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Research article

n-quasi-A-(m, q)-isometry on a Banach space

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Abstract: In this paper, we introduce the class of n-quasi-A-(m, q)-isometry operators on a Banach space X, which represents a generalization of the n-quasi-(m, q)-isometry on a Banach space and the n-quasi-(A, m)-isometry on a Hilbert space. After giving some basic properties of this class of operators, we study the product and the power of such operators in this class.

Keywords: m-isometry; (m, q)-isometry; n-quasi-A-(m, q)-isometry
Mathematics Subject Classification: 47B99, 47A05

1. Introduction

The class of m-isometry operators was introduced in 1990 by Agler in [1] and was developed in 1995 by Agler and Stankas in [2–4]. A bounded linear operator $T \in \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is called an m-isometry, for a positif integer m (that is $m \geq 1$), if

$$\beta_m(T) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k} T^k = 0,$$

where T^* denotes the adjoint operator of T. This latter is equivalent to

$$\Delta_m(T, x) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^k x\|^2 = 0, \ \forall \ x \in \mathcal{H}.$$

Some generalizations of this class of operators exist in the literature, like the (A, m)-isometry, which was introduced in 2012 by Saddi and Sid Ahmed [9], for a positif operator A, by

$$\beta_m(T, A) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k} A T^k = 0$$

or equivalently

$$\Delta_m(T, A, x) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} \|A^{\frac{1}{2}} T^k x\|^2 = 0, \ \forall\ x \in \mathcal{H}.$$

We also mention the class of n-quasi-m-isometry on a Hilbert space defined by

$$\beta_{m,n}(T) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k+n} T^{k+n} = 0$$

or in an equivalent manner

$$\Delta_{m,n}(T, x) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^{n+k} x\|^2 = 0, \ \forall\ x \in \mathcal{H}.$$

For more details about these class, please see references [10, 11].

The n-quasi-(A, m)-isometries (which are particular cases of n-quasi-(m, q)-isometries) were thoroughly studied by Agler and Stankus in a series of three papers in which the authors employed the theory of periodic distributions to derive a function theory model for m-isometrics, a disconjugacy theory for a subclass of Toeplitz operators. In addition, they introduced a class of 2-isometrics operators arising from a class of non stationary stochastic processes related to Brownian motion.

Recently, Sid Ahmed et al. [7] combined these two classes and introduced the n-quasi-(A, m)-isometry on a Hilbert space. Noting that all these works are in a Hilbert space $\mathcal{H}$, a generalization of those on a Banach space X was developed. For example, we can mention the work of Bayart [5], who introduced the (m, q)-isometry, for an integer $q \geq 1$, by

$$\Delta_m^q(T) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^k x\|^q = 0, \ \forall\ x \in X.$$

For $q = 2$, the $(m, 2)$-isometry coincide with the m-isometry defined on a Hilbert space.

We recall that the (A, m)-isometry was first introduced by Duggal in [6], for any operator $A \in \mathcal{L}(X)$, by

$$\Delta_m^q(T, A, x) := \sum_{k=0}^{m} (-1)^k \binom{m}{k} \|A T^k x\|^q = 0, \ \forall\ x \in X.$$

In this paper, we will generalize this last class by introducing the n-quasi-A-(m, q)-isometry on a Banach space and present some properties of this class like the product and the powers.

The main motivation for developing this work lies in the fact of knowing whether the different properties (spectral, product and power) of the n- quasi-(m, q)-isometry on a Banach space and the n-quasi-(A, m)-isometry on a Hilbert space are valid for operators in the new generalized class introduced in Definition 2.1. Note that in our study, we remove the condition of positivity on the operator A, which exists in the Hilbert case.

The paper is organized as follows. In Section 2, we define our class of n-quasi-A-(m, q)-isometry operators and present its basic properties. The power and the product of such operator belonging to this class are discussed in the last section.

2. Some basics properties

In this section, we define our new class and give its basic properties.

