Partial domination of network modelling

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Abstract: Partial domination was proposed in 2017 on the basis of domination theory, which has good practical application background in communication network. Let $G = (V, E)$ be a graph and $\mathcal{F}$ be a family of graphs, a subset $S \subseteq V$ is called an $\mathcal{F}$-isolating set of $G$ if $G[V \setminus N_G[S]]$ does not contain $F$ as a subgraph for all $F \in \mathcal{F}$. The subset $S$ is called an isolating set of $G$ if $\mathcal{F} = \{K_2\}$ and $G[V \setminus N_G[S]]$ does not contain $K_2$ as a subgraph. The isolation number of $G$ is the minimum cardinality of an isolating set of $G$, denoted by $\iota(G)$. The hypercube network and $n$-star network are the basic models for network systems, and many more complex network structures can be built from them. In this study, we obtain the sharp bounds of the isolation numbers of the hypercube network and $n$-star network.

Keywords: partial domination; isolation number; hypercube network; $n$-star network

Mathematics Subject Classification: 05C07, 05C69

1. Introduction

In this paper, all graphs considered are non-empty, finite, undirected and simple. Additionally, for standard graph theory terminology not given here, we refer to [1]. Let $G$ be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$, and $|V(G)| = n$, $|E(G)| = m$. For any $v \in V(G)$, the degree $d_G(v)$ of $v$ is the number of neighbors of $v$. The minimum and maximum degree of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The open neighborhood $N_G(v)$ of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is the set $N_G[S] = N_G(v) \cup \{v\}$. For any $S \subseteq V(G)$, the open neighborhood of $S$ is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$ and the closed neighborhood of $S$ is the set $N_G[S] = N_G(S) \cup S$. Furthermore, we define $N_S(v) = N_G(v) \cap S$ and $N_S[v] = N_S(v) \cup \{v\}$. The subgraph of $G$ induced by $S$ is denoted by $G[S]$. For a graph $H$, we say that $G$ is $H$-free if $G$ does not contain $H$ as a subgraph. The cycle and clique on $n$ vertices are denoted as $C_n$ and $K_n$.Abbreviate $\{1, 2, \ldots, n\}$ to $[n]$ and say “$i$” is a symbol of $[n]$, where $n \in \mathbb{N}^*$ and $i \in [n]$. 
In recent years, with the rapid development of information technology, the domination theory has been widely used in computer technology, cryptography, social network, communication network and many other subjects. In 1958, Claude Berge [2] first proposed the concept of the domination number. A vertex subset \( S \subseteq V(G) \) is a dominating set of \( G \) if \( N_G(S) = V(G) \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality of a dominating set of \( G \), i.e \( \gamma(G) = \min\{|S| : S \text{ is a dominating set of } G\} \).

In 2014, Caro and Hansberg [3] extended the domination to the partial domination, and proposed the concept of a \( \mathcal{F} \)-isolating set of a graph \( G \) for the first time. Let \( G = (V, E) \) be a graph and \( \mathcal{F} \) be a family of graphs.

**Definition 1.1.** [3] A subset \( S \subseteq V \) is called an \( \mathcal{F} \)-isolating set of \( G \) if \( G[V \setminus N_G(S)] \) does not contain \( F \) as a subgraph for all \( F \in \mathcal{F} \). The \( \mathcal{F} \)-isolation number of \( G \) is the minimum cardinality of an \( \mathcal{F} \)-isolating set of \( G \), denoted by \( \iota(G, \mathcal{F}) \).

In particular, if \( \mathcal{F} = \{K_k\} \), the set \( S \) is called a \( \{K_k\} \)-isolating set of \( G \) if \( G[V \setminus N_G(S)] \) does not contain \( K_k \) as a subgraph, and the \( \{K_k\} \)-isolation number of \( G \) is the minimum cardinality of a \( \{K_k\} \)-isolating set of \( G \), denoted by \( \iota(G, k) \). For any positive integer \( k \), if \( \mathcal{F} = \{K_{1,k+1}\} \), the set \( S \) is called a \( k \)-isolating set of \( G \) if \( G[V \setminus N_G(S)] \) does not contain \( K_{1,k+1} \) as a subgraph, and the \( k \)-isolation number of \( G \) is the minimum cardinality of a \( k \)-isolating set of \( G \), denoted by \( \iota_k(G) \). Especially, when \( k = 0 \), the set \( S \) is called an isolating set of \( G \), and the minimum cardinality of an isolating set of \( G \) is the isolation number of \( G \), denoted by \( \gamma(G) \).

