Research article

On a class of one-dimensional superlinear semipositone \((p, q)\)-Laplacian problem

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Abstract: We study the existence of positive solutions for a class of one-dimensional superlinear \((p, q)\)-Laplacian with Sturm-Liouville boundary conditions. We allow the reaction term to be singular at 0 with infinite semipositone behavior. Our approach depends on Amann’s fixed point theorem.

Keywords: \((p, q)\)-Laplacian; superlinear; positive solutions

Mathematics Subject Classification: Primary 34B15; Secondary 34B18

1. Introduction

In this paper, we investigate positive solutions for the one-dimensional BVP

\[
\begin{aligned}
&-\phi_\varepsilon(u')' = -\lambda g(u) + f(t, u), \ t \in (0, 1), \\
du(0) - bu'(0) = 0, \ cu(1) + du'(1) = 0,
\end{aligned}
\]

(1.1)

where \(\varepsilon \geq 0\), \(\phi_\varepsilon(s) = |s|^{p-2}s + \varepsilon |s|^{q-2}s\), \(p > q > 1\), \(a, b, c, d\) are nonnegative constants with \(ac + ad + bc > 0\), \(f : (0, 1) \times [0, \infty) \rightarrow \mathbb{R}\), \(g : (0, \infty) \rightarrow [0, \infty)\), and \(\lambda\) is a nonnegative parameter.

For \(\varepsilon = 0\), \(-\phi_\varepsilon(u')'\) is the usual \(p\)-Laplacian while for \(\varepsilon > 0\), the operator is referred to as the \((p, q)\)-Laplacian. We are focusing on the case when \(f (\cdot, u)\) is \(p\)-superlinear, and \(g\) is allowed to exhibit semipositone structure i.e., \(-g(0^+) \in (-\infty, 0)\). For a rich literature on semipositone problems and their applications, see [9]. Using Amann’s Fixed Point Theorem, we shall establish here the existence of a positive solution to (1.1) for \(\lambda \geq 0\) small when \(f (\cdot, u)\) is \(p\)-superlinear at 0 and \(\infty\), and the superlinearity is involved with the first eigenvalue of the \(p\)-Laplacian operator when \(\varepsilon = 0\). Our result in the \(p\)-Laplacian case improves previous ones in [3, 4, 8, 10, 12] (see Remark 1.1 below), while producing a new existence criteria in the \((p, q)\)-Laplacian case. We refer to [6, 7, 13] and the references therein for related existence results to (1.1) in the superlinear/sublinear cases when \(\varepsilon = 0\).
Let $\lambda_1$ be the principal eigenvalue of $-(\phi_0(u'))'$ on $(0, 1)$ with Sturm-Liouville boundary condition in (1.1), (see [2, 5]).

We consider the following hypotheses:

(A1) $g : (0, \infty) \to [0, \infty)$ is continuous, non-increasing, and integrable near 0.

(A2) $f : (0, 1) \times [0, \infty) \to \mathbb{R}$ is a Carathéodory function, and there exists $\gamma \in L^1(0, 1)$ such that

$$\inf_{z \in (0, \infty)} \frac{f(t, z)}{z^{p-1}} \geq -\gamma(t),$$

for a.e. $t \in (0, 1)$.

(A3) $\sup_{z \in \{0, \sigma\}} |f(t, z)|$ is integrable on $(0, 1)$ for all $c > 0$.

(A4) There exists a number $\sigma > 0$ such that

$$f(t, z) \leq \lambda_1 z^{p-1},$$

for $z \in (0, \sigma]$ and a.e. $t \in (0, 1)$, and in addition $f(t, z) \equiv \lambda_1 z^{p-1}$ on any subinterval of $[0, \sigma]$ if $\varepsilon = 0$.

(A5) $\lim_{z \to \infty} \frac{f(t, z)}{z^{p-1}} = \infty$ if $\varepsilon > 0$, and $\lim_{z \to \infty} \frac{f(t, z)}{z^{p-1}} < \lambda_1$ if $\varepsilon = 0$, where the limits are uniform for a.e. $t \in (0, 1)$.

Let $p(t) = \min(t, 1-t)$. By a positive solution of (1.1), we mean a function $u \in C^1[0, 1]$ with $\inf_{(0,1)}^\varepsilon > 0$ and satisfying (1.1).

Our main result is

**Theorem 1.1.** Let (A1)–(A5) hold. Then there exists a number $\lambda_0 > 0$ such that (1.1) has a positive solution for $0 \leq \lambda < \lambda_0$.

**Remark 1.1.** (i) When $\varepsilon = 0$, the existence of a positive solution to (1.1) was established in [3, 4], where $g \equiv 0$ with Sturm-Liouville condition in [3], and $g(u) = u^{-\delta}$, $\delta \in (0, 1)$ with Dirichlet boundary condition in [4], under the assumption

$$\limsup_{z \to 0^+} \frac{f(t, z)}{z^{p-1}} < \lambda_1 < \liminf_{z \to \infty} \frac{f(t, z)}{z^{p-1}}.$$ 

The results in [3, 4] provided extensions of the work in [8, 10, 12]. Note that our condition (A4) allows the case $\limsup_{z \to 0^+} \frac{f(t, z)}{z^{p-1}} = \lambda_1$.

(ii) In the case $bd = 0$, the proof of Theorem 1.1 shows that when $\varepsilon = 0$, (A4) can be replaced by the weaker condition $f(t, z) \leq \lambda_1 z^{p-1}$ and $f(t, z) \equiv \lambda_1 z^{p-1}$ on $[0, \sigma]$ for a.e. $t \in (0, 1)$.

**Example 1.1.** Let $\delta, \nu \in (0, 1)$ with $\delta + \nu < 1$, and $r > p - 1$. By Theorem 1.1, the following problems have a positive solution for $\lambda \geq 0$ small:

(i)  

$$
\begin{align*}
-(\phi_\nu(u'))' &= -\frac{\lambda_1}{u^\nu \ln(1+u)} + \lambda_1 u^{p-1} + u' - u^r, & t \in (0, 1), \\
au(0) - bu'(0) &= 0, & cu(1) + du'(1) = 0,
\end{align*}
$$

where $\varepsilon > 0$ and $r > s > p - 1$. Indeed, here

$$f(t, z) = \lambda_1 z^{p-1} + z' - z^s \leq \lambda_1 z^{p-1} \text{ for } z \leq 1,$$
i.e., (A4) holds. Since
\[ z^{1-p}f(t, z) = \lambda_1 + z^{-(p-1)} - z^{-\sigma} \geq \lambda_1 - 1 \]
for \( z \in (0, \infty) \), (A2) holds. Clearly (A1), (A3), and (A5) are satisfied.

