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Research article

The existence of positive solutions for high order fractional differential equations with sign changing nonlinearity and parameters

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Abstract: By constructing an auxiliary boundary value problem, the difficulty caused by sign changing nonlinearity terms is overcome by means of the linear superposition principle. Using the Guo-Krasnosel'skii fixed point theorem, the results of the existence of positive solutions for boundary value problems of high order fractional differential equation are obtained in different parameter intervals under a more relaxed condition compared with the existing literature. As an application, we give two examples to illustrate our results.

Keywords: nonlinear fractional differential equations; Riemann-Liouville derivatives; boundary value problems; sign changing nonlinearity; positive solutions
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1. Introduction

Fractional differential equation theory comes with fractional calculus and is an abstract form of many engineering and physical problems. It has been widely used in system control, system identification, grey system theory, fractal and porous media dispersion, electrolytic chemistry, semiconductor physics, condensed matter physics, viscoelastic systems, biological mathematics, statistics, diffusion and transport theory, chaos and turbulence and non-newtonian fluid mechanics. Fractional differential equation theory has attracted the attention of the mathematics and natural science circles at home and abroad, and has made a series of research results. It has become one of the international hot research directions and has very important theoretical significance and application value.

As an important research area of fractional differential equation, boundary value problems have attracted a great deal of attention in the last ten years, especially in terms of the existence of positive solutions, and have achieved a lot of results (see [1–20]). When the nonlinear term changes sign, the research on the existence of positive solutions progresses slowly, and relevant research results are not many (see [21–33]).

In [21], using a fixed point theorem in a cone, Agarwal et al. obtained the existence of positive solutions for the Sturm-Liouville boundary value problem

$$\begin{cases} (p(t)u'(t))' + \lambda f(t,u(t)) = 0, t \in (0,1), \\ \alpha_1 u(0) - \beta_1 p(0)u'(0) = 0, \\ \alpha_2 u(1) + \beta_2 p(0)u'(1) = 0, \end{cases}$$

where $\lambda > 0$ is a parameter, $p(t) \in C((0,1),[0,\infty))$, $\alpha_i,\beta_i \geq 0$ for $i = 1,2$ and $\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 > 0$; $f \in C((0,1) \times [0,\infty), R)$ and $f \geq -M$, for $M > 0, \forall t \in [0,1], u \geq 0$ (M is a constant).

In [22], Weigao Ge and Jingli Ren studied the Sturm-Liouville boundary value problem

$$\begin{cases} (p(t)u'(t))' + \lambda a(t)f(t,u(t)) = 0, t \in (0,1), \\ \alpha_1 u(0) - \beta_1 p(0)u'(0) = 0, \\ \alpha_2 u(1) + \beta_2 p(0)u'(1) = 0, \end{cases}$$

where $a(t) \geq 0$ and $\lambda > 0$ is a parameter. They removed the restriction $f \geq -M$, using Krasnosel'skii theorem, obtained some new existence theorems for the Sturm-Liouville boundary value problem.

In [23], Weigao Ge and Chunyan Xue studied the same Sturm-Liouville boundary value problem again. Without the restriction that f is bounded below, by the excision principle and area addition principle of degree, they obtained three theorems and extended the Krasnosel'skii's compression-expansion theorem in cones.

In [25], Yongqing Wang et al. considered the nonlinear fractional differential equation boundary value problem with changing sign nonlinearity

$$\begin{cases} D^{\alpha}_{0^+}u(t) + \lambda f(t,u(t)) = 0, t \in (0,1), \\ u(0) = u'(0) = u(1) = 0, \end{cases}$$

where $2 < \alpha \leq 3$, $\lambda > 0$ is a parameter, $D^{\alpha}_{0^+}$ is the standard Riemann-Liouville fractional derivative. f is allowed to change sign and may be singular at $t = 0,1$ and $-r(t) \leq f \leq z(t)g(x)$ for some given nonnegative functions r,z,g. By using Guo-Krasnosel'skii fixed point theorem, the authors obtained the existence of positive solutions.

In [28], J. Henderson and R. Luca studied the existence of positive solutions for a nonlinear Riemann-Liouville fractional differential equation with a sign-changing nonlinearity

$$\begin{cases} D^{\alpha}_{0^+}u(t) + \lambda f(t,u(t)) = 0, t \in (0,1), \\ \quad u(0) = u'(0) \cdots = u^{(n-2)}(0) = 0, \\ \quad D^{p}_{0^+}u(t)\Big|_{t=1} = \sum\limits_{i=1}^{m} a_i D^{q}_{0^+}u(t)\Big|_{t=\xi_i}, \end{cases}$$

where λ is a positive parameter, $\alpha \in (n-1,n], n \in N, n \geq 3, \xi_i \in R$ for all $i = 1, \ldots m,$ $(m \in N), 0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1, p,q \in R, p \in [1,n-2], q \in [0,p]$, $D^{\alpha}_{0^+}$ is the standard Riemann-Liouville fractional derivative. With the restriction that f may be singular at $t = 0,1$ and $-r(t) \leq f \leq z(t)g(t,x)$ for some given nonnegative functions r,z,g, applying Guo-Krasnosel'skii fixed point theorem, the existences of positive solutions are obtained.