Given a Banach space X, we denote by $\mathcal{L}(X)$ the class of all the (linear bounded) operators on X. Hereafter, $I = I_X$, $\mathcal{R}(T)$ and $\sigma_{ap}(T)$ denote the identity operator, the range and the approximate spectrum of an operator $T \in \mathcal{L}(X)$, respectively. We define the class of n-quasi-A-(m, q)-isometry operators by:

Definition 2.1. Let $A, T \in \mathcal{L}(X)$. T is called n-quasi-A-(m, q)-isometry if and only if

$$Q^q_{m,n}(T, A, x) := \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|AT^{n+k}x\|^q = 0, \ \forall\ x \in X.$$

Remark 2.1. Let $A, T \in \mathcal{L}(X)$. Then T is a n-quasi-A-(m, q)-isometry if and only if T is a A-(m, q)-isometry on $\overline{\mathcal{R}(T^n)}$.

Indeed, T is a n-quasi-A-(m, q)-isometry if and only if

$$\begin{aligned} 0 &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|AT^{n+k}x\|^q, \quad \forall\ x \in X \\ &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|AT^kT^nx\|^q, \quad \forall\ x \in X \\ &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|AT^ky\|^q, \quad \forall\ y \in \overline{\mathcal{R}(T^n)}. \end{aligned}$$

In the following proposition, we give some spectral properties of the n-quasi-A-(m, q)-isometry operators.

Proposition 2.1. *Let $A, T \in \mathcal{L}(X)$ such that T is a n-quasi-A-(m, q)-isometry. If $0 \notin \sigma_{ap}(A)$, then $\sigma_{ap}(T) = \zeta(0, 1) \cup \{0\}$, where*

$$\zeta(0, 1) = \{x \in X,\ \|x\| = 1\}.$$

Proof. Let $(x_p)_p$ such that $\|x_p\| = 1$ and $\lim\limits_{p\to\infty}(T - \lambda I)x_p = 0$.

Since T is a n-quasi-A-(m, q)-isometry, then

$$0 = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|AT^{n+k}x_p\|^q$$

$$= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|A(T^{n+k}-\lambda^{n+k})x_p + A\lambda^{n+k}x_p\|^q.$$

As $\lim_{p\to\infty}(T-\lambda I)x_p = 0$, then $\lim_{p\to\infty}(T^{n+k}-\lambda^{n+k}I)x_p = 0$, for all $k = 0, 1, \cdots, m$.

Therefore

$$\begin{aligned} 0 &= \lim_{p\to\infty}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\|A\lambda^{n+k}x_p\|^q \\ &= |\lambda|^{nq}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(|\lambda|^q)^k\lim_{p\to\infty}\|Ax_p\|^q \\ &= |\lambda|^{nq}(|\lambda|^q-1)^m\lim_{p\to\infty}\|Ax_p\|^q. \end{aligned}$$

Since $0 \notin \sigma_{ap}(A)$, then $\lambda = 0$ or $|\lambda| = 1$. □

Proposition 2.2. *Let $A, T \in \mathcal{L}(X)$. If T is an n-quasi-A-(m,q)-isometry, then T is a n_1-quasi-A-(m,q)-isometry, for all $n_1 \geq n$.*

Proof. Let T be a n-quasi-A-(m,q)-isometry on X. By Remark 2.1, T is a A-(m,q)-isometry on $\overline{\mathcal{R}(T^n)}$. Since $\overline{\mathcal{R}(T^n)} \supset \overline{\mathcal{R}(T^{n_1})}$ for all $n_1 \geq n$, therefore T is a A-(m,q)-isometry on $\overline{\mathcal{R}(T^{n_1})}$. According to Remark 2.1, we obtain that T is a n_1-quasi-A-(m,q)-isometry, for all $n_1 \geq n$. □

In the following proposition, thanks to a suitable condition, we give the inverse sense of the Proposition 2.2.

