Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 27573–27588.
DOI: 10.3934/math.20231410
Received: 01 July 2023
Revised: 21 August 2023
Accepted: 19 September 2023
Published: 28 September 2023

Research article

Symmetric n-derivations on prime ideals with applications

Shakir Ali[1,*], **Amal S. Alali**[2], **Sharifah K. Said Husain**[3] **and Vaishali Varshney**[4]

[1] Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India
[2] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O.Box-84428, Riyadh-11671, Saudi Arabia
[3] Laboratory Cryptography, Analysis and Structure, Institute for Mathematical Research & Department of Mathematics and Statistics, Faculty of Science, University Putra Malaysia, Malaysia
[4] Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

* **Correspondence:** Email: shakir50@rediffmail.com.

Abstract: Let $\mathfrak{S}$ be a ring. The main objective of this paper is to analyze the structure of quotient rings, which are represented as $\mathfrak{S}/\mathfrak{P}$, where $\mathfrak{S}$ is an arbitrary ring and $\mathfrak{P}$ is a prime ideal of $\mathfrak{S}$. The paper aims to establish a link between the structure of these rings and the behaviour of traces of symmetric n-derivations satisfying some algebraic identities involving prime ideals of an arbitrary ring $\mathfrak{S}$. Moreover, as an application of the main result, we investigate the structure of the quotient ring $\mathfrak{S}/\mathfrak{P}$ and traces of symmetric n-derivations.

Keywords: derivation; prime ideal; prime ring; derivation; symmetric derivation; symmetric n-derivation
Mathematics Subject Classification: 16N60, 16R50, 16W25

1. Introduction

This paper provides a concise exploration of different expansions in the concept of derivations within associative rings. Among these extensions, the most comprehensive and significant is the concept of symmetric n-derivations. Moreover, we aim to identify connections and relationships between symmetric n-derivations and other key algebraic concepts, revealing new insights into the structural nature of rings. Throughout, $\mathfrak{S}$ will be an associative ring with $\mathcal{Z}(\mathfrak{S})$, as its center. A ring $\mathfrak{S}$ is said to be prime if $\varrho\mathfrak{S}\xi = \{0\}$ implies that either $\varrho = 0$ or $\xi = 0$, and semiprime if $\varrho\mathfrak{S}\varrho = \{0\}$ implies that $\varrho = 0$, where $\varrho, \xi \in \mathfrak{S}$. The symbols $[\varrho, \xi]$ and $\varrho \circ \xi$ denote the commutator, $\varrho\xi - \xi\varrho$ and the anti-commutator, $\varrho\xi + \xi\varrho$, respectively, for any $\varrho, \xi \in \mathfrak{S}$. A ring $\mathfrak{S}$ is said to be n-torsion free if $n\varrho = 0$ implies that $\varrho = 0$ $\forall$ $\varrho \in \mathfrak{S}$. If $\mathfrak{S}$ is m-torsion free, then it is d-torsion free for every divisor d

of m. Recall that an ideal $\mathfrak{P}$ of $\mathfrak{S}$ is said to be prime if, $\mathfrak{P} \neq \mathfrak{S}$ and for $\varrho, \xi \in \mathfrak{S}$, $\varrho\mathfrak{S}\xi \subseteq \mathfrak{P}$ implies that $\varrho \in \mathfrak{P}$ or $\xi \in \mathfrak{P}$. An additive mapping $\mathscr{D} : \mathfrak{S} \to \mathfrak{S}$ is called a derivation, if $\mathscr{D}(\varrho\xi) = \mathscr{D}(\varrho)\xi + \varrho\mathscr{D}(\xi)$ holds for all $\varrho, \xi \in \mathfrak{S}$. Maksa [12] expanded the scope of derivation by introducing symmetric bi-derivations on rings and Vukman explored these derivations in greater depth in [19, 21]. Later, several authors have studied symmetric bi-derivations and its relative mappings on rings (see [3, 12, 13, 16] and references therein) that generated very beneficial results. Let $n \geq 2$ be a fixed integer and $\mathfrak{S}^n = \underbrace{\mathfrak{S} \times \mathfrak{S} \times \cdots \times \mathfrak{S}}_{n-\text{times}}$. A map $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ is said to be symmetric (permuting) if $\mathfrak{D}(\xi_1, \xi_2, \ldots, \xi_n) = \mathfrak{D}(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \ldots, \xi_{\sigma(n)})$, for all permutations $\sigma(m) \in S_n$ and $\xi_m \in \mathfrak{S}$, where $m = 1, 2, \ldots, n$. The notion of symmetric n-derivation was defined by Park [15] as follows: an n-additive (i.e., additive in each slot) map $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ is said to be symmetric n-derivation if $\mathfrak{D}$ is symmetric and $\mathfrak{D}(\xi_1, \xi_2, \ldots, \xi_i\xi_i', \ldots, \xi_n) = \mathfrak{D}(\xi_1, \xi_2, \ldots, \xi_i, \ldots, \xi_n)\xi_i' + \xi_i\mathfrak{D}(\xi_1, \xi_2, \ldots, \xi_i', \ldots, \xi_n)$. A map $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ defined by $\mathscr{d}(\varrho) = \mathfrak{D}(\varrho, \varrho, \ldots, \varrho)$ is called the trace of $\mathfrak{D}$. If $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ is symmetric and n-additive, then the trace $\mathscr{d}$ of $\mathfrak{D}$ satisfies the relation $\mathscr{d}(\varrho + \xi) = \mathscr{d}(\varrho) + \mathscr{d}(\xi) + \sum_{k=1}^{n-1} {}^nC_k\ \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-k)-\text{times}}, \underbrace{\xi, \ldots, \xi}_{k-\text{times}})$ for all $\varrho, \xi \in \mathfrak{S}$. Many authors have studied various identities involving traces of n-derivations and have obtained several interesting results (see [1, 4, 6, 7, 16] and references therein). A 1-derivation is clearly a derivation, a 2-derivation is a symmetric bi-derivation, and for $n = 3$, $\mathfrak{D}$ is a symmetric 3-derivation (or tri-derivation) on rings.

Posner's discovery [17], that a prime ring becomes commutative if it has a nonzero centralizing derivation marked the beginning of the theory of centralizing (commuting) maps on prime rings. Since then, several authors have conducted extensive research in this area. Notable among them are Bre$\hat{s}$ar [8], Deng-Bell [9], Lanski [11], Vukman [20] and others, whose works are cited in these publications. A new study in this direction was proposed by Almahdi et al. in [2] by involving prime ideals of an arbitrary rings. The focus of their research is on the quotient ring $\mathfrak{S}/\mathfrak{P}$, where $\mathfrak{S}$ is an arbitrary ring and $\mathfrak{P}$ is a prime ideal of $\mathfrak{S}$ and studied various differential identities involving prime ideals (see also [1, 10, 14] for recent work). Inspired by the idea of centralizing and commuting mappings, Idrissi and Oukhtite [10] introduced the concepts of $\mathscr{W}$-centralizing and $\mathscr{W}$-commuting mappings, where $\mathscr{W}$ is an ideal of a ring $\mathfrak{S}$. Basically, they investigated the relationship between the structure of the ring $\mathfrak{S}/\mathfrak{P}$ and generalized derivations of $\mathfrak{S}$ that are $\mathfrak{P}$-centralizing, where $\mathfrak{P}$ is a prime ideal of $\mathfrak{S}$.

In the current investigation, we will utilize the above-mentioned approach and explore algebraic identities associated with the trace of symmetric n-derivations acting on prime ideals $\mathfrak{P}$ of $\mathfrak{S}$, but without imposing the assumption of primeness on the ring under consideration. Moreover, we extend the Posner's second theorem [17] for the trace of symmetric n-derivations which involves prime ideals. Precisely, we prove that for any fixed integer $n \geq 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If there exists a non-zero symmetric n-derivation with trace $\mathscr{d}$ on $\mathfrak{S}$ such that $[\mathscr{d}(\varrho), \varrho] \in \mathfrak{P}$, for all $\varrho \in \mathfrak{S}$, then either $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain or $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$. Further, we prove this result for Jordan product (Theorem 1.2).

