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Research article

A Comprehensive study on (α,β)-multi-granulation bipolar fuzzy rough sets under bipolar fuzzy preference relation

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Abstract: The rough set (RS) and multi-granulation RS (MGRS) theories have been successfully extended to accommodate preference analysis by substituting the equivalence relation (ER) with the dominance relation (DR). On the other hand, the bipolar fuzzy sets (BFSs) are effective tools for handling bipolarity and fuzziness of the data. In this study, with the description of the background of risk decision-making problems in reality, we present (α,β)-optimistic multi-granulation bipolar fuzzified preference rough sets ($(\alpha,\beta)^o$-MG-BFPRSs) and (α,β)-pessimistic multi-granulation bipolar fuzzified preference rough sets ($(\alpha,\beta)^p$-MG-BFPRSs) using bipolar fuzzy preference relation (BFPR). Subsequently, the relevant properties and results of both $(\alpha,\beta)^o$-MG-BFPRSs and $(\alpha,\beta)^p$-MG-BFPRSs are investigated in detail. At the same time, a relationship among the (α,β)-BFPRSs, $(\alpha,\beta)^o$-MG-BFPRSs and $(\alpha,\beta)^p$-MG-BFPRSs is given.

Keywords: rough set; bipolar fuzzy preference relation; $(\alpha,\beta)^o$-MG-BFPRSs; $(\alpha,\beta)^p$-MG-BFPRSs
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1. Introduction

Due to various uncertainties in real-world issues, classical mathematical techniques are only sometimes successful. The concept of fuzzy set (FS) was initiated by Zadeh [73] as an extension of traditional set theory, which opened the doors for researchers to capture the uncertainty of the data. FS theory depends on the fuzzy membership function (MF), through which we can evaluate items' membership degree (MD) in a set. This theory has been extended to contain the non-membership function under different ranks of importance of choice values [5, 9, 13].

Pawlak proposed RS theory [46] as a valuable mathematical tool to combat the uncertainty and granularity of information systems and data processing. RS theory has received a lot of attention in recent decades, and its effectiveness has been successfully confirmed and implemented in several fields, like pattern recognition, conflict analysis, knowledge discovery, data mining, image processing, medical diagnosis, neural network and so on. The essential notion of RS theory is the equivalence relation (ER), which represents the indiscernibility relation between arbitrary objects. Although RS theory has been applied successfully in various domains, certain shortcomings may limit the application domain of the RS theory. These shortcomings could result from inaccurate information regarding the objects under consideration. Sometimes, an ER is challenging to find in incomplete information. Therefore, under different scenarios, the RS model has accomplished several exciting generalizations, which include the RS model based on tolerance relations [55], RS model based on arbitrary relations [6], RS based on neighborhood operators [71], RS based on topological structures [7, 10–12], fuzzy RS (FRS) model [17, 39, 40], rough FS (RFS) model [18], dominance-based RSs [21, 25], fuzzy dominance-based RSs [22], dominance-based neighborhood RS [15], variable precision RS [83] and covering-based RSs [74].

It is important to remember that every matter has two sides and bipolarity and fuzziness are inherent aspects of human cognition. Bipolar reasoning is vital in human cognitive processes, according to research in cognitive psychology. Positive and negative effects do not appear in the same part of the brain. Experts in various fields, including database querying, decision-making, and classification have noticed the importance of bipolarity.

Bipolarity and fuzziness are independent but complementary notions devised to model different aspects of human thinking. The former focuses on linguistic imprecision, whereas the latter emphasizes the relevance and polarity of data. Bipolar fuzzy sets (BFSs) are extensions of FSs given by Zhang [77], whose MD range is $[-1, 1]$. The idea that underlies this representation is related to the concept of bipolar information, which refers to both the positive and negative information in the given data. BFSs, therefore, impact various fields, like artificial intelligence, computer science, data science, machine learning, information science, cognitive science, decision analysis, management science, economics, neural science, quantum computing, and medical science.

BFSs have been used in other application domains, including bipolar fuzzy graphs [1, 54], computational psychiatry [80], physics and philosophy [81], and bipolar fuzzy logic [82]. Bipolar fuzzy TOPSIS was studied by Han et al. [27]. In the bipolar fuzzy context, some decision-making problems were resolved using aggregation operators [26, 60]. Many attempts have been made to combine RSs and BFSs [28, 69, 70]. Wei et al. [59] discussed a multiple-attribute decision-making method using interval-valued bipolar fuzzy information. Gul and Shabir [23] studied the roughness of a crisp set using (α, β)-indiscernibility of a bipolar fuzzy relation (BFR). Ali et al. [2] offered attribute reductions of BFR decision systems. Al-shami [8, 14] described various degrees of belong relations that associated an ordinary point with bipolar soft sets.

1.1. Research progress on multi-granulation RSs

RS theory and most of its extensions are based on a single relation defined on a given universe, called single granulation RS models. There are some limitations to the single granulation RS in some real-world applications. For instance, in a comprehensive evaluation decision-making process, the decision-makers often need to acquire the evaluation results of all items in the universe w.r.t. different evaluation indices and then select the optimal number of evaluation indices. The optimal combination of the chosen evaluation indices is a multiple granularity structure of all items in the universe. For this reason, the existing single granulation RS models cannot tackle this sort of decision-making problem. Therefore, Qian et al. [47] proposed the multi-granulation RS (MGRS) model to make up for the deficiency of the existing RS models. In the MGRS model, a target concept's set approximations are constructed by multiple ERs over universe.

So far, the MGRS theory has progressed promptly and has attracted a broad range of studies from theoretical and applied points of view. For example, in Qian et al.'s [47] MGRS theory, there are two basic models: the optimistic MGRS and the pessimistic MGRS [48]. Xu et al. [65] discussed two kinds of MGRS models. Yang et al. [67] developed the hierarchical structural properties of MGRSs. She and He [52] investigated the topological characteristics of MGRSs. Following the approach provided by Qian et al., Yang et al. [68] expanded MGRSs into the multi-granulation FRSs (MGFRSs). Sun et al. [57] constructed an MGFRS model over two universes with a decision-making application. She et al. [53] studied a multiple-valued logic strategy for MGRS. Kong et al. [32] proposed attribute reduction of multi-granulation information systems. Liu et al. [38] analyzed multi-granulation FRSs using fuzzy preference relations. Xu et al. [64] established the concept of generalized MGRSs. Mubarak et al. [43] proposed the pessimistic multi-granulation rough BFS model with application in medical diagnosis.

Zhan and Xu [75] suggested covering-based multi-granulation RFSs. Zhan et al. [76] presented covering-based multi-granulation FRSs and their related decision-making applications. An innovative neighborhood-based MGRS model was developed by Lin et al. [37]. Sun et al. [56] introduced multi-granulation vague RS over dual universes with decision-making applications. Qian et al. [49] projected three multi-granulation decision-theoretic RS models. Feng and Mi [19] analyzed variable precision multi-granulation fuzzy decision-theoretic RSs. Li et al. [34] originated a double-quantitative multi-granulation decision-theoretic RFS model. Zhang et al. [78] provided the non-dual MGRs and hybrid MGRs in addition to four constructive ways of rough approximations from existing RSs. Lin et al. [36] initiated a two-grade fusion strategy involved in the evidence theory and MGRSs and constructed three types of covering-based MGRSs whose set approximations were characterized by various covering approximation operators. Pan et al. [45] studied an MGRS model using preference relation for an ordinal system. Mandal and Ranadive [41] created fuzzy multi-granulation decision-theoretic RSs using fuzzy preference relation. Zhang et al. [79] suggested multi-granulation hesitant FRSs with decision-making applications. Huang et al. [30] created an intuitionistic fuzzy MGRS (IFMGRS), and three IFMGRS models that are generalizations of existing intuitionistic FRS models were constructed. Liang et al. [35] offered an efficient rough feature selection algorithm for large-scale data using MGRSs. Ali et al. [4] proposed new types of dominance-based MGRSs with applications in conflict analysis. Hu et al. [29] pioneered dynamic dominance-based MGRS approaches with evolving ordered data. You et al. [72] studied the relative reduction of neighborhood-covering pessimistic MGRS using evidence theory. Xue et al. [66] established three-way decisions based on multi-granulation support intuitionistic fuzzy probabilistic RSs. Qian et al. [50] introduced multi-granulation sequential three-way decisions based on multiple thresholds. Mandal and Ranadive [42] introduced multi-granulation bipolar-valued fuzzy probabilistic RSs and their corresponding three-way decisions over two universes. Gul and Shabir [24] proposed (α,β)-multi-granulation bipolar fuzzified RS using a finite family of bipolar fuzzy tolerance relations. Kang et al. [31] initiated the grey

MGRSs model. Multi-criteria optimization and compromise solution (abbreviated by VIKOR) method is one of the famous MCDM methods that ranks alternatives and determines the compromise solution that is the closest to the "ideal". Tufail and Shabir [58] studied VIKOR method for multiple criteria decision making (MCDM) based on bipolar fuzzy soft β-covering based bipolar fuzzy RS model and its application to site selection of solar power plants.

1.2. Knowledge gap and motivations and contributions of our research

As a generalization of FS theory, BFS theory makes the representations of real world more realistic, practical and accurate in scenarios, making it very promising. Based on the above contents, the research gaps, motivations and novelty of our research are listed as follows:

(1) Preference relation (PR) is a valuable tool to model decision-making problems, where decision-makers articulate their preference information over alternatives via pairwise comparisons. With various representations of preference information, numerous kinds of PRs have been put forth and investigated, such as the multiplicative PR [51], fuzzy PR (FPR) [16, 44] and BFPRs [25]. At the same time, MGRS theory has received significant attention in recent eras. It offers a formal theoretical framework to solve complicated problems in the context of multiple binary relations. However, according to the best of our knowledge, there does not exist any study where the hybridization of MGRS theory and BFPRs have been discussed for acquiring knowledge. Therefore, this article fills this research gap by establishing the ideas of $(\alpha,\beta)^o$-MG-BFPRS and $(\alpha,\beta)^p$-MG-BFPRS models by use of BFPR.

(2) Moreover, in the present literature, there have been many studies about MGRS models in the context of BFSs, where BFRs are applied to established fuzzy approximations. However, even with the help of BFRs, the researchers could not determine the crisp approximations. Naturally, the question arises whether we can acquire the crisp approximations using BFRs. The certifiable answer to this issue has driven the present authors to the construction of $(\alpha,\beta)^o$-MG-BFPRS and $(\alpha,\beta)^p$-MG-BFPRS models. Furthermore, the approximations defined based on two models serve as a bridge between BFRs and a crisp set.

1.3. The organization of this paper

This article is structured as follows:

(1) We present some fundamental knowledge in Section 2.

(2) The idea of $(\alpha,\beta)^o$-MG-BFPRS model is proposed, and its related properties are investigated in Section 3.

(3) Section 4 establishes the notion of $(\alpha,\beta)^p$-MG-BFPRS model and their relevant properties.

(4) In Section 5, we investigate the connection among the (α,β)-BFPRS, $(\alpha,\beta)^o$-MG-BFPRS, and $(\alpha,\beta)^p$-MG-BFPRS models.

(5) Finally, we conclude our study and present some topics for future research in Section 7.

2. Preliminaries

Some cardinal terminologies, inclusive of RSs, MGRSs, FSs, BFSs, and BFRSs, are described in this section.

2.1. RS theory

ER plays a crucial function in the RS theory [46] to cope with uncertainty, which categorizes the universe into classes that are known as information granules. Thus, in RS theory, we must deal with groups of objects rather than a single item.

Definition 2.1. *[46] An approximation space (AS) is a structure of the form* $(\mho, \vartheta)$*, where* $\mho$ *is a non-void universe and* ϑ *is an ER on* $\mho$*. Given any subset* $\mathcal{T}$ *of* $\mho$*,* $\mathcal{T}$ *may or may not be written as a union of some equivalence classes induced by* ϑ*. If it is possible to write* $\mathcal{T}$ *as the union of some equivalence classes, it is called definable; if not, it is termed an RS. If* $\mathcal{T}$ *is an RS, then it can be approximated by the following two definable sets:*

$$\left.\begin{aligned} \underline{\mathcal{T}}_{\vartheta} &= \{r \in \mho : [r]_{\vartheta} \subseteq \mathcal{T}\}, \\ \overline{\mathcal{T}}^{\vartheta} &= \{r \in \mho : [r]_{\vartheta} \cap \mathcal{T} \neq \emptyset\}, \end{aligned}\right\} \tag{1}$$

which are called lower and upper approximations of $\mathcal{T}$*, respectively, where*

$$[r]_{\vartheta} = \{s \in \mho : (r, s) \in \vartheta\}. \tag{2}$$

Furthermore, the set

$$\mathcal{B}nd_{\vartheta}(\mathcal{T}) = \overline{\mathcal{T}}^{\vartheta} - \underline{\mathcal{T}}_{\vartheta}, \tag{3}$$

is called the boundary region of $\mathcal{T} \subseteq \mho$*.*

RS theory uses a single ER. Using a finite collection of ERs, Qian et al. [47] laid the foundation of the MGRS. In Qian et al.'s MGRS theory, two strategies have been formulated. The first one is the optimistic MGRS (OMGRS), and the second one is the pessimistic MGRS (PMGRS).