Proposition 2.3. *Let $1 \leq p \leq n-1$ such that $\overline{\mathcal{R}(T^p)} = \overline{\mathcal{R}(T^{p+1})}$. If T is a n-quasi-A-(m,q)-isometry, then T is a p-quasi-A-(m,q)-isometry.*

Proof. Thanks to the hypothesis $\overline{\mathcal{R}(T^p)} = \overline{\mathcal{R}(T^{p+1})}$, it follows that $\overline{\mathcal{R}(T^p)} = \overline{\mathcal{R}(T^n)}$. Since T is a n-quasi-A-(m,q)-isometry on X, then T is a A-(m,q)-isometry on $\overline{\mathcal{R}(T^n)} = \overline{\mathcal{R}(T^p)}$. Therefore, T is a p-quasi-A-(m,q)-isometry. □

As in the Hilbert case, we have the following result.

Proposition 2.4. *Let $T \in \mathcal{L}(X)$ be a n-quasi-A-(m,q)-isometry. Then T is a n-quasi-A-(ℓ,q)-isometry for all $\ell \geq m$.*

Proof.

$$\begin{aligned} Q^q_{m+1,n}(T,A,x) &= \sum_{k=0}^{m+1}(-1)^{m+1-k}\binom{m+1}{k}\|AT^{n+k}x\|^q \\ &= (-1)^{m+1}\|AT^nx\|^q + \|AT^{n+m+1}x\|^q \\ &\quad + \sum_{k=1}^{m}(-1)^{m+1-k}\left[\binom{m}{k}+\binom{m}{k-1}\right]\|AT^{n+k}x\|^q \\ &= -(-1)^m\|AT^nx\|^q - \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\|AT^{n+k}x\|^q \end{aligned}$$

$$
\begin{aligned}
& + \sum_{k=1}^{m} (-1)^{m+1-k} \binom{m}{k-1} \|AT^{n+k}x\|^q + \|AT^{n+m+1}x\|^q \\
= \;& -Q^q_{m,n}(T,A,x) + \sum_{k=0}^{m-1} (-1)^{m-k} \binom{m}{k} \|AT^{n+1+k}x\|^q \\
& + \|AT^{n+1+m}x\|^q \\
= \;& Q^q_{m,n+1}(T,A,x) - Q^q_{m,n}(T,A,x) \\
= \;& 0.
\end{aligned}
$$

□

Example 2.1. Let $T, A \in \mathcal{L}(X)$, where $X = \ell^q(\mathbb{N})$, defined by

$$T\alpha_n = w_n\alpha_n \text{ and } A\alpha_n = \alpha_{n+1},$$

where $w_n = \left(\frac{n+1}{n}\right)^{\frac{1}{q}}$. By simple calculations, we get that T is a 2-quasi-A-$(2, q)$-isometry but it is not a 2-quasi-A-$(1, q)$-isometry. Indeed, we have

$$
\begin{aligned}
Q^q_{2,2}(T,A,\alpha_n) &= \|AT^4\alpha_n\|^q - 2\|AT^3\alpha_n\|^q + \|AT^2\alpha_n\|^q \\
&= \sum_{n\geq 1} \left(|w_n w_{n+1} w_{n+2} w_{n+4}|^q - 2|w_n w_{n+1} w_{n+2}|^q + |w_n w_{n+1}|^q\right) |\alpha_{n+1}|^q \\
&= \sum_{n\geq 1} \left(\frac{n+4}{n} - 2\frac{n+3}{n} + \frac{n+2}{n}\right) |\alpha_{n+1}|^q \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
Q^q_{1,2}(T,A,\alpha_n) &= \|AT^3\alpha_n\|^q - \|AT^2\alpha_n\|^q \\
&= \sum_{n\geq 1} \left(|w_n w_{n+1} w_{n+2}|^q - |w_n w_{n+1}|^q\right) |\alpha_{n+1}|^q \\
&= \sum_{n\geq 1} \left(\frac{n+3}{n} - \frac{n+2}{n}\right) |\alpha_{n+1}|^q \\
&= \sum_{n\geq 1} \frac{|\alpha_{n+1}|^q}{n} \\
&\neq 0.
\end{aligned}
$$

Proposition 2.5. *Let $A, T \in \mathcal{L}(X)$ such that T is a n-quasi-A-(m, q)-isometry. Then, for all $p \geq 0$, we have*