With respect to this problem, Borg et al. [4] studied the \( \iota(G, k) \) of a connected graph \( G \) is at most \( \frac{n}{k+1} \) unless \( G \cong K_k \), or \( k = 2 \) and \( G \cong C_5 \). Caro and Hansberg [3] investigated that \( \iota(G) \leq \frac{n}{3} \) for \( n \geq 6 \), \( \iota_k(T) \leq \frac{n}{k+3} \) of a tree \( T \) which is different from \( K_{1,k+1} \), \( \iota_k(G) \leq \frac{n}{4} \) of a maximal outerplanar graph \( G \) with \( n \geq 4 \) and so on. Tokunaga et al. [5] studied that if \( G \) is a maximal outerplanar graph of order \( n \) with \( n_2 \) vertices of degree 2, then \( \iota(G) \leq \frac{n+n_2}{3} \) when \( n_2 \leq \frac{n}{4} \), and \( \iota(G) \leq \frac{n-n_2}{3} \) otherwise. Borg and Kaemawichanurat [6] showed that if \( G \) is a maximal outerplanar graph with \( n \geq 5 \), then \( \iota_k(G) \leq \frac{n}{3} \), they also showed that \( \iota_1(G) \leq \frac{n+n_2}{6} \) when \( n_2 \leq \frac{n}{3} \), and \( \iota_1(G) \leq \frac{n-n_2}{3} \) otherwise, where \( n_2 \) is the number of vertices of degree 2. Borg and Kaemawichanurat [7] obtained that \( \iota_k(G) \leq \min\{\frac{n}{k+1}, \frac{n+n_2}{5}, \frac{n-n_2}{3}\} \) for a maximal outerplanar graph \( G \) and \( k \geq 0 \), where \( n_2 \) is the number of vertices of degree 2. Vazquez-Araujo [8] analyzed that \( \iota(T) = \frac{n}{3} \) implies \( \iota(T) = \gamma(T) \) for a tree \( T \), and proposed simple algorithms to build these trees from the connections of stars.

For a \( \{K_k\} \)-isolating set \( S \), in 2021, Favaron and Kaemawichanurat [9] restricted the induced subgraph \( G[S] \) to be an independent set and introduced the concept of the independent \( \{K_k\} \)-isolation number of \( G \). The vertex subset \( S \subseteq V \) is said to be independent \( \{K_k\} \)-isolating if \( S \) is an \( \{K_k\} \)-isolation set of \( G \) and \( G[S] \) has no edge. The independent \( \{K_k\} \)-isolation number of \( G \) is the minimum cardinality of an independent \( \{K_k\} \)-isolating set of \( G \), denoted by \( \iota_l(G, \mathcal{F}) \).

**Definition 1.2.** A subset \( S \subseteq V \) is called an independent \( \mathcal{F} \)-isolating set of \( G \) if \( S \) is an \( \mathcal{F} \)-isolating set of \( G \) and \( S \) is an independent set. The independent \( \mathcal{F} \)-isolation number of \( G \) is the minimum cardinality of an independent \( \mathcal{F} \)-isolating set of \( G \), denoted by \( \iota_l(G, \mathcal{F}) \).

**Definition 1.3.** A subset \( S \subseteq V \) is called an independent isolating set of \( G \) if \( S \) is an isolating set of \( G \) and \( G[S] \) has no edge. The independent isolation number of \( G \) is the minimum cardinality of an independent isolating set of \( G \), denoted by \( \iota_l(G) \).

In this paper, we investigate respectively the sharp bounds of the isolation number and the independent isolation number of the hypercube network and \( n \)-star network.
2. Main results

2.1. Isolation number of the hypercube network

The hypercube network is the basic model for interconnection networks, and it is a popular network because of its attractive properties, including regularity, symmetry, small diameter, strong connectivity, recursive construction, partitionability and relatively low link complexity [10, 11]. In general, a network model can be modeled as a graph. Let \( n \) be a positive integer. The hypercube network of dimension \( n \), denoted by \( Q_n \), is the simple graph whose vertices are the \( n \)-tuples with entries in \( \{0, 1\} \) and whose edges are the pairs of \( n \)-tuples that differ in exactly one position (see Figure 1). A \( m \)-dimensional subcube of \( Q_n \) is isomorphic to \( Q_m \) for any positive integer \( 1 \leq m \leq n \). A vertex of \( V(Q_n) \) is an even vertex if the number of 1s is even in its all symbols. A vertex of \( V(Q_n) \) is an odd vertex if the number of 1s is odd in its all symbols.