2.2. OMGRS model

Each ER can induce a partition in the universe, regarded as a granulation space. Thus, a family of ERs can generate a family of granulation spaces. In optimistic multi-granulation lower approximation, the term "optimistic" means that in multi-independent granulation spaces, we need only at least one of the granulation spaces to satisfy the inclusion condition between the equivalence class and the approximated target. The upper approximation of optimistic MGRS is defined by the complement of the optimistic multi-granulation lower approximation.

Definition 2.2. *[47] Let* $\Theta = \{\vartheta_1, \vartheta_2, \cdots, \vartheta_n\}$ *be a collection of n independent ERs over* $\mho$ *and* $\mathcal{T} \subseteq \mho$*. The optimistic multi-granulation lower and upper approximations of* $\mathcal{T} \subseteq \mho$ *are respectively described as:*

$$\left.\begin{aligned}&\underline{\Theta}_{opt}(\mathcal{T}) = \{r \in \mho : [r]_{\vartheta_i} \subseteq \mathcal{T} \text{ for some } i = 1, 2, \cdots, n\},\\&\overline{\Theta}^{opt}(\mathcal{T}) = (\underline{\Theta}_{opt}(\mathcal{T}^c))^c,\end{aligned}\right\} \tag{4}$$

where $\mathcal{T}^c$ is the complement of the set $\mathcal{T}$. If $\underline{\Theta}_{opt}(\mathcal{T}) \neq \overline{\Theta}^{opt}(\mathcal{T})$, then $\mathcal{T}$ is referred to as an OMGRS; else, it is an optimistic definable. The boundary region of $\mathcal{T} \subseteq \mho$ under the OMGRS environment is given as follows:

$$Bnd_{\Theta}^{opt}(\mathcal{T}) = \overline{\Theta}^{opt}(\mathcal{T}) - \underline{\Theta}_{opt}(\mathcal{T}). \tag{5}$$

2.3. PMGRS model

In the PMGRS, the target is still approximated via a family of ERs. However, the pessimistic case is different from the optimistic case. In pessimistic multi-granulation lower approximation, the term "pessimistic" means we need the granulation spaces to satisfy the inclusion condition between the equivalence class and the approximated target. The upper approximation of PMGRS is still characterized by the complement of the pessimistic multi-granulation lower approximation.

Definition 2.3. *[48] Let $\Theta = \{\vartheta_1, \vartheta_2, \cdots, \vartheta_n\}$ be a collection of n independent ERs over $\mho$ and $\mathcal{T} \subseteq \mho$. The pessimistic multi-granulation lower and upper approximations of $\mathcal{T} \subseteq \mho$ are defined as:*

$$\left.\begin{aligned}&\underline{\Theta}_{pes}(\mathcal{T}) = \{r \in \mho : [r]_{\vartheta_i} \subseteq \mathcal{T} \text{ for all } i = 1, 2, \cdots, n\},\\&\overline{\Theta}^{pes}(\mathcal{T}) = (\underline{\Theta}_{pes}(\mathcal{T}^c))^c.\end{aligned}\right\} \tag{6}$$

If $\underline{\Theta}_{pes}(\mathcal{T}) \neq \overline{\Theta}^{pes}(\mathcal{T})$, then $\mathcal{T}$ is called a PMGRS. Otherwise it is a pessimistic definable. The boundary region of $\mathcal{T} \subseteq \mho$ under the PMGRS environment is defined as:

$$Bnd_{\Theta}^{pes}(\mathcal{T}) = \overline{\Theta}^{pes}(\mathcal{T}) - \underline{\Theta}_{pes}(\mathcal{T}). \tag{7}$$

2.4. FS, BFS, BFRs and some cardinal terminologies

Definition 2.4. *[73] An FS $\mathfrak{F}$ on $\mho$ is a map $\mathfrak{F} : \mho \longrightarrow [0, 1]$. For each $r \in \mho$, the value $\mathfrak{F}(r)$ refers to the MD of r.*

Definition 2.5. *[77] A BFS ζ over $\mho$ is an object of the form:*

$$\zeta = \{\langle r, \zeta^P(r), \zeta^N(r)\rangle : r \in \mho\}, \tag{8}$$

where $\zeta^P : \mho \longrightarrow [0, 1]$ and $\zeta^N : \mho \longrightarrow [-1, 0]$ are called positive MD and negative MD, respectively. The positive MD $\zeta^P(r)$ denotes the satisfaction degree of an element r to the property and the negative MD $\zeta^N(r)$ represents the satisfaction degree of r to the somewhat implicit counter-property.

From now on, we will use $\mathcal{BF}(\mho)$ to symbolize the collection of all BFSs over $\mho$.

Definition 2.6. *[77] Let $\lambda, \zeta \in \mathcal{BF}(\mho)$. Then for all $r \in \mho$, we have*

(i) $\lambda \subseteq \zeta$, if $\lambda^P(r) \leq \zeta^P(r)$ and $\lambda^N(r) \geq \zeta^N(r)$;

(ii) $(\lambda \cap \zeta)(r) = \{\langle r, min(\lambda^P(r), \zeta^P(r)), max(\lambda^N(r), \zeta^N(r))\rangle\}$;

(iii) $(\lambda \cup \zeta)(r) = \{\langle r, max(\lambda^P(r), \zeta^P(r)), min(\lambda^N(r), \zeta^N(r))\rangle\}$;

(iv) $\lambda^c(r) = \{\langle r, 1 - \lambda^P(r), -1 - \lambda^N(r)\rangle\}$.

Definition 2.7. *[33] The whole BFS over $\mho$ is symbolized by $\mathfrak{U} = \langle \mathfrak{U}^P, \mathfrak{U}^N\rangle$ and is descried as $\mathfrak{U}^P(r) = 1$ and $\mathfrak{U}^N(r) = 0$, for all $r \in \mho$. The null BFS over $\mho$ is symbolized by $\Theta = \langle \Theta^P, \Theta^N\rangle$ and is given as $\Theta^P(r) = 0$ and $\Theta^N(r) = -1$, for all $r \in \mho$.*

Definition 2.8. *[67] A BFR $\mathcal{B}$ over $\mho$ can be described as:*

$$\mathcal{B} = \{\langle (q, r), \mu^P_{\mathcal{B}}(q, r), \mu^N_{\mathcal{B}}(q, r)\rangle : (q, r) \in \mho \times \mho\}, \tag{9}$$

where $\mu^P_{\mathcal{B}} : \mho \times \mho \longrightarrow [0, 1]$ and $\mu^N_{\mathcal{B}} : \mho \times \mho \longrightarrow [-1, 0]$.

For a BFR $\mathcal{B}$ over $\mho$, $\mu^P_{\mathcal{B}}(q, r)$ is the positive MD, which shows the satisfaction degree of an object (q, r) to the property corresponding to $\mathcal{B}$, and its negative MD $\mu^N_{\mathcal{B}}(q, r)$ represents the satisfaction degree to some implicit counter-property associated with $\mathcal{B}$.

Definition 2.9. *[23] Let $\mathcal{B} = \langle \mu^P_{\mathcal{B}}(q, r), \mu^N_{\mathcal{B}}(q, r)\rangle$ be a BFR over $\mho = \{x_1, x_2, \cdots, x_n\}$. By taking $a_{ij} = \mu^P_{\mathcal{B}}(q_i, r_j)$ and $b_{ij} = \mu^N_{\mathcal{B}}(q_i, r_j)$, $i = 1, 2, ..., n$; $j = 1, 2, ..., n$, the BFR $\mathcal{B}$ can be expressed as:*

$$\mathcal{B} = \left(\langle \mu^P_{\mathcal{B}}, \mu^N_{\mathcal{B}}\rangle\right)_{n\times n} = \begin{pmatrix} \langle a_{11}, b_{11}\rangle & \langle a_{12}, b_{12}\rangle & \cdots & \langle a_{1n}, b_{1n}\rangle \\ \langle a_{21}, b_{21}\rangle & \langle a_{22}, b_{22}\rangle & \cdots & \langle a_{2n}, b_{2n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_{n1}, b_{n1}\rangle & \langle a_{n2}, b_{n1}\rangle & \cdots & \langle a_{nn}, b_{nn}\rangle \end{pmatrix}, \text{ where } a_{ij} \in [0, 1] \text{ and } b_{ij} \in [-1, 0].$$

Recently, Gul and Shabir [25] initiated the idea of BFPR, which is stated as follows:

Definition 2.10. *[25] A BFPR $\mathfrak{B}$ over $\mho$ is a BFS over $\mho \times \mho$, which is described by its positive and negative MFs given as $\mu^P_{\mathfrak{B}} : \mho \times \mho \longrightarrow [0, 1]$ and $\mu^N_{\mathfrak{B}} : \mho \times \mho \longrightarrow [-1, 0]$. For $\mho = \{x_1, x_2, \cdots, x_n\}$, we can express it by an $n \times n$ matrix as:*

$$\mathfrak{B} = \left(\langle a_{ij}, b_{ij}\rangle\right)_{n\times n} = \begin{matrix} \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \langle a_{11}, b_{11}\rangle & \langle a_{12}, b_{12}\rangle & \cdots & \langle a_{1n}, b_{1n}\rangle \\ \langle a_{21}, b_{21}\rangle & \langle a_{22}, b_{22}\rangle & \cdots & \langle a_{2n}, b_{2n}\rangle \\ \vdots & \ddots & \vdots & \\ \langle a_{n1}, b_{n1}\rangle & \langle a_{n2}, b_{n2}\rangle & \cdots & \langle a_{nn}, b_{nn}\rangle \end{pmatrix},$$

where $\langle a_{ij}, b_{ij}\rangle$ denotes the bipolar fuzzy preference degree (BFPD) of alternative x_i over alternative x_j, $a_{ij} \in [0, 1]$, $b_{ij} \in [-1, 0]$. Moreover, a_{ij} and b_{ij} satisfy the following conditions, $a_{ij} + a_{ji} = 1, b_{ij} + b_{ji} = -1, a_{ii} = 0.5$ and $b_{ii} = -0.5$ $\forall i, j = 1, 2, \cdots, n$. Particularly,

- *$a_{ij} = 0.5, b_{ij} = -0.5$ indicates indifference between alternatives x_i and x_j;*

- $a_{ij} > 0.5, b_{ij} > -0.5$ *demonstrates that alternative* x_i *is better than alternative* x_j*;*
- $a_{ij} < 0.5, b_{ij} < -0.5$ *indicates that alternative* x_j *is better than alternative* x_i*;*
- $a_{ij} = 1, b_{ij} = 0$ *shows that alternative* x_i *is absolutely better than alternative* x_j*;*
- $a_{ij} = 0, b_{ij} = -1$ *means alternative* x_j *is absolutely better than alternative* x_i*.*

Definition 2.11. *[25] A BFPR* $\mathfrak{B} = \left(\langle a_{ij}, b_{ij}\rangle\right)_{n\times n}$ *is said to be an additive consistent if* $\forall i, j, k \in \{1, 2, \cdots, n\}$ *the following conditions hold:*

(1) $a_{ij} = a_{ik} - a_{jk} + 0.5$*,*

(2) $b_{ij} = b_{ik} - b_{jk} + 0.5$*.*

Definition 2.12. *[25] Let* $\mho = \{x_i : i = 1, 2, \cdots, n\}$ *be a non-empty universe of n objects and* $\mathfrak{C} = \{\mathfrak{C}_k : k = 1, 2, \cdots, m\}$ *be a non-empty set of m criteria. Let* $f : \mho \times \mathfrak{C} \longrightarrow [0, 1]$ *and* $g : \mho \times \mathfrak{C} \longrightarrow [-1, 0]$ *be positive and negative MFs, respectively. Then, we define the transfer functions to compute the BFPD of any two objects* $x_i, x_j \in \mho$ *about the criterion* $\mathfrak{C}_k$ *as follows:*

$$a_{ij}^{\mathfrak{C}_k} = \frac{f(x_i, \mathfrak{C}_k) - f(x_j, \mathfrak{C}_k) + 1}{2}, \tag{10}$$

$$b_{ij}^{\mathfrak{C}_k} = \frac{g(x_j, \mathfrak{C}_k) - g(x_i, \mathfrak{C}_k) - 1}{2}. \tag{11}$$

For a BFPR $\mathfrak{B}_{\mathfrak{C}_k}(x_i, x_j) = \left(\langle a_{ij}^{\mathfrak{C}_k}, b_{ij}^{\mathfrak{C}_k}\rangle\right)_{n\times n}$ *on the criteria* $\mathfrak{C}_k$*, the above transfer functions (10) and (11) satisfy the following properties for* $x_i, x_j, x_k \in \mho$*:*

(1) $a_{ii}^{\mathfrak{C}_k} = 0.5$ *and* $b_{ii}^{\mathfrak{C}_k} = -0.5$*.*

(2) $a_{ij}^{\mathfrak{C}_k} + a_{ji}^{\mathfrak{C}_k} = 1$ *and* $b_{ij}^{\mathfrak{C}_k} + b_{ji}^{\mathfrak{C}_k} = -1$*.*

(3) $a_{ij}^{\mathfrak{C}_k} + a_{j\ell}^{\mathfrak{C}_k} = a_{i\ell}^{\mathfrak{C}_k} + 0.5$ *and* $b_{ij}^{\mathfrak{C}_k} + b_{j\ell}^{\mathfrak{C}_k} = b_{i\ell}^{\mathfrak{C}_k} - 0.5$*.*

Example 2.13. *Table 1 depicts a bipolar fuzzy information matrix, where* $\mho = \{x_1, x_2, x_3, x_4, x_5\}$ *and* $\mathfrak{C} = \{\mathfrak{C}_1, \mathfrak{C}_2\}$.

