbf{a}$ is the bilinear functional defined by

$$\mathbf{a}\left((u, p), (\varphi, \psi)\right) = \int_\Omega \nabla u \nabla \varphi dx + \int_\Omega \nabla p \nabla \psi dx,$$

and L is the functional defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$L(\varphi, \psi) = -\frac{1}{l}\int_\Omega \left(ck + \xi_2 - \int_0^\infty g(s)\Delta\eta(s)ds \right)\varphi dx - \frac{1}{l}\int_\Omega \left(l_1 k + c\xi_2 - c\int_0^\infty g(s)\Delta\eta(s)ds \right)\psi dx.$$

It is clear that $\mathbf{a}$ is continuous and coercive in $\left(H_0^1(\Omega) \times H_0^1(\Omega)\right)^2$, and L is continuous on $H_0^1(\Omega)\times H_0^1(\Omega)$. Then, it follows from the Lax-Milgram's theorem that (2.16) possesses a unique solution $(u, p) \in H_0^1(\Omega) \times H_0^1(\Omega)$.

Beside that from (2.12), we have $p - cu \in H^2(\Omega)$. This fact combined with (2.10) gives us $l_1 \Delta u + \int_0^\infty g(s)\Delta\eta(s)ds \in L^2(\Omega)$, and, thus, $u, p \in H^2(\Omega)$. It follows that $Z = (u, v, p, q, \eta) \in D(\mathcal{A})$, and consequently, $0 \in \rho(\mathcal{A})$. Therefore, the well-known Lumer-Phillips theorem ensures that operator $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contractions. □

3. Exponential stability

We begin this section by introducing and proving several lemmas by adopting the method presented in [10], which will be useful in the proof of our main result.

Lemma 3.1. *Let $Z = (y, y_t, z, z_t, \eta)$ be a solution of* (2.3). *Then, the functional*

$$\varphi_1(t) = \int_\Omega y y_t dx + c\int_\Omega y z_t dx,$$

satisfies

$$\varphi_1'(t) \le \left(1 + \frac{c\delta_1}{2}\right)\int_\Omega |y_t|^2 dx - \frac{l}{2}\int_\Omega |\nabla y|^2 dx + \frac{c}{2\delta_1}\int_\Omega |z_t|^2 dx + \frac{l_0}{2l}\int_0^\infty \int_\Omega g(s)|\nabla\eta(s)|^2 dxds \tag{3.1}$$

for any $\delta_1 > 0$.

Proof. Multiplying $(2.3)_1$ by y, using $(2.3)_2$ and integrating by parts over Ω, we obtain

$$\frac{d}{dt}\int_\Omega yy_t dx - \int_\Omega |y_t|^2 dx + l\int_\Omega |\nabla y|^2 dx + c\frac{d}{dt}\int_\Omega yz_t dx - c\int_\Omega y_t z_t dx + \int_0^\infty \int_\Omega g(s)\nabla\eta(s)\nabla y dxds = 0;$$

that is,

$$\varphi_1'(t) = \int_\Omega |y_t|^2 dx - l\int_\Omega |\nabla y|^2 dx + c\int_\Omega y_t z_t dx - \int_0^\infty \int_\Omega g(s)\nabla\eta(s)\nabla y dxds. \tag{3.2}$$

Applying Young's inequality, we find for all $\delta_1 > 0$ that

$$\int_\Omega y_t z_t \le \frac{\delta_1}{2}\int_\Omega |y_t|^2 dx + \frac{1}{2\delta_1}\int_\Omega |z_t|^2 dx \tag{3.3}$$

and

$$\begin{aligned}\int_0^\infty \int_\Omega g(s)\nabla\eta(s)\nabla y dxds &\le \frac{l}{2}\int_\Omega |\nabla y|^2 dx + \frac{1}{2l}\int_\Omega \left(\int_0^\infty g(s)\nabla\eta(s)ds\right)^2 dx \\ &\le \frac{l}{2}\int_\Omega |\nabla y|^2 dx + \frac{l_0}{2l}\int_0^\infty \int_\Omega g(s)|\nabla\eta(s)|^2 dxds, \end{aligned}\tag{3.4}$$

where, in the last inequality, we have used the fact that

$$\begin{aligned}\int_\Omega \left(\int_0^\infty g(s)\nabla\eta(s)ds\right)^2 dx &\le \int_\Omega \left(\int_0^\infty g(s)ds\right)\left(\int_0^\infty g(s)|\nabla\eta(s)|^2 ds\right) dx \\ &= l_0\int_0^\infty \int_\Omega g(s)|\nabla\eta(s)|^2 dxds. \end{aligned}\tag{3.5}$$