The hypercube network have many classic and fascinating topological structural properties, such as those below.

**Lemma 2.1.** [1] Hypercube network satisfies the following properties:

(a) \( |V(Q_n)| = 2^n, |E(Q_n)| = n \cdot 2^{n-1} \);
(b) Each edge of \( Q_n \) has an even endvertex and an odd endvertex;
(c) \( Q_n \) is a \( n \)-regular bipartite graph;
(d) Every perfect matching of \( Q_n \) has \( 2^{n-1} \) edges;
(e) If \( n \geq 3 \), then \( Q_n \) has \( 2^{n-3} \) disjoint 3-dimensional subcubes of \( Q_n \).

![Figure 1](image_url)

**Figure 1.** (a) \( Q_2 \); (b) \( Q_3 \); (c) \( Q_4 \).
Theorem 2.2. \( \iota(Q_2) = 1, \iota(Q_3) = \iota(Q_4) = 2 \).

Proof. According to the structure of \( Q_n \), we know that \( Q_2 \cong C_4 \), so \( \iota(Q_2) = \iota(C_4) = 1 \).

It is easy to know \( \{(0, 0, 0), (1, 1, 1)\} \) is an isolating set of \( Q_3 \) (see Figure 1(b)), so \( \iota(Q_3) \leq 2 \). Let \( S_1 \) be a minimum isolating set of \( Q_3 \), clearly, \( |S_1| \geq 1 \). Suppose that \( |S_1| = 1 \). If the vertex of \( S_1 \) is an even vertex, without loss of generality, let \( S_1 = \{(1, 1, 0)\} \), then \( V(Q_3) \setminus N_{Q_3}[S_1] = \{(0, 0, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1)\} \). Hence, \( V(Q_3) \setminus N_{Q_3}[S_1] \) is not an independent set of \( Q_3 \), which is a contradiction. If the vertex of \( S_1 \) is an odd vertex, without loss of generality, let \( S_1 = \{(0, 0, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1)\} \), then \( V(Q_3) \setminus N_{Q_3}[S_1] = \{(0, 0, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1)\} \). Hence, \( V(Q_3) \setminus N_{Q_3}[S_1] \) is not an independent set of \( Q_3 \), which is a contradiction. Thus, \( |S_1| \neq 1 \), furthermore, \( \iota(Q_3) = |S_1| \geq 2 \). Hence, \( \iota(Q_3) = 2 \).

Additionally, it is easy to know \( \{(0, 0, 0, 0), (1, 1, 1, 1)\} \) is an isolating set of \( Q_4 \) (see Figure 1(c)), so \( \iota(Q_4) \leq 2 \). Let \( Q_4^1 \) and \( Q_4^2 \) be two disjoint 3-dimensional subcubes of \( Q_4 \) by Lemma 2.1(e), and \( S_2 \) be a minimum isolating set of \( Q_4 \). Obviously, \( |S_2| \geq 1 \). If \( |S_2| = 1 \) and \( S_2 = \{v\} \), then \( v \in Q_4^i \) for any \( i \in \{1, 2\} \). Since \( Q_4^j \subset G[V(Q_3)] \setminus N_{Q_3} \) for \( j \in \{1, 2\} \), \( V(Q_4) \setminus N_{Q_4}[S_2] \) is not an independent set of \( Q_4 \), which is a contradiction. Thus, \( |S_2| \neq 1 \), furthermore, \( \iota(Q_4) = |S_2| \geq 2 \). Hence, \( \iota(Q_4) = 2 \). \( \square \)

Theorem 2.3. Let \( n \) be a positive integer and \( n \geq 4 \), then \( \frac{2^{n-1}}{n} \leq \iota(Q_n) \leq 2^{n-3} \). Moreover, the bounds are sharp.

