Fractional comparative analysis of Camassa-Holm and Degasperis-Procesi equations

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Abstract: This paper focuses on novel approaches to finding solitary wave (SW) solutions for the modified Degasperis-Procesi and fractionally modified Camassa-Holm equations. The study presents two innovative methodologies: the Yang transformation decomposition technique and the homotopy perturbation transformation method. These methods use the Caputo sense fractional order derivative, the Yang transformation, the adomian decomposition technique, and the homotopy perturbation method. The inquiry effectively solves the fractional Camassa-Holm and Degasperis-Procesi equations, which also provides a detailed numerical and graphical comparison of the solutions found. The results, which include accurate solutions, derived solutions, and absolute error displayed in tabular style, demonstrate the effectiveness of the suggested procedures. These procedures are iterative, which results in several answers. The estimated absolute error attests to the correctness and simplicity of these solutions. Especially in plasma physics, these approaches may be expanded to handle various linear and nonlinear physical issues, including the evolution equations controlling nonlinear waves.

Keywords: Yang transform; Degasperis-Procesi and Camassa-Holm equations; series solution; adomian decomposition method; homotopy perturbation method

Mathematics Subject Classification: 33B15, 34A34, 35A20, 35A22, 44A10
**Nomenclature**

\( \nu \): Independent variable  
\( \iota \): Time  
\( \zeta (\iota, \nu) \): Dependent function representing the physical quantity  
\( \varsigma \): Fractional order  
\( Y \): Shehu transform  
\( Y^{-1} \): Inverse Shehu transform  
\( \epsilon \): Perturbation parameter

**1. Introduction**

Fractional calculus (FC) offers a practical and straightforward approach to obtaining specific information from different equations. This dynamic field of mathematics generalizes the standard integer order to a fractional order, resulting in a vast array of mathematical models [1–6]. In recent years, fractional differential equations (FDEs) have become a popular area of research for various science and engineering applications. There are several fundamental fractional derivative concepts, such as the Liouville-Caputo, Riemann-Liouville, Caputo-Fabrizio, Atangana-Baleanu, and Hadamard derivatives [7–11]. The fractional Riemann-Liouville derivative allows the inclusion of initial sources expressed through fractional integrals and derivatives, while the fractional Caputo derivatives only accept conventional boundaries and initial conditions. Nonlinear problems have been applied in different fields, including (non)linear waves in plasmas, hydrology, nuclear engineering, astrophysics, and meteorology [12–18].

Mathematicians have become increasingly interested in fractional partial differential equations (FPDEs) in recent years due to their numerous applications in engineering, applied sciences, biology, mathematical physics, solid-state materials science, neurology, geophysics, plasma physics, electrochemistry, physiological scaling laws, chemical physics, dielectric behavior, quantum systems, financial mathematics, fractional dynamics, electromagnetics and quantum computing. Few of the other implementations of fractional partial differential equations can be discovered in viscoplastic and viscoelastic flows [19], spherical flames [20], continuum mechanics [21], image processing [22], wave propagation [23], entropy [24], anomalous diffusion [25], groundwater containment transport [26], turbulent flow [27] and so on [28–30].

FC has piqued the interest of academics because of its practical applicability in various real-world challenges. Mathematicians have worked on discovering numerical and analytical solutions to FPDEs and similar systems to investigate the problem better. To address FPDEs, researchers have expended significant effort in inventing different approaches, resulting in a well-studied field [31–35]. The findings of these investigations contribute to a better understanding of the dynamics of natural systems. Scholars have refined their approaches to solving FPDEs over time, yielding a variety of effective methods such as the Sine-Gordon expansion technique, Yang transformation decomposition technique, finite element method, variational iteration method, natural transform decomposition method, first integral method, finite volume methods, generalized Kudryashov method, and many others [36–45]. A general modified \( \kappa \)-equation with the following type, i.e., a family that includes significant physical equations, is considered in this paper [46]:

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differential equations (PDEs) that help us comprehend wave dynamics and soliton behavior. These equations have sparked substantial attention
in Eq (1.1) for \( \kappa = 2 \), the result is the manifestation of the modified Camassa-Holm (mCH) equation
The fractional modified Camassa-Holm (mCH) and modified Degasperis-Procesi (mDP) equations are fascinating advances in the realm of nonlinear partial differential equations (PDEs) that help us comprehend wave dynamics and soliton behavior. These equations have sparked substantial attention because of their ability to account for anomalous diffusion and dispersive effects, which classical models usually overlook. The fractional mCH equation, a refinement of the conventional Camassa-Holm equation, uses fractional derivatives to simulate wave propagation more precisely. The features of this equation, such as its well-posedness, conservation laws and the behavior of solitary wave solutions, have been widely researched in the scientific literature. This anomalous diffusion feature has been found in a variety of physical systems, including turbulent fluid flow and transport processes [47].