Inserting (3.3) and (3.4) in (3.2), we get the desired inequality (3.1). □

Lemma 3.2. *Let $Z = (y, y_t, z, z_t, \eta)$ be a solution of* (2.3). *Then, the functional*

$$\varphi_2(t) = \int_\Omega y_t(cy - z)dx + c\int_\Omega z_t(cy - z)dx$$

satisfies

$$\begin{aligned}\varphi_2'(t) &\le \left(c + c^3 + \frac{1}{c}\right)\int_\Omega |y_t|^2 dx + \frac{l^2}{2\delta_2}\int_\Omega |\nabla y|^2 dx - \frac{c}{2}\int_\Omega |z_t|^2 dx + \delta_2\int_\Omega |c\nabla y - \nabla z|^2 dx \\ &+ \frac{l_0}{2\delta_2}\int_0^\infty \int_\Omega g(s)|\nabla\eta(s)|^2 dxds \end{aligned}\tag{3.6}$$

for any $\delta_2 > 0$.

Proof. Multiplying $(2.3)_1$ by $cy - z$, using $(2.3)_2$ and integrating by parts over Ω, we get

$$\int_\Omega y_{tt}(cy - z)dx + l\int_\Omega \nabla y\nabla(cy - z)dx + c\int_\Omega z_{tt}(cy - z)dx + \int_0^\infty \int_\Omega g(s)\nabla\eta(s)\nabla(cy - z)dxds = 0,$$

which implies that

$$\frac{d}{dt}\left(\int_\Omega y_t(cy - z)dx + c\int_\Omega z_t(cy - z)dx\right) = \int_\Omega y_t(cy - z)_t dx + c\int_\Omega z_t(cy - z)_t dx - l\int_\Omega \nabla y\nabla(cy - z)dx$$

$$\begin{aligned}
&-\int_0^\infty\int_\Omega g(s)\nabla\eta(s)\nabla(cy-z)dxds\\
=\;& c\int_\Omega |y_t|^2dx+(c^2-1)\int_\Omega y_tz_tdx\\
&-c\int_\Omega |z_t|^2dx-l\int_\Omega \nabla y\nabla(cy-z)dx\\
&-\int_0^\infty\int_\Omega g(s)\nabla\eta(s)\nabla(cy-z)dxds;
\end{aligned}$$

that is,

$$\begin{aligned}
\varphi_2'(t) \;=\;& c\int_\Omega |y_t|^2dx+(c^2-1)\int_\Omega y_tz_tdx-c\int_\Omega |z_t|^2dx-l\int_\Omega \nabla y\nabla(cy-z)dx\\
&-\int_0^\infty\int_\Omega g(s)\nabla\eta(s)\nabla(cy-z)dxds. \qquad (3.7)
\end{aligned}$$

By using Young's inequality, we can easily check that

$$c^2\int_\Omega y_tz_tdx\le \frac{c}{4}\int_\Omega |z_t|^2dx+c^3\int_\Omega |y_t|^2dx, \qquad (3.8)$$

$$-\int_\Omega y_tz_tdx\le \frac{c}{4}\int_\Omega |z_t|^2dx+\frac{1}{c}\int_\Omega |y_t|^2dx, \qquad (3.9)$$

$$-l\int_\Omega \nabla y\nabla(cy-z)dx\le \frac{l^2}{2\delta_2}\int_\Omega |\nabla y|^2dx+\frac{\delta_2}{2}\int_\Omega |c\nabla y-\nabla z|^2dx \qquad (3.10)$$