Proof. Let \( S \) be a minimum \( \{K_2\} \)-isolating set of \( Q_n \). Since every perfect matching of \( Q_n \) has \( 2^{n-1} \) edges and \( V(Q_n) \setminus N_{Q_n}[S] \) is an independent set, every edge of a perfect matching of \( Q_n \) has at least one vertex in \( N_{Q_n}(S) \), that is, \( |N_{Q_n}(S)| \geq \frac{2^{n-1}}{2} = 2^{n-2} \). By Lemma 2.1(c), we have \( |N_{Q_n}(S)| \leq \Delta(Q_n)|S| = d(v)|S| = n|S| \) for any vertex \( v \in S \). Thus, \( \iota(Q_n) = |S| \geq \frac{|N_{Q_n}(S)|}{d(v)} \geq \frac{2^{n-1}}{n} \).

By Lemma 2.1, we know that \( Q_n \) has \( 2^{n-3} \) disjoint 3-dimensional subcubes of \( Q_n \) for \( n \geq 4 \) and each edge of \( Q_n \) has one even endvertex and one odd endvertex, so each \( Q_j \) has four odd vertices and four even vertices, and every even vertex is adjacent to three odd vertices in \( Q_3 \). Without loss of generality, let \( x \in V(Q_i^j) \) be an even vertex and \( y \in V(Q_i^j) \setminus N_{Q_i^j}[x] \) be an odd vertex of \( Q_i^j \). Since \( n \geq 4 \), there exists a \( Q_i^j (2 \leq i \leq 2^{n-3}) \) such that \( |N_{Q_i^j}(x) \cap V(Q_i^j)| = 1 \). Let \( N_{Q_i^j}(x) \cap V(Q_i^j) = \{x'\} \) and \( y' \in V(Q_i^j) \setminus N_{Q_i^j}[x'] \) be an even vertex of \( Q_i^j \). By the structure of \( Q_n \), we know that \( N(x) \cap N(y) = \{x', y'\} \). Since each edge of \( Q_n \) has an even endvertex and an odd endvertex, the set \( V(Q_n) \setminus N_{Q_n}[S] \) is an independent set of \( Q_n \), thus the set \( S \) is an isolating set of \( Q_n \). Since \( |S \cap V(Q_i^j)| = 1 \) for \( 1 \leq i \leq 2^{n-3} \), we obtain \( \iota(Q_n) \leq |S| = 2^{n-3} \). In conclusion, \( \frac{2^{n-1}}{n} \leq \iota(Q_n) \leq 2^{n-3} \) for \( n \geq 4 \).

Especially, if \( \frac{2^{n-1}}{n} = 2^{n-3} \), then \( n = 4 \). So the upper bound is equal to the lower bound if and only if \( n = 4 \). If \( n = 4 \), then \( \iota(Q_4) = 2 = 2^{n-3} = \frac{2^{n-1}}{n} \) by Theorem 2.2. Thus, the upper and lower bounds are sharp. \( \square \)
Corollary 2.4. Let \( n \) be a positive integer and \( n \geq 4 \), then \( \frac{2^{n-1}}{n} \leq \iota(Q_n) \leq 2^{n-3} \). Moreover, the bounds are sharp.

Proof. Let \( S_I \) be a minimum independent isolating set of \( Q_n \). Obviously, \( S_I \) is an isolating set of \( Q_n \). Thus, by Theorem 2.2, we have \( \iota(Q_n) \geq \frac{2^{n-1}}{n} \). Let \( \{Q'_3, Q''_3, \cdots, Q'_{2^{n-3}}\} \) be the set of 2\(^{n-3}\) disjoint 3-dimensional subcubes of \( Q_n \). Choose one even vertex in each \( Q'_3(1 \leq i \leq 2^{n-3}) \) as the set \( S \) such that all vertices of \( V(Q_n) \setminus N_G[S] \) are even vertices. According to the proof of Theorem 2.2, the set \( S \) is an isolating set of \( Q_n \) and \( \iota(Q_n) \leq 2^{n-3} \). Since all vertices of \( S \) are even vertices, the set \( S \) is an independent isolating set of \( Q_n \). Thus \( \iota(Q_n) \leq 2^{n-3} \). In conclusion, \( \frac{2^{n-1}}{n} \leq \iota(Q_n) \leq 2^{n-3} \) for \( n \geq 4 \).

Especially, if \( n = 4 \), then \( \iota(Q_n) = 2 = 2^{n-3} = \frac{2^{n-1}}{n} \) by Theorem 2.2. Thus, the upper and lower bounds are sharp.