(ii) \[
\begin{align*}
-\phi_e(u'(t))' & = -\frac{1}{\phi\ln(1+\phi)} + \lambda_1 u_{p-1} e^{-u} + u', \ t \in (0, 1), \\
au(0) - bu'(0) & = 0, \ cu(1) + du'(1) = 0,
\end{align*}
\]
where \( \alpha \in (0, r - p + 1) \). Note that (A4) with \( \varepsilon = 0 \) is equivalent to
\[ \lambda_1 (1 - e^{-\varepsilon^2}) \geq z^{-(p-1)} \]
on \([0, \sigma]\) and \( \lambda_1 (1 - e^{-\varepsilon}) \neq z^{-(p-1)} \) on any subinterval of \([0, \sigma]\) for some \( \sigma > 0 \). This is true since
\[
\lim_{z \to 0^+} \frac{1 - e^{-\varepsilon^2}}{z^{p-1}} = \infty.
\]
Clearly the remaining conditions are satisfied.

Note that \( \lim_{z \to 0^+} \frac{f(u, z)}{z^{p-1}} = \lambda_1 \) in both examples.

2. Preliminaries

Let \( 0 \leq \alpha < \beta \leq 1 \). In what follows, \( \gamma \in L^1(\alpha, \beta) \) with \( \gamma \geq 0 \) and we shall denote the norm in \( L^0(\alpha, \beta) \) and \( C^1(\alpha, \beta) \) by \( \| \cdot \|_\gamma \) and \( | \cdot |_1 \) respectively.

Lemma 2.1. Let \( u, v \in C^1(\alpha, \beta) \) satisfy
\[
\begin{align*}
-\phi_e(u'(t))' + \gamma(t)\phi_e(u) & \geq -\phi_e(v'(t))' + \gamma(t)\phi_e(v) \quad a.e \ on \ (\alpha, \beta), \\
au(\alpha) - bu'(\alpha) & \geq av(\alpha) - bv'(\alpha), \ cu(\beta) + du'(\beta) \geq cv(\beta) + dv'(\beta).
\end{align*}
\]
Then \( u \geq v \) on \([\alpha, \beta]\).

Proof. Suppose \( u(t_0) < v(t_0) \) for some \( t_0 \in (\alpha, \beta) \). Let \( I = (\alpha_0, \beta_0) \subset (\alpha, \beta) \) be the largest open interval containing \( t_0 \) such that \( u < v \) on \( I \). Then \( u(\alpha_0) = v(\beta_0) \) if \( \alpha_0 > \alpha \) and \( u(\beta_0) = v(\alpha) \) if \( \beta_0 < \beta \). Multiplying the inequality in (2.1) by \( u - v \) and integrating on \( I \) gives
\[
\int_I (\phi_e(u'(t)) - \phi_e(v'(t)))(u'(t) - v'(t)) \leq 0,
\]
since \( \gamma \geq 0 \) and \( -(\phi_e(u'(t)) - \phi_e(v'(t)))(u - v)(t) \geq 0 \) in view of the boundary conditions at \( \alpha, \beta \). Since \( \phi_e \) is increasing, it follows that \( u' = v' \) on \( I \) and hence \( u = v + \sigma \) on \( I \), where \( \sigma \) is a negative constant. If \( \alpha_0 > \alpha \) or \( \beta_0 < \beta \) then \( \sigma = 0 \), a contradiction. On the other hand, if \( \alpha_0 = \alpha \) and \( \beta_0 = \beta \) then the boundary conditions in (2.1) gives \( ac\sigma, c\sigma \geq 0 \) and thus \( a = c = 0 \), a contradiction and hence the result follows.

Lemma 2.2. Let \( k \in L^1(\alpha, \beta) \). Then the problem
\[
\begin{align*}
-\phi_e(z'(t))' + \gamma(t)\phi_e(z) & = k(t) \quad on \ (\alpha, \beta), \\
a(\alpha) - b\phi_e'(\alpha) & = 0, \ c\phi_e(\beta) + d\phi_e'(\beta) = 0
\end{align*}
\]
has a unique solution \( z \equiv T_k(k) \in C^1(\alpha, \beta) \) with
\[
|z|_1 \leq K\phi_e^{-1}(|k|_1),
\]
where the constant \( K \) is independent of \( k, \alpha, \beta, z, \varepsilon \). In addition, the map \( T_k : L^1(\alpha, \beta) \to C(\alpha, \beta) \) is completely continuous.
**Proof.** Suppose first that \( \gamma \equiv 0 \).

By integrating, we see that the solution of (2.2) is given by

\[
z(t) = C_1 - \int_{\alpha}^{t} \phi^{-1}_e(C + \int_{\alpha}^{s} k) \, ds,
\]

where the constants \( C, C_1 \) satisfy

\[
\begin{cases}
aC_1 + b\phi^{-1}_e(C) = 0, \\c\left(C_1 - \int_{\alpha}^{\beta} \phi^{-1}_e(C + \int_{\alpha}^{s} k) \, ds\right) - d\phi^{-1}_e(C + \int_{\alpha}^{\beta} k) = 0.
\end{cases}
\]

(2.5)

Note that (2.5) has a unique solution \((C, C_1)\) since if \( a = 0 \) then \( C = 0 \) and

\[
C_1 = \frac{\alpha}{c} \phi^{-1}_e(\int_{\alpha}^{\beta} k) + \frac{\beta}{c} \int_{\alpha}^{\beta} \phi^{-1}_e(\int_{\alpha}^{s} k) \, ds,
\]

(2.6)

while if \( a > 0 \) then \( C_1 = \frac{b}{a} \phi^{-1}_e(C) \), where \( C \) is the unique solution of

\[
g_e(C) \equiv bC\phi^{-1}_e(C) + ac \int_{\alpha}^{\beta} \phi^{-1}_e(C + \int_{\alpha}^{s} k) \, ds + ad\phi^{-1}_e(C + \int_{\alpha}^{\beta} k) = 0.
\]

(2.7)

Indeed, \( g_e(C) > 0 \) for \( C > ||k||_1 \) and \( g_e(C) < 0 \) for \( C < -||k||_1 \). Thus (2.7) has a unique solution \( C \) with \( |C| \leq ||k||_1 \) since \( g_e \) is continuous and increasing.