Furthermore, as an application of our main result, we investigate the structure of quotient ring $\mathfrak{S}/\mathfrak{P}$ and the traces of symmetric n-derivations via $\mathfrak{P}$-centralizing mappings. Precisely, we prove that for a fixed integer $n \geq 2$, let $\mathfrak{P}$ be a prime ideal of an arbitrary ring $\mathfrak{S}$ such that the quotient ring $\mathfrak{S}/\mathfrak{P}$ is $(n+1)!$-torsion free and $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ be a nonzero symmetric n-derivation with trace $\mathscr{d}$ on $\mathfrak{S}$. If $\mathscr{d}$

is $\mathfrak{P}$-centralizing on $\mathfrak{S}$, then either $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain (Theorem 2.1).

In order to support the development of our main findings, we rely on the utilization of a well-established lemma, which serves as a valuable tool for our proof within the context of our discussion.

Lemma 1.1. *[4, Theorem 5.3.1.] For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be an $n!$-torsion free semiprime ring and I a nonzero ideal of $\mathfrak{S}$. Let $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ be a symmetric n-derivation such that $\mathfrak{D}(I, I, \ldots, I) \neq \{0\}$ and $[\delta(\varrho), \varrho] = 0$ for all $\varrho \in I$, where δ is the trace of $\mathfrak{D}$. Then $\mathfrak{S}$ contains a nonzero central ideal.*

Lemma 1.2. *[16, Lemma 2.3] For a fixed positive integer n, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$, such that $R/\mathfrak{P}$ is $n!$-torsion free. Suppose that $l_1, l_2, \ldots, l_n \in \mathfrak{S}$ satisfy $\alpha l_1 + \alpha^2 l_2 + \cdots + \alpha^n l_n \in \mathfrak{P}$ for $\alpha = 1, 2, \ldots, n$. Then $l_t \in \mathfrak{P}$ for $t = 1, 2, \ldots, n$.*

Lemma 1.3. *[18] Let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$. If one of the following conditions is satisfied, then $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain:*

(1) $[\varrho, \xi] \in \mathfrak{P} \;\; \forall \;\; \varrho, \xi \in \mathfrak{S}$,

(2) $\varrho \circ \xi \in \mathfrak{P} \;\; \forall \;\; \varrho, \xi \in \mathfrak{S}$.

The first main result of this paper is the following theorem:

Theorem 1.1. *For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free and $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ be a nonzero symmetric n-derivation on $\mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$. If $[\mathscr{d}(\varrho), \varrho] \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$, then we have one of the following assertions:*

(1) $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$,

(2) $\mathfrak{S}/\mathfrak{P}$ *is a commutative integral domain.*

Proof. We are given that

$$[\mathscr{d}(\varrho), \varrho] \in \mathfrak{P} \; \forall \; \varrho \in \mathfrak{S}. \tag{1.1}$$

Replace ϱ by $\varrho + q\xi$ for $\xi \in \mathfrak{S}$ and $1 \leq q \leq n-1$ leads to

$$[\mathscr{d}(\varrho + q\xi), (\varrho + q\xi)] \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S},$$

thereby obtaining

$$[\mathscr{d}(\varrho), \varrho] + [\mathscr{d}(\varrho), q\xi] + [\mathscr{d}(q\xi), \varrho] + [\mathscr{d}(q\xi), q\xi] + \left[\sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{q\xi, \ldots, q\xi}_{t-\text{times}}), \varrho\right] + \left[\sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{q\xi, \ldots, q\xi}_{t-\text{times}}), q\xi\right] \in \mathfrak{P},$$

$\forall \;\; \varrho, \xi \in \mathfrak{S}$. Using the relation (1.1), we obviously find that

$$q[\mathscr{d}(\varrho), \xi] + q^n[\mathscr{d}(\xi), \varrho] + \left[\sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{q\xi, \ldots, q\xi}_{t-\text{times}}), \varrho\right] + \left[\sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{q\xi, \ldots, q\xi}_{t-\text{times}}), q\xi\right] \in \mathfrak{P} \; \forall \; \varrho, \xi \in \mathfrak{S},$$

which implies that,

$$qA_1(\varrho;\xi) + q^2A_2(\varrho;\xi) + \cdots + q^nA_n(\varrho;\xi) \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S},$$

where $A_t(\varrho;\xi)$ represents the term in which ξ appears t-times.

The application of Lemma 1.2 yields

$$[\mathscr{d}(\varrho), \xi] + n[\mathfrak{D}(\varrho, \ldots, \varrho, \xi), \varrho] \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}. \tag{1.2}$$

Replacing ξ by $\varrho\xi$, we can see that

$$[\mathscr{d}(\varrho), \varrho\xi] + n[\mathfrak{D}(\varrho, \ldots, \varrho, \varrho\xi), \varrho] \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}.$$

On further solving

$$\varrho\{[\mathscr{d}(\varrho), \xi] + n[\mathfrak{D}(\varrho, \ldots, \varrho, \xi), \varrho]\} + [\mathscr{d}(\varrho), \varrho]\xi + n[\mathscr{d}(\varrho), \varrho]\xi + n\mathscr{d}(\varrho)[\xi, \varrho] \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}.$$

By using (1.2) and using the hypothesis, we get

$$n\mathscr{d}(\varrho)[\xi, \varrho] \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}.$$

Since $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free, we get

$$\mathscr{d}(\varrho)[\xi, \varrho] \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}. \tag{1.3}$$

Taking ξ by $r\xi$ in the above yields,

$$\mathscr{d}(\varrho)r[\xi, \varrho] \in \mathfrak{P} \ \forall \ \varrho, \xi, r \in \mathfrak{S}. \tag{1.4}$$

Replacing ϱ by $\varrho + qw_1$ for $1 \leq q \leq n-1$, $w_1 \in \mathfrak{S}$, we obtain

$$\mathscr{d}(\varrho + qw_1)r[\xi, \varrho + qw_1] \in \mathfrak{P} \ \forall \ \varrho, \xi, r, w_1 \in \mathfrak{S}.$$

After simplification, we find that

$$\begin{aligned} &\mathscr{d}(\varrho)r[\xi, \varrho] + \mathscr{d}(\varrho)r[\xi, qw_1] + \mathscr{d}(qw_1)r[\xi, \varrho] + \mathscr{d}(qw_1)r[\xi, qw_1] + \\ &\quad \sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{qw_1, \ldots, qw_1}_{t-\text{times}})r[\xi, \varrho] + \sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{qw_1, \ldots, qw_1}_{t-\text{times}})r[\xi, qw_1] \in \mathfrak{P}, \end{aligned}$$

$\forall \ \varrho, \xi, r, w_1 \in \mathfrak{S}$. Application of (1.3) and Lemma 1.2 gives

$$\mathscr{d}(\varrho)r[\xi, w_1] + n\mathfrak{D}(w_1, \varrho, \ldots, \varrho)r[\xi, \varrho] \in \mathfrak{P} \ \forall \ \varrho, \xi, r, w_1 \in \mathfrak{S}. \tag{1.5}$$

Replacing r by $r[\xi, \varrho]$ and using (1.4) and using torsion restriction, we get

$$\mathfrak{D}(w_1, \varrho, \ldots, \varrho)r[\xi, \varrho]^{2^1} \in \mathfrak{P} \ \forall \ \varrho, \xi, r, w_1 \in \mathfrak{S}. \tag{1.6}$$

