AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(11): 26731–26744.
DOI: 10.3934/math.20231368
Received: 09 August 2023
Revised: 06 September 2023
Accepted: 10 September 2023
Published: 19 September 2023

Research article

Global attractive periodic solutions of neutral-type neural networks with delays in the leakage terms

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Abstract: In this paper, we introduce a class of neutral-type neural networks with delay in the leakage terms. Using coincidence degree theory, Lyapunov functional method and the properties of neutral operator, we establish some new sufficient criteria for the existence and global attractiveness of periodic solutions. Finally, an example demonstrates our findings.

Keywords: periodic solution; neural networks; neutral-type; leakage terms
Mathematics Subject Classification: 34D23, 92B20

1. Introduction

In this paper, we are devoted to investigating a class of neutral-type neural networks with delay in the leakage terms as follows:

$$(x_i(t) - c_i(t)x_i(t-\sigma))' = -a_i x_i(t-\tau) - b_i \int_0^{\infty} h_i(s)x_i(t-s)ds + \sum_{j=1}^{n} d_{ij} f_j(x_j(t-\gamma(t))) + I_i(t), \quad (1.1)$$

where $i = 1, 2, \cdots, n,\ a_i, b_i$ and d_{ij} are positive constants, $\sigma, \tau > 0$ are constant delays, $c_i(t)$, $\gamma(t)$ and $I_i(t)$ are ω-periodic continuous functions, $h_i \in C(\mathbb{R}^+, \mathbb{R}^+)$, $f_j(\cdot) \in C(\mathbb{R}, \mathbb{R})$, $\mathbb{R}^+ = [0, \infty)$. For more detailed practical significance of system (1.1), refer to [1–3].

Leakage delay or forgetting delay widely exists in network systems, and its research has important practical value. In 2007, Gopalsamy [2] studied the following BAM neural networks with constant delays in the leakage terms:

$$\begin{cases} x_i'(t) = -a_i x_i(t-\tau_i^{(1)}) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t-\sigma_j^{(2)})) + I_i, \\ y_i'(t) = -b_i y_i(t-\tau_i^{(2)}) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\sigma_j^{(1)})) + j_i, \end{cases} \quad (1.2)$$

where $x_i(t-\tau_i^{(1)})$ and $y_i(t-\tau_i^{(2)})$ are leakage terms, $\tau_i^{(1)}$ and $\tau_i^{(2)}$ are called leakage delays. Li and Cao [4] considered delay-dependent stability of neutral-type neural networks with delay in the leakage term as follows:

$$\begin{cases} x_i'(t) &= -Cx(t-\sigma) + Af(x(t-\tau(t))) + B\int_{-\infty}^{t} K(t-s)f(x(s))ds \\ & \quad +Dx'(t-h(t)) + J, \ \ t > 0, \\ x(s) &= \phi(s), \ \ s \in (-\infty, 0], \end{cases} \tag{1.3}$$

where $x(t-\sigma)$ is leakage term, σ is leakage delay. System (1.2) only contains constant delay, system (1.3) contains mixed delays including time-varying delays and continuously distributed delays. Peng [5] investigated the existence and global attractivity of periodic solutions for BAM neural networks with continuously distributed delays in the leakage terms:

$$\begin{cases} x_i'(t) &= -a_i\int_0^{\infty} h_i^{(1)}(s)x_i(t-s)ds + \sum_{j=1}^{p} a_{ij}f_j\Big(\int_0^{\infty} h_{ij}(s)y_j(t-s)ds\Big) + I_i(t), \\ y_j'(t) &= -b_j\int_0^{\infty} h_j^{(2)}(s)y_j(t-s)ds + \sum_{i=1}^{m} b_{ji}g_i\Big(\int_0^{\infty} l_{ji}(s)x_i(t-s)ds\Big) + J_j(t), \end{cases}$$

where $x_i(t-s)$ and $y_j(t-s)$ are leakage terms. In [6], the authors considered existence, uniqueness and the global asymptotic stability of fuzzy cellular neural networks with mixed delays by using the Lyapunov method and the linear matrix inequality approach. For more results about neural networks with delays in the leakage terms, see e.g [7–11].

The dynamic characteristics of the neutral-type neural networks with delay in the leakage terms have been widely studied and can be applied in various fields such as artificial neural networks, intelligent fuzzy recognition, automatic control, etc. In addition, due to the limited speed of information processing, the existence of time delays often leads to oscillation, divergence, or instability of neural networks. Therefore, It is important to consider the impact of delay on network stability. Many authors have extensively studied networks with various types of delays, see [12–16]. Neutral-type neural networks is a system which its delay term contains derivative. In fact, many dynamic systems can be modelled as neutral-type differential systems. Due to the fact that neutral-type systems are more extensive and complex than non neutral systems, they have always been a hot topic of research. Kong and Zhu [17] discussed the finite-time stabilization of a class of discontinuous fuzzy neutral-type neural networks with multiple time-varying delays. Zhang et al. [18] studied synchronization control of neutral-type neural networks with sampled-data via adaptive event-triggered communication scheme. Karthick et al. [19] investigated memory feedback finite-time control for memristive neutral-type neural networks with quantization by using proper Lyapunov Krasovskii functional and linear matrix inequalities. System (1.1) shows the neutral character by the operator $(A_ix_i)(t) = x_i(t) - c_i(t)x_i(t-\sigma)$, $i = 1, \cdots, n$, which is a D-operator form, see [20]. System (1.3) shows the neutral character by the term $Dx'(t-h(t))$ which is different from one in system (1.1). Since the D-operator $(A_ix_i)(t)$ has many important properties, we can conveniently use these properties to study system (1.1). For more results about neutral-type neural networks, see [21–24].