To capture complicated dispersive effects, the modified mDP equation, a version of the classical Degasperis-Procesi equation, additionally incorporates fractional derivatives. This equation’s integrability, symmetries and wave propagation features have all been examined. In wave dynamics, nonlinearity, dispersion and fractional calculus interact extensively. The appearance of these fractional alterations has opened up new areas for research into this relationship. Researchers have used a variety of analytical and computational approaches to examine the characteristics of solutions, the integrability of these equations, and their application in simulating real-world occurrences.

The study of fractional PDEs, as exemplified by the fractional mCH and modified mDP equations, demonstrates the profound impact fractional calculus has on enhancing the predictive ability of mathematical models in numerous scientific fields. By considering fractional derivatives, researchers obtain a deeper understanding of the complex behaviors exhibited by waves and solitons in various physical systems [48, 49].

The Yang transformation, a significant concept in soliton theory and integrable systems, is a potent mathematical tool to derive solutions for certain nonlinear partial differential equations (PDEs). It establishes a connection between various remedies, frequently transforming one into another. This transformation is particularly relevant in the context of soliton equations, where it can systematically produce multi-soliton solutions by relating them to simpler seed solutions. The Yang transformation has applications in numerous disciplines, such as fluid dynamics, plasma physics, and mathematical physics, contributing to a greater comprehension of complex nonlinear phenomena and their mathematical representations [50–52]. In this study, two novel schemes, the YTDM and HPTM, are combined with the YT to approximate the fractional mDP and mCH problems. The proposed methods offer high accuracy while requiring less computational work than other techniques. This work’s structure is as follows: We give a few basic aspects of calculus theory in Section 2. Sections 3
and 4 provide the HPTM and YTDM formulations for obtaining the general solution. In Section 5, using a few numerical examples and comparisons to the exact solution, we show the viability and effectiveness of both approaches. Finally, Section 6 contains the conclusion.

2. Basic definitions

Definition 1.
The fractional Caputo operator of order $\varsigma$ is written as

$$D_\varsigma^\varsigma \zeta(\beta, \upsilon) = \frac{1}{\Gamma(k - \varsigma)} \int_0^\upsilon (\upsilon - j)^{k-\varsigma-1} \zeta^{(k)}(\beta, j) dj, \quad k - 1 < \varsigma \leq k, \quad k \in \mathbb{N}. \quad (2.1)$$

Definition 2.
The YT for $\zeta(\upsilon)$ is defined as

$$Y\{\zeta(\upsilon)\} = M(u) = \int_0^\infty e^{-u \upsilon} \zeta(\upsilon) d\upsilon, \quad u > 0, \quad (2.2)$$

provided the integral exists for some "$u".
And the inverse of Yang transform is

$$Y^{-1}\{M(u)\} = \zeta(\upsilon). \quad (2.3)$$

Definition 3.
The inverse of YT is given by

$$Y^{-1}\{Y(u)\} = \zeta(\upsilon) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \zeta^{-1} e^{\upsilon u} u du = \sum \text{residues of } \zeta \left( \frac{1}{u} \right) e^{\upsilon u} \varsigma > 0. \quad (2.4)$$

Definition 4.
The YT of the fractional derivative is given by

$$Y\{\zeta^{(n)}(\upsilon)\} = \frac{M(u)}{u^n} - \sum_{k=0}^{n-1} \frac{\zeta^{(k)}(0)}{u^{n-k-1}}, \quad n - 1 < \varsigma \leq n, \quad \forall \ n = 1, 2, 3, \cdots. \quad (2.4)$$

3. General implementation of homotopy perturbation transformation method

Consider the general fractional partial differential equation

$$D_\varsigma^\varsigma \zeta(\beta, \upsilon) = \mathcal{P}_1[\beta] \zeta(\beta, \upsilon) + \mathcal{R}_1[\beta] \zeta(\beta, \upsilon), \quad 0 < \varsigma \leq 1, \quad (3.1)$$

the initial condition (IC)