In [31], Liu and Zhang studied the existence of positive solutions to the boundary value problem for a high order fractional differential equation with delay and singularities including changing sign nonlinearity

$$\begin{cases} D_{0^+}^{\alpha}x(t) + f(t, x(t-\tau)) = 0,\ t \in (0,1)\backslash\{\tau\}, \\ x(t) = \eta(t),\ t \in [-\tau, 0], \\ x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0,\ n \geq 3, \\ x^{(n-2)}(1) = 0, \end{cases}$$

where $n-1 < \alpha \leq n, n = [\alpha]+1, D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative. The restriction on the nonlinearity f is as follows: there exists a nonnegative function $\rho \in C(0,1) \cap L(0,1)$, $\rho(t) \not\equiv 0$, such that $f(t,x) \geq -\rho(t)$ and $\varphi_2(t)h_2(x) \leq f(t, v(t)x) + \rho(t) \leq \varphi_1(t)(g(x) + h_1(x))$, for $\forall\ (t,x) \in (0,1) \times R^+$, where $\varphi_1,\ \varphi_2 \in L(0,1)$ are positive, $h_1,\ h_2 \in C(R_0^+, R^+)$ are nondecreasing, $g \in C(R_0^+, R^+)$ is nonincreasing, $R_0^+ = [0, +\infty)$, and

$$v(t) = \begin{cases} 1,\ t \in (0, \tau], \\ (t-\tau)^{\alpha-2n+1},\ t \in (\tau, 1). \end{cases}$$

By Guo-krasnosel'skii fixed point theorem and Leray-Schauder's nonlinear alternative theorem, some existence results of positive solutions are obtained, respectively.

In [33], Tudorache and Luca considered the nonlinear ordinary fractional differential equation with sequential derivatives

$$\begin{cases} D_{0^+}^{\beta}(q(t)D_{0^+}^{\gamma}u(t)) = \lambda f(t, u(t)),\ t \in (0,1), \\ u^{(j)}(0) = 0, j = 0,\ 1 \cdots,\ n-2,\ D_{0^+}^{\gamma}u(0) = 0, \\ q(1)D_{0^+}^{\gamma}u(1) = \int_0^1 q(t)D_{0^+}^{\gamma}u(t)d\eta_0(t),\ D_{0^+}^{\alpha_0}u(1) = \sum_{i=1}^{p}\int_0^1 D_{0^+}^{\alpha_i}u(t)d\eta_i(t), \end{cases}$$

where $\beta \in (1,2],\ \gamma \in (n-1, n],\ n \in N, n \geq 3,\ p \in N, \alpha_i \in R, i = 0, 1 \cdots p,\ 0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leq \alpha_0 < \gamma - 1,\ \alpha_0 \geq 1,\ \lambda > 0,\ q : [0,1] \mapsto (0, \infty)$ is a continuous function, $f \in C((0,1) \times [0,\infty), R)$ may be singular at $t = 0$ and/or $t = 1$, and there exist the functions $\xi, \phi \in C((0,1),\ [0,\infty))$, $\varphi \in C((0,1) \times [0,\infty),\ [0,\infty))$ such that $-\xi(t) \leq f(t,x) \leq \phi(t)\varphi(t,x),\ \forall t \in (0,1),\ x \in (0,\infty)$ with $0 < \int_0^1 \xi(s)ds < \infty, 0 < \int_0^1 \phi(s)ds < \infty$. By the Guo-Krasnosel'skii fixed point theorem, the existence of positive solutions are obtained.

As can be seen from the above research results, fixed point theorems are still common tools to solve the existence of positive solutions to boundary value problems with sign changing nonlinearity, especially the Guo-Krasnosel'skii fixed point theorem. In addition, for boundary value problems of ordinary differential equations, Weigao Ge et al. removed the restriction that the nonlinear item bounded below. However, for fractional boundary value problems, from the existing literature, there are still many restrictions on nonlinear terms.

Our purpose of this paper is to establish the existence of positive solutions of boundary value problems (BVPs for short) of the nonlinear fractional differential equation as follows

$$\begin{cases} D_{0^+}^{\alpha}u(t) + \lambda f(t, u(t)) = 0,\ t \in (0,1), \\ u(0) = u'(0) \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0,\ n \geq 3, \end{cases} \tag{1.1}$$

where $n-1<\alpha<n$, $\lambda>0$, $f:[0,1]\times[0,+\infty)\to\mathbb{R}$ is a known continuous nonlinear function and allowed to change sign, and $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative.

In this paper, by the Guo-Krasnosel'skii fixed point theorem, the sufficient conditions for the existence of positive solutions for BVPs (1.1) are obtained under a more relaxed condition compared with the existing literature, as follows. Throughout this paper, we suppose that the following conditions are satisfied.

H_0: There exists a known function $\omega\in C(0,1)\cap L(0,1)$ with $\omega(t)>0$, $t\in(0,1)$ and $\int_0^1(1-s)^{\alpha-2}\omega(s)ds<+\infty$, such that $f(t,u)>-\omega(t)$, for $t\in(0,1)$, $u\in\mathbb{R}$.