We give the main contributions of this paper as follows:

(1) We first study a class neutral-type neural networks with time delay in the leakage term and D-operator form which is deferent from the existing neutral-type neural networks, see [18, 19].

(2) System (1.1) includes leakage constant delays, time-varying delays and continuously distributed delays which generalizes the corresponding ones of [3–5]. The neural networks with leakage delays has extensive applications in real world. Therefore, our study has important practical value.

(3) Using properties of neutral-type operators, coincidence degree theory, Lyapunov functional method we obtain existence and global attractivity of periodic solutions to system (1.1). Our main results are also valid for the case of non neutral-type neural networks. The research methods in this paper can be used other type dynamic systems and neural networks.

The remainder of this paper are organized as follows: Section 2 gives some basic lemmas which can be used in this paper. In Section 3, we obtain some sufficient conditions for existence of periodic solution of system (1.1). Section 4 gives some sufficient conditions for guaranteeing the global attractivity of periodic solution of system (1.1). In Section 5, an example is given to show the effectiveness of the main results. Finally, some conclusions and discussions are given.

2. Preliminaries

Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces. Let $\mathcal{L} : D(\mathcal{L}) \subset \mathbb{X} \to \mathbb{Y}$ be a Fredholm operator with index zero which means that $Im\mathcal{L}$ is closed in $\mathbb{Y}$ and $dimKer\mathcal{L} = codimIm\mathcal{L} < +\infty$. Let projectors $P : \mathbb{X} \to \mathbb{X}$, $Q : \mathbb{Y} \to \mathbb{Y}$ such that $ImP = Ker\mathcal{L}$, $Im\mathcal{L} = KerQ$. Furthermore, $\mathcal{L}_{D(\mathcal{L})\cap KerP} : (I - P)\mathbb{X} \to Im\mathcal{L}$ is invertible. Denote by K_p the inverse of $\mathcal{L}_P$.

Let Ω be an open bounded subset of $\mathbb{X}$. Let the operator $\mathcal{N} : \bar{\Omega} \to \mathbb{Y}$ be $\mathcal{L}$-compact in $\bar{\Omega}$ which means that $Q\mathcal{N}(\bar{\Omega})$ is bounded and the operator $K_p(I - Q)\mathcal{N}(\bar{\Omega})$ is relatively compact. We first give the famous Mawhin's continuation theorem.

Lemma 2.1. *[25] Assume that $\mathbb{X}$ and $\mathbb{Y}$ are two Banach spaces, and $\mathcal{L} : D(\mathcal{L}) \subset \mathbb{X} \to \mathbb{Y}$, is a Fredholm operator with index zero. Furthermore, $\Omega \subset \mathbb{X}$ is an open bounded set and $\mathcal{N} : \bar{\Omega} \to \mathbb{Y}$ is L-compact on $\bar{\Omega}$. If all the following conditions hold:*

(1) $\mathcal{L}x \neq \lambda\mathcal{N}x, \forall x \in \partial\Omega \cap D(\mathcal{L}), \forall\lambda \in (0, 1)$,
(2) $\mathcal{N}x \notin Im\mathcal{L}, \forall x \in \partial\Omega \cap Ker\mathcal{L}$,
(3) $deg\{Q\mathcal{N}, \Omega \cap Ker\mathcal{L}, 0\} \neq 0$.

Then equation $\mathcal{L}x = \mathcal{N}x$ has a solution on $\bar{\Omega} \cap D(\mathcal{L})$.

Let $C_T = \{x : x \in C(\mathbb{R}, \mathbb{R}), x(t + T) \equiv x(t)\}$ with the norm $\|x\| = \max_{t\in[0,T]} |x(t)|$. Obviously, C_T is a Banach space.

Lemma 2.2. *[26] Let*

$$A : C_T \to C_T, \ \ [Ax](t) = x(t) - c(t)x(t - \tau), \ \ \forall t \in \mathbb{R},$$

where C_T is a $T-$periodic continuous function space, $c(t) \in C_T$, $\tau > 0$ is a constant. If $|c(t)| \neq 1$, then operator A has continuous inverse A^{-1} on C_T, satisfying

(1)

$$[A^{-1}f](t) = \begin{cases} f(t) + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c(t-(i-1)\tau)f(t-j\tau), \ c_0 < 1, \ \forall f \in C_T, \\ -\frac{f(t+\tau)}{c(t+\tau)} - \sum_{j=1}^{\infty}\prod_{i=1}^{j+1}\frac{1}{c(t+i\tau)}f(t+j\tau+\tau), \ \sigma > 1, \ \forall f \in C_T, \end{cases}$$

(2)

$$\int_0^T |[A^{-1}f](t)|dt \leq \begin{cases} \frac{1}{1-c_0}\int_0^T |f(t)|dt, \ c_0 < 1, \ \forall f \in C_T, \\ \frac{1}{\sigma-1}\int_0^T |f(t)|dt, \ \sigma > 1, \ \forall f \in C_T, \end{cases}$$

(3)

$$|A^{-1}f|_0 \le \begin{cases} \frac{1}{1-c_0}|f|_0, \ c_0 < 1, \ \forall f \in C_T, \\ \frac{1}{\sigma-1}|f|_0, \ \sigma > 1, \ \forall f \in C_T, \end{cases}$$

where $c_0 = \max_{t\in[0,T]} |c(t)|$, $\sigma = \min_{t\in[0,T]} |c(t)|$. *Throughout this paper, let*

$$\mathbb{X} = \mathbb{Y} = \{x \in C(\mathbb{R}, \mathbb{R}^n) : x(t+T) = x(t)\}$$

with the norm $\|x\| = \max_{i=1,\cdots,n} |x_i|_0, \quad x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n, \ |x_i|_0 = \max_{t\in[0,T]} |x_i(t)|$.