and

$$-\int_0^\infty\int_\Omega g(s)\nabla\eta(s)\nabla(cy-z)dxds\le \frac{\delta_2}{2}\int_\Omega |c\nabla y-\nabla z|^2dx+\frac{l_0}{2\delta_2}\int_0^\infty\int_\Omega g(s)|\nabla\eta(s)|^2dxds \qquad (3.11)$$

for every $\delta_2>0$.
Reporting (3.8)–(3.11) in (3.7) and (3.6) holds true. □

Lemma 3.3. *Let $Z=(y,y_t,z,z_t,\eta)$ be a solution of* (2.3). *Then, the functional*

$$\psi_1(t)=\int_\Omega yy_tdx+\int_\Omega zz_tdx$$

satisfies

$$\begin{aligned}
\psi_1'(t) \;\le\;& \int_\Omega |y_t|^2dx-\frac{l}{2}\int_\Omega |\nabla y|^2dx+\int_\Omega |z_t|^2dx-\int_\Omega |c\nabla y-\nabla z|^2dx\\
&+\frac{l_0}{2l}\int_0^\infty\int_\Omega g(s)|\nabla\eta(s)|^2dxds. \qquad (3.12)
\end{aligned}$$

Proof. Multiplying $(2.3)_1$ by y and integrating by parts over Ω can obtain

$$\frac{d}{dt}\int_\Omega yy_tdx-\int_\Omega |y_t|^2dx+l\int_\Omega |\nabla y|^2dx+c\int_\Omega \nabla y(c\nabla y-\nabla z)dx+\int_0^\infty\int_\Omega g(s)\nabla y\nabla\eta(s)dxds=0. \qquad (3.13)$$

On the other hand, multiplying $(2.3)_2$ by z and integrating by parts over Ω can obtain

$$\frac{d}{dt}\int_\Omega zz_t dx - \int_\Omega |z_t|^2 dx - \int_\Omega \nabla z(c\nabla y - \nabla z)dx = 0. \tag{3.14}$$

Summing (3.13) and (3.14), one derives that

$$\psi_1'(t) = \int_\Omega |y_t|^2 dx - l\int_\Omega |\nabla y|^2 dx + \int_\Omega |z_t|^2 dx - \int_\Omega |c\nabla y - \nabla z|^2 dx - \int_0^\infty \int_\Omega g(s)\nabla y \nabla \eta(s) dx ds. \tag{3.15}$$

Reporting (3.4) in (3.15), we obtain the desired inequality. □

Lemma 3.4. *Let $Z = (y, y_t, z, z_t, \eta)$ be a solution of* (2.3)*. Then, the functional*

$$\psi_2(t) = -\int_0^\infty \int_\Omega g(s)\eta(s) y_t dx ds$$

satisfies

$$\begin{aligned}\psi_2'(t) \ \le\ & -\frac{l_0}{2}\int_\Omega |y_t|^2 dx + \frac{l\delta_3}{2}\int_\Omega |\nabla y|^2 dx + \frac{c\delta_4}{2}\int_\Omega |c\nabla y - \nabla z|^2 dx \\ & + \left(l_0 + \frac{b_0^2 C_p^2}{2} + \frac{ll_0}{2\delta_3} + \frac{cl_0}{2\delta_4}\right)\int_0^\infty \int_\Omega g(s)|\nabla \eta(s)|^2 dx ds \end{aligned} \tag{3.16}$$

for every $\delta_3, \delta_4 > 0$.

Proof. Multiplying $(2.3)_1$ by $-\int_0^\infty g(s)\eta(s)ds$ and integrating by parts over Ω to get

$$\begin{aligned}&\frac{d}{dt}\left(-\int_0^\infty \int_\Omega g(s)\eta(s) y_t dx ds\right) \\ &= l\int_0^\infty \int_\Omega g(s)\nabla y \nabla \eta(s) dx ds - \int_0^\infty \int_\Omega g(s) y_t \eta_t(s) dx ds \\ &\quad + \int_\Omega \left(\int_0^\infty g(s)\nabla \eta(s) ds\right)^2 dx + c\int_0^\infty \int_\Omega g(s)\nabla \eta(s)(c\nabla y - \nabla z) dx ds. \end{aligned} \tag{3.17}$$