Table 1. Bipolar fuzzy information matrix.

$\mho/\mathfrak{C}$	$\mathfrak{C}_1$	$\mathfrak{C}_2$
x_1	(0.5, - 0.25)	(0.8, - 0.7)
x_2	(0.25, - 0.8)	(0.9, - 0.4)
x_3	(0.33, - 0.25)	(0.75, - 0.4)
x_4	(0.65, - 0.6)	(0.3, - 0.75)
x_5	(1, - 0.5)	(0.4, - 0.35)

Based on criteria $\mathfrak{C}_1$ and $\mathfrak{C}_2$, we can construct the BFPRs of alternative x_i to the alternative $x_j (i, j = 1, 2, \cdots, 5)$ by using formulas (10) and (11), we obtain:

$$\mathfrak{B}_{\mathfrak{C}_1}(x_i, x_j) = \begin{pmatrix} \langle 0.500, -0.500\rangle & \langle 0.625, -0.775\rangle & \langle 0.585, -0.500\rangle & \langle 0.425, -0.675\rangle & \langle 0.250, -0.625\rangle \\ \langle 0.375, -0.225\rangle & \langle 0.500, -0.500\rangle & \langle 0.460, -0.225\rangle & \langle 0.300, -0.400\rangle & \langle 0.125, -0.350\rangle \\ \langle 0.415, -0.500\rangle & \langle 0.540, -0.775\rangle & \langle 0.500, -0.500\rangle & \langle 0.340, -0.675\rangle & \langle 0.165, -0.625\rangle \\ \langle 0.575, -0.325\rangle & \langle 0.700, -0.600\rangle & \langle 0.660, -0.325\rangle & \langle 0.500, -0.500\rangle & \langle 0.325, -0.450\rangle \\ \langle 0.750, -0.375\rangle & \langle 0.875, -0.650\rangle & \langle 0.835, -0.375\rangle & \langle 0.675, -0.550\rangle & \langle 0.500, -0.500\rangle \end{pmatrix}, \tag{12}$$

$$\mathfrak{B}_{\mathfrak{C}_2}(x_i, x_j) = \begin{pmatrix} \langle 0.500, -0.500\rangle & \langle 0.450, -0.350\rangle & \langle 0.525, -0.350\rangle & \langle 0.750, -0.525\rangle & \langle 0.700, -0.325\rangle \\ \langle 0.550, -0.650\rangle & \langle 0.500, -0.500\rangle & \langle 0.575, -0.500\rangle & \langle 0.800, -0.675\rangle & \langle 0.750, -0.475\rangle \\ \langle 0.475, -0.650\rangle & \langle 0.425, -0.500\rangle & \langle 0.500, -0.500\rangle & \langle 0.725, -0.675\rangle & \langle 0.675, -0.475\rangle \\ \langle 0.250, -0.475\rangle & \langle 0.200, -0.325\rangle & \langle 0.275, -0.325\rangle & \langle 0.500, -0.500\rangle & \langle 0.450, -0.300\rangle \\ \langle 0.300, -0.675\rangle & \langle 0.250, -0.525\rangle & \langle 0.325, -0.525\rangle & \langle 0.550, -0.700\rangle & \langle 0.500, -0.500\rangle \end{pmatrix}. \tag{13}$$

In [25], Gul and Shabir adopted the transfer functions (10) and (11) to originate the idea (α,β)-bipolar fuzzified preference RS ((α,β)-BFPRS), given as follows:

Definition 2.14. *[25] Let $\mathfrak{B}_{\mathfrak{C}}(x_i, x_j) = \left[(a^{\mathfrak{C}}_{ij}, b^{\mathfrak{C}}_{ij})\right]_{n\times n}$ be a BFPR over $\mho$ on the criteria $\mathfrak{C}$ with positive and negative MFs given as $\mu^P_{\mathfrak{B}_{\mathfrak{C}}} : \mho \times \mho \longrightarrow [0, 1]$ and $\mu^N_{\mathfrak{B}_c} : \mho \times \mho \longrightarrow [-1, 0]$. For any $\alpha \in [0.5, 1)$ and $\beta \in (-1, -0.5]$, the lower and upper (α,β)-BFPR-approximations for any $\mathcal{X} \subseteq \mho$ w.r.t. $\mathfrak{B}_{\mathfrak{C}}$ are formulated as:*

$$\left.\begin{aligned} \underline{BFP}_{(\alpha,\beta)}(\mathcal{X}) &= \left(\underline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\alpha}, \underline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\beta}\right), \\ \overline{BFP}_{(\alpha,\beta)}(\mathcal{X}) &= \left(\overline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\alpha}, \overline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\beta}\right), \end{aligned}\right\} \tag{14}$$

where,

$$\left.\begin{aligned} \underline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\alpha} &= \{x_i \in \mho : a^{\mathfrak{C}}_{ij} < 1 - \alpha \text{ for all } x_j \in \mathcal{X}^c\}, \\ \overline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\alpha} &= \{x_i \in \mho : a^{\mathfrak{C}}_{ij} \geq 1 - \alpha \text{ for some } x_j \in \mathcal{X}\}, \\ \underline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\beta} &= \{x_i \in \mho : b^{\mathfrak{C}}_{ij} \leq -1 - \beta \text{ for some } x_j \in \mathcal{X}\}, \\ \overline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\beta} &= \{x_i \in \mho : b^{\mathfrak{C}}_{ij} > -1 - \beta \text{ for all } x_j \in \mathcal{X}^c\}, \end{aligned}\right\} \tag{15}$$

are said to be the α-lower, α-upper, β-lower and β-upper approximations of $\mathcal{X}$, respectively. Moreover, when $\underline{BFP}_{(\alpha,\beta)}(\mathcal{X}) \neq \overline{BFP}_{(\alpha,\beta)}(\mathcal{X})$, then $\mathcal{X}$ is titled as (α,β)-BFPRS w.r.t. $\mathfrak{B}_{\mathfrak{C}}$; else, it is (α,β)-bipolar fuzzified preference definable w.r.t. $\mathfrak{B}_{\mathfrak{C}}$.

The boundary region under (α,β)-BFPR-approximations are given as:

$$\mathcal{BND}_{(\alpha,\beta)}(\mathcal{X}) = \left(\overline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\alpha} - \underline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\alpha}, \underline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\beta} - \overline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\beta}\right). \tag{16}$$

Definition 2.15. *[25] Let $\mathfrak{B}_{\mathfrak{C}}$ be a BFPR over $\mho$ on the criteria $\mathfrak{C}$, $\alpha \in [0.5, 1)$ and $\beta \in (-1, -0.5]$. The measure of accuracy of $\mathcal{X}$ under (α,β)-BFPRSs is defined as:*

$$\mathcal{A}^{\mathfrak{B}_{\mathfrak{C}}}_{(\alpha,\beta)}(\mathcal{X}) = (\mathfrak{X}^{\alpha}, \mathfrak{X}^{\beta}), \tag{17}$$

where

$$\mathfrak{X}^{\alpha} = \frac{\left|\underline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\alpha}\right|}{\left|\overline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\alpha}\right|} \text{ and } \mathfrak{X}^{\beta} = \frac{\left|\overline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\beta}\right|}{\left|\underline{\mathfrak{B}_{\mathfrak{C}}(\mathcal{X})}_{\beta}\right|},$$

where $\emptyset \neq \mathcal{X} \subseteq \mho$ and $|\bullet|$ denotes the set's cardinality.

The measure of roughness $\mathcal{R}^{\mathfrak{B}_{\mathfrak{C}}}_{(\alpha,\beta)}(\mathcal{X})$ of $\mathcal{X}$ under (α,β)-BFPRSs is given as:

$$\mathcal{R}^{\mathfrak{B}_{\mathfrak{C}}}_{(\alpha,\beta)}(\mathcal{X}) = (1, 1) - \mathcal{A}^{\mathfrak{B}_{\mathfrak{C}}}_{(\alpha,\beta)}(\mathcal{X}) = (1 - \mathfrak{X}^{\alpha}, 1 - \mathfrak{X}^{\beta}). \tag{18}$$

Clearly, $(0, 0) \leq \mathcal{A}^{\mathfrak{B}_{\mathfrak{C}}}_{(\alpha,\beta)}(\mathcal{X}), \mathcal{R}^{\mathfrak{B}_{\mathfrak{C}}}_{(\alpha,\beta)}(\mathcal{X}) \leq (1, 1)$ for any $\mathcal{X} \subseteq \mho$, $\alpha \in [0.5, 1)$ and $\beta \in (-1, -0.5]$.

3. (α,β)-optimistic multi-granulation bipolar fuzzified preference rough sets ($(\alpha,\beta)^o$-MG-BFPRSs)

In this portion, we generalize the idea of (α,β)-BFPRSs to $(\alpha,\beta)^o$-MG-BFPRSs. This generalization is based on a finite collection of BFPRs instead of a single BFPR. Moreover, we examine some axiomatic systems of $(\alpha,\beta)^o$-MG-BFPRSs with several constructive illustrations.

Definition 3.1. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots, \mathfrak{B}_{\mathfrak{C}_m}$ described by its positive and negative MFs given as $\mu^P_{\mathfrak{B}_{\mathfrak{C}_t}} : \mho \times \mho \longrightarrow [0, 1]$ and $\mu^N_{\mathfrak{B}_{\mathfrak{C}_t}} : \mho \times \mho \longrightarrow [-1, 0]$; $t = 1, 2, \cdots, m$. For any $\alpha \in [0.5, 1)$ and $\beta \in (-1, -0.5]$, the lower and upper $(\alpha,\beta)^o$-MG-BFPR-approximations for any $\mathcal{T} \subseteq \mho$ w.r.t. Υ are described as:*

$$\left.\begin{aligned} \underline{MBFP}_{(\alpha,\beta)^o}(\mathcal{T}) &= \left(\Big(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\alpha}(\mathcal{T}), \Big(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\beta}(\mathcal{T})\right), \\ \overline{MBFP}_{(\alpha,\beta)^o}(\mathcal{T}) &= \left(\Big(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\alpha}(\mathcal{T}), \Big(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\beta}(\mathcal{T})\right), \end{aligned}\right\} \tag{19}$$

where,

$$\Big(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\alpha}(\mathcal{T}) = \bigcup_{t=1}^{m} \Big\{x_i \in \mho : a^{\mathfrak{C}_t}_{ij} < 1 - \alpha \text{ for all } x_j \in \mathcal{T}^c\Big\}, \tag{20}$$

$$\Big(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\alpha}(\mathcal{T}) = \bigcap_{t=1}^{m} \Big\{x_i \in \mho : a^{\mathfrak{C}_t}_{ij} \geq 1 - \alpha \text{ for some } x_j \in \mathcal{T}\Big\}, \tag{21}$$

$$\Big(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\beta}(\mathcal{T})=\bigcap_{t=1}^{m}\Big\{x_i\in\mho : b_{ij}^{\mathfrak{C}_t}\leq -1-\beta \textit{ for some } x_j\in\mathcal{T}\Big\}, \tag{22}$$

$$\Big(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\beta}(\mathcal{T})=\bigcup_{t=1}^{m}\Big\{x_i\in\mho : b_{ij}^{\mathfrak{C}_t}> -1-\beta \textit{ for all } x_j\in\mathcal{T}^c\Big\}, \tag{23}$$

are said to be the α-optimistic lower, α-optimistic upper, β-optimistic lower and β-optimistic upper multi-granulation rough approximations of $\mathcal{T}$, respectively. Moreover, if $\underline{MBFP}_{(\alpha,\beta)^o}(\mathcal{T})\neq\overline{MBFP}_{(\alpha,\beta)^o}(\mathcal{T})$, then $\mathcal{T}$ is titled as $(\alpha,\beta)^o$-MG-BFPRSs w.r.t. $\mathfrak{B}_{\mathfrak{C}_t}$; else, it is named (α,β)-optimistic multi-granulation bipolar fuzzified preference definable w.r.t. $\mathfrak{B}_{\mathfrak{C}_t}$.