This paper is organized as follows. In Section 2, we introduce some definitions and lemmas to prove our major results. In Section 3, some sufficient conditions for the existence of at least one and two positive solutions for BVPs (1.1) are investigated. As applications, some examples are presented to illustrate our major results in Section 4.

2. Preliminaries

In this section, we give out some important definitions, basic lemmas and the fixed point theorem that will be used to prove the major results.

Definition 2.1. (see [1]) *Let $\varphi(x)\in L^1(a,b)$. The integrals*

$$(I_{a+}^{\alpha}\varphi)(x)\overset{def}{=}\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}\varphi(t)\,dt,\ x>a,$$

$$(I_{b-}^{\alpha}\varphi)(x)\overset{def}{=}\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}\varphi(t)\,dt,\ x<a,$$

where $\alpha>0$, are called the Riemann-Liouville fractional integrals of the order α. They are sometimes called left-sided and right-sided fractional integrals respectively.

Definition 2.2. (see [1]) *For functions $f(x)$ given in the interval $[a,b]$, each of the expressions*

$$(D_{a+}^{\alpha}f)(x)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_a^x(x-t)^{n-\alpha-1}f(t)\,dt,n=[\alpha]+1,$$

$$(D_{b-}^{\alpha}f)(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_x^b(t-x)^{n-\alpha-1}f(t)\,dt,n=[\alpha]+1$$

is called Riemann-Liouville derivative of order α, $\alpha>0$, left-handed and right-handed respectively.

Definition 2.3. (see [2]) *Let E be a real Banach space. A nonempty, closed, and convex set $P\subset E$ is called a cone if the following two conditions are satisfied:*

(1) if $x\in P$ and $\mu\geq0$, then $\mu x\in P$;

(2) if $x\in P$ and $-x\in P$, then $x=0$.

Every cone $P\subset E$ induces the ordering in E given by $x_1\leq x_2$ if and only if $x_2-x_1\in P$.

Lemma 2.1. (see [3]) *Let $\alpha>0$, assume that $u, D_{0^+}^{\alpha}u\in C(0,1)\cap L^1(0,1)$, then,*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_nt^{\alpha-n}$$

holds for some $C_i\in\mathbb{R}, i=1,2,\ldots,n$, where $n=[\alpha]+1$.

Lemma 2.2. *Let $y\in C[0,1]$ and $n-1<\alpha<n$. Then, the following BVPs*

$$\begin{cases} D_{0^+}^{\alpha}u(t) + y(t) = 0, 0 < t < 1, \\ u(0) = u'(0) \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0, n \geq 3 \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-n+1} - (t-s)^{\alpha-1}, 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-n+1}, 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Proof. From Definitions 2.1 and 2.2, Lemma 2.1, we know

$$\begin{aligned} u(t) &= -I_{0+}^{\alpha}y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n} \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}, \end{aligned}$$

where $C_i \in R, i = 1, 2 \cdots n$.
From $u(0) = u'(0) \cdots = u^{(n-2)}(0) = 0$, we get $C_i = 0,\ i = 2, 3 \cdots n$, such that

$$\begin{aligned} u^{(n-2)}(t) &= -\frac{1}{\Gamma(\alpha-n+2)}\int_0^t (t-s)^{\alpha-n+1}y(s)ds + C_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-n+2)}t^{\alpha-n+2}, \\ u^{(n-2)}(1) &= -\frac{1}{\Gamma(\alpha-n+2)}\int_0^1 (1-s)^{\alpha-n+1}y(s)ds + C_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-n+2)}. \end{aligned}$$

From $u^{(n-2)}(1) = 0$, we get $C_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-n+1}y(s)ds$, so that

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-n+1}y(s)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t [t^{\alpha-1}(1-s)^{\alpha-n+1} - (t-s)^{\alpha-1}]y(s)ds + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-n+1}y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds. \end{aligned}$$

The proof is completed. □

Lemma 2.3. *Let* $n-1 < \alpha < n$. *The function* $G(t,s)$ *defined by (2.2) is continuous on* $[0,1]\times[0,1]$ *and satisfies* $0 \leq G(t,s) \leq G(1,s)$ *and* $G(t,s) \geq t^{\alpha-1}G(1,s)$ *for* $t, s \in [0,1]$.

Proof. From the definition (2.2), it's easy to know $G(t,s)$ is continuous on $[0,1]\times[0,1]$. Next, we prove that $G(t,s)$ satisfies $0 \leq G(t,s) \leq G(1,s)$.

For $0 \leq s \leq t \leq 1$,

$$\frac{\partial G(t,s)}{\partial t} = \frac{1}{\Gamma(\alpha)}(\alpha-1)(t-s)^{\alpha-2}\left[\frac{t^{\alpha-2}(1-s)^{\alpha-n+1}}{t^{\alpha-2}(1-\frac{s}{t})^{\alpha-2}} - 1\right]$$

$$
\begin{aligned}
&\geq \frac{1}{\Gamma(\alpha)}(\alpha-1)(t-s)^{\alpha-2}[(1-s)^{3-n}-1] \\
&\geq 0(n \geq 3).
\end{aligned}
$$

For $0 \leq t \leq s \leq 1$, obviously, $\frac{\partial G(t,s)}{\partial t} \geq 0$. Such that, $G(t,s)$ is an increasing function of t and satisfies $0 \leq G(t,s) \leq G(1,s)$.