Now again replacing ϱ by $\varrho + qw_2$ in above for $1 \le q \le n-1$, for all $w_2 \in \mathfrak{S}$, we get

$$\mathfrak{D}(w_1, \varrho + qw_2, \ldots, \varrho + qw_2)r[\xi, \varrho + qw_2]^{2^1} \in \mathfrak{P} \ \forall \ \varrho, \xi, r, w_1, w_2 \in \mathfrak{S}$$

which on further solving, we obtain

$$qA_1(\varrho;\xi;w_1;w_2) + q^2A_2(\varrho;\xi;w_1;w_2) + \cdots + q^nA_n(\varrho;\xi;w_1;w_2) \in \mathfrak{P} \ \forall \ \varrho, \xi, w_1, w_2 \in \mathfrak{S},$$

where $A_t(\varrho;\xi;w_1;w_2)$ denotes the sum of the terms in which w_2 appears t-times. Application of Lemma 1.2, we have

$$(n-1)\mathfrak{D}(w_1, w_2, \varrho, \ldots, \varrho)r[\xi, \varrho]^{2^1} + \mathfrak{D}(w_1, \varrho, \ldots, \varrho)r[\xi, w_2][\xi, \varrho] + \mathfrak{D}(w_1, \varrho, \ldots, \varrho)r[\xi, \varrho][\xi, w_2] \in \mathfrak{P} \ \forall \ \varrho, \xi, w_1, w_2 \in \mathfrak{S}.$$

For $r = r[\xi, \varrho]^2$ and using (1.6), we get

$$\mathfrak{D}(w_1, w_2, \varrho, \ldots, \varrho)r[\xi, \varrho]^{2^2} \in \mathfrak{P} \ \forall \ \varrho, \xi, w_1, w_2 \in \mathfrak{S}.$$

Proceeding in the similar manner, after some steps we arrive at

$$\mathfrak{D}(w_1, w_2, w_3, \ldots, w_n)r[\xi, \varrho]^{2^n} \in \mathfrak{P} \ \forall \ \varrho, \xi, w_i \in \mathfrak{S}, \ 1 \le i \le n.$$

On taking account of primeness of $\mathfrak{P}$, we get either $\mathfrak{D}(w_1, w_2, w_3, \ldots, w_n) \in \mathfrak{P}$ or $[\xi, \varrho]^{2^n} \in \mathfrak{P}$ for all $\xi, \varrho, w_i \in \mathfrak{S}$, $1 \le i \le n$. Let us suppose $\mathfrak{D}(w_1, w_2, w_3, \ldots, w_n) \in \mathfrak{P}$. In particular, when $w_1 = w_2 = \cdots = w_n = w$, we get $\mathscr{d}(w) \in \mathfrak{P}$ for all $w \in \mathfrak{S}$ and hence $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$. If for all $\varrho, \xi \in \mathfrak{S}$, $[\xi, \varrho]^{2^n} \in \mathfrak{P}$ implies $[\xi, \varrho] \in \mathfrak{P}$. Using Lemma 1.3, we conclude that $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

Corollary 1.1. *[15, Theorem 2.3] Let $n \ge 2$ be a fixed integer and let $\mathfrak{S}$ be a noncommutative $n!$-torsion free prime ring. Suppose that there exists a symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ such that the trace $\mathscr{d}$ of $\mathfrak{D}$ is commuting on $\mathfrak{S}$. Then $\mathfrak{D} = 0$.*

Proof. It is given that

$$[\mathscr{d}(\varrho), \varrho] = 0 \ \forall \ \varrho \in \mathfrak{S}. \tag{1.7}$$

Since $\mathfrak{S}$ is a prime ring, then $\mathfrak{P} = \{0\}$ is a prime ideal of $\mathfrak{S}$. Hence, (1.7) can be written as

$$[\mathscr{d}(\varrho), \varrho] \in \mathfrak{P} \ \forall \ \varrho \in \mathfrak{S}.$$

Application of Theorem 1.1 yields that either $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P} = \{0\}$ or $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain which contradicts the commutativity of $\mathfrak{S}$. Hence $\mathscr{d}(\mathfrak{S}) = 0$, making $\mathfrak{D} = 0$.

Theorem 1.2. *For a fixed integer $n \ge 2$, let $\mathfrak{S}$ be any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free and $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ be a nonzero symmetric n-derivation on $\mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$. If $\mathscr{d}(\varrho) \circ \varrho \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$ then one of the following holds:*

(1) $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$.

(2) $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

Proof. Given that

$$\mathscr{d}(\varrho) \circ \varrho \in \mathfrak{P}\ \forall\ \varrho \in \mathfrak{S}. \tag{1.8}$$

Replacing ϱ by $\varrho + q\xi$ for $\xi \in \mathfrak{S}$ and $1 \leq q \leq n-1$, we obtain

$$\mathscr{d}(\varrho + q\xi) \circ (\varrho + q\xi) \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}.$$

Solving further, we get

$$\mathscr{d}(\varrho) \circ \varrho + \mathscr{d}(\varrho) \circ q\xi + \mathscr{d}(q\xi) \circ \varrho + \mathscr{d}(q\xi) \circ q\xi + \\ \sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{q\xi, \ldots, q\xi}_{t-\text{times}}) \circ \varrho + \sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{q\xi, \ldots, q\xi}_{t-\text{times}}) \circ q\xi \in \mathfrak{P},$$

$\forall\ \varrho, \xi \in \mathfrak{S}$. On taking account of the given condition, we find that

$$q(\mathscr{d}(\varrho) \circ \xi) + q^n(\mathscr{d}(\xi) \circ \varrho) + \sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{q\xi, \ldots, q\xi}_{t-\text{times}}) \circ \varrho + \\ \sum_{t=1}^{n-1} {}^nC_t \mathfrak{D}(\underbrace{\varrho, \ldots, \varrho}_{(n-t)-\text{times}}, \underbrace{q\xi, \ldots, q\xi}_{t-\text{times}}) \circ q\xi \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S},$$

which implies that,

$$qA_1(\varrho; \xi) + q^2 A_2(\varrho; \xi) + \cdots + q^n A_n(\varrho; \xi) \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S},$$

where $A_t(\varrho; \xi)$ represents the term in which ξ appears t-times.

Using Lemma 1.2, it follows that

$$\mathscr{d}(\varrho) \circ \xi + n\mathfrak{D}(\varrho, \ldots, \varrho, \xi) \circ \varrho \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}. \tag{1.9}$$

Replacing ξ by $\varrho\xi$, we can see that

$$\varrho\{\mathscr{d}(\varrho) \circ \xi + n\mathfrak{D}(\varrho, \ldots, \varrho, \xi) \circ \varrho\} + [\mathscr{d}(\varrho), \varrho]\xi + n(\mathscr{d}(\varrho) \circ \varrho)\xi + n\mathscr{d}(\varrho)[\xi, \varrho] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}.$$

By using (1.9) and using the hypothesis, one can obviously verify that

$$[\mathscr{d}(\varrho), \varrho]\xi + n\mathscr{d}(\varrho)[\xi, \varrho] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}. \tag{1.10}$$

Putting $r\xi$ instead of ξ in (1.10), we see that

$$n\mathscr{d}(\varrho)r[\xi, \varrho] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}.$$

Since $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free, we get

$$\mathscr{d}(\varrho)r[\xi, \varrho] \in \mathfrak{P}\ \forall\ \varrho, \xi \in \mathfrak{S}. \tag{1.11}$$

which is same as (1.4). Hence, proceeding in the same way as after (1.4), we conclude.

2. Applications

In this section, we delve into the applications of the results that were established in Section 2. Inspired by the concept of centralizing and commuting maps, which is extensively discussed in [17,20], Idrissi and Oukhtite introduced the notions of $\mathscr{W}$-centralizing and $\mathscr{W}$-commuting maps where $\mathscr{W}$ is an ideal of the considered ring $\mathfrak{S}$ in their work [10]. For more comprehensive information, please refer to the aforementioned references.

Definition 2.1. *[10] Let $\mathscr{W}$ be an ideal of a ring $\mathfrak{S}$.*

(1) A map $f : \mathfrak{S} \to \mathfrak{S}$ is $\mathscr{W}$-centralizing if $\overline{[f(\varrho), \varrho]} \in \mathscr{Z}(\mathfrak{S}/\mathscr{W})$ for all $\varrho \in \mathfrak{S}$ or equivalently $[[f(\varrho), \varrho], \xi] \in \mathscr{W}$ for all $\varrho, \xi \in \mathfrak{S}$.