Using the inequality (see Proposition A(iii) in Appendix)

\[
\phi^{-1}_e(mx) \leq m^{\frac{1}{\gamma}} \phi^{-1}_e(x)
\]

for \( m \geq 1, x \geq 0 \), and (2.4)–(2.6), we get

\[
|z(t)| + |z'(t)| \leq |C_1| + 2\phi^{-1}_e(2||k||_1) \leq \left(c_0 + 2\frac{\alpha}{\gamma}\right) \phi^{-1}_e(||k||_1),
\]

for all \( t \in [\alpha, \beta] \), where \( c_0 = (d/c + 1) \) if \( a = 0 \), \( c_0 = b/a \) if \( a > 0 \), from which (2.3) follows.

Next, we consider the general case \( \gamma \in L^1(\alpha, \beta) \) with \( \gamma \geq 0 \). In view of the above, there exist \( z_1, z_2 \in C^1[\alpha, \beta] \) satisfying

\[
-(\phi_e(z'_1))' = |k(t)| \text{ on } (\alpha, \beta), \quad -(\phi_e(z'_2))' = |k(t)| \text{ on } (\alpha, \beta),
\]

with Sturm-Liouville boundary conditions.

By Lemma 2.1, \( z_1 \leq 0 \leq z_2 \) on \( (\alpha, \beta) \), which implies

\[
-(\phi_e(z'_1))' + \gamma(t)\phi_e(z_1) \leq -|k(t)| \leq k(t) \text{ on } (\alpha, \beta)
\]

and

\[
-(\phi_e(z'_2))' + \gamma(t)\phi_e(z_2) \geq |k(t)| \geq k(t) \text{ on } (\alpha, \beta),
\]

i.e., \((z_1, z_2)\) is a pair of sub- and supersolution of (2.2) with \( z_1 \leq z \leq z_2 \) on \( (\alpha, \beta) \). Thus (2.2) has a solution \( z \in C^1[\alpha, \beta] \) with \( z_1 \leq z \leq z_2 \) on \( (\alpha, \beta) \). The solution is unique due to Lemma 2.1.
Since
\[-(\phi_e(z'))' = k(t) - \gamma(t)\phi_e(z) \text{ on } (\alpha, \beta)\]
and \(|z|_{\infty} \leq \max(||z_1||_{\infty}, ||z_2||_{\infty}) \leq K\phi_{e_1}^{-1}(||k||_1)\) in view of (2.3) when \(\gamma = 0\), it follows that
\[||k(t) - \gamma(t)\phi_e(z)||_1 \leq ||k||_1 + ||\gamma||_1\phi_e(\alpha, \beta^{-1}(||k||_1)) \leq K_2||k||_1,\]
where \(K_1 = \max(K, 1)\) and \(K_2 = 1 + \alpha^{-1}||\gamma||_1\). Here we have used Proposition A(iii) in Appendix. Consequently, it is
\[||z||_1 \leq K\phi_{e_1}^{-1}(K_2||k||_1) \leq KK\phi_{e_1}^{-1}(||k||_1),\]
where we have used Proposition A(ii) in Appendix. Thus (2.3) holds. Next, we verify that \(T_e\) is continuous. Let \((k_n) \subset L^1(\alpha, \beta)\) and \(k \in L^1(\alpha, \beta)\) be such that \(||k_n - k||_1 \to 0\). Let \(u_n = T_e k_n\) and \(u = T_e k\).

Multiplying the equation
\[-(\phi_e(u_n') - \phi_e(u'))' + \gamma(t)(\phi_e(u_n) - \phi_e(u)) = k_n - k \text{ on } (\alpha, \beta)\]
by \(u_n - u\) and integrating between \(\alpha\) and \(\beta\), we obtain
\[c_n + \int_{\alpha}^{\beta} (\phi_e(u_n') - \phi_e(u'))(u_n' - u') \leq ||k_n - k||_1||u_n - u||_\infty,\]
(2.8)
where \(c_n = -(\phi_e(u_n') - \phi_e(u'))(u_n - u)\big|_{\alpha}^{\beta} \geq 0\). By [11, Lemma 30],
\[(\phi_e(x) - \phi_e(y))(x - y) \geq (|x|^{p-2}x - |y|^{p-2}y)(x - y) \geq c_0|x - y|^{\max(p, 2)}\]
(2.9)
for all \(x, y \in \mathbb{R}\) with \(|x| + |y| \leq 2M\), where \(c_0 > 0\) is a constant depending only on \(p\) and \(M\). Applying (2.9) with \(x = u_n', y = u'\) and note that \(|u_n' + u'1 \leq 2M\), where \(M = K \max\left(\phi^{-1}(||u_n||_1), \phi^{-1}(||k||_1)\right)\), we obtain from (2.8) that
\[c_n + c_0 \int_{\alpha}^{\beta} |u_n' - u'|^{\max(p, 2)} \leq 2M||k_n - k||_1 \to 0 \text{ as } n \to \infty.\]
(2.10)
If \(b = 0\) then \((u_n - u)(\alpha) = 0\) and the Mean Value Theorem implies that
\[|u_n(t) - u(t)| \leq \int_{\alpha}^{t} |u_n' - u'| \leq \left(\int_{\alpha}^{\beta} |u_n' - u'|^{\max(p, 2)}\right)^{\frac{1}{\max(p, 2)}} \to 0 \text{ as } n \to \infty\]
for \(t \in [\alpha, \beta]\). Hence \(||u_n - u||_\infty \to 0\) as \(n \to \infty\) in view of (2.10).

If \(b > 0\) then \(u_n(\alpha) = \frac{b}{a}u_n(\alpha), u' = \frac{b}{a}u(\alpha)\), and since (2.9) gives
\[\frac{bc_0}{a} \left(\frac{a}{b}\right)^{\max(p, 2)} |u_n(\alpha) - u(\alpha)|^{\max(p, 2)} \leq \left(\phi_e(\frac{a}{b}u_n(\alpha)) - \phi_e(\frac{a}{b}u(\alpha))\right)(u_n(\alpha) - u(\alpha)) \leq c_n,
\]
it follows from the Mean Value Theorem and (2.10) that
\[||u_n - u||_\infty \leq |u_n(\alpha) - u(\alpha)| + \left(\int_{\alpha}^{\beta} |u_n' - u'|^{\max(p, 2)}\right)^{\frac{1}{\max(p, 2)}} \to 0 \text{ as } n \to \infty.\]
Hence \(T_e\) is continuous. Since \((u_n)\) is bounded in \(C^1[\alpha, \beta]\), \(T_e\) is completely continuous, which completes the proof. \(\square\)
Lemma 2.3. Let $k \in L^1(0, 1)$ with $k \geq 0$, and $u \in C^1[0, 1]$ satisfy
\[
\begin{cases}
-(\phi_v(u'))' + \gamma(t)\phi_v(u) \geq -k(t) \quad \text{on } (0, 1), \\
u(t) - bu(0) \geq 0, \quad cu(1) + du'(1) \geq 0.
\end{cases}
\]

Then there exist constants $\kappa, C > 0$ independent of $u, k, \varepsilon$ such that if $\|u\|_\infty \geq C\phi_\varepsilon^{-1}(\|k\|_1)$ then
\[u(t) \geq \kappa\|u\|_\infty p(t)\]
for $t \in [0, 1]$.