At last, we prove that $G(t,s)$ satisfies $G(t,s) \geq t^{\alpha-1}G(1,s)$.

For $0 \leq s \leq t \leq 1$,

$$
\begin{aligned}
& G(t,s) - t^{\alpha-1}G(1,s) \\
= & \frac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-n+1} - (t-s)^{\alpha-1}] - \frac{t^{\alpha-1}}{\Gamma(\alpha)}[(1-s)^{\alpha-n+1} - (1-s)^{\alpha-1}] \\
= & \frac{1}{\Gamma(\alpha)}[(t-ts)^{\alpha-1} - (t-s)^{\alpha-1}] \\
\geq & 0.
\end{aligned}
$$

For $0 \leq t \leq s \leq 1$,

$$
\frac{G(t,s)}{G(1,s)} = \frac{t^{\alpha-1}(1-s)^{\alpha-n+1}}{(1-s)^{\alpha-n+1} - (1-s)^{\alpha-1}} \geq \frac{t^{\alpha-1}(1-s)^{\alpha-n+1}}{(1-s)^{\alpha-n+1}} = t^{\alpha-1}.
$$

The proof is completed. □

At the end of this section, we present the Guo-Krasnosel'skii fixed point theorem that will be used in the proof of our main results.

Lemma 2.4. (see [34]) *Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. Let $F : P \to P$ be a comletely continuous operator such that either*

1) $\|Fx\| \leq \|x\|, x \in P \cap \partial\Omega_1, \|Fx\| \geq \|x\|, x \in P \cap \partial\Omega_2$; *or*

2) $\|Fx\| \geq \|x\|, x \in P \cap \partial\Omega_1, \|Fx\| \leq \|x\|, x \in P \cap \partial\Omega_2$;

holds. Then, F has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Existence of the positive solution

By a positive solution of BVPs (1.1), we mean a function $u : [0,1] \to [0,+\infty)$ such that $u(t)$ satisfies (1.1) and $u(t) > 0$ for $t \in (0,1)$.

Let Banach space $E = C[0,1]$ be endowed with $\|x\| = \max_{t\in[0,1]} |x(t)|$. Let $I = [0,1]$, define the cone $P \subset E$ by

$$
P = \{x \in E : x(t) \geq t^{\alpha-1}\|x\|, t \in I\}.
$$

Lemma 3.1. *Let $\lambda > 0$, $\omega \in C(0,1) \cap L(0,1)$ with $\omega(t) > 0$ on $(0,1)$, and $n-1 < \alpha < n$. Then, the following boundary value problem of fractional differential equation*

$$
\begin{cases} D_{0^+}^{\alpha} v(t) + \lambda\omega(t) = 0, 0 < t < 1, \\ v(0) = v'(0) \cdots = v^{(n-2)}(0) = v^{(n-2)}(1) = 0, n \geq 3 \end{cases} \tag{3.1}
$$

has a unique solution

$$v(t) = \lambda \int_0^1 G(t,s)\omega(s)ds \tag{3.2}$$

and

$$0 \le v(t) \le \lambda t^{\alpha-1} M, \tag{3.3}$$

where

$$M = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-n+1}\omega(s)\,ds.$$

Proof. From Lemma 2.2, let $y(t) = \lambda\omega(t)$, we have (3.2) immediately. In view of Lemma 2.3, we obtain

$$\begin{aligned} 0 \le v(t) &= \lambda \int_0^1 G(t,s)\omega(s)ds \\ &= \lambda \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-n+1}\omega(s)\,ds - \lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\omega(s)\,ds \\ &\le \lambda \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-n+1}\omega(s)\,ds \\ &= \lambda t^{\alpha-1} M. \end{aligned} \tag{3.4}$$

From (3.4), (3.3) holds.

The proof is completed. □

Lemma 3.2. *Suppose that $v = v(t)$ is the solution of BVPs (3.1) and define the function $g(t,u(t))$ by*

$$g(t,u(t)) = f(t,u(t)) + \omega(t). \tag{3.5}$$

Then, $u(t)$ is the solution of BVPs (1.1), if and only if $x(t) = u(t) + v(t)$ is the solution of the following BVPs

$$\begin{cases} D_{0^+}^{\alpha} x(t) + \lambda g(t, x(t) - v(t)) = 0 \\ x(0) = x'(0) \cdots = x^{(n-2)}(0) = x^{(n-2)}(1) = 0, \ n \ge 3. \end{cases} \tag{3.6}$$

And when $x(t) > v(t)$, $u(t)$ is a positive solution of BVPs(1.1).