(1) $\|AT^{n+p}x\|^q = \sum_{k=0}^{m-1} \binom{p}{k} Q^q_{k,n}(T,A,x).$

(2) $Q^q_{m-1,n}(T,A,x) = \lim_{p\to\infty} \frac{\|AT^{n+p}x\|^q}{\binom{p}{m-1}} \geq 0.$

Proof. (1) By induction, we prove that, for all $p \geq 0$,

$$\|AT^{n+p}x\|^q = \sum_{k=0}^{p} \binom{p}{k} Q^q_{k,n}(T, A, x).$$

For $p = 0$, we infer that

$$\begin{aligned} \sum_{k=0}^{0} \binom{0}{k} Q^q_{k,n}(T, A, x) &= Q^q_{0,n}(T, A, x) \\ &= \|AT^n x\|^q. \end{aligned}$$

We suppose that $\|AT^{n+j}x\|^q = \sum\limits_{k=0}^{j} \binom{j}{k} Q^q_{k,n}(T, A, x)$ for all $j \leq p$.

We know that

$$\begin{aligned} &\|AT^{n+p+1}x\|^q \\ = \;& Q^q_{p+1,n}(T, A, x) - \sum_{k=0}^{p} (-1)^{p+1-k} \binom{p+1}{k} \|AT^{n+k}x\|^q \\ = \;& Q^q_{p+1,n}(T, A, x) - \sum_{k=0}^{p} (-1)^{p+1-k} \binom{p+1}{k} \sum_{j=0}^{k} \binom{k}{j} Q^q_{j,n}(T, A, x) \\ = \;& Q^q_{p+1,n}(T, A, x) - \sum_{j=0}^{p} Q^q_{j,n}(T, A, x) \sum_{k=j}^{p} (-1)^{p+1-k} \binom{p+1}{k} \binom{k}{j} \\ = \;& Q^q_{p+1,n}(T, A, x) - \sum_{j=0}^{p} \binom{p+1}{j} Q^q_{j,n}(T, A, x) \underbrace{\sum_{k=j}^{p} (-1)^{p+1-k} \binom{p+1-j}{k-j}}_{=-1} \\ = \;& Q^q_{p+1,n}(T, A, x) + \sum_{j=0}^{p} \binom{p+1}{j} Q^q_{j,n}(T, A, x) \\ = \;& \sum_{j=0}^{p+1} \binom{p+1}{j} Q^q_{j,n}(T, A, x). \end{aligned}$$

Then, for all $p \geq 0$, we have that $\|AT^{n+p}x\|^q = \sum\limits_{k=0}^{p} \binom{p}{k} Q^q_{k,n}(T, A, x)$.

Since T is a n-quasi-A-(m, q)-isometry on X, then, by Proposition 2.4, we obtain that

$$Q^q_{k,n}(T, A, x) = 0, \quad \text{for all } k \geq m.$$

Hence, for all $p \geq 0$, we get that

$$\|AT^{n+p}x\|^q = \sum_{k=0}^{m-1} \binom{p}{k} Q^q_{k,n}(T, A, x).$$

(2) We know, by assertion (1), that

$$\begin{aligned}\|AT^{n+p}x\|^q &= \sum_{k=0}^{m-1}\binom{p}{k}Q^q_{k,n}(T,A,x)\\ &= \binom{p}{m-1}Q^q_{m-1,n}(T,A,x)+\sum_{k=0}^{m-2}\binom{p}{k}Q^q_{k,n}(T,A,x).\end{aligned}$$

Dividing both sides by $\binom{p}{m-1}\neq 0$, we see that

$$Q^q_{m-1,n}(T,A,x)=\frac{1}{\binom{p}{m-1}}\|AT^{n+p}x\|^q-\frac{1}{\binom{p}{m-1}}\sum_{k=0}^{m-2}\binom{p}{k}Q^q_{k,n}(T,A,x).$$

Upon taking the limit as $p\to\infty$, we know that $\lim_{p\to\infty}\frac{\binom{p}{k}}{\binom{p}{m-1}}=0$, for all $k=0,1,...,m-2$. Therefore, since $\|AT^{n+p}x\|^q\geq 0$, it holds that

$$Q^q_{m-1,n}(T,A,x)=\lim_{p\to\infty}\frac{\|AT^{n+p}x\|^q}{\binom{p}{m-1}}\geq 0.$$

□

3. Product and powers of n-quasi-A-(m,q)-isometry operators

In this section, we study the product and power of an n-quasi-A-(m,q)-isometry operators.