(2) A map $f : \mathfrak{S} \to \mathfrak{S}$ is $\mathscr{W}$-commuting if $[f(\varrho), \varrho] \in \mathscr{W}$ for all $\varrho \in \mathfrak{S}$.

Remark 2.1. *From the definition it is clear that every $\mathscr{W}$-commuting map is necessarily $\mathscr{W}$-centralizing but the converse need not be true in general.*

Example 2.1. *Let us consider $\mathfrak{S} = \left\{\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c \in \mathbb{Z}\right\}$. Of course $\mathfrak{S}$ with matrix addition and matrix multiplication is a ring. Consider $\mathscr{W} = \left\{\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c \in 2\mathbb{Z}\right\}$ be an ideal of $\mathfrak{S}$. Define $f : \mathfrak{S} \to \mathfrak{S}$ by $f(M) = \begin{bmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}$ for any $M \in \mathfrak{S}$. Then, it is easy to see that f is $\mathscr{W}$-centralizing, but not $\mathscr{W}$-commuting.*

The objective of this section is to extend the well-known Posner's second theorem, originally presented in [17]. This theorem establishes that the existence of a nonzero centralizing derivation in a prime ring leads to the ring being commutative. In [10], Idrissi and Oukhtite extended the Posner's second theorem for the prime ideals. In fact, they proved that if $\mathfrak{P}$ is a prime ideal of ring $\mathfrak{S}$ and F a generalized derivation of $\mathfrak{S}$ associated with a derivation d such that F is $\mathfrak{P}$-centralizing then $d(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain. Our next result is motivated by the above mentioned result, and we take the traces of symmetric n-derivations instead of derivations under suitable torsion restrictions. Precisely, we prove the following result:

Theorem 2.1. *For a fixed integer $n \geq 2$, let $\mathfrak{S}$ by any ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $(n+1)!$-torsion free and $\mathfrak{D}$ be a nonzero symmetric n-derivation of $\mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$. If $\mathscr{d}$ is $\mathfrak{P}$-centralizing then one of the following holds:*

(1) $\mathscr{d}(\mathfrak{S}) \subseteq \mathfrak{P}$

(2) $\mathfrak{S}/\mathfrak{P}$ is a commutative integral domain.

Proof. We are given that

$$\overline{[\mathscr{d}(\varrho), \varrho]} \in \mathscr{Z}(\mathfrak{S}/\mathfrak{P}) \ \forall \ \varrho \in \mathfrak{S}.$$

This can also be re-written as
$$[[\mathscr{d}(\varrho),\varrho],\xi] \in \mathfrak{P}\ \forall\ \varrho,\xi \in \mathfrak{S}. \tag{2.1}$$
Replacing ϱ by $\varrho + q\omega$ for $\omega \in \mathfrak{S}$ and $1 \leq q \leq n+1$, we obtain
$$[[\mathscr{d}(\varrho + q\omega),(\varrho + q\omega)],\xi] \in \mathfrak{P}\ \forall\ \varrho,\xi,\omega \in \mathfrak{S}.$$
As we continue to solve, we find that
$$\begin{aligned}&[[\mathscr{d}(\varrho),\varrho],\xi] + [[\mathscr{d}(q\omega),\varrho],\xi] + [[\mathscr{d}(\varrho),q\omega],\xi] + [[\mathscr{d}(q\omega),q\omega],\xi]\\ &\quad + [[\sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{q\omega,\ldots,q\omega}_{t-\text{times}}),\varrho],\xi] + [[\sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{q\omega,\ldots,q\omega}_{t-\text{times}}),q\omega],\xi] \in \mathfrak{P},\end{aligned}$$
$\forall\ \varrho,\xi,\omega \in \mathfrak{S}$. Using hypothesis, we have
$$\begin{aligned}&q^n[[\mathscr{d}(\omega),\varrho],\xi] + q[[\mathscr{d}(\varrho),\omega],\xi] + [[\sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{q\omega,\ldots,q\omega}_{t-\text{times}}),\varrho],\xi]\\ &\qquad + [[\sum_{t=1}^{n-1} {}^{n}C_t \mathfrak{D}(\underbrace{\varrho,\ldots,\varrho}_{(n-t)-\text{times}},\underbrace{q\omega,\ldots,q\omega}_{t-\text{times}}),q\omega],\xi] \in \mathfrak{P}\ \forall\ \varrho,\xi,\omega \in \mathfrak{S},\end{aligned}$$
which implies that
$$qA_1(\varrho,\xi,\omega) + q^2A_2(\varrho,\xi,\omega) + \cdots + q^{n-1}A_{n-1}(\varrho,\xi,\omega) \in \mathfrak{P},$$
$\forall\ \varrho,\xi,\omega \in \mathfrak{S}$, where $A_t(\varrho,\xi,\omega)$ represents the term in which ω appears t-times.

On taking account of Lemma 1.2, we see that
$$[[\mathscr{d}(\varrho),\omega],\xi] + n[[\mathfrak{D}(\varrho,\ldots,\varrho,\omega),\varrho],\xi] \in \mathfrak{P}\ \forall\ \varrho,\xi,\omega \in \mathfrak{S}. \tag{2.2}$$
Substituting $\varrho\omega$ for ω in (2.2), we deduce that
$$[[\mathscr{d}(\varrho),\varrho]\omega + \varrho[\mathscr{d}(\varrho),\omega],\xi] + n[[\varrho\mathfrak{D}(\varrho,\ldots,\varrho,\omega) + \mathfrak{D}(\varrho,\ldots,\varrho)\omega,\varrho],\xi] \in \mathfrak{P},$$
$\forall\ \varrho,\xi,\omega \in \mathfrak{S}$. On further solving, we get
$$\begin{aligned}&[\mathscr{d}(\varrho),\varrho][\omega,\xi] + [[\mathscr{d}(\varrho),\varrho],\xi]\omega + \varrho[[\mathscr{d}(\varrho),\omega],\xi] + [\varrho,\xi][\mathscr{d}(\varrho),\omega] + n\varrho[[\mathfrak{D}(\varrho,\ldots,\varrho,\omega),\varrho],\xi]\\ &\quad + n[\varrho,\xi][\mathfrak{D}(\varrho,\ldots,\varrho,\omega),\varrho] + n\mathscr{d}(\varrho)[[\omega,\varrho],\xi] + n[\mathscr{d}(\varrho),\xi][\omega,\varrho] + n[\mathscr{d}(\varrho),\varrho][\omega,\xi]\\ &\qquad + n[[\mathscr{d}(\varrho),\varrho],\xi]\omega \in \mathfrak{P}\ \forall\ \varrho,\xi,\omega \in \mathfrak{S}.\end{aligned}$$
Using the hypothesis and (2.2), we have
$$\begin{aligned}&[\varrho,\xi][\mathscr{d}(\varrho),\omega] + (n+1)[\mathscr{d}(\varrho),\varrho][\omega,\xi] + n[\varrho,\xi][\mathfrak{D}(\varrho,\ldots,\varrho,\omega),\varrho]\\ &\qquad + n\mathscr{d}(\varrho)[[\omega,\varrho],\xi] + n[\mathscr{d}(\varrho),\xi][\omega,\varrho] \in \mathfrak{P}\ \forall\ \varrho,\xi,\omega \in \mathfrak{S}.\end{aligned} \tag{2.3}$$

Taking $\xi = \varrho$ in (2.3), we get

$$(2n+1)[\mathscr{d}(\varrho),\varrho][\omega,\varrho] + n\mathscr{d}(\varrho)[[\omega,\varrho],\varrho] \in \mathfrak{P} \ \forall \ \varrho,\omega \in \mathfrak{S}. \tag{2.4}$$

Similarly, replacing ω by $\omega\varrho$ in (2.2), we obtain

$$[\mathscr{d}(\varrho),\omega][\varrho,\xi] + (n+1)[\omega,\xi][\mathscr{d}(\varrho),\varrho] + n[\mathfrak{D}(\varrho,\ldots,\varrho,\omega),\varrho][\varrho,\xi] + n[[\omega,\varrho],\xi]\mathscr{d}(\varrho) + n[\omega,\varrho][\mathscr{d}(\varrho),\xi] \in \mathfrak{P} \ \forall \ \varrho,\xi,\omega \in \mathfrak{S}. \tag{2.5}$$