Proof. In view of Lemma 2.2, if $u(t)$ and $v(t)$ are the solutions of BVPs (1.1) and BVPs (3.1), respectively, we have

$$\begin{aligned} D_{0^+}^{\alpha}(u(t) + v(t)) &= D_{0^+}^{\alpha} u(t) + D_{0^+}^{\alpha} v(t) \\ &= -\lambda f(t,u(t)) - \lambda\omega(t) \\ &= -\lambda[f(t,u(t)) + \omega(t)] \\ &= -\lambda g(t,u(t)), \end{aligned}$$

such that

$$D_{0^+}^{\alpha}(u(t) + v(t)) + \lambda g(t,u(t)) = 0.$$

Let $x(t) = u(t) + v(t)$, we have $u(t) = x(t) - v(t)$ and

$$D_{0^+}^{\alpha}x(t) + \lambda g(t, x(t) - v(t)) = 0.$$

It is easily to obtain $x(0) = x'(0) = x'(1) = 0$ from the boundary conditions of BVPs (1.1) and BVPs (3.1).

Hence, $x(t)$ is the solution of BVPs (3.6).

On the other hand, if $v(t)$ and $x(t)$ are the solution of BVPs (3.1) and BVPs (3.6), respectively. Similarly, $u(t) = x(t) - v(t)$ is the solution of BVPs (1.1). Obviously, when $x(t) > v(t)$, $u(t) > 0$ is a positive solution of BVPs (1.1).

The proof is completed.

□

Lemma 3.3. *Let $T : P \to E$ be the operator defined by*

$$Tx(t) := \lambda \int_0^1 G(t, s)g(s, x(s) - v(s))ds. \tag{3.7}$$

Then, $T : P \to P$ is comletely continuous.

Proof. In view of the definition of the function $g(t, u(t))$, we know that $g(t, x(t)-v(t)) > 0$ is continuous from the continuity of $x(t)$ and $v(t)$.

By Lemma 2.3, we obtain

$$\|Tx\| = \max_{t\in[0,1]} |\lambda \int_0^1 G(t, s)g(s, x(s) - v(s))ds| = \lambda \int_0^1 G(1, s)g(s, x(s) - v(s))ds.$$

So that, for $t \in [0, 1]$,

$$Tx(t) = \lambda \int_0^1 G(t, s)g(s, x(s) - v(s))ds \geq t^{\alpha-1}\lambda \int_0^1 G(1, s)g(s, x(s) - v(s))ds = t^{\alpha-1}\|Tx\|.$$

Thus, $T(P) \subset P$.

As the continuity and nonnegativeness of $G(t, s)$ and H_0 implies T is a continuous operator.

Let $\Omega \subset P$ be bounded, there exists a positive constant $r > 0$, such that $|x| \leq r$, for all $x \in \Omega$. Set $M_0 = \max_{0\leq x\leq r, t\in I} | f(t, x(t) - v(t)) |$, then,

$$|g(t, x(t) - v(t))| \leq |f(t, x(t) - v(t))| + |\omega(t)| \leq M_0 + \omega(t).$$

So, for $x \in \Omega$ and $t \in [0, 1]$, we have

$$\begin{aligned} |Tx(t)| &= \left|\lambda \int_0^1 G(t, s)g(s, x(s) - v(s))ds\right| \\ &\leq \lambda\left(M_0 \int_0^1 G(1, s)ds + \int_0^1 G(1, s)\omega(s)ds\right) \\ &\leq \lambda\left(M_0 \int_0^1 G(1, s)ds + \frac{1}{\Gamma(\alpha)} \int_0^1 \omega(s)ds\right). \end{aligned}$$

Hence, T is uniformly bounded.

On the other hand, since $G(t,s) \in C([0,1]\times[0,1])$, for $\varepsilon > 0$, exists $\delta > 0$, for $t_1, t_2 \in [0,1]$ with $| t_1 - t_2 | \leq \delta$, implies $|G(t_1,s) - G(t_2,s)| < \frac{\varepsilon}{\lambda\left(M_0+\int_0^1 \omega(s)ds\right)}$, for $s \in [0,1]$.

Then, for all $x \in \Omega$:

$$\begin{aligned}
&|Tx(t_1) - Tx(t_2)| \\
&= \left|\lambda \int_0^1 G(t_1,s)g(s,x(s)-v(s))ds - \lambda \int_0^1 G(t_2,s)g(s,x(s)-v(s))ds\right| \\
&= \left|\lambda \int_0^1 (G(t_1,s)-G(t_2,s))g(s,x(s)-v(s))ds\right| \\
&\leq \lambda \int_0^1 |G(t_1,s)-G(t_2,s)|\,|g(s,x(s)-v(s))|ds \\
&\leq \lambda \int_0^1 |G(t_1,s)-G(t_2,s)|\,(M_0+\omega(s))\,ds \\
&< \lambda \int_0^1 \frac{\varepsilon}{\lambda\left(M_0+\int_0^1 \omega(s)ds\right)}\,(M_0+\omega(s))\,ds \\
&\leq \lambda \frac{\varepsilon}{\lambda\left(M_0+\int_0^1 \omega(s)ds\right)} \int_0^1 (M_0+\omega(s))\,ds = \varepsilon.
\end{aligned}$$

Hence, $T(\Omega)$ is equicontinuous. By Arzelà-Ascoli theorem, we have $T : P \to P$ is completely continuous.