**Proof.** Let $v \in C^1[0, 1]$ satisfy
\[
\begin{cases}
-(\phi_v(v'))' + \gamma(t)\phi_v(v) = -k(t) \quad \text{on } (0, 1), \\
 av(0) - bv(0) = 0, \quad cv(1) + dv'(1) = 0.
\end{cases}
\]

By Lemma 2.2, $|v|_1 \leq K\phi_\varepsilon^{-1}(\|k\|_1)$, where $K$ is independent of $k$. By Lemma 2.1, $u \geq v$ on $[0, 1]$. Suppose $\|u\|_\infty > K\phi_\varepsilon^{-1}(\|k\|_1)$, and $\|u\|_\infty = \|u(t)\|$ for some $t \in [0, 1]$. Then $u(t) > 0$ because if $u(t) \leq 0$ then $\|u\|_\infty = -u(\tau) \leq -v(t) \leq K\phi_\varepsilon^{-1}(\|k\|_1)$, a contradiction. In what follows, we may increase $K$ without mentioning if needed.

Suppose first that $\tau \in (0, 1)$. Let $z \in C^1[0, \tau]$ satisfying
\[
\begin{cases}
-(\phi_v(z'))' + \gamma(t)\phi_v(z) = -k(t) \quad \text{on } (0, \tau), \\
 az(0) - bz'(0) = 0, \quad z(\tau) = \|u\|_\infty.
\end{cases}
\] (2.11)

Note that $z_0$ is a subsolution of (2.11) and $z_0 + \|u\|_\infty$ is a supersolution of (2.11), where $z_0$ satisfies
\[
\begin{cases}
-(\phi_v(z_0'))' + \gamma(t)\phi_v(z_0) = -k(t) \quad \text{on } (0, \tau), \\
 az_0(0) - bz_0'(0) = 0, \quad z_0(\tau) = 0,
\end{cases}
\]
from which the existence of $z$ follows. By Lemma 2.1, $u \geq z \geq -K\phi_\varepsilon^{-1}(\|k\|_1)$ on $[0, \tau]$. Define $z_1(t) = z(t) + K\phi_\varepsilon^{-1}(\|k\|_1)$. Then $z_1 \geq 0$ on $[0, 1]$ and
\[z_1(0) \geq -K_1\phi_\varepsilon^{-1}(\|k\|_1),\]
where $K_1 = K$ if $b = 0$ and $K_1 = K(1 + a/b)$ if $b > 0$. Indeed, if $b = 0$ then $z(0) = v(0) = 0$ and so $z_1'(0) = z'(0) \geq v'(0) \geq -K\phi_\varepsilon^{-1}(\|k\|_1)$, while if $b > 0$ then $z_1'(0) = (a/b)z(0) \geq -K(a/b)\phi_\varepsilon^{-1}(\|k\|_1)$.

Since $z \leq z_1$ on $(0, \tau)$ and $z_1'(0) + K_1\phi_\varepsilon^{-1}(\|k\|_1) \geq 0$, it follows upon integrating the equation
\[(\phi_v(z'))' = \gamma(t)\phi_v(z) + k(t) \quad \text{on } (0, \tau)\]
that
\[
\begin{align*}
z_1(t) &= z_1(0) + \int_0^t \phi_\varepsilon^{-1}\left(\phi_v(z_1'(0)) + \int_0^s (\gamma(\xi)\phi_v(z) + k(\xi))d\xi\right)ds \\
&\leq z_1(0) + \int_0^t \phi_\varepsilon^{-1}\left(\phi_v(z_1'(0)) + K_1\phi_\varepsilon^{-1}(\|k\|_1)\right) + \int_0^s (\gamma(\xi)\phi_v(z_1) + k(\xi))d\xi ds \\
&\leq z_1(0) + \int_0^t \phi_\varepsilon^{-1}\left(\phi_v(z_1'(0)) + K_1\phi_\varepsilon^{-1}(\|k\|_1)\right) + \int_0^s (\gamma(\xi)\phi_v(z_1) + k(\xi))d\xi ds
\end{align*}
\]
where $t \in K$.

It follows that

\[ z_1(t) + \phi_e^{-1}\left( \phi_e(z'_1(0) + K_1 \phi_e^{-1}(||k||_1)) + \int_0^t (\gamma(\xi)\phi_e(\xi) + k(\xi))d\xi \right). \]

(2.12)

Applying $\phi_e$ on both sides of (2.12) and using the inequality (see Proposition A(i) in Appendix)

\[ \phi_e(x + y) \leq M(\phi_e(x) + \phi_e(y)) \quad \forall x, y \geq 0, \]

where $M = 2^{\max(p-2,0)}$, we obtain

\[ \phi_e(z_1(t)) \leq M[\phi_e(z_1(0)) + \phi_e(z'_1(0) + K_1 \phi_e^{-1}(||k||_1)) + ||k||_1] + M \int_0^t \gamma(\xi)\phi_e(\xi)d\xi. \]