Let $n^{(k)}$ be the (descending Pochhammer) symbol defined by:

$$n^{(k)}=\begin{cases}0 & if\ n=0,\\ 0 & if\ n>0\ and\ k>n,\\ k!\binom{n}{k} & if\ n>0\ and\ k\leq n.\end{cases}$$

Proposition 3.1. *T is a n-quasi-A-(m,q)-isometry if and only if we have*

$$\|AT^{n+p}x\|=\sum_{j=0}^{m-1}(-1)^{m-j-1}\frac{p(p-1)\cdots\widehat{(p-j)}\cdots(p-m+1)}{j!(m-j-1)!}\|AT^{j+n}x\|^q,$$

for all $p\geq 0$ and all $x\in X$, where $\widehat{(p-j)}$ denotes that the factor $(p-j)$ is omitted.

Proof. T is a n-quasi-A-(m,q)-isometry if and only if

$$\begin{aligned}\|AT^{n+p}x\|^q &= \sum_{k=0}^{m-1}\binom{p}{k}Q^q_{k,n}(T,A,x)\\ &= \sum_{k=0}^{m-1}\binom{p}{k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\|AT^{n+j}x\|^q\end{aligned}$$

$$= \sum_{j=0}^{m-1} \|AT^{n+j}x\|^q \sum_{k=j}^{m-1} (-1)^{k-j} \binom{p}{k}\binom{k}{j}.$$

By [12, Lemma 2.3], we have

$$\sum_{k=j}^{m-1}(-1)^{k-j}\binom{p}{k}\binom{k}{j} = (-1)^{m-j-1}\frac{p(p-1)\cdots\widehat{(p-j)}\cdots(p-m+1)}{j!(m-j-1)!}.$$

Then,

$$\|AT^{n+p}x\| = \sum_{j=0}^{m-1}(-1)^{m-j-1}\frac{p(p-1)\cdots\widehat{(p-j)}\cdots(p-m+1)}{j!(m-j-1)!}\|AT^{j+n}x\|^q.$$

□

Lemma 3.1. *Let T be a n-quasi-A-(m,q)-isometry and $\ell > m \geq 1$. For all $t \in \{0,\cdots,\ell-2\}$, we have*

$$\sum_{j=0}^{m+\ell-1}(-1)^{m+\ell-1-j}\binom{m+\ell-1}{j}\prod_{i=0}^{t}(j-i)\|AT^{j+n}x\|^q = 0.$$

Proof. Let $t \in \{0,\cdots,\ell-2\}$, we have

$$\begin{aligned}\binom{m+\ell-1}{j}\prod_{i=0}^{t}(j-i) &= \frac{(m+\ell-1)!}{j!(m+\ell-1-j)!}j(j-1)\cdots(j-t)\\ &= \frac{(m+\ell-t-2)!\prod_{i=0}^{t}(m+\ell+i)}{(j-t-1)!(m+\ell-1-j)!}\\ &= \binom{m+\ell-t-2}{j-t-1}\prod_{i=1}^{t+1}(m+\ell-i).\end{aligned}$$

Then

$$\begin{aligned}&\sum_{j=0}^{m+\ell-1}(-1)^{m+\ell-1-j}\binom{m+\ell-1}{j}\prod_{i=0}^{t}(j-i)\|AT^{j+n}x\|^q\\ =\ &\sum_{j=0}^{m+\ell-1}(-1)^{m+\ell-1-j}\binom{m+\ell-t-2}{j-t-1}\prod_{i=1}^{t+1}(m+\ell-i)\|AT^{j+n}x\|^q\\ =\ &\prod_{i=1}^{t+1}(m+\ell-i)\left(\sum_{j=t+1}^{m+\ell-1}(-1)^{m+\ell-1-j}\binom{m+\ell-t-2}{j-t-1}\|AT^{j+n}x\|^q\right)\\ =\ &\prod_{i=1}^{t+1}(m+\ell-i)\left(\sum_{j=0}^{m+\ell-t-2}(-1)^{m+\ell-t-2-j}\binom{m+\ell-t-2}{j}\|AT^{j+n}(T^{t+1}x)\|^q\right)\\ =\ &0.\end{aligned}$$