The proof is completed. □

A function $x(t)$ is said to be a solution of BVPs (3.6) if $x(t)$ satisfies BVPs (3.6). In addition, if $x(t) > 0$, for $t \in (0,1)$, $x(t)$ is said to be a positive solution of BVPs (3.6). Obviously, if $x(t) \in P$, and $x(t) \neq 0$ is a solution of BVPs (3.6), by $x(t) \geq t^{\alpha-1}|x|$, then $x(t)$ is a positive solution of BVPs (3.6). By Lemma 3.2, if $x(t) > v(t)$, $u(t) = x(t) - v(t)$ is a positive solution of BVPs (1.1).

Next, we give some sufficient conditions for the existence of positive solutions.

Theorem 3.1. *For a given* $0 < \eta < 1$*, let* $I_\eta = [\eta, 1]$*. If*
$\mathbf{H}_1$: $\lim\limits_{x\to+\infty} \inf\limits_{t\in I_\eta} \frac{f(t,x)}{x} = +\infty$
holds, there exists $\lambda^* > 0$, for any $0 < \lambda < \lambda^*$, the BVPs (1.1) has at least one positive solution.

Proof. By Lemma 3.2, if BVPs (3.6) has a positive solution $x(t)$ and $x(t) > v(t)$, BVPs (1.1) has a positive solution $u(t) = x(t) - v(t)$. We will apply Lemma 2.4 to prove the theorem.

In view of the definition of $g(t,u(t))$, we have $g(t,u(t)) \geq 0$, so that BVPs (3.6) has a positive solution, if and only if the operator T has a fixed point in P.

Define

$$g_1(r_1) = \sup_{t\in I, 0\leq x\leq r_1} g(t,x),$$

where $r_1 > 0$.

By the definition of $g_1(r_1)$ and H_1, we have

$$\lim_{r_1\to+\infty} \frac{r_1}{g_1(r_1)} = 0.$$

Then, there exists $R_1 > 0$, such that

$$\frac{R_1}{g_1(R_1)} = \max_{r_1>0} \frac{r_1}{g_1(r_1)}.$$

Let $L = g_1(R_1)$, $\lambda^* = \min\{\frac{R_1}{M}, \frac{(\alpha-1)\Gamma(\alpha+1)R_1}{L}\}$, where $\int_0^1 G(1,s)ds = \frac{1}{(\alpha-1)\Gamma(\alpha+1)}$.

In order to apply Lemma 2.4, we separate the proof into the following two steps.

Step 1:

For every $0 < \lambda < \lambda^*$, $t \in I$ let $\Omega_1 = \{x \in E : \|x\| < R_1\}$. Suppose $x \in P \cap \partial\Omega_1$, we obtain

$$\begin{aligned} R_1 &\geq x(t) - v(t) \geq t^{\alpha-1}\|x\| - \lambda t^{\alpha-1}M \\ &> t^{\alpha-1}R_1 - \frac{R_1}{M}t^{\alpha-1}M \\ &> 0. \end{aligned}$$

So that

$$g(t, x(t) - v(t)) \leq g_1(R_1) = L$$

and

$$\begin{aligned} Tx(t) &= \lambda \int_0^1 G(t,s)g(s, x(x) - v(s))ds \\ &\leq \lambda \int_0^1 G(1,s)g(s, x(s) - v(s))ds \\ &\leq \lambda^* \int_0^1 G(1,s)g_1(R_1)ds = \lambda^* L \int_0^1 G(1,s)ds \\ &< \frac{(\alpha-1)\Gamma(\alpha+1)R_1}{L} \frac{L}{(\alpha-1)\Gamma(\alpha+1)} \\ &= R_1. \end{aligned}$$

Therefore,

$$\|Tx\| < \|x\|, \ x \in P \cap \partial\Omega_1.$$

Step 2:

From H_1, we know that

$$\lim_{x\to+\infty} \inf_{t\in I_\eta} \frac{g(t,x)}{x} = \lim_{x\to+\infty} \inf_{t\in I_\eta} \frac{f(t,x)+\omega(t)}{x} = +\infty.$$

Then, there exists $R_2 > (1+\eta^{1-\alpha})R_1 > R_1$, such that for all $t \in I_\eta$, when $x > \frac{R_2}{1+\eta^{1-\alpha}}$,

$$g(t,x) > \delta x,$$

where $\delta > \frac{1+\eta^{1-\alpha}}{\lambda N} > 0$, $N = \int_\eta^1 G(1,s)ds$.