By Gronwall’s inequality,

\[ \phi_e(z_1(t)) \leq M[\phi_e(z_1(0)) + \phi_e(z'_1(0) + K_1 \phi_e^{-1}(||k||_1)) + ||k||_1]e^{M||y||_i} \]

for $t \in [0, \tau]$. In particular when $t = \tau$, we obtain

\[ \phi_e(z_1(0)) + \phi_e(z'_1(0) + K_1 \phi_e^{-1}(||k||_1)) + ||k||_1 \geq 2K_2 \phi_e(||u||_\infty), \]

where $K_2 = (2M)^{-1} \epsilon^{-M||y||_i}$. Since $\phi_e(x) + \phi_e(y) \leq 2\phi_e(x + y)$ for $x, y \geq 0$, this implies

\[ \phi_e(z_1(0) + z'_1(0) + K_1 \phi_e^{-1}(||k||_1)) \geq K_2 \phi_e(||u||_\infty) - \frac{||k||_1}{2} \geq K_3 \phi_e(||u||_\infty) \geq \phi_e(K_4||u||_\infty), \]

where $K_3 = K_2/2 < 1$ and $K_4 = K_3^{\frac{1}{\gamma}}$, provided that $\phi_e(||u||_\infty) \geq ||k||_1/K_2$ which is true if $||u||_\infty \geq (1/K_2)^{\frac{1}{\gamma}} \phi_e^{-1}(||k||_1)$. Consequently,

\[ z_1(0) + z'_1(0) + K_1 \phi_e^{-1}(||k||_1) \geq K_4||u||_\infty, \]

which implies

\[ z(0) + z'(0) \geq K_4||u||_\infty - (K + K_1) \phi_e^{-1}(||k||_1) \geq K_5||u||_\infty, \]

(2.13)

where $K_5 = K_4/2$, provided that $||u||_\infty \geq \frac{2K_1 + K_1}{K_4} \phi_e^{-1}(||k||_1)$. Since

\[ (\phi_e(z'))' = \gamma(t)\phi_e(z) + k \geq -\gamma(t)\phi_e(K\phi_e^{-1}(||k||_1)) \geq -K^{\frac{1}{\gamma}}||k||_1 \gamma(t) \quad \text{on} \quad (0, \tau), \]

it follows that

\[ \phi_e(z'(t)) \geq \phi_e(z'(0)) - K^{\frac{1}{\gamma}}||k||_1 \gamma(t) \]

(2.14)

for $t \in [0, \tau]$. If $b = 0$ then $z(0) = 0$ and (2.13) becomes $z'(0) \geq K_5||u||_\infty$, from which (2.14) implies

\[ \phi_e(z'(t)) \geq \phi_e(K_5||u||_\infty) - K^{\frac{1}{\gamma}}||k||_1 \gamma(t) \geq \frac{\phi_e(K_5||u||_\infty)}{2} \geq \phi_e(K_6||u||_\infty), \]

where $K_6 = 2^{1-a}K_5$, provided that $\phi_e(K_5||u||_\infty) \geq 2K^{\frac{1}{\gamma}}||k||_1$ which is true if $||u||_\infty \geq K_5^{-1}\left(2K^{\frac{1}{\gamma}}||k||_1\right)^{\frac{1}{\gamma}} \phi_e^{-1}(||k||_1)$. Consequently,

\[ z'(t) \geq K_6||u||_\infty \quad \text{on} \quad (0, \tau), \]
which implies upon integrating that
\[
  u(t) \geq z(t) \geq \frac{K_\delta}{a + b} ||u||_\infty t \quad \text{for } t \in [0, \tau].
\]  
(2.15)

If \( b > 0 \) then \( z'(0) = (a/b)z(0) \) and (2.13) becomes
\[
z(0) \geq \frac{K_\delta b}{a + b} ||u||_\infty.
\]  
(2.16)

Since \( z'(0) \geq 0 \), (2.14) gives
\[
z'(t) \geq -\phi_\zeta^{-1}\left(K\frac{1}{\gamma}||\gamma||_1||k||_1\right) \geq -\tilde{K}\phi_\zeta^{-1}(||k||_1) \quad \text{on } (0, \tau),
\]
where \( \tilde{K} = \left(K\frac{1}{\gamma}||\gamma||_1\right)^{\frac{1}{\gamma}} \). This, together with (2.16), implies
\[
z(t) \geq z(0) - \tilde{K}\phi_\zeta^{-1}(||k||_1) \geq \frac{K_\delta b}{a + b} ||u||_\infty - \tilde{K}\phi_\zeta^{-1}(||k||_1).
\]
Hence
\[
u(t) \geq z(t) \geq K_\tau ||u||_\infty \quad \text{for } t \in [0, \tau],
\]  
(2.17)

where \( K_\tau = \frac{K_\delta b}{2(a+b)} \), provided that \( ||u||_\infty \geq 2\frac{K(a+b)}{K_\delta b}\phi_\zeta^{-1}(||k||_1) \).

Combining (2.15) and (2.17), we obtain
\[
u(t) \geq \kappa_0 ||u||_\infty t, \quad \forall t \in [0, \tau],
\]  
(2.18)

where \( \kappa_0 = \min(K_\delta, K_\tau) \).

Next, let \( w \in C^1[\tau, 1] \) be the solution of
\[
\left\{ 
\begin{array}{l}
-(\phi_u(w'))' + \gamma(t)\phi_u(w) = -k(t) \quad \text{on } (\tau, 1), \\
w(\tau) = ||u||_\infty, \quad c\phi(1) + dw'(1) = 0.
\end{array}
\right.
\]

Then \( u \geq w \) on \([\tau, 1] \), and using similar arguments as above, we obtain
\[
u(t) \geq \kappa_1 ||u||_\infty (1 - t) \quad \forall t \in [\tau, 1],
\]  
(2.19)

where \( \kappa_1 > 0 \) is a constant independent of \( k \), provided that \( ||u||_\infty > C\phi_\zeta^{-1}(||k||) \) for some large constant \( C \) independent of \( u \).

Combining (2.18) and (2.19), we see that Lemma 2.3 holds with \( \kappa = \min(\kappa_0, \kappa_1) \). If \( \tau = 0 \) then (2.19) holds on \([0, 1] \), and if \( \tau = 1 \) then (2.17) holds on \([0, 1] \), which completes the proof. \( \square \)

3. Proof of the main result

Let \( E = C[0, 1] \) be with the usual sup-norm.