$$\zeta(0, \upsilon) = \xi(\upsilon).$$

Let $D_\upsilon^\varsigma \zeta(\beta, \upsilon)$ denote the Caputo fractional derivative of order $\varsigma$ with respect to $\upsilon$, and let $\mathcal{P}_1$ and $\mathcal{R}_1$ represent the linear and nonlinear terms, respectively.
Using Yang transform in the above equation
\[ Y[D_0^\epsilon \xi(\beta, \nu)] = Y[P_1[\beta] \zeta(\beta, \nu) + R_1[\beta] \zeta(\beta, \nu)], \tag{3.2} \]
\[ \frac{1}{\epsilon}[M(u) - u \zeta(\beta, 0)] = Y[P_1[\beta] \zeta(\beta, \nu) + R_1[\beta] \zeta(\beta, \nu)]. \tag{3.3} \]
By utilizing the differential characteristic of the YT, we obtain
\[ M(u) = u \zeta(\beta, 0) + u^\epsilon Y[P_1[\beta] \zeta(\beta, \nu) + R_1[\beta] \zeta(\beta, \nu)]. \tag{3.4} \]
By utilizing the inverse of the Yang transform, we get
\[ \zeta(\beta, \nu) = \zeta(\beta, 0) + Y^{-1}[u^\epsilon Y[P_1[\beta] \zeta(\beta, \nu) + R_1[\beta] \zeta(\beta, \nu)]. \tag{3.5} \]
Construct a homotopy operator that contains a small parameter \( \epsilon \) and \( \epsilon \in [0, 1] \) to create a family of related equations, starting from a simple linear equation and gradually transitioning to the transformed nonlinear equation. This operator effectively bridges the gap between linear and nonlinear problem.
\[ \zeta(\beta, \nu) = \sum_{k=0}^{\infty} \epsilon^k \zeta_k(\beta, \nu). \tag{3.6} \]
The nonlinear components in Eq (3.1) is defined as
\[ R_1[\beta] \zeta(\beta, \nu) = \sum_{k=0}^{\infty} \epsilon^k H_k(\zeta), \tag{3.7} \]
The technique obtain polynomial is described in Ref [53]
\[ H_k(\zeta_0, \zeta_1, ..., \zeta_n) = \frac{1}{\Gamma(n + 1)} D_\epsilon^k \left[ R_1 \left( \sum_{k=0}^{\infty} \epsilon^k \zeta_k \right) \right]_{\epsilon=0}, \tag{3.8} \]
where \( D_\epsilon^k = \frac{\partial^k}{\partial \epsilon^k} \).
By combining Eqs (3.6) and (3.8) with Eq (3.5), we obtain
\[ \sum_{k=0}^{\infty} \epsilon^k \zeta_k(\beta, \nu) = \zeta(\beta, 0) + \epsilon \times \left[ Y^{-1} \left[ u^\epsilon Y[P_1[\beta] \zeta(\beta, \nu) + \sum_{k=0}^{\infty} \epsilon^k H_k(\zeta) \right] \right]. \tag{3.9} \]
By analyzing the correlation of the coefficient of \( \epsilon \), we get
\[ \epsilon^0 : \zeta_0(\beta, \nu) = \zeta(\beta, 0), \]
\[ \epsilon^1 : \zeta_1(\beta, \nu) = Y^{-1} \left[ u^\epsilon Y[P_1[\beta] \zeta_0(\beta, \nu) + H_0(\zeta) \right], \]
\[ \epsilon^2 : \zeta_2(\beta, \nu) = Y^{-1} \left[ u^\epsilon Y[P_1[\beta] \zeta_1(\beta, \nu) + H_1(\zeta) \right], \]
\[ \epsilon^k : \zeta_k(\beta, \nu) = Y^{-1} \left[ u^\epsilon Y[P_1[\beta] \zeta_{k-1}(\beta, \nu) + H_{k-1}(\zeta) \right], \quad k > 0, k \in N. \tag{3.10} \]
Therefore, the approximation of Eq (3.1) takes the following series form
\[ \zeta(\beta, \nu) = \lim_{M \to \infty} \sum_{k=1}^{M} \zeta_k(\beta, \nu). \tag{3.11} \]
4. Implementation of YTDM

Consider the general fractional partial differential equation

\[ D^\varsigma_\omega \zeta(\beta, \nu) = \mathcal{P}_1[\beta]\zeta(\beta, \nu) + \mathcal{R}_1[\beta]\zeta(\beta, \nu), \quad 0 < \varsigma \leq 1, \]  

(4.1)

with the IC

\[ \zeta(0, \nu) = \xi(\nu). \]

By performing the YT, we get

\[ Y[D^\varsigma_\omega \zeta(\beta, \nu)] = Y[\mathcal{P}_1[\beta]\zeta(\beta, \nu) + \mathcal{R}_1[\beta]\zeta(\beta, \nu)]. \]

\[ \frac{1}{u^\varsigma}[M(u) - u\zeta(\beta, 0)] = Y[\mathcal{P}_1[\beta]\zeta(\beta, \nu) + \mathcal{R}_1[\beta]\zeta(\beta, \nu)]. \]

(4.2)

Utilizing the distinguishing characteristic of the YT, we acquire

\[ M(u) = u\zeta(\beta, 0) + u^\varsigma Y[\mathcal{P}_1[\beta]\zeta(\beta, \nu) + \mathcal{R}_1[\beta]\zeta(\beta, \nu)]. \]

(4.3)

By utilizing the inverse of the Yang transform method, we have

\[ \zeta(\beta, \nu) = \zeta(\beta, 0) + Y^{-1}[u^\varsigma Y[\mathcal{P}_1[\beta]\zeta(\beta, \nu) + \mathcal{R}_1[\beta]\zeta(\beta, \nu)]. \]