Let $\Omega_2 = \{x \in E : \|x\| < R_2\}$, for all $x \in P \cap \partial\Omega_2$, $t \in I_\eta$ we have

$$\begin{aligned} x(t) - v(t) &\geq t^{\alpha-1}R_2 - \lambda t^{\alpha-1}M \\ &> t^{\alpha-1}R_2 - \lambda^* t^{\alpha-1}M \\ &\geq t^{\alpha-1}R_2 - t^{\alpha-1}R_1 \\ &\geq \eta^{\alpha-1}(R_2 - R_1) \end{aligned}$$

$$= \frac{R_2}{1+\eta^{1-\alpha}} > 0.$$

So that

$$g(t, x(t) - v(t)) > \delta(x(t) - v(t)) > \delta \tfrac{R_2}{1+\eta^{1-\alpha}}$$

and

$$\begin{aligned}
\|Tx\| &= \max_{t\in I} \lambda \int_0^1 G(t,s) g(s, x(s) - v(s)) ds \\
&= \lambda \int_0^1 G(1,s) g(s, x(s) - v(s)) ds \\
&> \lambda \int_\eta^1 G(1,s) g(s, x(s) - v(s)) ds \\
&> \lambda\delta \frac{R_2}{1+\eta^{1-\alpha}} \int_\eta^1 G(1,s) ds \\
&= \lambda\delta \frac{R_2}{1+\eta^{1-\alpha}} N \\
&> \lambda \frac{1+\eta^{1-\alpha}}{\lambda N} \frac{R_2}{1+\eta^{1-\alpha}} N \\
&= R_2.
\end{aligned}$$

Thus, $\|Tx\| > \|x\|$, for $x \in P \cap \partial\Omega_2$.

Therefore, by the Lemma 2.4, the BVPs (3.6) has at least one positive solution $x \in P \cap (\overline{\Omega_2} \setminus \Omega_1)$, and $R_1 \leq \|x\| \leq R_2$. From $x(t) - v(t) > 0$, we know that BVPs (1.1) has at least one positive solution $u(t) = x(t) - v(t)$.

The proof is completed. □

Theorem 3.2. *Suppose*
H$_2$: $\lim_{x\to+\infty} \inf_{t\in I_\eta} f(t,x) = +\infty$;
H$_3$: $\lim_{x\to+\infty} \sup_{t\in I} \frac{f(t,x)}{x} = 0$;
hold, there exists $\lambda^* > 0$, for all $\lambda > \lambda^*$, the BVPs (1.1) has at least one positive solution.

Proof. Let $\sigma = 2\frac{M}{N}$. From H_2, we have

$$\lim_{x\to+\infty} \inf_{t\in I_\eta} g(t,x) = \lim_{x\to+\infty} \inf_{t\in I_\eta} (f(t,x) + \omega(t)) = +\infty,$$

such that for the above σ, there exists $X > 0$, when $x > X$, for all $t \in I_\eta$, we obtain

$$g(t,x) > \sigma.$$

Let $\lambda^* = \max\{\frac{N}{\eta^{\alpha-1}M}, \frac{X}{M}\}$, $R_1 = 2\lambda M\eta^{1-\alpha}$, where $\lambda > \lambda^*$. Let $\Omega_1 = \{x \in E : \|x\| < R_1\}$, if $x \in P \cap \partial\Omega_1$, $t \in I_\eta$, we have

$$\begin{aligned} x(t) - v(t) &\geq t^{\alpha-1}R_1 - \lambda t^{\alpha-1}M \\ &= \eta^{\alpha-1}R_1 - \lambda M \\ &= \eta^{\alpha-1} \cdot 2\lambda M\eta^{1-\alpha} - \lambda M = \lambda M \\ &> \lambda^* M \geq X, \end{aligned}$$

such that

$$g(t, x(t) - v(t)) > \sigma$$

and

$$\begin{aligned} \|Tx\| &= \max_{t\in I} \lambda \int_0^1 G(t,s)g(s, x(s) - v(s))ds \\ &= \lambda \int_0^1 G(1,s)g(s, x(s) - v(s))ds \\ &> \lambda \int_\eta^1 G(1,s)g(s, x(s) - v(s))ds \\ &= \lambda N\sigma = 2\lambda\frac{M}{N}N = 2\lambda M > R_1 \\ &= \|x\|. \end{aligned}$$

Hence, $\|Tx\| > \|x\|$, $x \in P \cap \partial\Omega_1$.

On the other hand, from H_3, we know that there exists $\varepsilon_0 = \frac{(\alpha-1)\Gamma(\alpha+1)}{2\lambda} > 0$, $R_0 > R_1$, for $t \in [0, 1]$, $x > R_0$, $f(t, x) < \varepsilon_0 x$ holds.

Because of $f \in C([0, 1] \times [0, +\infty), \mathbb{R})$, let $\overline{M} = \max\limits_{(t,x)\in I\times[0,R_0]} \{f(t, x)\}$, then, for $t \in [0, 1]$, $x \in [0, +\infty)$, $f(t, x) \leq \overline{M} + \varepsilon_0 x$ holds.