3. Existence of periodic solutions

For the sake of convenience, we list the following conditions.

(H_1) There exists $p_j > 0$ such that

$$|f_j(x)| \le p_j \ \text{ for } x \in \mathbb{R}, \ j = 1, \cdots, n.$$

(H_2) There exists a constant $D > 0$ such that

$$x_j f_j(x_j) < 0 \text{ for } |x_j| > D, \ j = 1, \cdots, n.$$

Theorem 3.1. *Suppose that* $\int_0^\infty h_i(s)ds = 1$, $\int_0^T I_i(s)ds = 0$, $\int_0^T \psi(s)\psi^T(s)ds \ne \mathbf{0}$, $|c_i(t)| \ne 1$, $1-\gamma'(t) > 0$ *for all* $t \in \mathbb{R}$, *and assumptions* (H_1)–(H_2) *hold, where* $\psi(t)$ *is defined by (3.6). Then system (1.1) has at least one* T*-periodic solution, if*

$$\frac{T(a_i + b_i + \tilde{c})}{1 - c_0} < 1 \quad \text{for } c_0 < \frac{1}{2}, \tag{3.1}$$

or if

$$\frac{T(a_i + b_i + \tilde{c})}{\breve{c} - 1} < 1 \quad \text{for } \breve{c} > 1, \tag{3.2}$$

where $c_0 = \max |c_i(t)|$, $\breve{c} = \min |c_i(t)|$, $\tilde{c} = \max |c_i'(t)|$, $t \in \mathbb{R}$, $i = 1, \cdots, n$.

Proof. Let $(A_i x_i)(t) = x_i(t) - c_i(t)x_i(t-\sigma)$, $i = 1, \cdots, n$. System (1.1) can be rewritten by the following system:

$$(A_i x_i)'(t) = -a_i x_i(t-\tau) - b_i \int_0^\infty h_i(s)x_i(t-s)ds + \sum_{j=1}^n d_{ij} f_j(x_j(t-\gamma(t))) + I_i(t). \tag{3.3}$$

Let

$$\mathcal{L} : D(\mathcal{L}) \subset \mathbb{X} \to \mathbb{Y}, \ \ (\mathcal{L}x)(t) = (Ax)'(t), \tag{3.4}$$

where

$$(\mathcal{L}x)(t) = ((\mathcal{L}_1 x_1)(t), \cdots, (\mathcal{L}_n x_n)(t))^T, \ (Ax)(t) = ((A_1 x_1)(t), \cdots, (A_n x_n)(t))^T, \ (\mathcal{L}_i x_i)(t) = (A_i x_i)'(t).$$

Define $\mathcal{N} : \bar{\Omega} \subset \mathbb{X} \to \mathbb{Y}$ by

$$(\mathcal{N}_i x_i)(t) = -a_i x_i(t-\tau) - b_i \int_0^\infty h_i(s)x_i(t-s)ds + \sum_{j=1}^n d_{ij} f_j(x_j(t-\gamma(t))) + I_i(t), \tag{3.5}$$

where $(\mathcal{N}x)(t) = ((\mathcal{N}_1 x_1)(t), \cdots, (\mathcal{N}_n x_n)(t))^T$. Thus, system (3.3) is equivalent to the following operator system:

$$(\mathcal{L}_i x_i)(t) = (\mathcal{N}_i x_i)(t), \ i = 1, \cdots, n,$$

where $\mathcal{L}_i$ and $\mathcal{N}_i$ are defined by (3.4) and (3.5), respectively. Obviously, $Im\mathcal{L} = \{y : y \in \mathbb{Y}, \ \int_0^T y(t)dt = \mathbf{0}\}$. For each $x \in Ker\mathcal{L}$, we get $\big(x(t) - c(t)x(t-\sigma)\big)' = \mathbf{0}$. Without loss of generality, let

$$x(t) - c(t)x(t-\sigma) = I, \tag{3.6}$$

where I is a $n\times n$ identity matrix. Let $\psi(t) \in \mathbb{R}^n$ be the unique $T-$periodic solution of (3.6), then $\psi(t) \neq \mathbf{0}$ and $Ker\mathcal{L} = \{a\psi(t) : a \in \mathbb{R}^n\}$. It is to see that $Im\mathcal{L}$ is closed in $\mathbb{Y}$ and $\text{dimKer}\mathcal{L} = \text{codimIm}\mathcal{L} = n$. So $\mathcal{L}$ is a Fredholm operator with index zero. Define continuous projectors P and Q by, respectively,

$$P : \mathbb{X} \to \text{Ker}\mathcal{L}, \ (Px)(t) = \frac{\int_0^T x(t)\psi(t)dt}{\int_0^T \psi(t)\psi^T(t)dt}\psi(t)$$

and

$$Q : \mathbb{Y} \to \mathbb{Y}/\text{Im}\mathcal{L}, \ Qy = \frac{1}{T}\int_0^T y(s)ds,$$

where ψ is defined by (3.6). Let

$$\mathcal{L}_P = \mathcal{L}|_{D(\mathcal{L})\cap\text{Ker}P} : D(\mathcal{L}) \cap \text{Ker}P \to \text{Im}\mathcal{L},$$

then

$$\mathcal{L}_P^{-1} = K_p : \text{Im}\mathcal{L} \to D(\mathcal{L}) \cap \text{Ker}P.$$