2.2. Isolation number of the \( n \)-star network

The \( n \)-star network was proposed by Akers, Harel and Krishnamurthy [12] as an attractive alternative to the hypercube network for interconnecting processors on a parallel computer. For a positive integer \( n(n \geq 2) \), the \( n \)-star network on \( n \) symbols, denoted by \( S_n \), is the graph with \( n! \) vertices, whose the vertex set \( V(S_n) \) is all permutations of symbols 1, 2, \( \cdots \), \( n \), and each vertex \( v \in V(S_n) \) is connected to \( n - 1 \) vertices which can be obtained by interchanging the first symbol of \( v \) with the \( i \)th symbol of \( v \) for \( 2 \leq i \leq n \) (\( S_4 \) is shown as an example in Figure 3).

Lemma 2.5. [13, 14] Let \( G \) be an \( n \)-vertex graph with minimum degree \( \delta(G) \). If \( \delta(G) \geq 1 \), then \( \gamma(G) \leq \frac{n}{2} \).

Theorem 2.6. Let \( n \) be a positive integer and \( n \geq 2 \), then \( \frac{n(n-2)!}{2} \leq \iota(S_n) \leq (n-1)! \). Moreover, the bounds are sharp.

Proof. Let \( S \) be a minimum isolating set of \( S_n \). According to the structure of \( S_n \), we know that \( S_n \) is a \( (n-1) \)-regular bipartite graph, and every perfect matching of \( S_n \) has \( \frac{n!}{2} \) edges. Note that \( S \) is a minimum isolating set of \( S_n \), and \( V(S_n) \setminus N_S[S] \) is an independent set, then every edge of a perfect matching of \( S_n \) has at least one endvertex in \( N_{S_n}(S) \), that is, \( |N_{S_n}(S)| \geq \frac{n!}{2} \). For any vertex \( v \in S \), we have \( |N_{S_n}(S)| \leq \frac{n!}{2} \).
**Figure 3.** $S_4$.

\[ \Delta(S_n) |S| = d(v)|S| = (n - 1)|S|. \]

Thus, \( \gamma(S_n) = |S| \geq \frac{|V_n(S)|}{d(v)} \geq \frac{n}{n-1} = \frac{n(n-2)!}{2}. \)

Inspired by Alon and Spencer [15] and Caro and Hansberg [3], we show that \( \gamma(S_n) \leq (n - 1)! \) by the probabilistic method. Since \( n \geq 2, d(v) = n - 1 \geq 1 \) for any vertex \( v \in V(S_n) \). Let \( p \in [0, 1] \), and we independently select a vertex subset \( A \subset V(S_n) \) at random such that \( P(v \in A) = p \). Let \( I \) be the set of the isolated vertices of \( V(S_n) \setminus A \). Meanwhile, let \( B = V(S_n) \setminus (N_{S_n}[A] \cup I) \) and \( D \) be a minimum dominating set of \( G[B] \). Since there is no isolated vertex in \( B \), \( \delta(G[B]) \geq 1 \), furthermore, \( \gamma(G[B]) = |D| \leq \frac{|B|}{2} \) by Lemma 2.5. Thus, \( A \cup D \) is an isolating set of \( S_n \). Note that the expected value \( E[|D|] \leq E[\frac{|B|}{2}] = \frac{1}{2}E[|B|] \). Hence, we have

\[
P(v \in B) = P(v \in B) = P(v \in V(S_n) \setminus (N_{S_n}[A] \cup I)) = P(v \in V(S_n) \setminus N_{S_n}[A])
= (1 - p)^{d(v)+1} = (1 - p)^{n-1} = (1 - p)^n.
\]

Thus, we obtain that

\[
E[A \cup D] \leq E[A] + \frac{1}{2}E[B] = p|V(S_n)| + \frac{1}{2}(1 - p)^n|V(S_n)| = (p + \frac{1}{2}(1 - p)^n) \cdot (n!).
\]

Considering the function \( f(p) = (p + \frac{1}{2}(1 - p)^n) \cdot (n!) \) and its derivative \( f'(p) = (1 - \frac{1}{2}n(1 - p)^{n-1}) \cdot (n!) \), we can see that \( f'(p) = 0 \) when \( p = 1 - (\frac{1}{2})^{\frac{1}{n}} \). It follows that

\[
\gamma(S_n) \leq E[A \cup D] \leq (p + \frac{1}{2}(1 - p)^n) \cdot (n!) \leq (1 - (\frac{2}{n})^{\frac{1}{n}}) + \frac{1}{2}(\frac{2}{n})^{\frac{1}{n}} \cdot (n!) \cdot (n!)
= (1 - (\frac{2}{n})^{\frac{1}{n}} + \frac{1}{2}(\frac{2}{n})^{\frac{1}{n}}) \cdot (n!).
\]
Since \( n \geq 2 \), we have \( \frac{(\frac{2}{n})^{n+1}}{\binom{n}{2}} \leq 1 \). Then \( \iota(S_n) \leq (1 - 1 + \frac{1}{2}(\frac{2}{n}) \cdot (n!)) = \frac{1}{n} \cdot (n!) = (n - 1)! \).