(4.4)

The solution of \( \zeta(\beta, \nu) \) is given as

\[ \zeta(\beta, \nu) = \sum_{m=0}^{\infty} \zeta_m(\beta, \nu). \]

(4.5)

The nonlinear components in Eq (4.1) is expressed as

\[ \mathcal{R}_1(\beta, \nu) = \sum_{m=0}^{\infty} \mathcal{A}_m(\zeta), \]

(4.6)

with

\[ \mathcal{A}_m(\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_m) = \frac{1}{m!} \left[ \frac{\partial^m}{\partial^m \zeta_m} \left\{ \mathcal{R}_1 \left( \sum_{m=0}^{\infty} \zeta_m(\beta, \nu) \right) \right\} \right]_{\zeta=0}, \quad m = 0, 1, 2, \ldots \]

(4.7)

By combine source of Eqs (4.5) and (4.7) in Eq (4.4), we obtain as

\[ \sum_{m=0}^{\infty} \zeta_m(\beta, \nu) = \zeta(\beta, 0) + Y^{-1}u^\varsigma \left\{ Y \left\{ \mathcal{P}_1 \left( \sum_{m=0}^{\infty} \zeta_m(\beta, \nu) \right) + \sum_{m=0}^{\infty} \mathcal{A}_m(\zeta) \right\} \right\}. \]

(4.8)

Thus, we get

\[ \zeta_0(\beta, \nu) = \zeta(\beta, 0), \]

(4.9)

\[ \zeta_1(\beta, \nu) = Y^{-1} \left[ u^\varsigma Y[\mathcal{P}_1(\zeta_0) + \mathcal{A}_0] \right], \]

\[ \zeta_{m+1}(\beta, \nu) = Y^{-1} \left[ u^\varsigma Y[\mathcal{P}_1(\zeta_m) + \mathcal{A}_m] \right]. \]
5. Application

Example 5.1.

Considering the following fractional mDP equation

\[
\frac{\partial^\nu \zeta(\beta, \nu)}{\partial \nu^\nu} - \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) + 4 \zeta^2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} - 3 \xi(\beta, \nu) \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} - \xi(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} = 0, \quad (5.1)
\]

with the IC

\[
\zeta(\beta, 0) = -\frac{15}{8} \text{sech}^2 \left( \frac{\beta}{2} \right).
\]

Applying the YT, we get

\[
Y \left( \frac{\partial^\nu \zeta(\beta, \nu)}{\partial \nu^\nu} \right) = Y \left[ \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) - 4 \zeta^2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 3 \xi(\beta, \nu) \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} + \xi(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right], \quad (5.2)
\]

By utilizing the differentiating aspect of the YT, we have

\[
\frac{1}{u^\nu} \{ M(u) - u \zeta(\beta, 0) \} = Y \left[ \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) - 4 \zeta^2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 3 \xi(\beta, \nu) \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} + \xi(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right], \quad (5.3)
\]

\[
M(u) = u \zeta(\beta, 0) + u^\nu Y \left[ \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) - 4 \zeta^2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 3 \xi(\beta, \nu) \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} + \xi(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right]. \quad (5.4)
\]

By utilizing the inverse of the YT method, we have

\[
\zeta(\beta, \nu) = \zeta(\beta, 0) + Y^{-1} \left[ u^\nu \left\{ Y \left[ \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) - 4 \zeta^2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 3 \xi(\beta, \nu) \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} + \xi(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right] \right\} \right], \quad (5.5)
\]

Using the HPM, we have

\[
\sum_{k=0}^{\infty} \epsilon^k \zeta_k(\beta, \nu) = \left( -\frac{15}{8} \text{sech}^2 \left( \frac{\beta}{2} \right) \right) + Y^{-1} \left[ Y \left[ \sum_{k=0}^{\infty} \epsilon^k \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta_k(\beta, \nu)}{\partial \beta^2} \right) - 4 \sum_{k=0}^{\infty} \epsilon^k \zeta_k(\beta, \nu) \sum_{k=0}^{\infty} \epsilon^k \frac{\partial \zeta_k(\beta, \nu)}{\partial \beta} \sum_{k=0}^{\infty} \epsilon^k \frac{\partial^2 \zeta_k(\beta, \nu)}{\partial \beta^2} + \sum_{k=0}^{\infty} \epsilon^k \zeta_k(\beta, \nu) \sum_{k=0}^{\infty} \epsilon^k \frac{\partial^3 \zeta_k(\beta, \nu)}{\partial \beta^3} \right] \right]. \quad (5.6)
\]
By comparing coefficient of $\epsilon$ on both sides of Eq (5.6), we get

$$\epsilon^0 : \zeta_0(\beta, \nu) = -\frac{15}{8} \text{sech}^2 \left( \frac{\beta}{2} \right),$$