Since K_p is an embedding operator, then K_p is a completely operator in $\text{Im}\mathcal{L}$. By the definitions of Q and $\mathcal{N}$, it knows that $Q\mathcal{N}(\bar{\Omega})$ is bounded on $\bar{\Omega}$. Hence nonlinear operator $\mathcal{N}$ is $\mathcal{L}$-compact on $\overline{\Omega}$.

For any $\lambda \in (0, 1), \ i = 1, \cdots, n$, consider the following operator system:

$$(\mathcal{L}_i x_i)(t) = \lambda(\mathcal{N}_i x_i)(t),$$

i.e.,

$$(A_i x_i)'(t) = -\lambda a_i x_i(t-\tau) - \lambda b_i \int_0^\infty h_i(s)x_i(t-s)ds + \lambda\sum_{j=1}^n d_{ij} f_j(x_j(t-\gamma(t))) + \lambda I_i(t). \tag{3.7}$$

Integrating both sides of (3.7) on $[0, T]$, one can see

$$\int_0^T \Big[-a_i x_i(t) - b_i \int_0^\infty h_i(s)x_i(t-s)ds + \sum_{j=1}^n \frac{d_{ij}}{1-\gamma'(\tilde{\gamma}(t))} f_j(x_j(t))\Big]dt = 0, \tag{3.8}$$

where $\tilde{\gamma}(t)$ is a inverse function of $t - \gamma(t)$. We claim that there exists a point $t_1 \in [0, T]$ such that

$$|x_i(t_1)| \leq D, \ \ i = 1, \cdots, n, \tag{3.9}$$

where D is defined by assumption (H_2). If $|x(t)| > D$ for all $t \in [0, T]$, then by assumption (H_2) we get

$$\int_0^T \Big[-a_i x_i(t) - b_i \int_0^\infty h_i(s) x_i(t-s) ds + \sum_{j=1}^n \frac{d_{ij}}{1-\gamma'(v(t))} f_j(x_j(t)) \Big] dt \neq 0,$$

which contradicts (3.8). Hence, (3.9) holds. From (3.9) and assumption (H_1), one can see

$$|x_i|_0 \leq D + \int_0^T |x_i'(s)| ds \tag{3.10}$$

and

$$|(A_i x_i)'|_0 \leq (a_i + b_i)|x_i|_0 + \sum_{j=1}^n d_{ij} p_j + |I_i|_0. \tag{3.11}$$

From $(A_i x_i)(t) = x_i(t) - c_i(t) x_i(t-\sigma)$, we have

$$(A_i x_i')(t) = (A_i x_i)'(t) + c_i'(t) x_i(t-\sigma). \tag{3.12}$$

If $c_0 < \frac{1}{2}$, in view of Lemma 2.2, (3.11) and (3.12), we have

$$\begin{aligned} \int_0^T |x_i'(t)| dt &= \int_0^T |(A_i^{-1} A_i x_i')(t)| dt \leq \int_0^T \frac{|(A_i x_i')(t)|}{1-c_0} dt \\ &= \int_0^T \frac{|(A_i x_i)'(t) + c_i'(t) x_i(t-\sigma)|}{1-c_0} dt \\ &\leq \frac{T}{1-c_0}\Big(|(A_i x_i)'|_0 + \tilde{c}|x_i|_0 \Big) \\ &\leq \frac{T(a_i + b_i + \tilde{c})}{1-c_0}|x_i|_0 + \frac{T}{1-c_0}\Big(\sum_{j=1}^n d_{ij} p_j + |I_i|_0 \Big). \end{aligned} \tag{3.13}$$

From (3.10) and (3.13), we have

$$|x_i|_0 \leq D + \frac{T(a_i + b_i + \tilde{c})}{1-c_0}|x_i|_0 + \frac{T}{1-c_0}\Big(\sum_{j=1}^n d_{ij} p_j + |I_i|_0 \Big). \tag{3.14}$$

It follows by (3.1) and (3.14) that

$$|x_i|_0 \leq \frac{(1-c_0)D}{1 - T(a_i + b_i + \tilde{c})} + \frac{T}{1 - T(a_i + b_i + \tilde{c})}\Big(\sum_{j=1}^n d_{ij} p_j + |I_i|_0 \Big) \leq M_1. \tag{3.15}$$