On taking $\xi = \varrho$ in (2.5), we get

$$(2n+1)[\omega,\varrho][\mathscr{d}(\varrho),\varrho] + n[[\omega,\varrho],\varrho]\mathscr{d}(\varrho) \in \mathfrak{P} \ \forall \ \varrho,\omega \in \mathfrak{S}. \tag{2.6}$$

Now replace ω by ωr in (2.4) to get

$$(2n+1)\{[\mathscr{d}(\varrho),\varrho]\omega[r,\varrho] + [\mathscr{d}(\varrho),\varrho][\omega,\varrho]r\} + n\mathscr{d}(\varrho)\omega[[r,\varrho],\varrho] + 2n\mathscr{d}(\varrho)[\omega,\varrho][r,\varrho] + n\mathscr{d}(\varrho)[[\omega,\varrho],\varrho]r \in \mathfrak{P} \ \forall \ \varrho,\omega,r \in \mathfrak{S}.$$

Using (2.4) in the above equation, we find that

$$(2n+1)[\mathscr{d}(\varrho),\varrho]\omega[r,\varrho] + n\mathscr{d}(\varrho)\omega[[r,\varrho],\varrho] + 2n\mathscr{d}(\varrho)[\omega,\varrho][r,\varrho] \in \mathfrak{P} \ \forall \ \varrho,\omega,r \in \mathfrak{S}.$$

Replace ω by $\mathscr{d}(\varrho)$ in the above relation to get

$$(2n+1)[\mathscr{d}(\varrho),\varrho]\mathscr{d}(\varrho)[r,\varrho] + n(\mathscr{d}(\varrho))^2[[r,\varrho],\varrho] + 2n\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho][r,\varrho] \in \mathfrak{P} \ \forall \ \varrho,r \in \mathfrak{S}. \tag{2.7}$$

Again using (2.4) in (2.7), we see that

$$(2n+1)[\mathscr{d}(\varrho),\varrho]\mathscr{d}(\varrho)[r,\varrho] - (2n+1)\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho][r,\varrho] + 2n\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho][r,\varrho] \in \mathfrak{P} \ \forall \ \varrho,r \in \mathfrak{S},$$

which on solving, we get

$$\{(2n+1)[\mathscr{d}(\varrho),\varrho]\mathscr{d}(\varrho) - \mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho]\}[r,\varrho] \in \mathfrak{P} \ \forall \ \varrho,r \in \mathfrak{S}.$$

Now, left multiply by $[r,\varrho]$ in the above equation, we get

$$(2n+1)[r,\varrho][\mathscr{d}(\varrho),\varrho]\mathscr{d}(\varrho)[r,\varrho] - [r,\varrho]\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho][r,\varrho] \in \mathfrak{P}, \tag{2.8}$$

$\forall \ \varrho,r \in \mathfrak{S}$. Similarly, using (2.6) one can easily obtain

$$[r,\varrho]\{(2n+1)\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho] - [\mathscr{d}(\varrho),\varrho]\mathscr{d}(\varrho)\} \in \mathfrak{P} \ \forall \ \varrho,r \in \mathfrak{S}.$$

Right multiply by $[r,\varrho]$ in the above equation

$$(2n+1)[r,\varrho]\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho][r,\varrho] - [r,\varrho][\mathscr{d}(\varrho),\varrho]\mathscr{d}(\varrho)[r,\varrho] \in \mathfrak{P}, \tag{2.9}$$

$\forall \ \varrho,r \in \mathfrak{S}$. On adding (2.8) and (2.9), we find that

$$2n[r,\varrho]\{[\mathscr{d}(\varrho),\varrho]\mathscr{d}(\varrho) + \mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho]\}[r,\varrho] \in \mathfrak{P} \ \forall \ \varrho,r \in \mathfrak{S}. \tag{2.10}$$

Since $2n$ divides $(n+1)!$, we find that $\mathfrak{S}/\mathfrak{P}$ is $2n$-torsion free and hence for all $\varrho, r \in \mathfrak{S}$, we have

$$[r,\varrho]\{[\mathscr{d}(\varrho),\varrho]\mathscr{d}(\varrho)+\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho]\}[r,\varrho]\in\mathfrak{P}. \tag{2.11}$$

Using (2.11) in (2.9), we get

$$(2n+2)[r,\varrho]\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho][r,\varrho]\in\mathfrak{P}\ \forall\ \varrho, r\in\mathfrak{S}.$$

Since $2(n+1)$ divides $(n+1)!$, we find that $\mathfrak{S}/\mathfrak{P}$ is $(2n+2)$-torsion free and hence for all $\varrho, r \in \mathfrak{S}$, we have

$$[r,\varrho]\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho][r,\varrho]\in\mathfrak{P}\ \forall\ \varrho, r\in\mathfrak{S}.$$

Premultiplying by $\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho]$ in above equation, we get

$$\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho][r,\varrho]\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho][r,\varrho]\in\mathfrak{P}\ \forall\ \varrho, r\in\mathfrak{S}. \tag{2.12}$$

Using the primeness of $\mathfrak{P}$, we get

$$\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho][r,\varrho]\in\mathfrak{P}\ \forall\ \varrho, r\in\mathfrak{S}. \tag{2.13}$$

Replace r by $r\omega$ and use (2.13) to get

$$\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho]r[\omega,\varrho]\in\mathfrak{P}\ \forall\ \varrho,\omega, r\in\mathfrak{S}.$$

Replacing ω by $\mathscr{d}(\varrho)$, we get

$$\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho]r[\mathscr{d}(\varrho),\varrho]\in\mathfrak{P}\ \forall\ \varrho, r\in\mathfrak{S}.$$

Now replace r by $r\mathscr{d}(\varrho)$ to obtain

$$\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho]r\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho]\in\mathfrak{P}\ \forall\ \varrho, r\in\mathfrak{S}.$$

In light of primeness of $\mathfrak{P}$, we get

$$\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho]\in\mathfrak{P}\ \forall\ \varrho\in\mathfrak{S}. \tag{2.14}$$

Replacing ϱ by $\varrho + q\xi$ in (2.14), where $1 \leq q \leq n+1$ and implementing Lemma 1.2, we get

$$\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\xi]+n\mathscr{d}(\varrho)[\mathfrak{D}(\varrho,\ldots,\varrho,\xi),\varrho]+n\mathfrak{D}(\varrho,\ldots,\varrho,\xi)[\mathscr{d}(\varrho),\varrho]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}. \tag{2.15}$$

Replacing ξ by $\xi\varrho$ in (2.15), we get

$$\begin{aligned}\mathscr{d}(\varrho)\xi[\mathscr{d}(\varrho),\varrho]+\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\xi]\varrho+n\mathscr{d}(\varrho)\xi[\mathscr{d}(\varrho),\varrho]+&\\ n\mathscr{d}(\varrho)[\xi,\varrho]\mathscr{d}(\varrho)+n\mathscr{d}(\varrho)[\mathfrak{D}(\varrho,\ldots,\varrho,\xi),\varrho]\varrho+n\xi\mathscr{d}(\varrho)[\mathscr{d}(\varrho),\varrho]+&\\ n\mathfrak{D}(\varrho,\ldots,\varrho,\xi)\varrho[\mathscr{d}(\varrho),\varrho]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}.&\end{aligned}$$

Use (2.14) and (2.15) in the above relation, we see that

$$(n+1)\mathscr{d}(\varrho)\xi[\mathscr{d}(\varrho),\varrho]+n\mathscr{d}(\varrho)[\xi,\varrho]\mathscr{d}(\varrho)-n\mathfrak{D}(\varrho,\ldots,\varrho,\xi)[[\mathscr{d}(\varrho),\varrho],\varrho]\in\mathfrak{P}\ \forall\ \varrho,\xi\in\mathfrak{S}$$

and hence, we get

$$(n+1)\mathscr{d}(\varrho)\xi[\mathscr{d}(\varrho),\varrho] + n\mathscr{d}(\varrho)[\xi,\varrho]\mathscr{d}(\varrho) \in \mathfrak{P}\ \forall\ \varrho,\xi \in \mathfrak{S}. \tag{2.16}$$