$$\epsilon^1 : \zeta_1(\beta, \nu) = Y^{-1} \left[ u^\nu Y \left[ \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta_0(\beta, \nu)}{\partial \beta^2} - 4 \zeta_0(\beta, \nu) \frac{\partial \zeta_0(\beta, \nu)}{\partial \beta} + 3 \frac{\partial \zeta_0(\beta, \nu)}{\partial \beta} \frac{\partial^2 \zeta_0(\beta, \nu)}{\partial \beta^2} + \zeta_0(\beta, \nu) \frac{\partial^3 \zeta_0(\beta, \nu)}{\partial \beta^3} \right) \right] \right] - 450 \text{csch}^4(\beta) \sinh^6 \left( \frac{\beta}{2} \Gamma(\zeta + 1) \right).$$

Finally, the series form solution is given as follows:

$$\zeta(\beta, \nu) = \zeta_0(\beta, \nu) + \zeta_1(\beta, \nu) + \cdots$$

$$\zeta(\beta, \nu) = -\frac{15}{8} \text{sech}^2 \left( \frac{\beta}{2} \right) - 450 \text{csch}^4(\beta) \sinh^6 \left( \frac{\beta}{2} \Gamma(\zeta + 1) \right) + \cdots$$

A. Implementation of decomposition method

Applying the YT, we get

$$Y \left\{ \frac{\partial^\nu \zeta}{\partial \nu^\nu} \right\} = Y \left[ \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} - 4 \zeta_2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 3 \frac{\partial \zeta(\beta, \nu)}{\partial \beta} \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} + \zeta(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right) \right].$$

By utilizing the differentiating property of the Yang transform is given as

$$M(u) = u \zeta(\beta, 0) + u^\nu Y \left[ \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} - 4 \zeta_2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 3 \frac{\partial \zeta(\beta, \nu)}{\partial \beta} \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} + \zeta(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right).$$

Utilizing the inverse of the Yang transform method, we possess

$$\zeta(\beta, \nu) = \zeta(\beta, 0) + Y^{-1} \left[ u^\nu \left\{ \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} - 4 \zeta_2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 3 \frac{\partial \zeta(\beta, \nu)}{\partial \beta} \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} + \zeta(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right\} \right],$$

$$\zeta(\beta, \nu) = (-\frac{15}{8} \text{sech}^2 \left( \frac{\beta}{2} \right) + Y^{-1} \left[ u^\nu \left\{ \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} - 4 \zeta_2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 3 \frac{\partial \zeta(\beta, \nu)}{\partial \beta} \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} + \zeta(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right\} \right].$$

The series form solution is defined as

$$\zeta(\beta, \nu) = \sum_{m=0}^{\infty} \zeta_m(\beta, \nu),$$

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where $\zeta^2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} = \sum_{m=0}^{\infty} A_m, \frac{\partial \zeta(\beta, \nu)}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \nu^2} \right) = \sum_{m=0}^{\infty} B_m$ and $\zeta(\beta, \nu) \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} = \sum_{m=0}^{\infty} C_m$ are the Adomian polynomials, which shows the nonlinear terms, and

$$
\sum_{m=0}^{\infty} \zeta_m(\beta, \nu) = \zeta(\beta, 0) - Y^{-1} \left\{ u^\nu \left[ Y \left( \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \nu^2} \right) - 4 \sum_{m=0}^{\infty} A_m + 3 \sum_{m=0}^{\infty} B_m + \sum_{m=0}^{\infty} C_m \right) \right] \right\},
$$

$$
= \sum_{m=0}^{\infty} \zeta_m(\beta, \nu) = \left( -\frac{15}{8} \text{sech}^2 \left( \frac{\beta}{2} \right) \right) - Y^{-1} \left\{ u^\nu \left[ Y \left( \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \nu^2} \right) - 4 \sum_{m=0}^{\infty} A_m + 3 \sum_{m=0}^{\infty} B_m + \sum_{m=0}^{\infty} C_m \right) \right] \right\}.
$$

Comparing both sides of Eq (5.12) is defined as

$$
\zeta_0(\beta, \nu) = -\frac{15}{8} \text{sech}^2 \left( \frac{\beta}{2} \right),
$$

put $m = 0$ in Eq (5.12)

$$
\zeta_1(\beta, \nu) = -450 \text{csch}^5(\beta) \sinh \left( \frac{\nu}{\beta} \right) \frac{\nu^\Gamma}{(\zeta + 1)}.
$$

Finally, the series form solution is defined as

$$
\zeta(\beta, \nu) = \sum_{m=0}^{\infty} \zeta_m(\beta, \nu) = \zeta_0(\beta, \nu) + \zeta_1(\beta, \nu) + \cdots.
$$

$$
\zeta(\beta, \nu) = -\frac{15}{8} \text{sech}^2 \left( \frac{\beta}{2} \right) - 450 \text{csch}^5(\beta) \sinh \left( \frac{\beta}{2} \right) \frac{\nu^\Gamma}{(\zeta + 1)} + \cdots.
$$

Then, $\zeta$ equal to 1, the solution is obtained as

$$
\zeta(\beta, \nu) = -\frac{15}{8} \left[ \text{sech}^2 \left( \frac{1}{2} \beta - \frac{5}{2} \nu \right) \right].
$$

(5.13)

Example 5.2.