Let $R_2 > \max\left\{R_0, \lambda M, \frac{2\lambda\left(\overline{M}+\int_0^1 \omega(s)ds\right)}{\Gamma(\alpha)}\right\}$, $\Omega_2 = \{x \in E : \|x\| < R_2\}$, for $x \in P \cap \partial\Omega_2$ and $t \in [0, 1]$, we have

$$x(t) - v(t) \geq t^{\alpha-1}R_2 - \lambda t^{\alpha-1}M = t^{\alpha-1}(R_2 - \lambda M) \geq 0.$$

So that,

$$\begin{aligned} &g(t, x(t) - v(t)) = f(t, x(t) - v(t)) + \omega(t) \\ &\leq \overline{M} + \varepsilon_0 (x(t) - v(t)) + \omega(t) \\ &\leq \overline{M} + \varepsilon_0 x(t) + \omega(t). \end{aligned}$$

Therefore,

$$\begin{aligned}
\|Tx\| &= \max_{t\in I} \lambda \int_0^1 G(t,s)g(s,x(s)-v(s))ds = \lambda \int_0^1 G(1,s)g(s,x(s)-v(s))ds \\
&\leq \lambda \int_0^1 G(1,s)\left(\overline{M}+\varepsilon_0 x(s)+\omega(s)\right)ds \\
&\leq \lambda\varepsilon_0 R_2 \int_0^1 G(1,s)ds + \lambda \int_0^1 G(1,s)\left(\overline{M}+\omega(s)\right)ds \\
&\leq \lambda\varepsilon_0 R_2 \tfrac{1}{(\alpha-1)\Gamma(\alpha+1)} + \tfrac{\lambda}{\Gamma(\alpha)}\left(\overline{M}+\int_0^1 \omega(s)ds\right) \\
&< \tfrac{\lambda R_2}{(\alpha-1)\Gamma(\alpha+1)}\tfrac{(\alpha-1)\Gamma(\alpha+1)}{2\lambda} + \tfrac{R_2}{2} \\
&= R_2 \\
&= \|x\|.
\end{aligned}$$

So, we get

$$\|Tx\| < \|x\|,\ x \in P \cap \partial\Omega_2.$$

Hence, from Lemma 2.4, we know that the operator T has at least one fixed point x, which satisfies $x \in P \cap (\overline{\Omega_2} \setminus \Omega_1)$ and $R_1 \leq \|x\| \leq R_2$. From $x(t) - v(t) > 0$, we know that BVPs (1.1) has at least one positive solution $u(t) = x(t) - v(t)$.

The proof is completed. □

4. Examples

In this section, we provide two examples to demonstrate the applications of the theoretical results in the previous sections.

Example 4.1. Consider the following BVPs

$$\begin{cases} D_{0^+}^{\frac{5}{2}} u + \lambda(u^2 - e^{\sin t} - 3t - 2e) = 0, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{4.1}$$

where $\alpha = \frac{5}{2}$, $f(t,u) = u^2 - e^{\sin t} - 2e$, $\omega(t) = \frac{e^{\sin 10^{-1}}}{\sqrt{10}} t^{\frac{-1}{2}} + 12e$.

Let $\eta = 10^{-1}$, then,

$$g(t,u) = u^2 - e^{\sin t} + \frac{e^{\sin 10^{-1}}}{\sqrt{10}} t^{\frac{-1}{2}} + 10e,$$

$$g_1(r) = \sup_{t\in I_\eta, 0\leq u\leq r} g(t,u) = r^2 + 10e,$$

and

$$\lim_{r\to+\infty} \frac{r}{g_1(r)} = \lim_{r\to+\infty} \frac{r}{r^2+10e} = 0,$$

$R_1 = \sqrt{10e}$, $L = g_1(R_1) = 20e$, $M = 16.7716$, $N = 0.26667$, $\frac{R_1}{M} = 0.310866$, $\frac{(\alpha-1)\Gamma(\alpha+1)R_1}{L} = 0.478069$, $\lambda^* = 0.310866$, $R_2 = 170.086$, $N = 0.196967$.

We can check that the condition of Theorem 3.1 is satisfied. Therefore, there exists at least one positive solution.

Example 4.2. Consider the following BVPs

$$\begin{cases} D_{0^+}^{\frac{7}{3}} u + \lambda(e^{-t}u^{\frac{2}{3}} - t - 10) = 0, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{4.2}$$

where $\alpha = \frac{7}{3}$, $f(t,u) = e^{-t}u^{\frac{2}{3}} - t - 10$, $\omega(t) = t^{\frac{-2}{3}} + 10$.

Let $\eta = 0.3$, such that

$$g(t,u) = e^{-t}u^{\frac{2}{3}} + t^{\frac{-2}{3}} - t,$$

and $M = 8.52480$, $N = 0.219913$, $\sigma = \frac{2M}{N} = 77.5290$, $\frac{N}{\eta^{\alpha-1}M} = 0.128451$, $R_1 = 2\lambda M\eta^{-\frac{4}{3}} = 84.8957\lambda > 10.9049$.

We can check that the conditions of Theorem 3.2 are satisfied. Therefore, there exists at least one positive solution.

5. Conclusions

In this paper, the constraint on the nonlinear term is weakened to $f(t,u) > -\omega(t)$(where $\omega(t) > 0$). Under similar conditions, by constructing an auxiliary boundary value problem and using the principle of linear superposition, the difficulty caused by sign-change of nonlinear terms is overcome. Under the condition of singularity of nonlinear terms, the existence conclusions of positive solutions are obtained based on the Guo-Krasnosel'skii fixed point theorem.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declares that they have no competing interest.

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