Substituting $\varrho\xi$ for ξ in (2.16), we get

$$(n+1)\mathscr{d}(\varrho)\varrho\xi[\mathscr{d}(\varrho),\varrho] + n\mathscr{d}(\varrho)\varrho[\xi,\varrho]\mathscr{d}(\varrho) \in \mathfrak{P}\ \forall\ \varrho,\xi \in \mathfrak{S}. \tag{2.17}$$

Left multiply (2.16) by ϱ to get

$$(n+1)\varrho\mathscr{d}(\varrho)\xi[\mathscr{d}(\varrho),\varrho] + n\varrho\mathscr{d}(\varrho)[\xi,\varrho]\mathscr{d}(\varrho) \in \mathfrak{P}\ \forall\ \varrho,\xi \in \mathfrak{S}. \tag{2.18}$$

Combining (2.17) and (2.18), we get

$$(n+1)[\mathscr{d}(\varrho),\varrho]\xi[\mathscr{d}(\varrho),\varrho] + n[\mathscr{d}(\varrho),\varrho][\xi,\varrho]\mathscr{d}(\varrho) \in \mathfrak{P}\ \forall\ \varrho,\xi \in \mathfrak{S}. \tag{2.19}$$

Replacing ω by ωr in (2.6), we get

$$(2n+1)[\omega r,\varrho][\mathscr{d}(\varrho),\varrho] + n[[\omega r,\varrho],\varrho]\mathscr{d}(\varrho) \in \mathfrak{P}\ \forall\ \varrho,\omega,r \in \mathfrak{S}, \tag{2.20}$$

which on further solving, we obtain

$$\begin{aligned}(2n+1)\omega[r,\varrho][\mathscr{d}(\varrho),\varrho] + (2n+1)[\omega,\varrho]r[\mathscr{d}(\varrho),\varrho] + n\omega[[r,\varrho],\varrho]\mathscr{d}(\varrho) + 2n[\omega,\varrho][r,\varrho]\mathscr{d}(\varrho)\\ + n[[\omega,\varrho],\varrho]r\mathscr{d}(\varrho) \in \mathfrak{P}\ \forall\ \varrho,\omega,r \in \mathfrak{S}.\end{aligned}$$

Now use (2.6) in above relation to get

$$(2n+1)[\omega,\varrho]r[\mathscr{d}(\varrho),\varrho] + 2n[\omega,\varrho][r,\varrho]\mathscr{d}(\varrho) + n[[\omega,\varrho],\varrho]r\mathscr{d}(\varrho) \in \mathfrak{P}\ \forall\ \varrho,\omega,r \in \mathfrak{S}.$$

Replacing ω by $\mathscr{d}(\varrho)$ in the above relation, we get

$$(2n+1)[\mathscr{d}(\varrho),\varrho]r[\mathscr{d}(\varrho),\varrho] + 2n[\mathscr{d}(\varrho),\varrho][r,\varrho]\mathscr{d}(\varrho) + n[[\mathscr{d}(\varrho),\varrho],\varrho]r\mathscr{d}(\varrho) \in \mathfrak{P}\ \forall\ \varrho,r \in \mathfrak{S}.$$

Using the hypothesis, we obtain

$$(2n+1)[\mathscr{d}(\varrho),\varrho]r[\mathscr{d}(\varrho),\varrho] + 2n[\mathscr{d}(\varrho),\varrho][r,\varrho]\mathscr{d}(\varrho) \in \mathfrak{P}\ \forall\ \varrho,r \in \mathfrak{S}. \tag{2.21}$$

Combining (2.19) and (2.21), we get

$$(2n+1)[\mathscr{d}(\varrho),\varrho]r[\mathscr{d}(\varrho),\varrho] - 2(n+1)[\mathscr{d}(\varrho),\varrho]r[\mathscr{d}(\varrho),\varrho] \in \mathfrak{P}\ \forall\ \varrho,r \in \mathfrak{S}.$$

On solving,

$$[\mathscr{d}(\varrho),\varrho]r[\mathscr{d}(\varrho),\varrho] \in \mathfrak{P}\ \forall\ \varrho,r \in \mathfrak{S}.$$

Invoking the primeness of $\mathfrak{P}$, it follows that

$$[\mathscr{d}(\varrho),\varrho] \in \mathfrak{P}\ \forall\ \varrho \in \mathfrak{S}.$$

Hence, by Theorem 1.1, we get the required result.

Corollary 2.1. *[7, Theorem 1] For any fixed integer $n \geq 2$, let $\mathfrak{S}$ be a $(n+1)!$-torsion free semiprime ring. Suppose that $\mathfrak{S}$ admits a nonzero symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \longrightarrow \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \longrightarrow \mathfrak{S}$. If $\mathscr{d}$ is centralizing on $\mathfrak{S}$, then $\mathscr{d}$ is commuting on $\mathfrak{S}$.*

Proof. For the non-trivial implication assume that $[\mathscr{d}(\varrho), \varrho] \in \mathcal{Z}(\mathfrak{S})$ or $[[\mathscr{d}(\varrho), \varrho], \xi] = 0 \ \forall \ \varrho, \xi \in \mathfrak{S}$. Since $\mathfrak{S}$ is a semiprime ring, there exists a family Γ of prime ideals of $\mathfrak{S}$ such that $\bigcap_{\mathfrak{P}\in\Gamma} \mathfrak{P} = (0)$ thereby obtaining $[[\mathscr{d}(\varrho), \varrho], \xi] \in \mathfrak{P}$ for all $\mathfrak{P} \in \Gamma$. Using the proof of Theorem 2.1, we obtain

$$[\mathscr{d}(\varrho), \varrho] r [\mathscr{d}(\varrho), \varrho] \in \mathfrak{P} \text{ for all } \varrho, r \in \mathfrak{S} \text{ and for all } \mathfrak{P} \in \Gamma,$$

and therefore,

$$[\mathscr{d}(\varrho), \varrho] r [\mathscr{d}(\varrho), \varrho] = 0 \text{ for all } \varrho, r \in \mathfrak{S}.$$

and the semiprimeness of $\mathfrak{S}$ yields that $[\mathscr{d}(\varrho), \varrho] = 0$ for all $\varrho \in \mathfrak{S}$. Hence, $\mathscr{d}$ is commuting on $\mathfrak{S}$.

Theorem 2.2. *For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be a ring and $\mathfrak{P}$ be a prime ideal of $\mathfrak{S}$ such that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free. If $\mathfrak{D}_1, \mathfrak{D}_2 : \mathfrak{S}^n \to \mathfrak{S}$ are n-derivations of $\mathfrak{S}$ with traces $\mathscr{d}_1, \mathscr{d}_2 : \mathfrak{S} \to \mathfrak{S}$ satisfying one of the following conditions:*

(1) $[\mathscr{d}_1(\varrho), \xi] + [\varrho, \mathscr{d}_2(\xi)] \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}$,

(2) $[\mathscr{d}_1(\varrho), \xi] + [\varrho, \mathscr{d}_2(\xi)] - [\varrho, \xi] \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}$,

then one of the following holds:

(a) $\mathscr{d}_1(\mathfrak{S}) \subseteq \mathfrak{P}$ *and* $\mathscr{d}_2(\mathfrak{S}) \subseteq \mathfrak{P}$,

(b) $\mathfrak{S}/\mathfrak{P}$ *is commutative integral domain.*

Proof. (*1*) Assume that

$$[\mathscr{d}_1(\varrho), \xi] + [\varrho, \mathscr{d}_2(\xi)] \in \mathfrak{P} \ \forall \ \varrho, \xi \in \mathfrak{S}. \tag{2.22}$$