On the other hand, if $\check{c} > 1$, similarly to the above proof, by (3.2), there exists a constant $M_2 > 0$ such that

$$|x_i|_0 \leq M_2. \tag{3.16}$$

From (3.15) and (3.16), we have

$$|x_i|_0 \leq \max\{M_1, M_2\} = M_3. \tag{3.17}$$

Let $\Omega_1 = \{x \in Ker\mathcal{L} : Q\mathcal{N}x = \mathbf{0}\}$. We show that Ω_1 is a bounded set. For each $x \in \Omega_1$, there exists $x_i = \eta_i\psi_i(t)$ such that

$$\int_0^T \Big[-a_i\eta_i\psi_i(t) - b_i\int_0^\infty h_i(s)\eta_i\psi_i(t)(t-s)ds + \sum_{j=1}^n \frac{d_{ij}}{1-\gamma'(\nu(t))} f_j(\eta_j\psi_j(t)(t))\Big]dt = 0, \tag{3.18}$$

where $i = 1, \cdots, n, \eta_i \in \mathbb{R}$. When $c_0 < \frac{1}{2}$, we have

$$\begin{aligned}\psi_i(t) = A_i^{-1}(1) &= 1 + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c_i(t-(i-1)\sigma) \\ &\le 1 + \sum_{j=1}^{\infty}\prod_{i=1}^{j} c_0 \\ &= 1 - \frac{c_0}{1-c_0} := \gamma_1 > 0.\end{aligned}$$

Then, we have $\eta_i \le \frac{D}{\gamma_1}$. Otherwise, for all $t \in [0, T]$, we have $\eta_i\psi_i(t) > D$ and

$$\int_0^T \Big[-a_i\eta_i\psi_i(t) - b_i\int_0^\infty h_i(s)\eta_i\psi_i(t)(t-s)ds + \sum_{j=1}^n \frac{d_{ij}}{1-\gamma'(\nu(t))} f_j(\eta_j\psi_j(t)(t))\Big]dt \ne 0,$$

which contradicts (3.18). On the other hand, when $\check{c} > 1$, we have

$$\begin{aligned}\psi_i(t) = A_i^{-1}(1) &= -\frac{1}{c_i(t+\tau)} - \sum_{j=1}^{\infty}\prod_{i=1}^{j+1} \frac{1}{c_i(t+i\tau)} \\ &\le -\frac{1}{\check{c}} - \sum_{j=1}^{\infty}\prod_{i=1}^{j+1} \frac{1}{\check{c}} \\ &= -\frac{1}{\check{c}-1} := \gamma_2 < 0.\end{aligned}$$

Then, we have $\eta_i \le -\frac{D}{\gamma_2}$. Otherwise, for all $t \in [0, T]$, we have $\eta_i\psi_i(t) < -D$ and

$$\int_0^T \Big[-a_i\eta_i\psi_i(t) - b_i\int_0^\infty h_i(s)\eta_i\psi_i(t)(t-s)ds + \sum_{j=1}^n \frac{d_{ij}}{1-\gamma'(\nu(t))} f_j(\eta_j\psi_j(t)(t))\Big]dt \ne 0,$$

which contradicts (3.18). Hence, Ω_1 is a bounded set. Denote

$$|\eta_i\psi_i(t)| \le M_4 \text{ and } M = \max\{M_3, M_4\} + 1,$$

where M_3 is defined by (3.17). Let $\Omega = \{x \in \mathbb{X} : \|x\| < M\}$. From the above proof, conditions (1) and (2) of Lemma 2.1 hold. Now, we show that condition (3) of Lemma 2.1 holds. Take the homotopy

$$H(x, \kappa) = -\kappa x + (1-\kappa)Q\mathcal{N}x,\ x \in \bar{\Omega} \cap Ker\mathcal{L},\ \kappa \in [0, 1].$$

We claim that

$$H(x, \kappa) \ne \mathbf{0} \text{ for all } x \in \partial\Omega \cap Ker\mathcal{L}. \tag{3.19}$$

If (3.19) does not hold, then

$$\kappa x_i = \frac{1-\kappa}{T}\int_0^T \Big[-a_ix_i(t) - b_i\int_0^\infty h_i(s)x_i(t-s)ds + \sum_{j=1}^n \frac{d_{ij}}{1-\gamma'(\tilde{\gamma}(t))} f_j(x_j(t))\Big]dt. \tag{3.20}$$

Since $|x_i| > D$, then (3.20) does not holds. Hence, (3.19) holds. So we have

$$\begin{aligned} deg\{Q\mathcal{N}, \Omega \cap Ker\mathcal{L}, 0\} &= deg\{H(\cdot, 0), \Omega \cap Ker\mathcal{L}, 0\} \\ &= deg\{H(\cdot, 1), \Omega \cap Ker\mathcal{L}, 0\} \neq 0. \end{aligned}$$

So, condition (3) of Lemma 2.1 holds. Applying Lemma 2.1, we reach the conclusion. □

Remark 3.1. *Lemma 2.1 is critical for estimating the prior bound of solution to system (1.1). In this section, we obtain the existence results of periodic solutions when the neutral operator is stable* ($|c_i(t)| < 1$) *or unstable* ($|c_i(t)| > 1$). *Therefore, the existence results of this article have broader theoretical and practical value.*

4. Global attractivity of periodic solutions

For any $t_0 \geq 0$, we give the initial conditions associated with system (1.1) as follows:

$$x_i(s) = \phi_i(s), \ -\infty < s \leq t_0, \ i = 1, \cdots, n, \tag{4.1}$$

where $\phi_i \in C((-\infty, t_0), \mathbb{R})$ and $\|\phi\| < \infty$.