Proof of Theorem 1.1. Let \( C, \kappa \) be given by Lemma 2.3 and define \( \sigma_0 = \kappa \sigma, h(t) = g(\sigma_0 p(t)) \). For \( v \in E, g(\max(v, \sigma_0 p)) \in L^1(0, 1) \) by (A1), and \( 0 \leq f(t, |v|) + \gamma(t)\phi_u(|v|) \in L^1(0, 1) \) by (A2) and (A3). Let \( \lambda \geq 0 \) be small so that \( C\phi_\zeta^{-1}(\lambda||h||_1) < \sigma \). Then the problem
\[
\left\{ 
\begin{array}{l}
-(\phi_u(u'))' + \gamma(t)\phi_u(u) = -\lambda g(\max(v, \sigma_0 p)) + f(t, |v|) + \gamma(t)\phi_u(|v|) \quad \text{on } (0, 1), \\
u(0) - bu'(0) = 0, \quad cu(1) + du'(1) = 0
\end{array}
\right.
\]
has a unique solution \( u = A_x v \in C^1[0,1] \) in view of Lemma 2.2. Since the operator \( S : E \to L^1(0,1) \)
defined by \( (Sv)(t) = -\lambda g(\max(v, \sigma_0 p)) + f(t, |v|) + \gamma(t)|v|^{p-1} \) is continuous, it follows from Lemma 2.2
that \( A_x : E \to E \) is completely continuous. We shall verify that

(i) \( u = \theta A_x u, \ \theta \in (0,1) \implies \|u\|_{\infty} \neq \sigma \).

Let \( u \in E \) satisfy \( u = \theta A_x u \) for some \( \theta \in (0,1) \) with \( \|u\|_{\infty} = \sigma \).

Suppose \( \varepsilon > 0 \). Then

\[
- \left( \phi_{\varepsilon} \left( \frac{u'}{\theta} \right) \right)' + \gamma(t) \phi_{\varepsilon} \left( \frac{u}{\theta} \right) = -\lambda g(\max(u, \sigma_0 p(t))) + f(t, |u|) + \gamma(t) \phi_{\varepsilon}(|u|)
\]
on \( (0,1) \), which implies upon multiplying by \( \theta^{p-1} \) that

\[
- (\phi_{\varepsilon \theta^{p-1}}(u'))' + \gamma(t) \phi_{\varepsilon \theta^{p-1}}(u) = \theta^{p-1} (-\lambda g(\max(u, \sigma_0 p(t))) + f(t, |u|) + \gamma(t) \phi_{\varepsilon}(|u|))
\]

\[
\geq -\lambda h(t) \quad \text{on} \ (0,1). \tag{3.1}
\]

Since \( \|u\|_{\infty} > C \phi^{-1}_{\varepsilon} (\lambda \|h\|_1 \), Lemma 2.3 gives

\[
u(t) \geq \kappa \|u\|_{\infty} p(t) \geq \sigma_0 p(t) > 0
\]

for \( t \in (0,1) \) (recall that \( \kappa \sigma = \sigma_0 \)). Hence it follows from (3.1) and (A4) that

\[
- (\phi_{\varepsilon \theta^{p-1}}(u'))' = \theta^{p-1} f(t, u) - \lambda \theta^{p-1} g(u) + \theta^{p-1} \gamma(t) \phi_{\varepsilon} \phi_{\varepsilon \theta^{p-1}}(u)
\]

\[
= \theta^{p-1} f(t, u) - \lambda \theta^{p-1} g(u) + \gamma(t)(\theta^{p-1} - 1) u^{p-1} + \varepsilon \gamma(t)(\theta^{p-1} - \theta^{-q}) u^{\theta^{-1}} \tag{3.2}
\]
on \( (0,1) \). Multiplying (3.2) by \( u \) and integrating gives

\[
- \phi_{\varepsilon \theta^{p-1}}(u'(1)) u(1) + \phi_{\varepsilon \theta^{p-1}}(u'(0)) u(0) + \int_0^1 \phi_{\varepsilon \theta^{p-1}}(u') u' \leq \lambda_1 \int_0^1 u^p.
\]

Since \( au(0) - bu'(0) = 0 = cu(1) + du'(1) \) and \( \varepsilon > 0 \), this implies

\[
- \phi_{0}(u'(1)) u(1) + \phi_{0}(u'(0)) u(0) + \int_0^1 |u'|^p < \lambda_1 \int_0^1 u^p, \tag{3.3}
\]

Consequently,

\[
\lambda_1 > -\frac{-\phi_{0}(u'(1)) u(1) + \phi_{0}(u'(0)) u(0) + \int_0^1 |u'|^p}{\int_0^1 u^p}.
\]

Since \( \lambda_1 \) is characterized by the Raleigh formula

\[
\lambda_1 = \inf_{v \in V} \frac{-\phi_{0}(v'(1)) v(1) + \phi_{0}(v'(0)) v(0) + \int_0^1 |v'|^p}{\int_0^1 |v|^p}, \tag{3.4}
\]

where \( V = \{ u \in C^1[0,1] : au(0) - bu'(0) = 0 = cu(1) + du'(1) \} \), we get a contradiction. Thus (i) holds.
Next, suppose $\varepsilon = 0$. Then the $<$ inequality in (3.3) is replaced by $\leq$, which together with (3.4) imply
\[
\lambda_1 = \frac{-\phi_0(u'(1))u(1) + \phi_0(u'(0))u(0) + \int_0^1 |u'|^p}{\int_0^1 |u|^p},
\]
i.e., $u$ is an eigenfunction corresponding to $\lambda_1$. Hence (3.2) gives
\[
\lambda_1 u^{p-1} \leq \theta^{p-1} f(t, u) \leq \theta^{p-1} \lambda_1 u^{p-1} \leq \lambda_1 u^{p-1} \quad \text{on} \ (0, 1),
\]
from which it follows that $f(t, u) = \lambda_1 u^{p-1}$ for a.e. $t \in (0, 1)$. Since $||u||_{\infty} = \sigma$, we get a contradiction with (A4) with $\varepsilon = 0$. If $bd = 0$, then $u(0) = 0$ or $u(1) = 0$, and since $||u||_{\infty} = \sigma$, we have $u[0, 1] = [0, \sigma]$, we get a contradiction if $f(t, z) \neq \lambda_1 z^{p-1}$ on $[0, \sigma]$ for a.e. $t \in (0, 1)$. Thus (i) holds.

Next, we verify that

(ii) There exists a constant $R > \sigma$ such that $u = A_4 u + \xi$, $\xi \geq 0 \implies ||u||_{\infty} \neq R$.