The positive, boundary, and negative regions under $(\alpha,\beta)^o$-MG-BFPR-approximations are specified as follows:

(i) $\mathcal{POS}_{(\alpha,\beta)^o}(\mathcal{T})=\left(\Big(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\alpha}(\mathcal{T}),\Big(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\beta}(\mathcal{T})\right),$

(ii) $\mathcal{BND}_{(\alpha,\beta)^o}(\mathcal{T})=\left(\Big(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\alpha}(\mathcal{T})-\Big(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\alpha}(\mathcal{T}),\Big(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\beta}(\mathcal{T})-\Big(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\beta}(\mathcal{T})\right),$

(iii) $\mathcal{NEG}_{(\alpha,\beta)^o}(\mathcal{T})=(\mho,\mho)-\left(\Big(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\alpha}(\mathcal{T}),\Big(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\beta}(\mathcal{T})\right)=\left(\left(\Big(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\alpha}(\mathcal{T})\right)^{c},\left(\Big(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^{o}_{\beta}(\mathcal{T})\right)^{c}\right).$

The information about an element $x\in\mho$ interpreted by the operators mentioned above is as follows:

- $(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{T})$ indicates the collection of objects $x_i\in\mho$ equivalent to all objects $x_j\in\mathcal{T}^c$ with a positive MD less than to a specific $\alpha\in[0.5,1)$ for some $t=1,2,\cdots,m$.

- $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{T})$ denotes the collection of objects $x_i\in\mho$ equivalent to at least one object $x_j\in\mathcal{T}$ with a positive MD greater than or equal to a specific $\alpha\in[0.5,1)$ for all $t=1,2,\cdots,m$.

- $(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta}(\mathcal{T})$ signifies the collection of objects $x_i\in\mho$ equivalent to at least one object $x_j\in\mathcal{T}$ with a negative MD less than or equal to a specific $\beta\in(-1,-0.5]$ for all $t=1,2,\cdots,m$.

- $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta}(\mathcal{T})$ represents the collection of objects $x_i\in\mho$ equivalent to all objects $x_j\in\mathcal{T}^c$ with a negative MD greater than a specific $\beta\in(-1,-0.5]$ for some $t=1,2,\cdots,m$.

Remark 3.2. *In the light of Definition 3.1, we have:*

(1) $\mathcal{T}\subseteq\mho$ is an $(\alpha,\beta)^o$-MG-BFPRSs w.r.t. $\mathfrak{B}_{\mathfrak{C}_t}$ if and only if $\mathcal{BND}_{(\alpha,\beta)^o}(\mathcal{T})=(\emptyset,\emptyset)$.

(2) If $\mathfrak{B}_{\mathfrak{C}_1}=\mathfrak{B}_{\mathfrak{C}_2}=\cdots=\mathfrak{B}_{\mathfrak{C}_m}=\mathfrak{B}_{\mathfrak{C}}$, then the operators given in Eqs (20) to (23) degenerates into (α,β)-BFPR-approximation operators of a set $\mathcal{T}$ given in Definition 2.14.

Example 3.3. *(Following Example 2.13) Consider the two BFPRs $\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}$ over $\mho$, where $\mho = \{x_1, x_2, x_3, x_4, x_5\}$ given in Example 2.13. If $\mathcal{T} = \{x_2, x_3\} \subseteq \mho$, then for $\alpha = 0.5$ and $\beta = -0.5$, we have*

$$(\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^o_\alpha(\mathcal{T}) = \{x_2, x_3\},$$

$$(\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^o_\alpha(\mathcal{T}) = \{x_1, x_2, x_3\},$$

$$(\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^o_\beta(\mathcal{T}) = \{x_2, x_3, x_5\},$$

$$(\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^o_\beta(\mathcal{T}) = \{x_2\}.$$

Hence, the lower and upper $(\alpha,\beta)^o$-MG-BFPR-approximations for $\mathcal{T}$ are given as follows:

$$\underline{MBFP}_{(\alpha,\beta)^o}(\mathcal{T}) = (\{x_2, x_3\}, \{x_2, x_3, x_5\}),$$

$$\overline{MBFP}_{(\alpha,\beta)^o}(\mathcal{T}) = (\{x_1, x_2, x_3\}, \{x_2\}).$$

Since, $\underline{MBFP}_{(\alpha,\beta)^o}(\mathcal{T}) \neq \overline{MBFP}_{(\alpha,\beta)^o}(\mathcal{T})$, so $\mathcal{T}$ is an $(\alpha,\beta)^o$-MG-BFPRSs w.r.t. $\mathfrak{B}_{\mathfrak{C}_1}$ and $\mathfrak{B}_{\mathfrak{C}_2}$. Furthermore,

$$\mathcal{POS}_{(\alpha,\beta)^o}(\mathcal{T}) = (\{x_2, x_3\}, \{x_2\}),$$
$$\mathcal{BND}_{(\alpha,\beta)^o}(\mathcal{T}) = (\{x_1\}, \{x_3, x_5\}),$$
$$\mathcal{NEG}_{(\alpha,\beta)^o}(\mathcal{T}) = (\{x_4, x_5\}, \{x_1, x_4\}).$$

Proposition 3.4. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots, \mathfrak{B}_{\mathfrak{C}_m}$ and $\alpha_1, \alpha_2 \in [0.5, 1)$ be such that $\alpha_1 \leq \alpha_2$. Then for any $\mathcal{T} \subseteq \mho$, we have*

(1) $(\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_{\alpha_2}(\mathcal{T}) \subseteq (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_{\alpha_1}(\mathcal{T})$;

(2) $(\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_{\alpha_1}(\mathcal{T}) \subseteq (\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_{\alpha_2}(\mathcal{T})$.

Proof. (1) For any $x_i \in (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_{\alpha_2}(\mathcal{T})$, we have $a_{ij}^{\mathfrak{C}_t} < 1-\alpha_2$ for all $x_j \in \mathcal{T}^c$ for some $t = 1, 2, \cdots, m$. But since, $\alpha_1 \leq \alpha_2$, so $1-\alpha_2 \leq 1-\alpha_1$. Thus, $a_{ij}^{\mathfrak{C}_t} < 1-\alpha_1$ for all $x_j \in \mathcal{T}^c$ for some $t = 1, 2, \cdots, m$. Therefore, $x_i \in (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_{\alpha_1}(\mathcal{T})$, showing that $(\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_{\alpha_2}(\mathcal{T}) \subseteq (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_{\alpha_1}(\mathcal{T})$.

(2) Let $x_i \in (\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_{\alpha_1}(\mathcal{T})$, then $a_{ij}^{\mathfrak{C}_t} \geq 1-\alpha_1$ for some $x_j \in \mathcal{T}$ for all $t = 1, 2, \cdots, m$. As $\alpha_1 \leq \alpha_2$, so $1-\alpha_1 \geq 1-\alpha_2$. Therefore, $a_{ij}^{\mathfrak{C}_t} \geq 1-\alpha_2$ for some $x_j \in \mathcal{T}$ for all $t = 1, 2, \cdots, m$. This indicates that $x_i \in (\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_{\alpha_2}(\mathcal{T})$. Hence, $(\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_{\alpha_1}(\mathcal{T}) \subseteq (\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_{\alpha_2}(\mathcal{T})$.

□

Proposition 3.5. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ and $\beta_1, \beta_2 \in (-1, -0.5]$ be such that $\beta_1 \le \beta_2$. Then for any $\mathcal{T} \subseteq \mho$, the subsequent properties hold:*

(1) $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta_2}(\mathcal{T}) \subseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta_1}(\mathcal{T})$;

(2) $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta_1}(\mathcal{T}) \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta_2}(\mathcal{T})$.

Proof. (1) Let $x_i \in (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta_2}(\mathcal{T})$, then $b_{ij}^{\mathfrak{C}_t} \le -1 - \beta_2$ for some $x_j \in \mathcal{T}$ for all $t = 1, 2, \cdots, m$. Since, $\beta_1 \le \beta_2$, so $-1 - \beta_2 \le -1 - \beta_1$. Thus, $b_{ij}^{\mathfrak{C}_t} < -1 - \beta_1$ for some $x_j \in \mathcal{T}$ for all $t = 1, 2, \cdots, m$. This shows that $x_i \in (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta_1}(\mathcal{T})$. Hence, $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta_2}(\mathcal{T}) \subseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta_1}(\mathcal{T})$.

(2) For $x_i \in (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta_1}(\mathcal{T})$, we have $b_{ij}^{\mathfrak{C}_t} > -1 - \beta_1$ for all $x_j \in \mathcal{T}^c$ for some $t = 1, 2, \cdots, m$. As $\beta_1 \le \beta_2$, so $-1 - \beta_1 \ge -1 - \beta_2$. Therefore, $b_{ij}^{\mathfrak{C}_t} \ge -1 - \beta_2$ for all $x_j \in \mathcal{T}^c$ for some $t = 1, 2, \cdots, m$. Therefore, $x_i \in (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta_2}(\mathcal{T})$ showing that $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta_1}(\mathcal{T}) \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta_2}(\mathcal{T})$.

□

Proposition 3.6. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots, \mathfrak{B}_{\mathfrak{C}_m}$ and $\alpha \in [0.5, 1)$. Then for each $\mathcal{X}, \mathcal{Y} \subseteq \mho$, we have*

(1) $\mathcal{X} \subseteq \mathcal{Y} \Longrightarrow (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{X}) \subseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{Y})$;

(2) $\mathcal{X} \subseteq \mathcal{Y} \Longrightarrow (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{X}) \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{Y})$.

Proof. (1) For any $x_i \in (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{X})$, we have $a_{ij}^{\mathfrak{C}_t} < 1 - \alpha$ for all $x_j \in \mathcal{X}^c$ for some $t = 1, 2, \cdots, m$. But since, $\mathcal{X} \subseteq \mathcal{Y}$, so $\mathcal{Y}^c \subseteq \mathcal{X}^c$. Thus in particular, $a_{ij}^{\mathfrak{C}_t} < 1 - \alpha$ for all $x_j \in \mathcal{Y}^c$ for some $t = 1, 2, \cdots, m$. Therefore, $x_i \in (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{Y})$ showing that $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{X}) \subseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{Y})$.

(2) Let $x_i \in (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{X})$, then $a_{ij}^{\mathfrak{C}_t} \ge 1 - \alpha$ for some $x_j \in \mathcal{X}$ for all $t = 1, 2, \cdots, m$. As $\mathcal{X} \subseteq \mathcal{Y}$, so $a_{ij}^{\mathfrak{C}_t} \ge 1 - \alpha$ for some $x_j \in \mathcal{X} \subseteq \mathcal{Y}$ for all $t = 1, 2, \cdots, m$. This implies that $x_i \in (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{Y})$. Hence, $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{X}) \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{Y})$.

□

Proposition 3.7. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ and $\beta \in (-1, -0.5]$. Then for each $\mathcal{X}, \mathcal{Y} \subseteq \mho$, we have*

(1) $\mathcal{X} \subseteq \mathcal{Y} \Longrightarrow (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta}(\mathcal{X}) \subseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta}(\mathcal{Y})$;

(2) $\mathcal{X} \subseteq \mathcal{Y} \Longrightarrow (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta}(\mathcal{X}) \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta}(\mathcal{Y})$.

Proof. (1) Let $x_i \in (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta}(\mathcal{X})$, then $b_{ij}^{\mathfrak{C}_t} \le -1 - \beta$ for some $x_j \in \mathcal{X}$ for all $t = 1, 2, \cdots, m$. Since, $\mathcal{X} \subseteq \mathcal{Y}$, so $b_{ij}^{\mathfrak{C}_t} < -1 - \beta$ for some $x_j \in \mathcal{X} \subseteq \mathcal{Y}$ for all $t = 1, 2, \cdots, m$. Thus, $x_i \in (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta}(\mathcal{Y})$ showing that $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta}(\mathcal{X}) \subseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta}(\mathcal{Y})$.

(2) For any $x_i \in (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X})$, we have $b^{\mathfrak{C}_t}_{ij} > -1-\beta$ for all $x_j \in \mathcal{X}^c$ for some $t = 1, 2, \cdots, m$. As $\mathcal{X} \subseteq \mathcal{Y}$, so $\mathcal{Y}^c \subseteq \mathcal{X}^c$. Therefore in particular, $b^{\mathfrak{C}_t}_{ij} \geq -1-\beta$ for all $x_j \in \mathcal{Y}^c$ for some $t = 1, 2, \cdots, m$. Thus, $x_i \in (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{Y})$. Hence, $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X}) \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{Y})$.

□

Theorem 3.8. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ and $\alpha \in [0.5, 1)$. Then for each $\mathcal{X} \subseteq \mho$, the following properties hold:*

(1) $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X}) \subseteq \mathcal{X} \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X})$;

(2) $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\varnothing) = \varnothing = (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\varnothing)$;

(3) $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mho) = \mho = (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mho)$;

(4) $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X}^c) = \left((\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X})\right)^c$;

(5) $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X}^c) = \left((\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X})\right)^c$.

Proof. (1) By definition, $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X}) \subseteq \mathcal{X}$ is trivial. For the next inclusion, let $x_i \in \mathcal{X} \subseteq \mho$. Then, we have $a^{\mathfrak{C}_t}_{ii} = 0.5 \geq 1-\alpha$ for some $x_i \in \mathcal{X}$ for all $t = 1, 2, \cdots, m$. This implies that $x_i \in (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X})$. Hence, $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X}) \subseteq \mathcal{X} \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X})$.