Definition 4.1. *Suppose that* $\bar{x}(t) = (\bar{x}_1(t), \cdots, \bar{x}_n(t))^T$ *is a periodic solution of system (1.1), and* $x(t) = (x_1(t), \cdots, x_n(t))^T$ *is a any solution of system (1.1). Let* $u_i(t) = x_i(t) - \bar{x}_i(t), \ i = 1, \cdots, n$. *If*

$$\lim_{t \to +\infty} \sum_{i=1}^{n} |u_i(t)| = 0,$$

we call $\bar{x}(t)$ *is globally attractive.*

Theorem 4.1. *Assume that all conditions of Theorem 3.1 hold and* $\int_0^\infty s h_i(s) ds < \infty$. *Suppose that*
(H_3) *There exists* $L_j > 0$ *such that*

$$|f_j(x) - f_j(y)| \leq L_j |x - y| \ \text{ for } x, y \in \mathbb{R}, \ j = 1, \cdots, n.$$

Then system (1.1) has an T-periodic solution which is globally attractive provided that

$$q_i = b_i - 3a_i - \alpha_i - 2b_i^2 \xi_i - 2b_i |c_i|_0 - |c_i|_0 \alpha_i - 3|c_i|_0 a_i - 2\xi_i a_i b_i - 2b_i^2 - a_i b_i - b_i \alpha_i - \rho_i > 0, \tag{4.2}$$

where $i = 1, \cdots, n, \ \xi_i = \int_0^\infty s h_i(s) ds, \ \alpha_i = \sum_{j=1}^n d_{ij} L_j$,

$$\rho_i = \max_{t \in \mathbb{R}} \sum_{j=1}^{n} \Big(d_{ij} L_j + |c_i|_0 + b_i \xi_i \Big) \omega(t),$$

$\omega(t) = \frac{1}{1 - \gamma'(\tilde{\gamma}(t))}$, $\tilde{\gamma}(t)$ *is an inverse function of* $t - \gamma(t)$.

Proof. From Theorem 3.1, system (1.1) has a $T-$periodic solution $\bar{x}(t) = (\bar{x}_1(t), \cdots, \bar{x}_n(t))^T$. Suppose that $x(t) = (x_1(t), \cdots, x_n(t))^T$ is a any solution of system (1.1). Let

$$u_i(t) = x_i(t) - \bar{x}_i(t), \ i = 1, \cdots, n.$$

For $t \geq t_0$, we have

$$(A_i u_i)'(t) = -a_i u_i(t-\tau) - b_i \int_0^\infty h_i(s) u_i(t-s) ds + \sum_{j=1}^{n} d_{ij} \Big[f_j(x_j(t-\gamma(t))) - f_j(\bar{x}_j(t-\gamma(t))) \Big]. \quad (4.3)$$

Let

$$V_i(t) = \Big[(A_i u_i)(t) - b_i \int_0^\infty h_i(s) \int_{t-s}^{t} u_i(\theta) d\theta ds \Big]^2, \; i = 1, \cdots, n. \quad (4.4)$$

Derivation of (4.4) along the solution of (4.3) gives

$$\begin{aligned} V_i'(t) &= 2\Big[(A_i u_i)(t) - b_i \int_0^\infty h_i(s) \int_{t-s}^{t} u_i(\theta) d\theta ds \Big] \Big[(A_i u_i)'(t) - b_i u_i + a_i \int_0^\infty h_i(s) u_i(t-s) ds \Big] \\ &= -2a_i u_i(t) u_i(t-\tau) + 2a_i c_i(t) u_i(t-\sigma) u_i(t-\tau) \\ &\quad - 2b_i u_i(t) \int_0^\infty h_i(s) u_i(t-s) ds + 2b_i c_i(t) u_i(t-\sigma) \int_0^\infty h_i(s) u_i(t-s) ds \\ &\quad + 2(u_i(t) - c_i(t) u_i(t-\sigma)) \sum_{j=1}^{n} d_{ij} \Big[f_j(x_j(t-\gamma(t))) - f_j(\bar{x}_j(t-\gamma(t))) \Big] \\ &\quad - 2b_i u_i^2(t) + 2b_i c_i(t) u_i(t) u_i(t-\sigma) + 2a_i u_i(t) \int_0^\infty h_i(s) u_i(t-s) ds \\ &\quad - 2c_i(t) a_i u_i(t-\sigma) \int_0^\infty h_i(s) u_i(t-s) ds + 2a_i b_i u_i(t-\tau) \int_0^\infty h_i(s) \int_{t-s}^{t} u_i(\theta) d\theta ds \\ &\quad + 2b_i^2 \int_0^\infty h_i(s) u_i(t-s) ds \int_0^\infty h_i(s) \int_{t-s}^{t} u_i(\theta) d\theta ds \\ &\quad - 2b_i \int_0^\infty h_i(s) \int_{t-s}^{t} u_i(\theta) d\theta ds \sum_{j=1}^{n} d_{ij} \Big[f_j(x_j(t-\gamma(t))) - f_j(\bar{x}_j(t-\gamma(t))) \Big] \\ &\quad + 2b_i^2 u_i(t) \int_0^\infty h_i(s) \int_{t-s}^{t} u_i(\theta) d\theta ds - 2a_i b_i \int_0^\infty h_i(s) \int_{t-s}^{t} u_i(\theta) d\theta ds \int_0^\infty h_i(s) u_i(t-s) ds. \end{aligned} \quad (4.5)$$