In conclusion, \( \frac{n(n-2)!}{2} \leq \iota(S_n) \leq (n - 1)! \) for any positive integer \( n \geq 2 \).

Especially, if \( n = 2 \), then \( S_2 \cong K_2 \), and \( \iota(K_2) = 1 = \frac{20!}{2} = \frac{n(n-2)!}{2} \). If \( n = 3 \), then \( S_3 \cong C_6 \), and \( \iota(C_6) = 2 = (3 - 1)! = (n - 1)! \). Hence, the bounds are sharp. \( \square \)

**Corollary 2.7.** Let \( n \) be a positive integer and \( n \geq 2 \), then \( \frac{n(n-2)!}{2} \leq \iota(S_n) \leq (n - 1)! \). Moreover, the bounds are sharp.

**Proof.** Let \( S_I \) be a minimum independent isolating set of \( S_n \). Obviously, \( S_I \) is also an isolating set of \( S_n \). By Theorem 2.6, we have \( \iota(S_n) \geq n(n-2)! \). Let \( V_I \) be the set of all vertices with the first symbol is 1.

Clearly, the set \( V_I \) is an independent set of \( S_n \). According the structure of \( S_n \), we know that any vertex of \( V_I \) has different \( n - 1 \) neighbors, and the first symbol of these \( n - 1 \) neighbors is 2, 3, \( \cdots \), \( n - 1 \), \( n \) respectively. Let \( x, y \in V_I \) and \( x \neq y \). If \( N_{S_n}(x) \cap N_{S_n}(y) = \emptyset \), then there exists a vertex to be adjacent to \( x \) and \( y \), which contradicts the definition of \( S_n \). Thus, \( N_{S_n}(x) \cap N_{S_n}(y) = \emptyset \), that is, the neighborhoods of any two vertices of \( V_I \) are disjoint. Hence, \( |N_{S_n}(V_I)| = (n - 1) \cdot |V_I| + |V_I| = n \cdot |V_I| = n \cdot (n - 1)! = n! = |V(S_n)| \), this means that \( G[V(S_n) \setminus N_{S_n}(V_I)] \) has no edge. Since \( V_I \) is independent, the set \( V_I \) is an independent isolating set of \( S_n \). Then \( \iota(S_n) \leq |V_I| = (n - 1)! \). In conclusion, \( \frac{n(n-2)!}{2} \leq \iota(S_n) \leq (n - 1)! \) for any positive integer \( n \geq 2 \).

Especially, if \( n = 2 \), then \( \{(1, 2)\} \) is an independent isolating set of \( S_2 \). Thus, \( \iota(S_2) = 1 = \frac{20!}{2} = \frac{n(n-2)!}{2} \). If \( n = 3 \), then \( \{(1, 2, 3), (1, 3, 2)\} \) is an independent isolating set of \( S_3 \). Thus, \( \iota(S_3) = 2 = (3 - 1)! = (n - 1)! \). Hence, the bounds are sharp. \( \square \)

3. Conclusions and problems

The hypercube network \( Q_n \) and \( n \)-star network \( S_n \) are both recursively constructed networks, and they have many known and attractive topological properties. This paper demonstrates the sharp bounds of the isolation number and the independent isolation number of the hypercube network and \( n \)-star network. In view of this research direction, there are still many academic issues worth studying:

**Problem 1.** Let \( m \geq 1 \) be a positive integer and \( \mathcal{F} = \{F_1, F_2, \cdots, F_m\} \). For any \( F_i \in \mathcal{F} \), if \( F_i \not\cong K_2 \), what \( F_i \) can be used to explore the \( \mathcal{F} \)-isolation number of the hypercube network or \( n \)-star network?

**Problem 2.** Consider the \( \mathcal{F} \)-isolation number of some other network models.

For future work, it would be interesting and meaningful to probe and research the \( \mathcal{F} \)-isolation number of some other network models.

**Use of AI tools declaration**

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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