(2) In light of Definition 3.1, we have

$$\Big(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^o_\alpha(\varnothing) = \bigcup_{t=1}^{m}\Big\{x_i \in \mho : a^{\mathfrak{C}_t}_{ij} < 1-\alpha \text{ for all } x_j \in (\varnothing)^c = \mho\Big\} = \varnothing.$$

And,

$$\Big(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^o_\alpha(\varnothing) = \bigcap_{t=1}^{m}\Big\{x_i \in \mho : a^{\mathfrak{C}_t}_{ij} \geq 1-\alpha \text{ for some } x_j \in \varnothing\Big\} = \varnothing.$$

Therefore, $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\varnothing) = \varnothing = (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\varnothing)$.

(3) By Definition 3.1,

$$\Big(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^o_\alpha(\mho) = \bigcup_{t=1}^{m}\Big\{x_i \in \mho : a^{\mathfrak{C}_t}_{ij} < 1-\alpha \text{ for all } x_j \in (\mho)^c = \varnothing\Big\} = \{x_i : x_i \in \mho\} = \mho.$$

Similarly,

$$\Big(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^o_\alpha(\mho) = \bigcap_{t=1}^{m}\Big\{x_i \in \mho : a^{\mathfrak{C}_t}_{ij} \geq 1-\alpha \text{ for some } x_j \in \mho\Big\} = \{x_i : x_i \in \mho\} = \mho.$$

Therefore, $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mho) = \mho = (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mho)$.

(4) For any $x_i \in \mho$,

$$\begin{aligned} x_i \in \Big(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^o_\alpha(\mathcal{X}^c) &\iff a^{\mathfrak{C}_t}_{ij} < 1-\alpha \text{ for all } x_j \in (\mathcal{X}^c)^c = \mathcal{X} \text{ for some } t = 1,2,\cdots,m \\ &\iff a^{\mathfrak{C}_t}_{ij} \ngeq 1-\alpha \text{ for any } x_j \in \mathcal{X} \text{ for some } t = 1,2,\cdots,m \\ &\iff x_i \notin \Big(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^o_\alpha(\mathcal{X}) \\ &\iff x_i \in \Big(\Big(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^o_\alpha(\mathcal{X})\Big)^c. \end{aligned}$$

Hence, $(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X}^c) = \Big((\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X})\Big)^c$.

(5) For any $x_i \in \mho$,

$$\begin{aligned} x_i \in \Big(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^o_\alpha(\mathcal{X}^c) &\iff a^{\mathfrak{C}_t}_{ij} \geq 1-\alpha \text{ for some } x_j \in \mathcal{X}^c \text{ for all } t = 1,2,\cdots,m \\ &\iff a^{\mathfrak{C}_t}_{ij} \nless 1-\alpha \text{ for any } x_j \in \mathcal{X}^c \text{ for all } t = 1,2,\cdots,m \\ &\iff x_i \notin \Big(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^o_\alpha(\mathcal{X}) \\ &\iff x_i \in \Big(\Big(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}}\Big)^o_\alpha(\mathcal{X})\Big)^c. \end{aligned}$$

Therefore, $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X}^c) = \Big((\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X})\Big)^c$.

□

Theorem 3.9. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ and $\beta \in (-1, -0.5]$. Then for any $\mathcal{X} \subseteq \mho$, the subsequent properties hold:*

(1) $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X}) \subseteq \mathcal{X} \subseteq (\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X})$;

(2) $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\varnothing) = \varnothing = (\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\varnothing)$;

(3) $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mho) = \mho = (\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mho)$;

(4) $(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X}^c) = \Big((\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X})\Big)^c$;

(5) $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X}^c) = \Big((\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X})\Big)^c$.

Proof. (1) By definition, $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X}) \subseteq \mathcal{X}$ is obvious. For the next inclusion, assume that $x_i \in \mathcal{X} \subseteq \mho$. Then, we have $b_{ii}^{\mathfrak{C}_t} = -0.5 \leq 1 - \beta$ for some $x_i \in \mathcal{X}$ for all $t = 1, 2, \cdots, m$. Thus, $x_i \in (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X})$ showing that $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X}) \subseteq \mathcal{X} \subseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X})$.

(2) According to Definition 3.1, we have

$$\left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^o_\beta(\varnothing) = \bigcup_{t=1}^{m} \left\{x_i \in \mho : b_{ij}^{\mathfrak{C}_t} > -1 - \beta \text{ for all } x_j \in (\varnothing)^c = \mho\right\} = \varnothing.$$

Also,

$$\left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^o_\beta(\varnothing) = \bigcap_{t=1}^{m} \left\{x_i \in \mho : b_{ij}^{\mathfrak{C}_t} \leq -1 - \beta \text{ for some } x_j \in \varnothing\right\} = \varnothing.$$

Hence, $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\varnothing) = \varnothing = (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\varnothing)$.

(3) In light of Definition 3.1,

$$\left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^o_\beta(\mho) = \bigcup_{t=1}^{m} \left\{x_i \in \mho : b_{ij}^{\mathfrak{C}_t} > -1 - \beta \text{ for all } x_j \in (\mho)^c = \varnothing\right\} = \{x_i : x_i \in \mho\} = \mho.$$

Also,

$$\left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^o_\beta(\mho) = \bigcap_{t=1}^{m} \left\{x_i \in \mho : b_{ij}^{\mathfrak{C}_t} \leq -1 - \beta \text{ for some } x_j \in \mho\right\} = \{x_i : x_i \in \mho\} = \mho.$$

Hence, $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mho) = \mho = (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mho)$.

(4) For any $x_i \in \mho$,

$$\begin{aligned} x_i \in \left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^o_\beta(\mathcal{X}^c) &\Longleftrightarrow b_{ij}^{\mathfrak{C}_t} \leq -1 - \beta \text{ for some } x_j \in \mathcal{X}^c \text{ for all } t = 1, 2, \cdots, m \\ &\Longleftrightarrow b_{ij}^{\mathfrak{C}_t} \ngtr -1 - \beta \text{ for all } x_j \in \mathcal{X}^c \text{ for all } t = 1, 2, \cdots, m \\ &\Longleftrightarrow x_i \notin \left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^o_\beta(\mathcal{X}) \\ &\Longleftrightarrow x_i \in \left(\left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^o_\beta(\mathcal{X})\right)^c. \end{aligned}$$

Therefore, $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X}^c) = \left((\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X})\right)^c$.

(5) For any $x_i \in \mho$,

$$x_i \in \left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^o_\beta(\mathcal{X}^c) \Longleftrightarrow b_{ij}^{\mathfrak{C}_t} > -1 - \beta \text{ for all } x_j \in (\mathcal{X}^c)^c = \mathcal{X} \text{ for some } t = 1, 2, \cdots, m$$

$$\Longleftrightarrow b_{ij}^{\mathfrak{C}_t} \nleq -1-\beta \text{ for any } x_j \in \mathcal{X} \text{ for some } t=1,2,\cdots,m$$
$$\Longleftrightarrow x_i \notin \underline{\Big(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}\Big)}_{\beta}^{o}(\mathcal{X})$$
$$\Longleftrightarrow x_i \in \Big(\underline{\Big(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}\Big)}_{\beta}^{o}(\mathcal{X})\Big)^{c}.$$

Hence, $\overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{X}^c) = \Big(\underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{X})\Big)^{c}$.

□

Theorem 3.10. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ and $\alpha \in [0.5, 1)$. Then for each $\mathcal{X}, \mathcal{Y} \subseteq \mho$, the following properties hold:*

(1) $\underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{X} \cup \mathcal{Y}) \supseteq \underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{X}) \cup \underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{Y})$;

(2) $\overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{X} \cup \mathcal{Y}) \supseteq \overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{X}) \cup \overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{Y})$;

(3) $\underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{X} \cap \mathcal{Y}) \subseteq \underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{X}) \cap \underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{Y})$;

(4) $\overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{X} \cap \mathcal{Y}) \subseteq \overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{X}) \cap \overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{Y})$.

Proof. It can be directly obtained by Proposition 3.6. □

Theorem 3.11. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ and $\beta \in (-1, -0.5]$. Then for each $\mathcal{X}, \mathcal{Y} \subseteq \mho$, the following properties hold:*

(1) $\underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{X} \cup \mathcal{Y}) \supseteq \underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{X}) \cup \underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{Y})$;

(2) $\overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{X} \cup \mathcal{Y}) \supseteq \overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{X}) \cup \overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{Y})$;

(3) $\underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{X} \cap \mathcal{Y}) \subseteq \underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{X}) \cap \underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{Y})$;

(4) $\overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{X} \cap \mathcal{Y}) \subseteq \overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{X}) \cap \overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\beta}^{o}(\mathcal{Y})$.

Proof. It can be directly obtained by Proposition 3.7. □

Proposition 3.12. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$, $\alpha \in [0.5, 1)$ and $\beta \in (-1, -0.5]$. Then for each $\mathcal{X} \subseteq \mho$, we have*

(1) $\underline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{X}) = \bigcup_{t=1}^{m} \underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\alpha}$;

(2) $\overline{(\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t})}_{\alpha}^{o}(\mathcal{X}) = \bigcap_{t=1}^{m} \overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\alpha}$;

(3) $\left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^{o}_{\beta}(\mathcal{X}) = \bigcap_{t=1}^{m} \underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\beta}$;

(4) $\left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^{o}_{\beta}(\mathcal{X}) = \bigcup_{t=1}^{m} \overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\beta}$.

Proof. Straightforward. □

Proposition 3.13. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$, $\alpha \in [0.5, 1)$ and $\beta \in (-1, -0.5]$. Then for each $\mathcal{X}, \mathcal{Y} \subseteq \mho$, we have*

(1) $\left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^{o}_{\alpha}(\mathcal{X} \cap \mathcal{Y}) = \bigcup_{t=1}^{m} \left(\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\alpha} \cap \underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{Y})}_{\alpha}\right)$;

(2) $\left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^{o}_{\alpha}(\mathcal{X} \cup \mathcal{Y}) = \bigcap_{t=1}^{m} \left(\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\alpha} \cup \overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{Y})}_{\alpha}\right)$;

(3) $\left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^{o}_{\beta}(\mathcal{X} \cup \mathcal{Y}) = \bigcap_{t=1}^{m} \left(\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\beta} \cup \underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{Y})}_{\beta}\right)$;

(4) $\left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^{o}_{\beta}(\mathcal{X} \cap \mathcal{Y}) = \bigcup_{t=1}^{m} \left(\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\beta} \cap \overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{Y})}_{\beta}\right)$.

Proof. It can be directly obtained by Proposition 3.12 and Theorem 4.7 and Theorem 4.9 of [25]. □

Definition 3.14. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$, $\alpha \in [0.5, 1)$ and $\beta \in (-1, -0.5]$. Then the accuracy measure $\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X})$ of $\mathcal{X} \subseteq \mho$ under $(\alpha,\beta)^o$-MG-BFPRSs is defined as:*

$$\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X}) = (\mathfrak{X}^o_{\alpha}, \mathfrak{X}^o_{\beta}), \tag{24}$$

where

$$\mathfrak{X}^o_{\alpha} = \frac{\left|\left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^{o}_{\alpha}(\mathcal{X})\right|}{\left|\left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^{o}_{\alpha}(\mathcal{X})\right|}, \tag{25}$$

and

$$\mathfrak{X}^o_{\beta} = \frac{\left|\left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^{o}_{\beta}(\mathcal{X})\right|}{\left|\left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)^{o}_{\beta}(\mathcal{X})\right|}. \tag{26}$$

The corresponding roughness measure $\mathcal{R}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X})$ of $\mathcal{X}$ under $(\alpha,\beta)^o$-MG-BFPRSs is defined as:

$$\mathcal{R}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X}) = (1, 1) - \mathcal{A}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X}) = (1 - \mathfrak{X}^o_{\alpha}, 1 - \mathfrak{X}^o_{\beta}). \tag{27}$$

Obviously, $(0, 0) \leq \mathcal{A}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X}), \mathcal{R}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X}) \leq (1, 1)$ *for any $\mathcal{X} \subseteq \mho$, $\alpha \in [0.5, 1)$ and $\beta \in (-1, -0.5]$.*

Example 3.15. *(Following Example 3.3) We can evaluate the accuracy measure and the roughness measure of $\mathcal{X} = \{x_2, x_3\} \subseteq \mho$ for $\alpha = 0.5$ and $\beta = -0.5$ under $(\alpha,\beta)^o$-MG-BFPRSs environment as follows:*

$$\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X}) = \left(\frac{2}{3}, \frac{1}{3}\right) = (0.666, 0.333),$$

$$\mathcal{R}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X}) = (1, 1) - (0.666, 0.333) = (0.333, 0.666).$$

Proposition 3.16. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over the universe $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$, $\alpha \in [0.5, 1)$ and $\beta \in (-1, -0.5]$. Then the accuracy measure $\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X})$ of $\mathcal{X} \subseteq \mho$ under $(\alpha,\beta)^o$-MG-BFPRSs own the following properties:*

(1) $\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X}) = (0, 0) \Longleftrightarrow (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha = \varnothing = (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta$;

(2) $\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X}) = (1, 1) \Longleftrightarrow (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha = (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha$ *and* $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta = (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta$;

(3) If $\mathcal{X} = \mho$ or $\mathcal{X} = \varnothing$, then $\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X}) = (1, 1)$.