□

Lemma 3.2. *Let T be a n-quasi-A-(m,q)-isometry, $p \geq 0$ and $\ell \geq m \geq 1$. Then, there exists a finite sequence $(a_{j,i})_{i=0}^{m-1}$ such that*

$$\|AT^{p+n}x\|^q = \sum_{k=0}^{m-1} \frac{(-1)^{m-1-k}}{k!(m-k-1)!}\left[a_{j,0} + \sum_{i=1}^{m-1} a_{j,i}\prod_{t=0}^{i-1}(p-t)\right]\|AT^{n+k}x\|^q,$$

for $j = 0, 1, ..., \ell - 1$.

Proof. By using [13], there exists a finite sequence $(a_{j,i})_{i=0}^{m-1}$ such that

$$p(p-1)\cdots\widehat{(p-j)}\cdots(p-m+1) = a_{j,0} + \sum_{i=1}^{m-1} a_{j,i}\prod_{t=0}^{i-1}(p-t).$$

Since T is a n-quasi-A-(m,q)-isometry, then by using Proposition 3.1, we obtain that

$$\begin{aligned}
&\|AT^{n+p}x\|^q \\
= &\sum_{j=0}^{m-1} \frac{(-1)^{m-j-1}}{j!(m-j-1)!}\left[p(p-1)\cdots\widehat{(p-j)}\cdots(p-m+1)\right]\|AT^{j+n}x\|^q \\
= &\sum_{j=0}^{m-1} \frac{(-1)^{m-j-1}}{j!(m-j-1)!}\left[a_{j,0} + \sum_{i=1}^{m-1} a_{j,i}\prod_{t=0}^{i-1}(p-t)\right]\|AT^{j+n}x\|^q.
\end{aligned}$$

□

Theorem 3.1. *Let n_1, n_2, m, l be positive integers and $T, S \in \mathcal{L}(X)$. If T is a n_1-quasi-A-(m,q)-isometry and S is a n_2-quasi-A-(l,q)-isometry such that $ST = TS$, then TS is a n-quasi-A-$(m+l-1,q)$-isometry, with $n = \max(n_1, n_2)$.*

Proof. We have

$$\begin{aligned}
Q^q_{m+l-1,n}(TS, A, x) &= \sum_{k=0}^{m+l-1} (-1)^{m+l-1-k}\binom{m+l-1}{k}\|A(TS)^{n+k}x\|^q \\
&= \sum_{k=0}^{m+l-1} (-1)^{m+l-1-k}\binom{m+l-1}{k}\|AT^{n+k}\left(S^{n+k}x\right)\|^q.
\end{aligned}$$

Using Lemma 3.2, we see that

$$\begin{aligned}
&Q^q_{m+l-1,n}(TS, A, x) = \sum_{k=0}^{m+l-1} (-1)^{m+l-1-k}\binom{m+l-1}{k}\sum_{j=0}^{m-1}\frac{(-1)^{m-j-1}}{j!(m-j-1)!} \\
&\times\left[a_{j,0} + \sum_{i=1}^{m-1} a_{j,i}\prod_{t=0}^{i-1}(p-t)\right]\|AT^{n+j}\left(S^{n+k}x\right)\|^q \\
= &\sum_{j=0}^{m-1}\frac{(-1)^{m-j-1}}{j!(m-j-1)!}\sum_{k=0}^{m+l-1} (-1)^{m+l-1-k}\binom{m+l-1}{k}\left[a_{j,0} + \sum_{i=1}^{m-1} a_{j,i}\prod_{t=0}^{i-1}(p-t)\right]
\end{aligned}$$

$$
\begin{aligned}
&\times\|AT^{n+j}\left(S^{n+k}x\right)\|^q\\
= &\sum_{j=0}^{m-1}\frac{(-1)^{m-j-1}}{j!(m-j-1)!}a_{j,0}\sum_{k=0}^{m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}\|AS^{n+k}\left(T^{n+j}x\right)\|^q\\
&+\sum_{j=0}^{m-1}\frac{(-1)^{m-j-1}}{j!(m-j-1)!}\sum_{i=1}^{m-1}a_{j,i}\sum_{k=0}^{m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}\\
&\times\prod_{t=0}^{i-1}(p-t)\|AS^{n+k}\left(T^{n+j}x\right)\|^q.
\end{aligned}
$$