Replace ξ by $\xi + q\omega$, $\omega \in \mathfrak{S}$ for $1 \leq q \leq n-1$ to get

$$[\mathscr{d}_1(\varrho), \xi + q\omega] + [\varrho, \mathscr{d}_2(\xi + q\omega)] \in \mathfrak{P} \ \forall \ \varrho, \xi, \omega \in \mathfrak{S},$$

which on solving and using the given condition, we find that

$$\sum_{t=1}^{n-1} {}^nC_t [\varrho, \mathfrak{D}_2(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{q\omega, \ldots, q\omega}_{t-\text{times}})] \in \mathfrak{P} \ \forall \ \varrho, \xi, \omega \in \mathfrak{S}.$$

Taking Lemma 1.2 into consideration and using the fact that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free, we obtain

$$[\varrho, \mathfrak{D}_2(\xi, \ldots, \xi, \omega)] \in \mathfrak{P} \ \forall \ \varrho, \xi, \omega \in \mathfrak{S}. \tag{2.23}$$

Put $\omega = \varrho\omega$ in (2.23) and use (2.23), we find that

$$[\varrho, \mathfrak{D}_2(\xi, \ldots, \xi, \varrho)]\omega + \mathfrak{D}_2(\xi, \ldots, \xi, \varrho)[\varrho, \omega] \in \mathfrak{P} \ \forall \ \varrho, \xi, \omega \in \mathfrak{S}.$$

If we put $\varrho = \xi$, we obtain

$$[\varrho, \mathscr{d}_2(\xi)]\omega + \mathscr{d}_2(\xi)[\varrho, \omega] \in \mathfrak{P} \ \forall \ \varrho, \xi, \omega \in \mathfrak{S}. \tag{2.24}$$

Replace ω by ωr in (2.24) and using (2.24) to get

$$\mathscr{d}_2(\xi)\omega[\xi, r] \in \mathfrak{P} \ \forall\ r, \xi, \omega \in \mathfrak{S},$$

which is same as (1.11). Hence proceeding in the same manner, we get $\mathscr{d}_2(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain. If $\mathscr{d}_2(\mathfrak{S}) \subseteq \mathfrak{P}$, the relation (2.22) reduces to $[\mathscr{d}_1(\varrho), \xi] \in \mathfrak{P}$ and by Theorem 1.1, we conclude that $\mathscr{d}_1(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.
(2) Assume that

$$[\mathscr{d}_1(\varrho), \xi] + [\varrho, \mathscr{d}_2(\xi)] - [\varrho, \xi] \in \mathfrak{P} \ \forall\ \varrho, \xi \in \mathfrak{S}. \tag{2.25}$$

Replace ξ by $\xi + q\omega$, $\omega \in \mathfrak{S}$ for $1 \leq q \leq n-1$ to get

$$[\mathscr{d}_1(\varrho), \xi + q\omega] + [\varrho, \mathscr{d}_2(\xi + q\omega)] - [\varrho, \xi + q\omega] \in \mathfrak{P} \ \forall\ \varrho, \xi, \omega \in \mathfrak{S}.$$

Further solving and taking account of the given condition, we get

$$\sum_{t=1}^{n-1} {}^{n}C_t[\varrho, \mathfrak{D}_2(\underbrace{\xi, \ldots, \xi}_{(n-t)-\text{times}}, \underbrace{q\omega, \ldots, q\omega}_{t-\text{times}})] \in \mathfrak{P} \ \forall\ \varrho, \xi, \omega \in \mathfrak{S}.$$

Taking Lemma 1.2 into consideration and using the fact that $\mathfrak{S}/\mathfrak{P}$ is $n!$-torsion free, we obtain

$$[\varrho, \mathfrak{D}_2(\xi, \ldots, \xi, \omega)] \in \mathfrak{P} \ \forall\ \varrho, \xi, \omega \in \mathfrak{S},$$

which is same as (2.23). Hence proceeding in the same way as before, we find that $\mathscr{d}_2(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain. If $\mathscr{d}_2(\mathfrak{S}) \subseteq \mathfrak{P}$, the relation (2.25) becomes

$$[\mathscr{d}_1(\varrho), \xi] - [\varrho, \xi] \in \mathfrak{P}. \tag{2.26}$$

As a special case of (2.26) when $\xi = \varrho$, we may write $[\mathscr{d}_1(\varrho), \varrho] \in \mathfrak{P}$ for all $\varrho \in \mathfrak{S}$ and by Theorem 1.1, we conclude that $\mathscr{d}_1(\mathfrak{S}) \subseteq \mathfrak{P}$ or $\mathfrak{S}/\mathfrak{P}$ is commutative integral domain.

Corollary 2.2. *Let $n \geq 2$ be a fixed integer and let $\mathfrak{S}$ be a $n!$-torsion free semiprime ring. If $\mathfrak{S}$ admits two nonzero symmetric n-derivations $\mathfrak{D}_1 : \mathfrak{S}^n \to \mathfrak{S}$ with trace $\mathscr{d}_1 : \mathfrak{S} \to \mathfrak{S}$ and $\mathfrak{D}_2 : \mathfrak{S}^n \to \mathfrak{S}$ with trace $\mathscr{d}_2 : \mathfrak{S} \to \mathfrak{S}$ satisfying $[\mathscr{d}_1(\varrho), \xi] + [\varrho, \mathscr{d}_2(\xi)] = 0$ for all $\varrho, \xi \in \mathfrak{S}$, then $\mathfrak{S}$ contains a nonzero two-sided central ideal.*

Proof. Assume that $[\mathscr{d}_1(\varrho), \xi] + [\varrho, \mathscr{d}_2(\xi)] = 0 \ \forall\ \varrho, \xi \in \mathfrak{S}$. According to semiprimeness, there exists a family Γ of prime ideals $\mathfrak{P}$ such that $\bigcap_{\mathfrak{P}\in\Gamma} \mathfrak{P} = (0)$ and therefore,

$$[\mathscr{d}_1(\varrho), \xi] + [\varrho, \mathscr{d}_2(\xi)] \in \mathfrak{P} \text{ for all } \varrho, \xi \in \mathfrak{S} \text{ and for all } \mathfrak{P} \in \Gamma.$$

Using the proof of Theorem 2.2(i), we obtain

$$\mathscr{d}_2(\xi)\omega[\xi, r] \in \mathfrak{P} \ \forall\ r, \xi, \omega \in \mathfrak{S} \text{ and for all } \mathfrak{P} \in \Gamma,$$

and therefore,

$$\mathscr{d}_2(\xi)\omega[\xi, r] = 0 \ \forall\ r, \xi, \omega \in \mathfrak{S}. \tag{2.27}$$

Putting $r = d_2(\xi)$ and left multiply by ξ in above equation, we get

$$\xi d_2(\xi)\omega[\xi, d_2(\xi)] = 0 \ \forall\ r, \xi, \omega \in \mathfrak{S}. \tag{2.28}$$

Replacing r by $d_2(\xi)$ and ω by $\xi\omega$ in (2.27), we arrive at

$$d_2(\xi)\xi\omega[\xi, d_2(\xi)] = 0 \ \forall\ r, \xi, \omega \in \mathfrak{S}. \tag{2.29}$$

Comparing the last two equations, we obtain

$$[\xi, d_2(\xi)]\omega[\xi, d_2(\xi)] = 0,$$

for all $\xi,\ \omega \in \mathfrak{S}$ and the semiprimeness of $\mathfrak{S}$ yields that $[\xi, d_2(\xi)] = 0$ for all $\xi \in \mathfrak{S}$. Therefore, $\mathfrak{S}$ contains a nonzero two-sided central ideal by Lemma 1.1.