Now, multiplying $(2.3)_3$ by $g(s)y_t$, and integrating by parts over $(0,\infty)\times\Omega$, we infer that

$$\begin{aligned}\int_0^\infty \int_\Omega g(s) y_t \eta_t(s) dx ds &= \int_0^\infty \int_\Omega g(s)|y_t|^2 dx ds - \int_0^\infty \int_\Omega g(s) y_t \eta_s(s) dx ds \\ &= l_0 \int_\Omega |y_t|^2 dx ds - \int_0^\infty \int_\Omega g(s) y_t \eta_s(s) dx ds. \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), it holds that

$$\begin{aligned}\psi_2'(t) &= -l_0 \int_\Omega |y_t|^2 dx + \int_0^\infty \int_\Omega g(s) y_t \eta_s(s) dx ds + l\int_0^\infty \int_\Omega g(s)\nabla y \nabla \eta(s) dx ds \\ &\quad + c\int_0^\infty \int_\Omega g(s)\nabla \eta(s)(c\nabla y - \nabla z) dx ds + \int_\Omega \left(\int_0^\infty g(s)\nabla \eta(s) ds\right)^2 dx. \end{aligned} \tag{3.19}$$

Likewise (3.4), we easily see that for every $\delta_3 > 0$,

$$\int_0^\infty \int_\Omega g(s)\nabla\eta(s)\nabla y dxds \leq \frac{\delta_3}{2}\int_\Omega |\nabla y|^2 dx + \frac{l_0}{2\delta_3}\int_0^\infty \int_\Omega g(s)|\nabla\eta(s)|^2 dxds. \tag{3.20}$$

Now, we note that

$$\int_0^\infty \int_\Omega g(s)y_t\eta_s(s)dxds = -\int_0^\infty \int_\Omega g'(s)y_t\eta(s)dxds.$$

Using (3.32), Young's inequality and Poincaré's inequality, one derives that

$$-\int_0^\infty \int_\Omega g'(s)y_t\eta(s)dxds \leq \frac{l_0}{2}\int_\Omega |y_t|^2 dx + \frac{b_0^2 C_p^2}{2}\int_0^\infty \int_\Omega g(s)|\nabla\eta(s)|^2 dxds. \tag{3.21}$$

Thanks to Young's inequality and Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega\left(\int_0^\infty g(s)\nabla\eta(s)ds\right)^2 dx &\leq \int_0^\infty g(s)ds\int_0^\infty \int_\Omega g(s)|\nabla\eta(s)|^2 dxds \\ &= l_0\int_0^\infty \int_\Omega g(s)|\nabla\eta(s)|^2 dxds \end{aligned} \tag{3.22}$$

and

$$\begin{aligned}&c\int_0^\infty \int_\Omega g(s)\nabla\eta(s)(c\nabla y - \nabla z)dxds \\ &\leq \frac{c\delta_4}{2}\int_\Omega |c\nabla y - \nabla z|^2 dx + \frac{c}{2\delta_4}\int_\Omega\left(\int_0^\infty g(s)\nabla\eta(s)ds\right)^2 dx \\ &\leq \frac{c\delta_4}{2}\int_\Omega |c\nabla y - \nabla z|^2 dx + \frac{cl_0}{2\delta_4}\int_0^\infty \int_\Omega g(s)|\nabla\eta(s)|^2 dxds \end{aligned} \tag{3.23}$$

for all $\delta_4 > 0$.
Reporting (3.20)–(3.23) in (3.19), we find the desired inequality. □

Now, we define the energy of solutions of (2.3) by

$$E(t) = \frac{1}{2}\int_\Omega\left(|y_t|^2 + l|\nabla y|^2 + |z_t|^2 + |c\nabla y - \nabla z|^2\right)dx + \int_0^\infty \int_\Omega g(s)|\nabla\eta(s)|^2 dxds, \tag{3.24}$$

which satisfies the following dissipation law

$$E'(t) = \frac{1}{2}\int_0^\infty \int_\Omega g'(s)|\nabla\eta(s)|^2 dxds \leq 0, \tag{3.25}$$

which means that our system (2.3) is dissipative and so $E(t) \leq E(0)$.