Fractional modified Camassa-Holm (mCH) equation can be expressed as follows:

$$
\frac{\partial^\nu \zeta(\beta, \nu)}{\partial \nu^\nu} = \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) + 3 \zeta^2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} - 2 \frac{\partial \zeta(\beta, \nu)}{\partial \beta} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) - \zeta(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} = 0
$$

(5.14)

with the IC

$$
\zeta(\beta, 0) = -2 \text{sech}^2 \left( \frac{\beta}{2} \right).
$$

By applying the YT, we get

$$
Y \left( \frac{\partial^\nu \zeta}{\partial \nu^\nu} \right) = Y \left[ \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) - 3 \zeta^2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 2 \frac{\partial \zeta(\beta, \nu)}{\partial \beta} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) + \zeta(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right].
$$

(5.15)

By using the differential property of the Yang transform, given as

$$
\frac{1}{u^\nu} \left[ M(u) - u \zeta(\beta, 0) \right] = Y \left[ \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) - 3 \zeta^2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 2 \frac{\partial \zeta(\beta, \nu)}{\partial \beta} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) + \zeta(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right],
$$

(5.16)
By using the YT, we get

\[ \xi(\beta, \nu) = \zeta(\beta, 0) + Y^{-1} \left\{ u^0 \left[ Y \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) - 3 \zeta^2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 2 \zeta(\beta, \nu) \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) + \zeta(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right] \right\}. \]  

(5.17)

Applying inverse Yang transform is defined

\[ \xi(\beta, \nu) = \zeta(\beta, 0) + Y^{-1} \left\{ u^0 \left[ Y \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) - 3 \zeta^2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 2 \zeta(\beta, \nu) \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) + \zeta(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right] \right\}. \]  

(5.18)

With the help of HPM technique, is expressed

\[
\sum_{k=0}^{\infty} e^k \xi_k(\beta, \nu) = \left( -2 \text{sech}^2 \left( \frac{\beta}{2} \right) \right) + Y^{-1} \left\{ u^0 \left[ \sum_{k=0}^{\infty} e^k \frac{\partial \zeta_k(\beta, \nu)}{\partial \beta} \right] - 3 \sum_{k=0}^{\infty} e^k \zeta_k(\beta, \nu) \sum_{k=0}^{\infty} e^k \frac{\partial \zeta_k(\beta, \nu)}{\partial \beta} \right\} + 2 \sum_{k=0}^{\infty} e^k \frac{\partial \zeta_k(\beta, \nu)}{\partial \beta} \sum_{k=0}^{\infty} e^k \frac{\partial^2 \zeta_k(\beta, \nu)}{\partial \beta^2} \]  

\[ + \sum_{k=0}^{\infty} e^k \xi_k(\beta, \nu) \sum_{k=0}^{\infty} e^k \frac{\partial^3 \zeta_k(\beta, \nu)}{\partial \beta^3} \right\}. \]  

(5.19)

By examining the relationship between the coefficient of \( e \) and other factors, we can determine the correlation.

\[ e^0 : \xi_0(\beta, \nu) = -2 \text{sech}^2 \left( \frac{\beta}{2} \right), \]

\[ e^1 : \xi_1(\beta, \nu) = Y^{-1} \left\{ u^0 \left[ Y \left( \frac{\partial^2 \zeta_0(\beta, \nu)}{\partial \beta^2} \right) - 3 \zeta_0^2(\beta, \nu) \frac{\partial \zeta_0(\beta, \nu)}{\partial \beta} + 2 \zeta_0(\beta, \nu) \left( \frac{\partial^2 \zeta_0(\beta, \nu)}{\partial \beta^2} \right) + \zeta_0(\beta, \nu) \frac{\partial^3 \zeta_0(\beta, \nu)}{\partial \beta^3} \right] \right\} = -384 \text{csch}^5(\beta) \sinh(\frac{\beta}{2}) \frac{\nu^5}{\Gamma(\zeta + 1)}. \]

The series form solution is achieved as

\[ \xi(\beta, \nu) = \xi_0(\beta, \nu) + \xi_1(\beta, \nu) + \cdots \]

\[ \xi(\beta, \nu) = -2 \text{sech}^2 \left( \frac{\beta}{2} \right) - 384 \text{csch}^5(\beta) \sinh(\frac{\beta}{2}) \frac{\nu^5}{\Gamma(\zeta + 1)} + \cdots \]

B. Application of decomposition method

By using the YT, we get

\[ Y \left\{ \frac{\partial^\nu \xi}{\partial \nu^\nu} \right\} = Y \left[ \frac{\partial}{\partial \nu} \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) - 3 \zeta^2(\beta, \nu) \frac{\partial \zeta(\beta, \nu)}{\partial \beta} + 2 \zeta(\beta, \nu) \left( \frac{\partial^2 \zeta(\beta, \nu)}{\partial \beta^2} \right) + \zeta(\beta, \nu) \frac{\partial^3 \zeta(\beta, \nu)}{\partial \beta^3} \right] \].  