Proof. Straightforward. □

4. (α,β)-pessimistic multi-granulation bipolar fuzzified preference rough sets ($(\alpha,\beta)^p$-MG-BFPRSs)

In this section, we propose the notion of the $(\alpha,\beta)^p$-MG-BFPRS model and study some of its significant properties.

Definition 4.1. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over the universe $\mho$ on the criteria $\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots, \mathfrak{B}_{\mathfrak{C}_m}$ described by its positive and negative MFs given as $\mu^P_{\mathfrak{B}_{\mathfrak{C}_t}} : \mho \times \mho \longrightarrow [0, 1]$ and $\mu^N_{\mathfrak{B}_{\mathfrak{C}_t}} : \mho \times \mho \longrightarrow [-1, 0]$; $t = 1, 2, \cdots, m$. For any $\alpha \in [0.5, 1)$ and $\beta \in (-1, -0.5]$, the lower and upper $(\alpha,\beta)^p$-MG-BFPR-approximations for any $\mathcal{T} \subseteq \mho$ w.r.t. Υ are characterized as:*

$$\left.\begin{aligned} \underline{MBFP}_{(\alpha,\beta)^p}(\mathcal{T}) &= \left(\Big(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^p_\alpha(\mathcal{T}), \Big(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^p_\beta(\mathcal{T})\right),\\ \overline{MBFP}_{(\alpha,\beta)^p}(\mathcal{T}) &= \left(\Big(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^p_\alpha(\mathcal{T}), \Big(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^p_\beta(\mathcal{T})\right), \end{aligned}\right\} \tag{28}$$

where,

$$\Big(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^p_\alpha(\mathcal{T}) = \bigcap_{t=1}^{m} \Big\{x_i \in \mho : a^{\mathfrak{C}_t}_{ij} < 1 - \alpha \text{ for all } x_j \in \mathcal{T}^c\Big\}, \tag{29}$$

$$\Big(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^p_\alpha(\mathcal{T}) = \bigcup_{t=1}^{m} \Big\{x_i \in \mho : a^{\mathfrak{C}_t}_{ij} \geq 1 - \alpha \text{ for some } x_j \in \mathcal{T}\Big\}, \tag{30}$$

$$\Big(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^p_\beta(\mathcal{T}) = \bigcup_{t=1}^{m} \Big\{x_i \in \mho : b^{\mathfrak{C}_t}_{ij} \leq -1 - \beta \text{ for some } x_j \in \mathcal{T}\Big\}, \tag{31}$$

$$\Big(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\Big)^p_\beta(\mathcal{T}) = \bigcap_{t=1}^{m} \Big\{x_i \in \mho : b^{\mathfrak{C}_t}_{ij} > -1 - \beta \text{ for all } x_j \in \mathcal{T}^c\Big\}, \tag{32}$$

are said to be the α-pessimistic lower, α-pessimistic upper, β-pessimistic lower and β-pessimistic upper multi-granulation rough approximations of $\mathcal{T}$, respectively. Moreover, if $\underline{MBFP}_{(\alpha,\beta)^p}(\mathcal{T}) \neq \overline{MBFP}_{(\alpha,\beta)^p}(\mathcal{T})$, then $\mathcal{T}$ is titled as $(\alpha,\beta)^p$-MG-BFPRSs w.r.t. $\mathfrak{B}_{\mathfrak{C}_t}$; else, it is called (α,β)-pessimistic multi-granulation bipolar fuzzified preference definable w.r.t. $\mathfrak{B}_{\mathfrak{C}_t}$.

The corresponding positive, boundary and negative regions under $(\alpha,\beta)^p$-MG-BFPR-approximations are listed as follows:

(i) $\mathcal{POS}_{(\alpha,\beta)^p}(\mathcal{T}) = \left(\left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)_{\alpha}^{p}(\mathcal{T}), \left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)_{\beta}^{p}(\mathcal{T}) \right),$

(ii) $\mathcal{BND}_{(\alpha,\beta)^p}(\mathcal{T}) = \left(\left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)_{\alpha}^{p}(\mathcal{T}) - \left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)_{\alpha}^{p}(\mathcal{T}), \left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)_{\beta}^{p}(\mathcal{T}) - \left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)_{\beta}^{p}(\mathcal{T}) \right),$

(iii) $\mathcal{NEG}_{(\alpha,\beta)^p}(\mathcal{T}) = (\mho,\mho) - \left(\left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)_{\alpha}^{p}(\mathcal{T}), \left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)_{\beta}^{p}(\mathcal{T}) \right) = \left(\left(\left(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)_{\alpha}^{p}(\mathcal{T}) \right)^{c}, \left(\left(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}}\right)_{\beta}^{p}(\mathcal{T}) \right)^{c} \right).$

The information concerning an element $x \in \mho$ interpreted by operators mentioned above is as follows:

- $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{T})$ signifies the collection of objects $x_i \in \mho$ equivalent to all objects $x_j \in \mathcal{T}^c$ with a positive MD less than a specific $\alpha \in [0.5, 1)$ for all $t = 1, 2, \cdots, m$.

- $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{T})$ denotes the collection of objects $x_i \in \mho$ equivalent to at least one object $x_j \in \mathcal{T}$ with a positive MD greater than or equal to a certain $\alpha \in [0.5, 1)$ for some $t = 1, 2, \cdots, m$.

- $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{T})$ expresses the collection of objects $x_i \in \mho$ equivalent to at least one object $x_j \in \mathcal{T}$ with a negative MD less than or equal to a specific $\beta \in (-1, -0.5]$ for some $t = 1, 2, \cdots, m$.

- $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{T})$ refers to the collection of objects $x_i \in \mho$ equivalent to all objects $x_j \in \mathcal{T}^c$ with a negative MD greater than a specific $\beta \in (-1, -0.5]$ for all $t = 1, 2, \cdots, m$.

Remark 4.2. *In light of Definition 4.1, it follows that:*

(1) $\mathcal{T} \subseteq \mho$ *is a* $(\alpha,\beta)^p$*-MG-BFPRSs w.r.t.* $\mathfrak{B}_{\mathfrak{C}_t}$ *if and only if* $\mathcal{BND}_{(\alpha,\beta)^p}(\mathcal{T}) = (\emptyset, \emptyset)$.

(2) If $\mathfrak{B}_{\mathfrak{C}_1} = \mathfrak{B}_{\mathfrak{C}_2} = \cdots = \mathfrak{B}_{\mathfrak{C}_m} = \mathfrak{B}_{\mathfrak{C}}$*, then the operators given in Eqs (29) to (32) reduce into* (α,β)*-BFPR-approximation operators of a set* $\mathcal{T}$ *given in Definition 2.14.*

Example 4.3. *(Following Example 2.13) Assume that* $\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}$ *are two BFPRs over* $\mho$*, where* $\mho = \{x_1, x_2, x_3, x_4, x_5\}$ *given in Example 2.13. If we take* $\mathcal{T} = \{x_1, x_2\} \subseteq \mho$*, then for* $\alpha = 0.5$ *and* $\beta = -0.5$*, we get*

$$(\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})_{\alpha}^{p}(\mathcal{T}) = \{\},$$

$$(\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^p_{\alpha}(\mathcal{T}) = \{x_1, x_2, x_3, x_4, x_5\},$$

$$(\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^p_{\beta}(\mathcal{T}) = \{x_1, x_2, x_3, x_4, x_5\},$$

$$(\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^p_{\beta}(\mathcal{T}) = \{\}.$$

Thus, the lower and upper $(\alpha,\beta)^p$-MG-BFPR-approximations for $\mathcal{T}$ are given as follows:

$$\underline{MBFP}_{(\alpha,\beta)^p}(\mathcal{T}) = (\{\}, \{x_1, x_2, x_3, x_4, x_5\}),$$

$$\overline{MBFP}_{(\alpha,\beta)^p}(\mathcal{T}) = (\{x_1, x_2, x_3, x_4, x_5\}, \{\}).$$

As, $\underline{MBFP}_{(\alpha,\beta)^p}(\mathcal{T}) \neq \overline{MBFP}_{(\alpha,\beta)^p}(\mathcal{T})$, so $\mathcal{T}$ is a $(\alpha,\beta)^p$-MG-BFPRSs w.r.t. $\mathfrak{B}_{\mathfrak{C}_1}$ and $\mathfrak{B}_{\mathfrak{C}_2}$. Moreover,

$$\begin{aligned}
\mathcal{POS}_{(\alpha,\beta)^p}(\mathcal{T}) &= (\{\}, \{\}),\\
\mathcal{BND}_{(\alpha,\beta)^p}(\mathcal{T}) &= (\{x_1, x_2, x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4, x_5\}),\\
\mathcal{NEG}_{(\alpha,\beta)^p}(\mathcal{T}) &= (\{\}, \{\}).
\end{aligned}$$

Proposition 4.4. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPR over $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ and $\alpha_1, \alpha_2 \in [0.5, 1)$ be such that $\alpha_1 \leq \alpha_2$. Then for any $\mathcal{T} \subseteq \mho$, we have*

(1) $(\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_{\alpha_2}(\mathcal{T}) \subseteq (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_{\alpha_1}(\mathcal{T})$;

(2) $(\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_{\alpha_1}(\mathcal{T}) \subseteq (\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_{\alpha_2}(\mathcal{T})$.

Proof. Analogous to the proof of Proposition 3.4. □

Proposition 4.5. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ and $\beta_1, \beta_2 \in (-1, -0.5]$ be such that $\beta_1 \leq \beta_2$. Then for each $\mathcal{T} \subseteq \mho$, the subsequent properties hold:*

(1) $(\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_{\beta_2}(\mathcal{T}) \subseteq (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_{\beta_1}(\mathcal{T})$;

(2) $(\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_{\beta_1}(\mathcal{T}) \subseteq (\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_{\beta_2}(\mathcal{T})$.

Proof. Analogous to the proof of Proposition 3.5. □

Proposition 4.6. *Let $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ be a finite collection of BFPRs over $\mho$ on the criteria $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ and $\alpha \in [0.5, 1)$. Then for each $\mathcal{X}, \mathcal{Y} \subseteq \mho$, we have*

(1) $\mathcal{X} \subseteq \mathcal{Y} \Longrightarrow (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_{\alpha}(\mathcal{X}) \subseteq (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_{\alpha}(\mathcal{Y})$;

(2) $\mathcal{X} \subseteq \mathcal{Y} \Longrightarrow (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X}) \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{Y})$.

Proof. Similar to the proof of Proposition 3.6. □

Proposition 4.7. *Let* $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ *be a finite collection of BFPRs over* $\mho$ *on the criteria* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ *and* $\beta \in (-1, -0.5]$. *Then for each* $\mathcal{X}, \mathcal{Y} \subseteq \mho$,

(1) $\mathcal{X} \subseteq \mathcal{Y} \Longrightarrow (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X}) \subseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{Y})$;

(2) $\mathcal{X} \subseteq \mathcal{Y} \Longrightarrow (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X}) \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{Y})$.

Proof. Similar to the proof of Proposition 3.7. □

Theorem 4.8. *Let* $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ *be a finite collection of BFPRs over* $\mho$ *on the criteria* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ *and* $\alpha \in [0.5, 1)$. *Then for each* $\mathcal{X}, \mathcal{Y} \subseteq \mho$, *the following properties hold:*

(1) $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X}) \subseteq \mathcal{X} \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X})$;

(2) $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\varnothing) = \varnothing = (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\varnothing)$;

(3) $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mho) = \mho = (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mho)$;

(4) $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X}^c) = \left((\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X})\right)^c$;

(5) $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X}^c) = \left((\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X})\right)^c$.

Proof. Analogous to the proof of Proposition 3.8. □

Theorem 4.9. *Let* $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ *be a finite collection of BFPRs over* $\mho$ *on the criteria* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ *and* $\beta \in (-1, -0.5]$. *Then for any* $\mathcal{X} \subseteq \mho$, *the subsequent statements hold:*

(1) $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X}) \subseteq \mathcal{X} \subseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X})$;

(2) $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\varnothing) = \varnothing = (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\varnothing)$;

(3) $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mho) = \mho = (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mho)$;

(4) $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X}^c) = \left((\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X})\right)^c$;

(5) $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X}^c) = \left((\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X})\right)^c$.

Proof. Similar to the proof of Proposition 3.9. □

Theorem 4.10. *Let* $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ *be a finite collection of BFPRs over* $\mho$ *on the criteria* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ *and* $\alpha \in [0.5, 1)$. *Then for each* $\mathcal{X}, \mathcal{Y} \subseteq \mho$, *the following axioms hold:*

(1) $(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X}\cup\mathcal{Y})\supseteq(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X})\cup(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{Y})$;

(2) $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X}\cup\mathcal{Y})\supseteq(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X})\cup(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{Y})$;

(3) $(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X}\cap\mathcal{Y})\subseteq(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X})\cap(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{Y})$;

(4) $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X}\cap\mathcal{Y})\subseteq(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X})\cap(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{Y})$.