From (4.5) and a lengthy simplification (by using $a^2 + b^2 \geq 2ab$), we have

$$\begin{aligned} V_i'(t) &\leq -(b_i - a_i - \alpha_i - b_i^2 \xi_i) u_i^2(t) + (b_i |c_i|_0 + |c_i|_0 \alpha_i + |c_i|_0 a_i) u_i^2(t-\sigma) \\ &\quad + (a_i + |c_i|_0 a_i + \xi_i a_i b_i) u_i^2(t-\tau) + \Big(b_i^2 + b_i |c_i|_0 + a_i + a_i |c_i|_0 + b_i^2 \xi_i + a_i b_i \xi_i \Big) \int_0^\infty h_i(s) u_i^2(t-s) ds \\ &\quad + \sum_{j=1}^{n} \Big(d_{ij} L_j + |c_i|_0 + b_i \xi_i \Big) |u_j(t-\gamma(t))|^2 + \Big(a_i b_i + b_i^2 + b_i \alpha_i \Big) \int_0^\infty h_i(s) \int_{t-s}^{t} u_i^2(\theta) d\theta ds. \end{aligned} \quad (4.6)$$

Let

$$V_{\sigma i}(t) = \int_{t-\sigma}^{t} (b_i |c_i|_0 + |c_i|_0 \alpha_i + |c_i|_0 a_i) u_i^2(s) ds, \quad (4.7)$$

$$V_{\tau i}(t) = \int_{t-\tau}^{t} (a_i + |c_i|_0 a_i + \xi_i a_i b_i) u_i^2(s) ds, \quad (4.8)$$

$$V_{\gamma_i}(t) = \sum_{j=1}^{n} \Big(d_{ij} L_j + |c_i|_0 + b_i \xi_i \Big) \int_{t-\gamma(t)}^{t} \omega(s) u_j^2(s) ds, \tag{4.9}$$

$$V_{hi}(t) = \Big(b_i^2 + b_i |c_i|_0 + a_i + a_i |c_i|_0 + b_i^2 \xi_i + a_i b_i \xi_i \Big) \int_0^{\infty} h_i(s) \int_{t-s}^{t} u_i^2(\theta) d\theta ds, \tag{4.10}$$

$$V_{\tilde{h}i}(t) = \Big(a_i b_i + b_i^2 + b_i \alpha_i \Big) \int_0^{\infty} h_i(s) \int_{t-s}^{t} \int_{\nu}^{t} u_i^2(\theta) d\theta d\nu ds. \tag{4.11}$$

From (4.7)–(4.11), we have

$$V'_{\sigma i}(t) = (b_i |c_i|_0 + |c_i|_0 \alpha_i + |c_i|_0 a_i)(u_i^2(t) - u_i^2(t-\sigma)), \tag{4.12}$$

$$V'_{\tau i}(t) = (a_i + |c_i|_0 a_i + \xi_i a_i b_i)(u_i^2(t) - (u_i^2(t-\tau)), \tag{4.13}$$

$$V'_{\gamma_i}(t) = \sum_{j=1}^{n} \Big(d_{ij} L_j + |c_i|_0 + b_i \xi_i \Big) (\omega(t) u_j^2(t) - u_j^2(t-\gamma(t))), \tag{4.14}$$

$$V'_{hi}(t) = \Big(b_i^2 + b_i |c_i|_0 + a_i + a_i |c_i|_0 + b_i^2 \xi_i + a_i b_i \xi_i \Big) \Big(u_i^2(t) - \int_0^{\infty} h_i(s) u_i^2(t-s) ds \Big), \tag{4.15}$$

$$V'_{\tilde{h}i}(t) = \Big(a_i b_i + b_i^2 + b_i \alpha_i \Big) \Big(\xi_i u_i^2(t) - \int_0^{\infty} h_i(s) \int_{t-s}^{t} u_i^2(\theta) d\theta ds \Big). \tag{4.16}$$

Construct the following Lyapunov functional:

$$V(t) = \sum_{i=1}^{n} \Big(V_i(t) + V_{\sigma i}(t) + V_{\tau i}(t) + V_{\gamma i}(t) + V_{hi}(t) + V_{\tilde{h}i}(t) \Big). \tag{4.17}$$

From (4.6) and (4.12)–(4.17), we have

$$V'(t) \le \sum_{i=1}^{n} -q_i u_i^2(t) \text{ for } t \ge t_0, \tag{4.18}$$

where q_i is defined by (4.2). In view of (4.1), we can see $V(t_0) < \infty$. Thus,

$$V(t) \le V(t_0)$$

and

$$\Big| (A_i u_i)(t) - b_i \int_0^{\infty} h_i(s) \int_{t-s}^{t} u_i(\theta) d\theta ds \Big| \le \sqrt{V(t_0)}.$$

Therefore,

$$\big| (A_i u_i)(t) \big| \le \frac{\sqrt{V(t_0)}}{1 - b_i \xi_i \Xi}$$

and

$$|u_i(t)| = \big| (A_i^{-1} A_i u_i)(t) \big| \le \frac{\Xi \sqrt{V(t_0)}}{1 - b_i \xi_i \Xi}, \tag{4.19}$$

where $\Xi = \max\{\frac{1}{1-c_0}, \frac{1}{\tilde{c}-1}\}$. Since $V(t_0)$ can be made arbitrarily small for sufficient small initial values, it follows by (4.19) that the solution of system (1.1) is uniformly bounded on $[t_0, \infty)$. By (4.19) we have

$$V(t) + \int_{t_0}^{t} \sum_{i=1}^{n} q_i u_i^2(s)ds \leq V(t_0) < \infty.$$

In view of Barbalat's lemma (see Gopalsamy [27]), we get

$$u_i(t) \to 0 \text{ as } t \to +\infty, \ i = 1, \cdots, n.$$

This completes the proof of the global attractivity of T-periodic solution of system (1.1). □