Let $u \in E$ satisfy $u = A_4 u + \xi$ for some $\xi \geq 0$. Then $u$ satisfies
\[
-(\phi_\varepsilon(u'))' + \gamma(t)\phi_\varepsilon(u - \xi) = -\lambda g(\max(u, \sigma_0 p(t))) + f(t, |u|) + \gamma(t)\phi_\varepsilon(|u|) \quad (3.5)
\]
on $(0, 1)$, which implies
\[
-(\phi_\varepsilon(u'))' + \gamma(t)\phi_\varepsilon(u) \geq -\lambda h(t) \quad (3.6)
\]
on $(0, 1)$. Note that
\[
au(0) - bu(0) = a\xi \geq 0, \quad cu(1) + du'(1) = c\xi \geq 0. \quad (3.7)
\]
Suppose $||u||_{\infty} = R > \sigma$. Then Lemma 2.3 gives
\[
u(t) \geq k||u||_{\infty} p(t) \geq kR p(t) \geq \sigma_0 p(t) \quad (3.8)
\]
for $t \in (0, 1)$. Using (3.8) in (3.5), we get
\[
-(\phi_\varepsilon(u'))' \geq -\lambda g(u) + f(t, u) \quad \text{on} \ (0, 1). \quad (3.9)
\]
Suppose $\varepsilon > 0$ and let $M > 0$. Since $\lim_{z \to \infty} \frac{f'(z) - \lambda g(z)}{\phi_\varepsilon(z)} = \infty$ by (A1) and (A5), there exists a positive constant $L$ such that
\[
f(t, z) - \lambda g(z) \geq M\phi_\varepsilon(z) \quad (3.10)
\]
for a.e. $t \in (0, 1)$ and $z > L$. By (3.8),
\[
u(t) \geq k\frac{||u||_{\infty}}{4} = \frac{kR}{4} > L \quad \text{for} \ t \in \{1/4, 3/4\}
\]
for $R$ large, from which (3.9) and (3.10) imply
\[
-(\phi_\varepsilon(u'))' \geq M\phi_\varepsilon(u) \geq M\phi_\varepsilon\left(\frac{||u||_{\infty}}{4}\right) \quad \text{on} \ \{1/4, 3/4\}.
\]
Since $u(1/4)$ and $u(3/4)$ are positive, the comparison principle gives $u \geq \bar{u}$ on $[1/4, 3/4]$, where $\bar{u}$ is the solution of
\[
\begin{cases}
-(\phi_\varepsilon(\bar{u}'))' = M\phi_\varepsilon\left(\frac{||\bar{u}||_{\infty}}{4}\right) & \text{on} \ (1/4, 3/4),
\bar{u}(1/4) = \bar{u}(3/4) = 0.
\end{cases}
\]
Let $\|\bar{u}\|_\infty = \bar{u}(\tau)$ for some $\tau \in (1/4, 3/4)$. If $\tau \leq 1/2$ then we have

$$\|u\|_\infty \geq \bar{u}(5/8) = \int_{5/8}^{3/4} \phi_\epsilon^{-1} \left( M \phi_\epsilon \left( \frac{\kappa \|u\|_\infty}{4} \right) (s - \tau) \right) ds \geq \frac{1}{8} \phi_\epsilon^{-1} \left( M \phi_\epsilon \left( \frac{\kappa \|u\|_\infty}{4} \right) \right),$$

while if $\tau > 1/2$,

$$\|u\|_\infty \geq \bar{u}(3/8) = \int_{1/4}^{3/8} \phi_\epsilon^{-1} \left( M \phi_\epsilon \left( \frac{\kappa \|u\|_\infty}{4} \right) (\tau - s) \right) ds \geq \frac{1}{8} \phi_\epsilon^{-1} \left( M \phi_\epsilon \left( \frac{\kappa \|u\|_\infty}{4} \right) \right).$$

Hence using Proposition A(iii) we see that in either case,

$$\phi_\epsilon(8\|u\|_\infty) \geq \frac{M}{8} \phi_\epsilon \left( \frac{\kappa \|u\|_\infty}{4} \right) \geq \phi_\epsilon \left( \frac{1}{8} \frac{M}{8} \phi_\epsilon \left( \frac{\kappa \|u\|_\infty}{4} \right) \right)$$

i.e., $\|u\|_\infty \geq \frac{4}{M(8/\epsilon)^{1/2}} \|u\|_\infty$, a contradiction if $M$ is large enough, which proves (ii).

Suppose next that $\epsilon = 0$. Since $\liminf_{z \to +\infty} \frac{f(z) - Lg(z)}{z^p} > \lambda_1$ uniformly for a.e. $t \in (0, 1)$, there exist positive constants $L_0, \tilde{\lambda}$ with $\tilde{\lambda} > \lambda_1$ such that

$$f(t, z) - \lambda g(z) \geq \tilde{\lambda} z^{p-1} \quad (3.11)$$

for a.e. $t \in (0, 1)$ and all $z \geq L_0$. For $\delta_1 \in (0, 1/2)$, let $\lambda_{1, \delta_1}$ be the first eigenvalue of the problem

$$\begin{cases} 
- (\phi_0(v'))' = \lambda_{1, \delta_1} \phi_0(v) & \text{on} \ (\delta_1, \delta_2), \\
\phi_1(0) - b \phi'_1(\delta_1) = 0, \ cv(\delta_2) + d \phi'_1(\delta_2) = 0,
\end{cases} \quad (3.12)$$

where $\delta_2 = 1 - \delta_1$. By the continuity of the first eigenvalue with respect to the domain, $\lambda_{1, \delta_1} \to \lambda_1$ as $\delta_1 \to 0$. Hence there exists $\delta > 0$ such that $\lambda_{1, \delta_1} < \tilde{\lambda}$ for $\delta_1 \leq \delta$.

Let $\delta_1 = \delta/2, \delta_2 = 1 - \delta/2$, and $\mu \in (\lambda_{1, \delta_1}, \tilde{\lambda})$. By decreasing $\delta$ if necessary, we have from (3.7) that

$$a \bar{u}(\delta_1) - b \bar{u}'(\delta_1) \geq 0 \text{ if } a > 0, \quad c \bar{u}(\delta_2) + d \bar{u}'(\delta_2) \geq 0 \text{ if } c > 0, \quad (3.13)$$

where $\bar{u} = u + 1$. By (3.8),

$$u(t) \geq \frac{\kappa R \delta}{4} \geq L_0 \quad (3.14)$$

for $t \in [\delta/4, 1 - \delta/4]$ for $R$ large. It follows from (3.9), (3.11) and (3.14) that

$$- (\phi_0(u'))' \geq - \lambda g(u) + f(t, u) \geq \tilde{\lambda} u^{p-1} \text{ on } [\delta/4, 1 - \delta/4]. \quad (3.15)$$