Proof. It can be directly obtained by Proposition 4.6. □

Theorem 4.11. *Let* $\Upsilon=\{\mathfrak{B}_{\mathfrak{C}_1},\mathfrak{B}_{\mathfrak{C}_2},\cdots\mathfrak{B}_{\mathfrak{C}_m}\}$ *be a finite collection of BFPRs over* $\mho$ *on the criteria* $\mathfrak{C}_1,\mathfrak{C}_2,\cdots,\mathfrak{C}_m$ *and* $\beta\in(-1,-0.5]$. *Then for each* $\mathcal{X},\mathcal{Y}\subseteq\mho$, *the following properties hold:*

(1) $(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X}\cup\mathcal{Y})\supseteq(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X})\cup(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{Y})$;

(2) $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X}\cup\mathcal{Y})\supseteq(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X})\cup(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{Y})$;

(3) $(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X}\cap\mathcal{Y})\subseteq(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X})\cap(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{Y})$;

(4) $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X}\cap\mathcal{Y})\subseteq(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X})\cap(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{Y})$.

Proof. It can be directly obtained by Proposition 4.7. □

Proposition 4.12. *Let* $\Upsilon=\{\mathfrak{B}_{\mathfrak{C}_1},\mathfrak{B}_{\mathfrak{C}_2},\cdots\mathfrak{B}_{\mathfrak{C}_m}\}$ *be a finite collection of BFPRs over* $\mho$ *on the criteria* $\mathfrak{C}_1,\mathfrak{C}_2,\cdots,\mathfrak{C}_m$, $\alpha\in[0.5,1)$ *and* $\beta\in(-1,-0.5]$. *Then for each* $\mathcal{X}\subseteq\mho$, *we have*

(1) $(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X})=\bigcap_{t=1}^{m}\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\alpha}$;

(2) $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X})=\bigcup_{t=1}^{m}\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\alpha}$;

(3) $(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X})=\bigcup_{t=1}^{m}\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\beta}$;

(4) $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\beta}^{p}(\mathcal{X})=\bigcap_{t=1}^{m}\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\beta}$.

Proof. Straightforward. □

Proposition 4.13. *Let* $\Upsilon=\{\mathfrak{B}_{\mathfrak{C}_1},\mathfrak{B}_{\mathfrak{C}_2},\cdots\mathfrak{B}_{\mathfrak{C}_m}\}$ *be a finite collection of BFPR over* $\mho$ *on the criteria* $\mathfrak{C}_1,\mathfrak{C}_2,\cdots,\mathfrak{C}_m$, $\alpha\in[0.5,1)$ *and* $\beta\in(-1,-0.5]$. *Then for each* $\mathcal{X},\mathcal{Y}\subseteq\mho$, *the following properties hold:*

(1) $(\underline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X}\cap\mathcal{Y})=\bigcap_{t=1}^{m}\left(\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\alpha}\cap\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{Y})}_{\alpha}\right)$;

(2) $(\overline{\sum_{t=1}^{m}\mathfrak{B}_{\mathfrak{C}_t}})_{\alpha}^{p}(\mathcal{X}\cup\mathcal{Y})=\bigcup_{t=1}^{m}\left(\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\alpha}\cup\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{Y})}_{\alpha}\right)$;

(3) $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^p_\beta(\mathcal{X} \cup \mathcal{Y}) = \bigcup_{t=1}^{m} \left(\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_\beta \cup \underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{Y})}_\beta\right)$;

(4) $(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^p_\beta(\mathcal{X} \cap \mathcal{Y}) = \bigcap_{t=1}^{m} \left(\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_\beta \cap \overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{Y})}_\beta\right)$.

Proof. It can be directly obtained by Proposition 4.12 and Theorem 4.7 and Theorem 4.9 of [25]. □

Definition 4.14. *Let* $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ *be a finite collection of BFPRs over* $\mho$ *on the criteria* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$, $\alpha \in [0.5, 1)$ *and* $\beta \in (-1, -0.5]$. *Then the accuracy measure* $\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X})$ *of* $\mathcal{X} \subseteq \mho$ *under* $(\alpha,\beta)^p$*-MG-BFPRSs is defined as:*

$$\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X}) = (\mathfrak{X}^p_\alpha, \mathfrak{X}^p_\beta), \tag{33}$$

where

$$\mathfrak{X}^p_\alpha = \frac{\left|(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^p_\alpha(\mathcal{X})\right|}{\left|(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^p_\alpha(\mathcal{X})\right|}, \tag{34}$$

and

$$\mathfrak{X}^p_\beta = \frac{\left|(\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^p_\beta(\mathcal{X})\right|}{\left|(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^p_\beta(\mathcal{X})\right|}. \tag{35}$$

The corresponding roughness measure $\mathcal{R}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X})$ *of* $\mathcal{X}$ *under* $(\alpha,\beta)^p$*-MG-BFPRSs is defined as:*

$$\mathcal{R}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X}) = (1, 1) - \mathcal{A}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X}) = (1 - \mathfrak{X}^p_\alpha, 1 - \mathfrak{X}^p_\beta). \tag{36}$$

Obviously, $(0, 0) \le \mathcal{A}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X}), \mathcal{R}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X}) \le (1, 1)$ *for any* $\mathcal{X} \subseteq \mho$, $\alpha \in [0.5, 1)$ *and* $\beta \in (-1, -0.5]$.

Example 4.15. *(Following Example 4.3) We can determine the accuracy measure and the roughness measure of* $\mathcal{X} = \{x_1, x_2\} \subseteq \mho$ *for* $\alpha = 0.5$ *and* $\beta = -0.5$ *under* $(\alpha,\beta)^p$*-MG-BFPRSs environment as follows:*

$$\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X}) = \left(\frac{0}{5}, \frac{0}{5}\right) = (0, 0),$$

$$\mathcal{R}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X}) = (1, 1) - (0, 0) = (1, 1).$$

Proposition 4.16. *Let* $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ *be a finite collection of BFPRs over the universe* $\mho$ *on the criteria* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$, $\alpha \in [0.5, 1)$ *and* $\beta \in (-1, -0.5]$. *Then the accuracy measure* $\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X})$ *of* $\mathcal{X} \subseteq \mho$ *under* $(\alpha,\beta)^p$*-MG-BFPRSs owns the following properties:*

(1) $\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X}) = (0, 0) \Longleftrightarrow (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^p_\alpha = \varnothing = (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^p_\beta$;

(2) $\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X}) = (1, 1) \Longleftrightarrow (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^p_\alpha = (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^p_\alpha$ *and* $(\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^p_\beta = (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^p_\beta$;

(3) *If* $\mathcal{X} = \mho$ *or* $\mathcal{X} = \varnothing$, *then* $\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X}) = (1, 1)$.

Proof. Straightforward. □

5. Relationship among the (α,β)-BFPRSs, $(\alpha,\beta)^o$-MG-BFPRSs and $(\alpha,\beta)^p$-MG-BFPRSs

In this part, the respective connections among the (α,β)-BFPRSs, $(\alpha,\beta)^o$-MG-BFPRSs and $(\alpha,\beta)^p$-MG-BFPRSs are studied.

Proposition 5.1. *Let* $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ *be a finite collection of BFPRs over* $\mho$ *on the criteria* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ *and* $\alpha \in [0.5, 1)$. *Then for each* $\mathcal{X} \subseteq \mho$*, the following properties hold:*

(1) $\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\alpha} \subseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{X})$;

(2) $\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\alpha} \supseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{p}_{\alpha}(\mathcal{X})$;

(3) $\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\alpha} \supseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\alpha}(\mathcal{X})$;

(4) $\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\alpha} \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{p}_{\alpha}(\mathcal{X})$;

Proof. It can be directly obtained by Definitions 2.14, 3.1 and 4.1. □

Proposition 5.1 shows the link of containment between α-lower approximation, α-upper approximation and α-optimistic lower, α-optimistic upper, α-pessimistic lower and α-pessimistic upper multi-granulation rough approximations of a subset $\mathcal{X}$ of $\mho$.

In other words, the α-optimistic lower multi-granulation rough approximation of $\mathcal{X} \subseteq \mho$ is finer than the α-lower approximation of $\mathcal{X} \subseteq \mho$. Moreover, the α-pessimistic lower multi-granulation of $\mathcal{X} \subseteq \mho$ is coarser than the α-lower approximation of $\mathcal{X} \subseteq \mho$. Similarly, the α-upper approximation of $\mathcal{X} \subseteq \mho$ is finer than the α-optimistic upper multi-granulation approximation of $\mathcal{X} \subseteq \mho$. Furthermore, the α-upper approximation of $\mathcal{X} \subseteq \mho$ is coarser than the α-pessimistic upper multi-granulation approximation of $\mathcal{X} \subseteq \mho$.

Proposition 5.2. *Let* $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ *be a finite collection of BFPRs over* $\mho$ *on the criteria* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ *and* $\beta \in (-1, -0.5]$. *Then for any* $\mathcal{X} \subseteq \mho$*, the subsequent axioms hold:*

(1) $\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\beta} \supseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta}(\mathcal{X})$;

(2) $\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\beta} \subseteq (\underline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{p}_{\beta}(\mathcal{X})$;

(3) $\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\beta} \subseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{o}_{\beta}(\mathcal{X})$;

(4) $\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_{\beta} \supseteq (\overline{\sum_{t=1}^{m} \mathfrak{B}_{\mathfrak{C}_t}})^{p}_{\beta}(\mathcal{X})$.

Proof. It can be directly obtained by Definitions 2.14, 3.1 and 4.1. □

The above proposition shows the connection of containment between β-lower approximation, β-upper approximation and β-optimistic lower, β-optimistic upper, β-pessimistic lower and β-pessimistic upper multi-granulation rough approximations of a subset $\mathcal{X}$ of $\mho$.

The results also reveal that, the β-optimistic lower multi-granulation rough approximation of $\mathcal{X} \subseteq \mho$ is coarser than the β-lower approximation of $\mathcal{X} \subseteq \mho$. Moreover, the β-pessimistic lower multi-granulation of $\mathcal{X} \subseteq \mho$ is finer than the β-lower approximation of $\mathcal{X} \subseteq \mho$. Similarly, the β-upper approximation of $\mathcal{X} \subseteq \mho$ is coarser than the β-optimistic upper multi-granulation approximation of $\mathcal{X} \subseteq \mho$. Furthermore, the β-upper approximation of $\mathcal{X} \subseteq \mho$ is finer than the β-pessimistic upper multi-granulation approximation of $\mathcal{X} \subseteq \mho$.

Proposition 5.3. *Suppose* $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ *is a finite collection of BFPRs over* $\mho$ *on the criteria* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ *such that* $\mathfrak{B}_{\mathfrak{C}_1} = \mathfrak{B}_{\mathfrak{C}_2} = \cdots = \mathfrak{B}_{\mathfrak{C}_m}$ *for any* $\alpha \in [0.5, 1)$ *and* $\beta \in (-1, -0.5]$. *Then for each* $\mathcal{X} \subseteq \mho$,

(1) $\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_\alpha = (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X}) = (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_\alpha(\mathcal{X})$;

(2) $\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_\alpha = (\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X}) = (\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_\alpha(\mathcal{X})$;

(3) $\underline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_\beta = (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X}) = (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_\beta(\mathcal{X})$;

(4) $\overline{\mathfrak{B}_{\mathfrak{C}_t}(\mathcal{X})}_\beta = (\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X}) = (\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_\beta(\mathcal{X})$.

Proof. Straightforward. □

The next result shows the relationship of containment between $(\alpha,\beta)^o$-MG-BFPR-approximations and $(\alpha,\beta)^p$-MG-BFPR-approximations for any $\mathcal{X} \subseteq \mho$.

Proposition 5.4. *Let* $\Upsilon = \{\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}, \cdots \mathfrak{B}_{\mathfrak{C}_m}\}$ *be a finite collection of BFPRs over* $\mho$ *on the criteria* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_m$ *and* $\alpha \in [0.5, 1)$ *and* $\beta \in (-1, -0.5]$. *Then for each* $\mathcal{X} \subseteq \mho$,

(1) $(\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X}) \supseteq (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_\alpha(\mathcal{X})$;

(2) $(\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_\alpha(\mathcal{X}) \subseteq (\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_\alpha(\mathcal{X})$;

(3) $(\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X}) \subseteq (\underline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_\beta(\mathcal{X})$;

(4) $(\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^o_\beta(\mathcal{X}) \supseteq (\overline{\sum_{t=1}^m \mathfrak{B}_{\mathfrak{C}_t}})^p_\beta(\mathcal{X})$.

Proof. It can be directly obtained by Definitions 3.1 and 4.1. □

The subsequent result signifies the connection among accuracy measures of (α,β)-BFPRSs, $(\alpha,\beta)^o$-MG-BFPRSs and $(\alpha,\beta)^p$-MG-BFPRSs for any $\mathcal{X} \subseteq \mho$.