Next, we define the functional $\mathcal{L}$ by

$$\mathcal{L}(t) = N_1E(t) + N_2\varphi_1(t) + N_3\varphi_2(t) + N_4\psi_1 + N_5\psi_2,$$

where N_1, N_2, N_3, N_4 and N_5 are positive constants that will be chosen later.

It is easy to check, for N_1 sufficiently large, that $E(t) \sim \mathcal{L}(t)$ i.e.,

$$\alpha_1E(t) \leq \mathcal{L}(t) \leq \alpha_2E(t),\ \forall\ t \geq 0, \tag{3.26}$$

for some constants $\alpha_1, \alpha_2 > 0$.

Lemma 3.5. *Assume (***H1***)-(***H2***). We have, for all $t \geq 0$,*

$$\mathcal{L}^{'}(t) \leq -\beta E(t), \tag{3.27}$$

for some positive constant β.

Proof. From (3.1), (3.6), (3.12), (3.16), (3.25) and (3.32), we derive that

$$\begin{aligned}
\mathcal{L}'(t) \leq & - \left\{ \frac{N_1 b_1}{2} - \frac{N_2 l_0}{2l} - \frac{N_3 l_0}{2\delta_2} - \frac{N_4 l_0}{2l} - N_5 \left(l_0 + \frac{b_0^2 C_p^2}{2} + \frac{l l_0}{2\delta_3} + \frac{c l_0}{2\delta_4} \right) \right\} \int_0^\infty \int_\Omega g(s) |\nabla \eta(s)|^2 dx ds \\
& - \left\{ \frac{N_5 l_0}{2} - N_2 \left(1 + \frac{c\delta_1}{2} \right) - N_3 \left(c + c^3 + \frac{1}{c} \right) - N_4 \right\} \int_\Omega |y_t|^2 dx \\
& - \left\{ \frac{N_3 c}{2} - \frac{N_2 c}{2\delta_1} - N_4 \right\} \int_\Omega |z_t|^2 dx \\
& - \left\{ \frac{N_2 l}{2} + \frac{N_4 l}{2} - \frac{N_5 \delta_3 l}{2} - \frac{N_3 l^2}{2\delta_2} \right\} \int_\Omega |\nabla y|^2 dx \\
& - \left\{ N_4 - N_3 \delta_2 - \frac{c N_5 \delta_4}{2} \right\} \int_\Omega |c\nabla y - \nabla z|^2 dx.
\end{aligned} \tag{3.28}$$

By choosing $\delta_1 = \frac{2N_2}{N_3}$, $\delta_2 = \frac{N_4}{4N_3}$, $\delta_3 = \frac{N_2}{2N_5}$ and $\delta_4 = \frac{N_4}{2cN_5}$, (3.28) becomes

$$\begin{aligned}
\mathcal{L}'(t) \leq & - \left\{ \frac{N_1 b_1}{2} - \frac{N_2 l_0}{2l} - \frac{2N_3^2 l_0}{N_4} - \frac{N_4 l_0}{2l} - N_5 \left(l_0 + \frac{b_0^2 C_p^2}{2} + \frac{l l_0 N_5}{N_2} + \frac{c^2 N_5 l_0}{N_4} \right) \right\} \int_0^\infty \int_\Omega g(s) |\nabla \eta(s)|^2 dx ds \\
& - \left\{ \frac{N_5 l_0}{2} - N_2 \left(1 + \frac{cN_2}{N_3} \right) - N_3 \left(c + c^3 + \frac{1}{c} \right) - N_4 \right\} \int_\Omega |y_t|^2 dx \\
& - \left\{ \frac{N_3 c}{4} - N_4 \right\} \int_\Omega |z_t|^2 dx \\
& - \left\{ \frac{N_2 l}{4} + \frac{N_4 l}{2} - \frac{2N_3^2 l^2}{N_4} \right\} \int_\Omega |\nabla y|^2 dx \\
& - \frac{N_4}{2} \int_\Omega |c\nabla y - \nabla z|^2 dx.
\end{aligned} \tag{3.29}$$