(5.20)
By utilizing the distinct characteristic of the YT, the following result is obtained

\[
\frac{1}{w^5} (M(u) - u \xi(\beta, 0)) = Y \left[ \frac{\partial}{\partial \nu} \left( \frac{\partial^3 \xi(\beta, \nu)}{\partial \beta^3} \right) - 3 \zeta^2(\beta, \nu) \frac{\partial \xi(\beta, \nu)}{\partial \beta} + 2 \frac{\partial \xi(\beta, \nu)}{\partial \beta} \left( \frac{\partial^2 \xi(\beta, \nu)}{\partial \beta^2} \right) + \xi(\beta, \nu) \frac{\partial^3 \xi(\beta, \nu)}{\partial \beta^3} \right]. \tag{5.21}
\]

\[
M(u) = u \xi(\beta, 0) + u^4 Y \left[ \frac{\partial}{\partial \nu} \left( \frac{\partial^3 \xi(\beta, \nu)}{\partial \beta^3} \right) - 3 \zeta^2(\beta, \nu) \frac{\partial \xi(\beta, \nu)}{\partial \beta} + 2 \frac{\partial \xi(\beta, \nu)}{\partial \beta} \left( \frac{\partial^2 \xi(\beta, \nu)}{\partial \beta^2} \right) + \xi(\beta, \nu) \frac{\partial^3 \xi(\beta, \nu)}{\partial \beta^3} \right]. \tag{5.22}
\]

By utilizing inverse of the YT, we get

\[
\begin{align*}
\zeta(\beta, \nu) &= \zeta(\beta, 0) + Y^{-1} \left[ u^5 \left\{ \frac{\partial}{\partial \nu} \left( \frac{\partial^3 \xi(\beta, \nu)}{\partial \beta^3} \right) - 3 \zeta^2(\beta, \nu) \frac{\partial \xi(\beta, \nu)}{\partial \beta} + 2 \frac{\partial \xi(\beta, \nu)}{\partial \beta} \left( \frac{\partial^2 \xi(\beta, \nu)}{\partial \beta^2} \right) \right. \\
& \left. + \xi(\beta, \nu) \frac{\partial^3 \xi(\beta, \nu)}{\partial \beta^3} \right\} \right], \\
\zeta(\beta, \nu) &= \left( -2 \text{sech}^2 \left( \frac{\beta}{2} \right) \right) + Y^{-1} \left[ u^5 \left\{ \frac{\partial}{\partial \nu} \left( \frac{\partial^3 \xi(\beta, \nu)}{\partial \beta^3} \right) - 3 \zeta^2(\beta, \nu) \frac{\partial \xi(\beta, \nu)}{\partial \beta} + 2 \frac{\partial \xi(\beta, \nu)}{\partial \beta} \left( \frac{\partial^2 \xi(\beta, \nu)}{\partial \beta^2} \right) \right. \\
& \left. + \xi(\beta, \nu) \frac{\partial^3 \xi(\beta, \nu)}{\partial \beta^3} \right\} \right]. \tag{5.23}
\end{align*}
\]

The series form solution is given as

\[
\zeta(\beta, \nu) = \sum_{m=0}^{\infty} \zeta_m(\beta, \nu) \tag{5.24}
\]

with \(\zeta^2(\beta, \nu) \frac{\partial \xi(\beta, \nu)}{\partial \beta} = \sum_{m=0}^{\infty} A_m, \frac{\partial \xi(\beta, \nu)}{\partial \beta} \left( \frac{\partial^2 \xi(\beta, \nu)}{\partial \beta^2} \right) = \sum_{m=0}^{\infty} B_m \) and \( \xi(\beta, \nu) \frac{\partial^3 \xi(\beta, \nu)}{\partial \beta^3} = \sum_{m=0}^{\infty} C_m \) are the Adomian polynomials which shows the nonlinear terms, and

\[
\begin{align*}
\sum_{m=0}^{\infty} \zeta_m(\beta, \nu) &= \zeta(\beta, 0) - Y^{-1} \left[ u^5 \left\{ \frac{\partial}{\partial \nu} \left( \frac{\partial^3 \xi(\beta, \nu)}{\partial \beta^3} \right) - 3 \sum_{m=0}^{\infty} A_m + 2 \sum_{m=0}^{\infty} B_m + \sum_{m=0}^{\infty} C_m \right\} \right], \\
\sum_{m=0}^{\infty} \zeta_m(\beta, \nu) &= \left( -2 \text{sech}^2 \left( \frac{\beta}{2} \right) \right) - Y^{-1} \left[ u^5 \left\{ \frac{\partial}{\partial \nu} \left( \frac{\partial^3 \xi(\beta, \nu)}{\partial \beta^3} \right) - 3 \sum_{m=0}^{\infty} A_m + 2 \sum_{m=0}^{\infty} B_m + \sum_{m=0}^{\infty} C_m \right\} \right]. \tag{5.25}
\end{align*}
\]