Proposition 5.5. *Let* $\mathcal{A}^{\mathfrak{B}_\mathfrak{C}}_{(\alpha,\beta)}(\mathcal{X}) = (\mathfrak{X}^\alpha, \mathfrak{X}^\beta)$, $\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^o}(\mathcal{X}) = (\mathfrak{X}^o_\alpha, \mathfrak{X}^o_\beta)$ *and* $\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^p}(\mathcal{X}) = (\mathfrak{X}^p_\alpha, \mathfrak{X}^p_\beta)$ *be the accuracy measures of* $\mathcal{X} \subseteq \mho$ *under* (α,β)*-BFPRSs,* $(\alpha,\beta)^o$*-MG-BFPRSs and* $(\alpha,\beta)^p$*-MG-BFPRSs environments, respectively. Then for* $\alpha \in [0.5, 1)$ *and* $\beta \in (-1, -0.5]$, *we have*

(1) $\mathfrak{X}^o_\alpha \geq \mathfrak{X}^\alpha$ *and* $\mathfrak{X}^o_\beta \geq \mathfrak{X}^\beta$;

(2) $\mathfrak{X}^p_\alpha \leq \mathfrak{X}^\alpha$ *and* $\mathfrak{X}^p_\beta \leq \mathfrak{X}^\beta$;

(3) $\mathfrak{X}^{o}_{\alpha} \geq \mathfrak{X}^{p}_{\alpha}$ *and* $\mathfrak{X}^{o}_{\beta} \geq \mathfrak{X}^{p}_{\beta}$.

Proof. Straightforward. □

To better comprehend the connection among the (α,β)-BFPRSs, $(\alpha,\beta)^o$-MG-BFPRSs and $(\alpha,\beta)^p$-MG-BFPRSs, we provide the following illustration.

Example 5.6. *(Continued Example 2.13) Consider the two BFPRs* $\mathfrak{B}_{\mathfrak{C}_1}, \mathfrak{B}_{\mathfrak{C}_2}$ *over* $\mho$*, where* $\mho = \{x_1, x_2, x_3, x_4, x_5\}$ *given in Example 2.13. If* $\mathcal{T} = \{x_2, x_3\} \subseteq \mho$*, then for* $\alpha = 0.5$ *and* $\beta = -0.5$*, the* α*-lower,* α*-upper,* β*-lower, and* β*-upper approximations of* $\mathcal{T}$ *w.r.t.* $\mathfrak{B}_{\mathfrak{C}_1}$ *by using Definition 2.14 are respectively given as:*

$$\begin{aligned}
\underline{\mathfrak{B}_{\mathfrak{C}_1}(\mathcal{T})}_{\alpha} &= \{x_2, x_3\},\\
\overline{\mathfrak{B}_{\mathfrak{C}_1}(\mathcal{T})}_{\alpha} &= \{x_1, x_2, x_3, x_4, x_5\},\\
\underline{\mathfrak{B}_{\mathfrak{C}_1}(\mathcal{T})}_{\beta} &= \{x_1, x_2, x_3, x_4, x_5\},\\
\overline{\mathfrak{B}_{\mathfrak{C}_1}(\mathcal{T})}_{\beta} &= \{x_2\}.
\end{aligned}$$

Similarly, in the light of Definition 2.14, the α*-lower,* α*-upper,* β*-lower, and* β*-upper approximations of* $\mathcal{T}$ *w.r.t.* $\mathfrak{B}_{\mathfrak{C}_2}$ *are respectively given as:*

$$\begin{aligned}
\underline{\mathfrak{B}_{\mathfrak{C}_2}(\mathcal{T})}_{\alpha} &= \{\},\\
\overline{\mathfrak{B}_{\mathfrak{C}_2}(\mathcal{T})}_{\alpha} &= \{x_1, x_2, x_3\},\\
\underline{\mathfrak{B}_{\mathfrak{C}_2}(\mathcal{T})}_{\beta} &= \{x_2, x_3, x_5\},\\
\overline{\mathfrak{B}_{\mathfrak{C}_2}(\mathcal{T})}_{\beta} &= \{\}.
\end{aligned}$$

Now, according to Definition 3.1, the α*-optimistic lower,* α*-optimistic upper,* β*-optimistic lower and* β*-optimistic upper multi-granulation rough approximations of* $\mathcal{T}$ *are calculated respectively as:*

$$\begin{aligned}
(\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\alpha}(\mathcal{T}) &= \{x_2, x_3\},\\
(\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\alpha}(\mathcal{T}) &= \{x_1, x_2, x_3\},\\
(\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\beta}(\mathcal{T}) &= \{x_2, x_3, x_5\},\\
(\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\beta}(\mathcal{T}) &= \{x_2\}.
\end{aligned}$$

Similarly, from Definition 4.1, the α*-pessimistic lower,* α*-pessimistic upper,* β*-pessimistic lower and* β*-pessimistic upper multi-granulation rough approximations of* $\mathcal{T}$ *are calculated respectively as:*

$$\begin{aligned}
(\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\alpha}(\mathcal{T}) &= \{\},\\
(\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\alpha}(\mathcal{T}) &= \{x_1, x_2, x_3, x_4, x_5\},
\end{aligned}$$

$$(\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\beta}(\mathcal{T}) = \{x_1, x_2, x_3, x_4, x_5\},$$
$$(\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\beta}(\mathcal{T}) = \{\}.$$

We observe that $\underline{\mathfrak{B}_{\mathfrak{C}_1}(\mathcal{T})}_{\alpha} \subseteq (\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\alpha}(\mathcal{T})$, $\underline{\mathfrak{B}_{\mathfrak{C}_2}(\mathcal{T})}_{\alpha} \subseteq (\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\alpha}(\mathcal{T})$ *and* $\underline{\mathfrak{B}_{\mathfrak{C}_1}(\mathcal{T})}_{\alpha} \supseteq (\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\alpha}(\mathcal{T})$, $\underline{\mathfrak{B}_{\mathfrak{C}_2}(\mathcal{T})}_{\alpha} \supseteq (\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\alpha}(\mathcal{T})$. *Further,* $\overline{\mathfrak{B}_{\mathfrak{C}_1}(\mathcal{T})}_{\alpha} \supseteq (\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\alpha}(\mathcal{T})$, $\overline{\mathfrak{B}_{\mathfrak{C}_2}(\mathcal{T})}_{\alpha} \supseteq (\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\alpha}(\mathcal{T})$ *and* $\overline{\mathfrak{B}_{\mathfrak{C}_1}(\mathcal{T})}_{\alpha} \subseteq (\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\alpha}(\mathcal{T})$, $\overline{\mathfrak{B}_{\mathfrak{C}_2}(\mathcal{T})}_{\alpha} \subseteq (\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\alpha}(\mathcal{T})$, *which verifies Proposition 5.1. Similarly,* $\underline{\mathfrak{B}_{\mathfrak{C}_1}(\mathcal{T})}_{\beta} \supseteq (\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\beta}(\mathcal{T})$, $\underline{\mathfrak{B}_{\mathfrak{C}_2}(\mathcal{T})}_{\beta} \supseteq (\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\beta}(\mathcal{T})$ *and* $\underline{\mathfrak{B}_{\mathfrak{C}_1}(\mathcal{T})}_{\beta} \subseteq (\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\beta}(\mathcal{T})$, $\underline{\mathfrak{B}_{\mathfrak{C}_2}(\mathcal{T})}_{\beta} \subseteq (\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\beta}(\mathcal{T})$. *Moreover,* $\overline{\mathfrak{B}_{\mathfrak{C}_1}(\mathcal{T})}_{\beta} \subseteq (\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\beta}(\mathcal{T})$, $\overline{\mathfrak{B}_{\mathfrak{C}_2}(\mathcal{T})}_{\beta} \subseteq (\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\beta}(\mathcal{T})$ *and* $\overline{\mathfrak{B}_{\mathfrak{C}_1}(\mathcal{T})}_{\beta} \supseteq (\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\beta}(\mathcal{T})$, $\overline{\mathfrak{B}_{\mathfrak{C}_2}(\mathcal{T})}_{\beta} \supseteq (\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\beta}(\mathcal{T})$, *which verifies Proposition 5.2. Also, one can see that* $(\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\alpha}(\mathcal{T}) \supseteq (\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\alpha}(\mathcal{T})$, $(\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\alpha}(\mathcal{T}) \subseteq (\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\alpha}(\mathcal{T})$, $(\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\beta}(\mathcal{T}) \subseteq (\underline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\beta}(\mathcal{T})$ *and* $(\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{o}_{\beta}(\mathcal{T}) \supseteq (\overline{\mathfrak{B}_{\mathfrak{C}_1} + \mathfrak{B}_{\mathfrak{C}_2}})^{p}_{\beta}(\mathcal{T})$, *which verifies Proposition 5.4. Moreover, the measure of accuracy of* $\mathcal{T}$ *under* (α,β)*-BFPRSs w.r.t.* $\mathfrak{B}_{\mathfrak{C}_1}$ *is evaluated as:*

$$\mathcal{A}^{\mathfrak{B}_{\mathfrak{C}}}_{(\alpha,\beta)}(\mathcal{T}) = (\mathfrak{X}^{\alpha}, \mathfrak{X}^{\beta}) = (0.4, 0.2).$$

Similarly, we can calculate the accuracy measure of $\mathcal{T}$ *under* $(\alpha,\beta)^{o}$*-MG-BFPRSs and* $(\alpha,\beta)^{p}$*-MG-BFPRSs environment is as follows:*

$$\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^{o}}(\mathcal{T}) = (\mathfrak{X}^{o}_{\alpha}, \mathfrak{X}^{o}_{\beta}) = (0.666, 0.333),$$

$$\mathcal{A}^{\Upsilon}_{(\alpha,\beta)^{p}}(\mathcal{T}) = (\mathfrak{X}^{p}_{\alpha}, \mathfrak{X}^{p}_{\beta}) = (0, 0).$$

Clearly, we can see that $\mathfrak{X}^{o}_{\alpha} > \mathfrak{X}^{\alpha}$, $\mathfrak{X}^{o}_{\beta} > \mathfrak{X}^{\beta}$, $\mathfrak{X}^{p}_{\alpha} < \mathfrak{X}^{\alpha}$, $\mathfrak{X}^{p}_{\beta} < \mathfrak{X}^{\beta}$, $\mathfrak{X}^{o}_{\alpha} > \mathfrak{X}^{p}_{\alpha}$, $\mathfrak{X}^{o}_{\beta} > \mathfrak{X}^{p}_{\beta}$, *which verifies Proposition 5.5.*

6. Comparative study and discussion

One desired direction in RS theory is MGRS, which approximates a target set via granular structures obtained by multiple binary relations. On the other hand, BFS is considered more appropriate to capture uncertainty because it provides two-sided information about alternatives. In the literature, there are various hybrid MGRS models. Each of these models has merits and demerits. The problem under consideration determines the capability of any model. For instance, Qian et al. [47,48] construct a framework of OMGRS and PMGRS by getting inspiration from multi-source datasets, and multiple granulations are needed by multi-scale data for set approximation [61]. Many things are different when we compare our proposed study with existing theories. For example:

(1) Our work is different from the existing study in [24] in term of MGRS. In [24], Gul and Shabir originated the idea (α,β)-multi-granulation bipolar fuzzified RS using a finite collection of bipolar fuzzy tolerance relations. While in our proposed work, we have used a finite collection of BFPRs.

(2) If we compare our proposed approach with the methods offered in [4, 15, 19, 21, 29, 36–38, 41], we conclude that these methods are unable to capture bipolarity in decision-making which is an essential part of human thinking and behavior.

(3) Some studied on the FSs and BFSs can also be found in [3, 5, 9, 13, 26, 27, 33], but the roughness of the proposed approaches is not studied in these papers. Our proposed study is a unification of MGRSs and BFPR. In this study, we implement the notions of multi-granulation roughness to the target set by using BFPR, which is the uniqueness and novelty of our study.

7. Conclusions and future work

MGRS theory is an extension of the classical RS theory, a mathematical framework for dealing with uncertainty and vagueness in data under multiple binary relations over the universe. It provides a new perspective based on multi-granulation analysis for knowledge acquisition and decision-making. On the other hand, bipolarity refers to an explicit handling of positive and negative aspects of data. Numerous human decisions are influenced by their positive and negative, or bipolar, assessments. In this paper, in terms of BFPR and MGRS theory, we have established $(\alpha,\beta)^o$-MG-BFPRSs and $(\alpha,\beta)^p$-MG-BFPRSs models. Several essential properties of the two models have been investigated in detail. At the same time, a relationship among the (α,β)-BFPRS, $(\alpha,\beta)^o$-MG-BFPRS and $(\alpha,\beta)^p$-MG-BFPRS models have been established.

We hope our investigations provide more insight into the foundations of MGRS theory and lead to more robust mathematical approaches to approximate reasoning in soft computing. Meanwhile, several avenues remain for further theoretical research in this direction. Generally, future lines of study include the following complementary issues:

(1) The attribute reduction of $(\alpha,\beta)^o$-MG-BFPRS and $(\alpha,\beta)^p$-MG-BFPRS models should be analyzed, and comprehensive experimental investigations and comparisons with existing methods should also be verified and explored.

(2) Further research may be conducted to develop effective algorithms for various decision-making problems.

(3) Another avenue is to look at the topological characteristics of the proposed models.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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