Since S is a n_2-quasi-A-(l,q)-isometry, then according to Proposition 2.4, we get that S is a n-quasi-A-$(m+l-1,q)$-isometry. Hence,

$$
\sum_{j=0}^{m-1}\frac{(-1)^{m-j-1}}{j!(m-j-1)!}a_{j,0}\underbrace{\sum_{k=0}^{m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}\|AS^{n+k}\left(T^{n+j}x\right)\|^q}_{=0}=0.
$$

Since $i=1,...,m-1$, then, by using Lemma 3.1, we infer that

$$
\sum_{k=0}^{m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}\prod_{t=0}^{i-1}(p-t)\|AS^{n+k}\left(T^{n+j}x\right)\|^q=0.
$$

Consequently, one obtains that

$$
\sum_{j=0}^{m-1}\frac{(-1)^{m-j-1}}{j!(m-j-1)!}\sum_{i=1}^{m-1}a_{j,i}\underbrace{\sum_{k=0}^{m+l-1}(-1)^{m+l-1-k}\binom{m+l-1}{k}\prod_{t=0}^{i-1}(p-t)\|AS^{n+k}\left(T^{n+j}x\right)\|^q}_{=0}=0,
$$

which gives that

$$
Q^q_{m+l-1,n}(TS,A,x)=0.
$$

□

The following example shows that Theorem 3.1 is not necessarily true if S and T are not commuting.

Example 3.1. We consider the operators on the two dimensional $\left(\mathbb{R}^2,\|.\|_2\right)$.

$$
T=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ S=\begin{pmatrix}1&0\\1&1\end{pmatrix}\text{ and }A=\begin{pmatrix}1&1\\1&2\end{pmatrix}.
$$

Note that $ST\neq TS$. Moreover, by a direct computation, we show that T and S are quasi-A-$(3,2)$-isometry. However neither TS nor ST is a quasi-A-$(5,2)$-isometry.

Corollary 3.1. *Let n_1, n_2, m, l be positive integers and $T,S,A_1,A_2\in\mathfrak{L}(X)$ such that $TS=ST, A_1A_2=A_2A_1, TA_1=A_1T$ and $SA_2=A_2S$. If T is a n_1-quasi-A_1-(m,q)-isometry and S is a n_2-quasi-A_2-(l,q)-isometry, then TS is a n-quasi-A_1A_2-$(m+l-1,q)$-isometry, with $n=\max(n_1,n_2)$.*

Proof. Following the same steps as in the proof of Theorem 3.1, we can prove that

$$Q^q_{m+l-1,n}(TS, A_1A_2, x) = 0.$$

Indeed, we have

$$\begin{aligned} Q^q_{m+l-1,n}(TS, A_1A_2, x) &= \sum_{k=0}^{m+l-1} (-1)^{m+l-1-k}\binom{m+l-1}{k}\|A_1A_2(TS)^{n+k}x\|^q \\ &= \sum_{k=0}^{m+l-1} (-1)^{m+l-1-k}\binom{m+l-1}{k}\|A_1T^{n+k}\left(A_2S^{n+k}x\right)\|^q. \end{aligned}$$

□

Theorem 3.2. *Let T be a n-quasi-A-(m,q)-isometry. Then, for each positive integer k, T^k is a n-quasi-A-(m,q)-isometry.*

Proof.

$$\begin{aligned} Q^q_{m,n}(T^k, A, x) &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\|A\left(T^k\right)^{n+j} x\|^q \\ &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\|AT^{kn+kj}x\|^q \\ &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{i=0}^{m-1}\binom{kj}{i}Q^q_{i,kn}(T, A, x) \\ &= \sum_{i=0}^{m-1}\frac{1}{i!}\left[\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(kj)^{(i)}\right]Q^q_{i,kn}(T, A, x). \end{aligned}$$

According to [14, Lemma 1], we have $\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(kj)^{(i)} = 0$ for each $i = 0, 1, ..., m-1$. It follows that $Q^q_{m,n}(T^k, A, x) = 0$. □

The converse of Theorem 3.2 is not true in general as shown in the following example.