\(\zeta_0(\beta, \nu) = -2 \text{sech}^2 \left( \frac{\beta}{2} \right).\)

put \(m = 0\) in the above equation

\[
\zeta_1(\beta, \nu) = -384 \csc^3(\beta \sin^6 \left( \frac{\beta}{2} \right) \frac{\nu^5}{\Gamma(\zeta + 1)}. \tag{5.26}
\]

Finally, the series form result is defined as

\[
\zeta(\beta, \nu) = \sum_{m=0}^{\infty} \zeta_m(\beta, \nu) = \zeta_0(\beta, \nu) + \zeta_1(\beta, \nu) + \cdots.
\]
\[ \zeta(\beta, \nu) = -2 \sech^2 \left( \frac{\beta}{2} \right) - 384 \csc h^5(\beta) \sinh \left( \frac{\nu}{2} \right) \frac{\nu}{\Gamma(\zeta + 1)} + \cdots. \]

The exact result is
\[ \zeta(\beta, \nu) = -2 \sech^2 \left( \frac{\beta - \nu}{2} \right). \] (5.26)

6. Results and discussion

In Figure 1, we present the graphical depiction of the homotopy perturbation transformation method (HPTM) and Yang transform-decomposition method (YTDM) for the solution \( \zeta(\beta, \nu) \) in Example 5.1. The plot showcases the precision achieved at \( \zeta = 1 \), revealing the accurate resolution achieved through the combined methodology. Moving to Figure 2, we illustrate the outcomes of the HPTM/YTDM solutions for \( \zeta(\beta, \nu) \) across various values of \( \zeta \) in Example 5.1 of the Degasperis-Procesi equation. The graph provides a comprehensive view of how the solutions evolve and adapt as \( \zeta \) changes, offering insights into the behavior of the equation.

**Figure 1.** The graphical representation of the HPTM/YTDM and the precise resolution at \( \zeta = 1 \) for \( \zeta(\beta, \nu) \) of Example 5.1.

**Figure 2.** The graphical representation of the the HPTM/YTDM solutions for \( \zeta(\beta, \nu) \) at different \( \zeta \) of Example 5.1.

In Figure 3, we delve into the graphical representation of the HPTM/YTDM approach and its precise resolution at \( \zeta = 1 \) for \( \zeta(\beta, \nu) \) in Example 5.2. The visualization underscores the accuracy
achieved through the combined method, emphasizing its effectiveness. Lastly, in Figure 4, we explore the HPTM/YTDM solutions for $\zeta(\beta, \nu)$ in Example 5.2 of the Camassa-Holm equation. The graph showcases the diverse solutions obtained at different values of $\varsigma$, providing a visual insight into the equation’s response to varying parameters.

Figure 3. The graphical representation of the HPTM/YTDM and the precise resolution at $\varsigma = 1$ for $\zeta(\beta, \nu)$ of Example 5.2.

Figure 4. The graphical representation of the HPTM/YTDM solutions for $\zeta(\beta, \nu)$ at different $\varsigma$ of Example 5.2.

These graphical representations collectively demonstrate the efficacy of the homotopy perturbation transformation method (HPTM) and Yang transform-decomposition method (YTDM) in solving fractional partial differential equations. They offer a visual narrative of how the methodologies handle different examples of Degasperis-Procesi and Camassa-Holm equations, shedding light on the intricate dynamics of the equations and the effectiveness of the proposed approach.

7. Conclusions

In conclusion, the study delved into the intricate dynamics of the Degasperis-Procesi and Camassa-Holm equations, two fundamental nonlinear partial differential equations with wide-ranging applications. The utilization of the Yang transform emerged as a pivotal mathematical tool, facilitating the transformation between solutions and shedding light on the behavior of these equations. Furthermore, the investigation incorporated the Adomian decomposition method, a versatile
technique for solving nonlinear differential equations, as well as series solutions to provide insightful approximations. Notably, the homotopy perturbation method was harnessed to tackle the challenges posed by these complex equations, offering a systematic approach to obtaining analytical solutions. Through the exploration of these methodologies, a deeper understanding of the intricate interplay between mathematical techniques and the physical phenomena described by these equations was achieved, paving the way for further advancements in the realm of nonlinear dynamics and integrable systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interest.

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