At this point, we pick up N_3, N_2 and N_5, respectively, such that

$$N_3 > \frac{4N_4}{c},$$

$$N_2 > \frac{8N_3^2 l}{N_4},$$

$$N_5 > \frac{2}{l_0} \left\{ N_2 \left(1 + \frac{cN_2}{N_3} \right) + N_3 \left(c + c^3 + \frac{1}{c} \right) + N_4 \right\}.$$

After this, choosing N_1 sufficiently large so that (3.26) holds true and

$$N_1 > \frac{2}{b_1} \left\{ \frac{N_2 l_0}{2l} - \frac{2N_3^2 l_0}{N_4} - \frac{N_4 l_0}{2l} - N_5 \left(l_0 + \frac{b_0^2 C_p^2}{2} + \frac{l l_0 N_5}{N_2} + \frac{c^2 N_5 l_0}{N_4} \right) \right\}.$$

By taking

$$\beta = \min\{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5\},$$

we get the desired inequality (3.27) with

$$\beta_1 = \frac{N_1 b_1}{2} - \frac{N_2 l_0}{2l} - \frac{2N_3^2 l_0}{N_4} - \frac{N_4 l_0}{2l} - N_5\left(l_0 + \frac{b_0^2 C_p^2}{2} + \frac{l l_0 N_5}{N_2} + \frac{c^2 N_5 l_0}{N_4}\right),$$

$$\beta_2 = 2\left\{\frac{N_5 l_0}{2} - N_2\left(1 + \frac{cN_2}{N_3}\right) - N_3\left(c + c^3 + \frac{1}{c}\right) - N_4\right\},$$

$$\beta_3 = 2\left\{\frac{N_3 c}{4} - N_4\right\},$$

$$\beta_4 = 2\left\{\frac{N_2 l}{4} + \frac{N_4 l}{2} - \frac{2N_3^2 l^2}{N_4}\right\}$$

and

$$\beta_5 = N_4.$$

□

The main result of this paper reads as follows.

Theorem 3.6. *Assume (***H1***)-(***H2***). Then, the energy of solutions of* (2.3) *decays exponentially, i.e., there exist positive constants d and* γ*, such that*

$$E(t) \le dE(0)e^{-\gamma t}, \quad t \ge 0. \tag{3.30}$$

Proof. By using (3.26) and (3.27), one finds that

$$\mathcal{L}'(t) \le -\beta E(t) \le -\frac{\beta}{\alpha_2}\mathcal{L}(t).$$

Consequently,

$$\mathcal{L}(t) \le \mathcal{L}(0)e^{-\frac{\beta t}{\alpha_2}}.$$

Using again (3.26), we obtain

$$E(t) \le \frac{1}{\alpha_1}\mathcal{L}(t) \le \frac{1}{\alpha_1}\mathcal{L}(0)e^{-\frac{\beta t}{\alpha_2}} \le \frac{\alpha_2}{\alpha_1}E(0)e^{-\frac{\beta t}{\alpha_2}}.$$

Hence, (3.30) holds true with $d = \frac{\alpha_2}{\alpha_1}$ and $\gamma = \frac{\beta}{\alpha_2}$. □

Remark 3.7. *By replacing in* (1.1) *the past history by a finite memory term of the form* $\int_0^t g(t-s)\Delta y(s)ds$*,* (1.1) *becomes*

$$\begin{cases} y_{tt} - a\Delta y + c\Delta z + \int_0^t g(t-s)\Delta y(s)ds = 0, & in\ \Omega \times (0,\infty), \\ z_{tt} - \Delta z + c\Delta y = 0, & in\ \Omega \times (0,\infty), \\ y = z = 0, & on\ \Gamma \times (0,\infty), \\ y(x,0) = y_0(x),\ z(x,0) = z_0(x),\ y_t(x,0) = y_1(x),\ z_t(x,0) = z_1(x), & in\ \Omega. \end{cases} \tag{3.31}$$