By (3.6) and (3.15),

$$- (\phi_0(u'))' \geq - \lambda h(t) - \gamma(t) \phi_0(u) \geq - \gamma(t), \quad (3.16)$$

for a.e. $t \in (0, 1)$, where $\gamma(t) = \lambda h(t) + \gamma(t) \phi_0(L) \geq 0$. We claim that the eigenvalue problem

$$\begin{cases} 
- (\phi_0(v'))' = \mu \phi_0(v) & \text{on} \ (\delta_1, \delta_2), \\
\phi_1(0) - b \phi'_1(\delta_1) = 0, \ cv(\delta_2) + d \phi'_1(\delta_2) = 0
\end{cases} \quad (3.17)$$

has a positive solution, provided that $R$ is large enough.
Let $\psi_1$ be the positive solution of (3.12) with $||\psi_1||_\infty = 1$. Then clearly $\psi_1$ is a subsolution of (3.17). Since (3.14) implies
\[
\frac{u}{u+1} \geq \frac{\kappa R_\delta/4}{1+\kappa R_\delta/4} \quad \text{on } [\delta/4, 1-\delta/4]
\]
for $R$ large and $\frac{\kappa R_\delta/4}{1+\kappa R_\delta/4} \to 1$ as $R \to \infty$, it follows from (3.15) that
\[
-(\phi_0(\bar{u}'))' \geq \tilde{\lambda} u^{p-1} = \tilde{\lambda} \bar{u}^{p-1} \left(\frac{u}{u+1}\right)^{p-1} \geq \mu \bar{u}^{p-1} \quad \text{on } (\delta_1, \delta_2),
\] (3.18)
for $R$ large.

**Case 1.** $a, c > 0$. Then $\bar{u}$ is a supersolution of (3.17) in view of (3.13) and (3.18).

**Case 2.** $ac = 0$. If $a = 0$ then (3.7) gives $u'(0) = 0$. Combining (3.14)–(3.16), we deduce that for $R$ large,
\[
-\phi_0(u'(\delta_1)) = -\int_0^{\delta_1} (\phi_0(u'))' \geq -\int_0^{\delta/4} \gamma L + \tilde{\lambda} \int_{\delta/4}^{\delta/2} u^{p-1} > 0
\]
i.e., $u'(\delta_1) < 0$. Similarly if $c = 0$ then $u'(1) = 0$, and
\[
\phi_0(u'(\delta_2)) = -\int_{\delta_2}^1 (\phi_0(u'))' \geq -\int_{1-\delta/4}^{1-\delta/2} \gamma L + \tilde{\lambda} \int_{1-\delta/2}^{1-\delta/4} u^{p-1} > 0
\]
i.e., $u'(\delta_2) > 0$. Since $a\bar{u}(\delta_1) - b\bar{u}'(\delta_1) > 0$ and $c\bar{u}(\delta_2) + d\bar{u}'(\delta_2) > 0$, it follows from (3.18) that $\bar{u}$ is a supersolution of (3.17).

Since $\psi_1 \leq 1 \leq \bar{u}$ on $[\delta_1, \delta_2]$, the existence of a solution $v$ to (3.17) with $\psi_1 \leq v \leq \bar{u}$ on $(\delta_1, \delta_2)$ follows, which is a contradiction. Thus (ii) holds. By Amann’s fixed point theorem [1, Theorem 12.3], $A_\varepsilon$ has a fixed point $u \in E$ with $||u||_\infty > \sigma$. Using $\xi = 0$ in (ii) and (3.8), we obtain $u(t) \geq \sigma_0 p(t)$ for $t \in [0, 1]$ i.e., $g(\max(u, \sigma_0 p(t))) = g(u)$ and therefore $u$ is a positive solution of (1.1), which completes the proof. \(\Box\)

**Use of AI tools declaration**

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

**Conflict of interest**

The authors declare no conflict of interest.

**References**


**Appendix A**

We provide here some inequalities regarding the operator $\phi_e$.

**Proposition A.**

(i) $\phi_e(x + y) \leq M(\phi_e(x) + \phi_e(y))$ for $x, y \geq 0$, where $M = 2^{\max(p-2,0)}$.

(ii) $\phi_e^{-1}(mx) \leq m^{r-1}\phi_e^{-1}(x)$ for $m \geq 1, x \geq 0$.

(iii) $\phi_e(cx) \leq c^{p-1}\phi_e(x)$ for $c \geq 1, x \geq 0$.

**Proof.** (i) Let $x, y \geq 0$. Since the function $z^r$ is convex on $[0, \infty)$ for $r \geq 1$,

$$\frac{(x + y)^r}{2} \leq \frac{x^r + y^r}{2},$$

i.e.,

$$(x + y)^r \leq 2^{r-1}(x^r + y^r).$$

On the other hand if $0 < r < 1$, we have

$$(x + y)^r \leq x^r + y^r.$$
Hence for $r > 0$,

$$(x + y)^r \leq 2^{\max(r-1,0)}(x^r + y^r),$$

which implies

$$\phi_{\varepsilon}(x + y) = (x + y)^{r-1} + \varepsilon(x + y)^{q-1} \leq 2^{\max(p-2,0)}(x^{p-1} + y^{p-1}) + \varepsilon 2^{\max(q-2,0)}(x^{q-1} + y^{q-1}) \leq 2^{\max(p-2,0)}(\phi_{\varepsilon}(x) + \phi_{\varepsilon}(y))$$

i.e., (i) holds.

(ii) Let $z \geq 0$ and $c \geq 1$. We claim that

$$\phi_{\varepsilon}(cz) \geq c^{q-1}\phi_{\varepsilon}(z). \quad (A.1)$$

Indeed,

$$\phi_{\varepsilon}(cz) = c^{p-1}z^{p-1} + \varepsilon c^{q-1}z^{q-1} \geq c^{q-1}\phi_{\varepsilon}(z)$$

i.e., (A.1) holds. Let $m \geq 1, x \geq 0$. Then by using (A.1) with $c = m^{\frac{1}{p-1}}$ and $z = \phi_{\varepsilon}^{-1}(x)$, we obtain

$$\phi_{\varepsilon}\left(m^{\frac{1}{p-1}}\phi_{\varepsilon}^{-1}(x)\right) \geq m\phi_{\varepsilon}(\phi_{\varepsilon}^{-1}(x)) = mx$$

i.e., (ii) holds.

(iii) Let $c \geq 1$ and $x \geq 0$. Then

$$\phi_{\varepsilon}(cx) = c^{p-1}x^{p-1} + \varepsilon c^{q-1}x^{q-1} \leq c^{p-1}(x^{p-1} + \varepsilon x^{q-1})$$

i.e., (iii) holds. □