Corollary 2.3. *Let $n \geq 2$ be a fixed integer and let $\mathfrak{S}$ be a $n!$-torsion free semiprime ring. If $\mathfrak{S}$ admits two nonzero symmetric n-derivations $\mathfrak{D}_1 : \mathfrak{S}^n \to \mathfrak{S}$ with trace $d_1 : \mathfrak{S} \to \mathfrak{S}$ and $\mathfrak{D}_2 : \mathfrak{S}^n \to \mathfrak{S}$ with trace $d_2 : \mathfrak{S} \to \mathfrak{S}$ satisfying $[d_1(\varrho), \xi] + [\varrho, d_2(\xi)] - [\varrho, \xi] = 0$ for all $\varrho, \xi \in \mathfrak{S}$, then $\mathfrak{S}$ contains a nonzero two-sided central ideal.*

Proof. Assume that $[d_1(\varrho), \xi] + [\varrho, d_2(\xi)] - [\varrho, \xi] = 0$ for all $\varrho, \xi \in \mathfrak{S}$. Since $\mathfrak{S}$ is semiprime, there exists a family Γ of prime ideals $\mathfrak{P}$ such that $\bigcap_{\mathfrak{P}\in\Gamma} \mathfrak{P} = (0)$ and thus,

$$[d_1(\varrho), \xi] + [\varrho, d_2(\xi)] - [\varrho, \xi] \in \mathfrak{P} \text{ for all } \varrho, \xi \in \mathfrak{S} \text{ and for all } \mathfrak{P} \in \Gamma.$$

Invoking the proof of Theorem 2.2(ii), we obtain

$$d_2(\xi)\omega[\xi, r] = 0 \ \forall\ r, \xi, \omega \in \mathfrak{S}. \tag{2.30}$$

Since the last equation is same as the (2.27), reasoning in the same manner as in the above mentioned corollary, we get the required result.

The following example shows that the condition "primeness of an ideal $\mathfrak{P}$" in Theorems 1.1, 1.2 and 2.1 cannot be omitted.

Example 2.2. *Consider the ring $\mathfrak{S} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}$. Let $\mathfrak{P} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$ be an ideal of $\mathfrak{S}$. Denote $A_i = \begin{bmatrix} 0 & a_i & b_i \\ 0 & 0 & c_i \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{S}$, $a_i, b_i, c_i \in \mathbb{Z}$, $1 \leq i \leq n$, and let us define $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ by $\mathfrak{D}(A_1, A_2, \ldots, A_n) = \begin{bmatrix} 0 & 0 & a_1a_2\cdots a_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with trace $d : \mathfrak{S} \to \mathfrak{S}$ define by $d\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & a^n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. One can easily see that $\mathfrak{D}$ is a symmetric n-derivation such that $[d(x), x] \in \mathfrak{P}$, $d(x) \circ x \in \mathfrak{P}$ and $[[d(x), x], y] \in \mathfrak{P}$ for $x, y \in \mathfrak{S}$. However, $d(\mathfrak{S}) \nsubseteq \mathfrak{P}$ and $\mathfrak{S}/\mathfrak{P}$ is noncommutative. Also, we observe that $\mathfrak{P}$ is not a prime ideal of $\mathfrak{S}$ as $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathfrak{S} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \subseteq \mathfrak{P}$, but $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin \mathfrak{P}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin \mathfrak{P}$.*

3. Conclusions

In this article, we discussed about the relationship between the structural aspects of the quotient rings $\mathfrak{S}/\mathfrak{P}$ and the behavioral patterns exhibited by traces of symmetric n-derivations satisfying some algebraic identities involving prime ideals of an arbitrary ring $\mathfrak{S}$. Further, we also investigated the structure of quotient ring $\mathfrak{S}/\mathfrak{P}$ and the traces of symmetric n-derivations via $\mathfrak{P}$-centralizing mappings.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the reviewers for their valuable suggestions and useful comments, which helps us to improve the presentation of manuscript. Moreover, the authors extend their appreciation to Princess Nourah bint Abdulrahman University (PNU), Riyadh, Saudi Arabia for funding this research under Researchers Supporting Project Number (PNURSP2023R231). Substantial part of this work was done when the author was a visiting Researcher at Institute for Mathematics Research(INSPEM), Universiti Putra Malaysia (UPM), Malaysia (August, 2023) hosted by Professor Sharifah Kartini S. Husain. The first author appreciates the gracious hospitality he received at INSPEM, UPM during his visit.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. S. Ali, T. M. Alsuraiheed, N. Parveen, V. Varshney, Action of n-derivations and n-multipliers on ideals of (semi)-prime rings, *AIMS Math.*, **8** (2023), 17208–17228. https://doi.org/10.3934/math.2023879
2. F. A. A. Almahdi, A. Mamouni, M. Tamekkante, A generalization of Posner's theorem on derivations in rings, *Indian J. Pure Appl. Math.*, **51** (2020), 187–194. https://doi.org/10.1007/s13226-020-0394-8
3. M. Ashraf, On symmetric bi-derivations in rings, *Rendiconti dell'Istituto di Matematica dell'Università di Trieste*, **31** (1999), 25–36.
4. M. Ashraf, M. R. Jamal, Traces of permuting n-additive maps and permuting n-derivations of rings, *Mediterr. J. Math.*, **11** (2014), 287–297. https://doi.org/10.1007/s00009-013-0298-5
5. M. Ashraf, N. Parveen, M. R. Jamal, Traces of permuting n-derivations and commutativity of rings, *S. E. Asian B. Math.*, **38** (2014), 321–332.
6. M. Ashraf, M. R. Jamal, M. R. Mozumder, On the traces of certain classes of permuting mappings in rings, *Georgian Math. J.*, **23** (2016), 15–23, https://doi.org/10.1515/gmj-2015-0051 .

7. M. Ashraf, A. Khan, M. R. Jamal, Traces of permuting generalized n-derivations of rings, *Miskolc Math. Notes*, **19** (2018), 731–740. https://doi.org/10.18514/MMN.2018.1851

8. M. Breŝar, Commuting maps: a survey, *Taiwanese J. Math.*, **8** (2004), 361–397. https://doi.org/10.11650/twjm/1500407660

9. Q. Deng, H. E. Bell, On derivations and commutativity in semiprime rings, *Commun. Algebra*, **23** (1995), 3705–3713. https://doi.org/10.1080/00927879508825427

10. M. A. Idrissi, L. Oukhtite Structure of a quotient ring R/P with generalized derivations acting on the prime ideal P and some applications, *Indian J. Pure Appl. Math.*, **53** (2022), 792–800. https://doi.org/10.1007/s13226-021-00173-x

11. C. Lanski, Differential identities, Lie ideals and Posner's theorems, *Pac. J. Math.*, **134** (1988), 275–297. https://doi.org/10.2140/pjm.1988.134.275

12. G. Maksa, A remark on symmetric bi-additive functions having nonnegative diagonalization, *Glasnik Math.*, **15** (1980), 279–282.

13. G. Maksa, On the trace of symmetric biderivations, *C. R. Math. Rep. Acad. Sci. Canada*, **9** (1987), 303–307.

14. A. Mamouni, L. Oukhtite, Z. Mohammed, On derivations involving prime ideals and commutativity in rings, *São Paulo J. Math. Sci.*, **14** (2020), 675–6898. https://doi.org/10.1007/s40863-020-00187-z

15. K. H. Park, On prime and semi-prime rings with symmetric n-derivations, *J. Chungcheong Math. Soc.*, **22** (2009), 451–458. https://doi.org/10.14403/jcms.2009.22.3.451

16. N. Parveen, Product of traces of symmetric bi-derivations in rings, *Palest. J. Math.*, **11** (2022), 210–216.

17. E. C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.*, **8** (1957), 1093–1100.

18. N. Rehman, H. Al-Noghashi, Action of prime ideals on generalized derivations, 2021. https://doi.org/10.48550/arXiv.2107.06769

19. J. Vukman, Symmetric bi-derivations on prime and semiprime rings, *Aeq. Math.*, **38** (1989), 245–254. https://doi.org/10.1007/BF01840009

20. J. Vukman, Commuting and centralizing mappings in prime rings, *Proc. Amer. Math. Soc.*, **109** (1990), 47–52. https://doi.org/10.2307/2048360

21. J. Vukman, Two results concerning symmetric bi-derivations on prime and semiprime rings, *Aeq. Math.*, **40** (1990), 181–189. https://doi.org/10.1007/BF02112294