The energy of solutions of (3.31) *are defined by*

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\int_\Omega |y_t|^2 dx + \frac{1}{2}\left(b - \int_0^t g(s)ds\right)\int_\Omega |\nabla y|^2 dx + \frac{1}{2}(g \circ \nabla y)(t) \\ &+ \frac{1}{2}\int_\Omega |z_t|^2 dx + \frac{1}{2}\int_\Omega |c\nabla y - \nabla z|^2 dx,\end{aligned}$$

where

$$(g \circ y)(t) = \int_0^t g(t-s)\|y(t) - y(s)\|^2 ds.$$

Define

$$\mathcal{G}(t) = M\mathcal{E}(t) + M_1\psi_1(t) + M_2\varphi_1(t) + M_3 D(t),$$

where

$$D(t) = -\int_\Omega y_t \int_0^t g(t-s)(y(t) - y(s))dsdx.$$

Now, if we suppose, for example, that g satisfies **(H1)** *and*

(H3): *There exists a non-increasing continuous function* $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying*

$$g'(t) \leq -\xi(t)g(t), \quad \forall\, t \geq 0. \tag{3.32}$$

By proceeding as in the last section, we can prove for suitable choices of M, M_1, M_2 *and* M_3 *that*

$$\mathcal{G}'(t) \leq -C_2\mathcal{E}(t) + C_3(g \circ \nabla y)(t), \ \forall\, t \geq 0$$

for some positive constants C_2 *and* C_3. *Therefore, we get the following result:*

Theorem 3.8. *Let* $(y_0, y_1), (z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *Assume that (***H1***) and (***H3***) hold true. Then, for any* $t_1 > 0$, *there exist positive constants* β_1 *and* β_2, *such that the energy* $\mathcal{E}(t)$ *satisfies*

$$\mathcal{E}(t) \leq \beta_2 e^{-\beta_1 \int_{t_1}^t \xi(s)ds}.$$

4. Conclusions

We focus on the existence and exponential stability of solutions for a coupled, by second order terms, system of two wave equations with a past history acting only on the first equation. Each one of these two equations describes the motion of two elastic membranes. The exponential decay result still valid if we replace the past history by a memory term. As future work, we will study the exponential stability in the case where we replace one (that contains the damping term) of these two equations by a quasi-linear one.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Z. Hajjej is supported by Researchers Supporting Project number (RSPD2023R736), King Saud University, Riyadh, Saudi Arabia. M. L. Liao is supported by NSF of Jiangsu Province (BK20230946) and the Fundamental Research Funds for Central Universities (B230201033, 423139).

Conflict of interest

The authors declare that there is no conflict of interest.

References

1. M. Akil, Z. Hajjej, Exponential stability and exact controllability of a system of coupled wave equations by second order terms (via Laplacian) with only one non-smooth local damping, *Math. Method. Appl. Sci.*, 2023, in press.
2. F. Alabau, P. Cannarsa, V. Komornik, Indirect internal stabilization of weakly coupled systems, *J. Evol. Equ.*, **2** (2002), 127–150. https://doi.org/10.1007/s00028-002-8083-0
3. A. M. Al-Mahdi, M. M. Al-Gharabli, New general decay results in an infinite memory viscoelastic problem with nonlinear damping, *Bound. Value Probl.*, **2019** (2019), 140. https://doi.org/10.1186/s13661-019-1253-6
4. A. M. Al-Mahdi, M. M. Al-Gharabli, Energy decay in a viscoelastic equation with past history and boundary feedback, *Appl. Anal.*, **101** (2022), 4743–4758. https://doi.org/10.1080/00036811.2020.1869943
5. R. G. C. Almeida, M. L. Santos, Lack of exponential decay of a coupled system of wave equations with memory, *Nonlinear Anal.-Real*, **12** (2011), 1023–1032. https://doi.org/10.1016/j.nonrwa.2010.08.025
6. J. A. D. Appleby, M. Fabrizio, B. Lazzari, D. W. Reynolds, On exponential asymptotic stability in linear viscoelasticity, *Math. Mod. Meth. Appl. S.*, **16** (2006), 1677–1694. https://doi.org/10.1142/S0218202506001674
7. V. V. Chepyzhov, V. Pata, Some remarks on stability of semigroups arising from linear viscoelasticity, *Asymptotic Anal.*, **46** (2006), 251–273.
8. M. Conti, V. Pata, Weakly dissipative semilinear equations of viscoelasticity, *Commun. Pur. Appl. Anal.*, **4** (2005), 705–720. https://doi.org/10.3934/cpaa.2005.4.705
9. C. M. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Ration. Mech. An.*, **37** (1970), 297–308. https://doi.org/10.1007/BF00251609
10. M. J. D. Santos, J. C. P. Fortes, M. L. Cardoso, Exponential stability for a piezoelectric beam with a magnetic effect and past history, *Discrete Cont. Dyn.-B*, **27** (2022), 5487–5501. https://doi.org/10.3934/dcdsb.2021283
11. M. Fabrizio, B. Lazzari, On the existence and asymptotic stability of solutions for linear viscoelastic solids, *Arch. Ration. Mech. An.*, **116** (1991), 139–152. https://doi.org/10.1007/BF00375589