Example 3.2. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. It is not difficult to prove that the operator $T = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ defined in $\left(\mathbb{R}^3, \|.\|_2\right)$ satisfies T^3 is a quasi-A-$(3,2)$-isometry but T is not a quasi-A-$(3,2)$-isometry.

Proposition 3.2. *Let $T \in \mathcal{L}(X)$ and n_1, n_2, r, s, m, l be positive integers. If T^r is a n_1-quasi-A-(m,q)-isometry and T^s is a n_2-quasi-A-(l,q)-isometry, then T^t is a n_0-quasi-A-(p,q)-isometry, where t is the greatest common divisor of r and s, $n_0 = \max(n_1, n_2)$ and $p = \min(m,\ l)$.*

Proof. Let's put $a_j = \|AT^{n+j}x\|^q, \ \forall \ j \geq 0$. Since T^r is a n_1-quasi-A-(m,q)-isometry and T^s is a n_2-quasi-A-(l,q)-isometry, then T^r is a n_0-quasi-A-(m,q)-isometry and T^s is a n_0-quasi-A-(l,q)-isometry. Hence,

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}a_{j+n_0r} = 0 \quad \text{and} \quad \sum_{j=0}^{l}(-1)^{l-j}\binom{l}{j}a_{j+n_0s} = 0.$$

By [15, Lemma 3.15], we infer that

$$\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}a_{j+n_0t} = 0,$$

which ends the proof. □

As an immediate consequence of Proposition 3.2, we have the following result.

Corollary 3.2. *Let $T, A \in \mathcal{L}(X)$ and r, s, m, n, l be positive integers. Then, the following properties hold.*

(1) If T is a n-quasi-A-(m,q)-isometry such that T^s is a n-quasi-A-$(1,q)$-isometry, then T is a n-quasi-A-$(1,q)$-isometry.
(2) If T^r and T^{r+1} are a n-quasi-A-(m,q)-isometries, then T is a n-quasi-A-(m,q)-isometry.
(3) If T^r is a n-quasi-A-(m,q)-isometry and T^{r+1} is a n-quasi-A-(l,q)-isometry with $m < l$, then T is a n-quasi-A-(m,q)-isometry.

As an immediate consequence of Proposition 3.2, Theorem 3.1 and Corollary 3.1, we have the following result.

Corollary 3.3. *Let n_1, n_2, r, s, m, l be positive integers and $T, S, A, A_1, A_2 \in \mathcal{L}(X)$. Let $n = \max(n_1, n_2)$. The following properties hold true.*

(1) If T is a n_1-quasi-A-(m,q)-isometry and S is a n_2-quasi-A-(l,q)-isometry such that $ST = TS$, then T^rS^s is a n-quasi-A-$(m+l-1,q)$-isometry.
(2) If T is a n_1-quasi-A_1-(m,q)-isometry and S is a n_2-quasi-A_2-(l,q)-isometry such that $TS = ST$, $A_1A_2 = A_2A_1$, $TA_1 = A_1T$ and $SA_2 = A_2S$, then T^rS^s is a n-quasi-A_1A_2-$(m+l-1,q)$-isometry.

4. Conclusions

We focus on some properties of a new class of operators called n-quasi-A-(m,q)-isometry operators. First, we give spectral properties and relationship between n-quasi-A-(m,q)-isometry and p-quasi-A-(m,q)-isometry. Second, the power and product of such operators have been investigated. As a future work, we can generalize our study on a metric, dislocated metric or dislocated quasi metric space (see references [8, 16, 17]).

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest

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