Remark 4.1. *Since there is no fixed method for constructing Lyapunov functional, constructing proper Lyapunov functional is very difficult. In this section, utilizing the properties of neutral operators, we construct a new Lyapunov functional which has important application for the proof of Theorem 4.1.*

5. Example

Consider the following two-neuron neural networks with delay in the leakage terms:

$$\begin{cases} (x_1(t) - 0.01 \sin t x_1(t-\pi))' = -0.02 x_1(t-\pi) - 0.5 \int_0^{\infty} h_1(s) x_1(t-s)ds + \sum_{j=1}^{2} d_{1j} f_j(x_j(t-\gamma(t))), \\ (x_2(t) - 0.01 \cos t x_2(t-\pi))' = -0.02 x_2(t-\pi) - 0.5 \int_0^{\infty} h_2(s) x_2(t-s)ds + \sum_{j=1}^{n} d_{2j} f_j(x_j(t-\gamma(t))), \end{cases} \tag{5.1}$$

where

$$c_1(t) = 0.01 \sin 10t, \ c_2(t) = 0.01 \cos 10t, \ \sigma = \tau = \pi, \ a_1 = a_2 = 0.02, \ b_1 = b_2 = 0.5,$$

$$d_{11} = 0.0025, \ d_{12} = 0.0015, \ d_{21} = 0.002, \ d_{22} = 0.001,$$
$$h_1(s) = h_2(s) = \frac{2}{\sqrt{\pi}} e^{-\frac{s^2}{2}}, \ f_1(x) = f_2(x) = \frac{-0.01x}{x^2+1}, \ \gamma(t) = 0.005 \cos 10t.$$

Obviously, we have

$$T = \frac{\pi}{5}, \ c_0 = |c_1|_0 = |c_2|_0 = 0.01, \ \tilde{c} = 0.1, \ L_1 = L_2 = 0.01, \ \xi_1 = \xi_2 \approx 0.55,$$

$$\alpha_1 = 4 \times 10^{-5}, \ \alpha_2 = 3 \times 10^{-5}, \ \eta_1 \approx 0.273, \ \eta_2 \approx 0.264.$$

When $c_0 = 0.01 < \frac{1}{2}$, we have

$$\frac{T(a_i + b_i + \tilde{c})}{1 - c_0} \approx 0.39 < 1, \ i = 1, 2.$$

Thus, condition (3.1) in Theorem 3.1 holds and system (5.1) exists a periodic solution $x = (0,0)^T$. Furthermore, we have $q_1 \approx 0.122 > 0$ and $q_2 \approx 0.113 > 0$ and condition (4.2) in Theorem 4.1 holds and the periodic solution $x = (0,0)^T$ of system (5.1) is globally attractive. Figure 1 shows that the state trajectory of the system (5.1). From Figure 1, we find that the periodic solution $x = (0,0)^T$ of system (5.1) is globally attractive which verifies the correctness of the conclusions of Theorems 3.1 and 4.1.

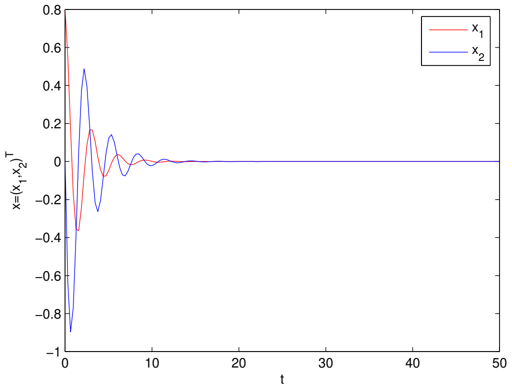

Figure 1. The states' trajectory of the system (5.1).

6. Conclusions

In this paper, we have dealt with the existence and global attractivity of periodic solution to a class of neutral-type neural networks with delay in the leakage terms. System (1.1) contains mixed delays including leakage constant delays, time-varying delays and continuously distributed delays. As is well known, delay has a fundamental impact on the dynamic behavior of network systems. The leakage delay is a special type of delay, and its research has important theoretical and practical value. From main results of this paper, we find that the leakage term has great impact on the dynamical properties of neural networks. our results greatly generalize the corresponding ones of [3–6]. Based on coincidence degree theory and the Lyapunov functional, some sufficient conditions ensuring the existence, and global attractivity of periodic solution have been presented. It should be pointed out that by utilizing the properties of neutral operators, we have constructed a new Lyapunov functional that can conveniently obtain the dynamic behavior of periodic solutions. A numerical example has been illustrated to demonstrate the usefulness of the proposed method. The research method in this article can be used to study various types of systems, neutral-type neural networks with delay in the leakage terms, for example, neutral-type neural networks with impulses and delay in the leakage terms, neutral-type neural networks with stochastic disturbances and delay in the leakage terms, and so on.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This paper is supported by Anhui University Natural Sciences fund: Kj2019A0703.

Conflict of Interest

The authors declare no conflict of interest.

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