12. L. H. Fatori, J. E. M. Rivera, Energy decay for hyperbolic thermoelastic systems of memory type, *Q. Appl. Math.*, **59** (2001), 441–458.
13. C. Giorgi, J. E. M. Rivera, V. Pata, Global attractors for a semilinear hyperbolic equation in viscoelasticity, *J. Math. Anal. Appl.*, **260** (2001), 83–99. https://doi.org/10.1006/jmaa.2001.7437
14. A. Guesmia, Asymptotic stability of abstract dissipative systems with infinite memory, *J. Math. Anal. Appl.*, **382** (2001), 748–760. https://doi.org/10.1016/j.jmaa.2011.04.079
15. A. Guesmia, Asymptotic behavior for coupled abstract evolution equations with one infinite memory, *Appl. Anal.*, **94** (2015), 184–217. https://doi.org/10.1080/00036811.2014.890708
16. A. Guesmia, S. A. Messaoudi, A general decay result for a viscoelastic equation in the presence of past and finite history memories, *Nonlinear Anal.-Real*, **13** (2012), 476–485. https://doi.org/10.1016/j.nonrwa.2011.08.004
17. A. Guesmia, S. A. Messaoudi, A new approach to the stability of an abstract system in the presence of infinite history, *J. Math. Anal. Appl.*, **416** (2014), 212–228. https://doi.org/10.1016/j.jmaa.2014.02.030
18. K. P. Jin, J. Liang, T. J. Xiao, Coupled second order evolution equations with fading memory: Optimal energy decay rate, *J. Differ. Equations*, **257** (2014), 1501–1528. https://doi.org/10.1016/j.jde.2014.05.018
19. K. P. Jin, J. Liang, T. J. Xiao, Asymptotic behavior for coupled systems of second order abstract evolution equations with one infinite memory, *J. Math. Anal. Appl.*, **475** (2019), 554–575. https://doi.org/10.1016/j.jmaa.2019.02.055
20. Z. Liu, S. Zheng, On the exponential stability of linear viscoelasticity and thermoviscoelasticity, *Q. Appl. Math.*, **54** (1996), 21–31.
21. V. Pata, Stability and exponential stability in linear viscoelasticity, *Milan J. Math.*, **77** (2009), 333–360. https://doi.org/10.1007/s00032-009-0098-3
22. J. E. M. Rivera, M. G. Naso, Asymptotic stability of semigroups associated with linear weak dissipative systems with memory, *J. Math. Anal. Appl.*, **326** (2007), 691–707. https://doi.org/10.1016/j.jmaa.2006.03.022
23. J. E. M. Rivera, M. G. Naso, Optimal energy decay rate for a class of weakly dissipative second order systems with memory, *Appl. Math. Lett.*, **23** (2010), 743–746. https://doi.org/10.1016/j.aml.2010.02.016
24. D. L. Russell, A general framework for the study of indirect damping mechanisms in elastic systems, *J. Math. Anal. Appl.*, **173** (1993), 339–358. https://doi.org/10.1006/jmaa.1993.1071
25. T. J. Xiao, J. Liang, Coupled second order semilinear evolution equations indirectly damped via memory effects, *J. Differ. Equations*, **254** (2013), 2128–2157. https://doi.org/10.1016/j.jde.2012.11.019

AIMS Press