New extrapolation projection contraction algorithms based on the golden ratio for pseudo-monotone variational inequalities

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Abstract: In real Hilbert spaces, for the purpose of trying to deal with the pseudo-monotone variational inequalities problem, we present a new extrapolation projection contraction algorithm based on the golden ratio in this study. Unlike ordinary inertial extrapolation, the algorithms are constructed based on a convex combined structure about the entire iterative trajectory. Extrapolation parameter $\psi$ is selected in a more relaxed range instead of only taking the golden ratio $\phi = \frac{\sqrt{5} + 1}{2}$ as the upper bound. Second, we propose an alternating extrapolation projection contraction algorithm to better increase the convergence effects of the extrapolation projection contraction algorithm based on the golden ratio. All our algorithms employ non-constantly decreasing adaptive step-sizes. The weak convergence results of the two algorithms are established for the pseudo-monotone variational inequalities. Additionally, the R-linear convergence results are investigated for strongly pseudo-monotone variational inequalities. Finally, we show the validity and superiority of the suggested methods with several numerical experiments. The numerical results show that alternating extrapolation does have obvious acceleration effect in practical application compared with no alternating extrapolation. Thus, the obvious effect of relaxing the selection range of parameter $\psi$ on our two algorithms is clearly demonstrated.

Keywords: golden ratio; variational inequalities; linear rate; pseudo-monotone operator; projection algorithms; weak convergence

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1. Introduction

In the present investigation, $H$ is a real Hilbert space and $C$ is a nonempty, closed, and convex subset of $H$, $A : H \to H$ is a continuous mapping. The variational inequality problem (abbreviated, VI($A, C$)) is of the form: find $z^* \in C$ satisfied with

$$\langle Az^*, z - z^* \rangle \geq 0, \; \forall z \in C.$$  (1.1)
Numerous domains have important uses for variational inequalities. Many academics have studied and come up with a multitude of findings [1–4].

The problem $\text{VI}(A, C)$ (1.1) is analogous to the problem of fixed points:

$$z^* = P_C(z^* - \lambda Az^*), \quad \lambda > 0.$$ 

As a result, $\text{VI}(A, C)$ (1.1) is possible solved by using the fixed point problem (see, e.g., [5, 6]). The following projection gradient algorithm is the simplest one:

$$z_{n+1} = P_C(z_n - \lambda A z_n).$$  (1.2)

However, this method’s convergence necessitates a moderately strong supposition that $A$ is a $\eta$-strongly monotone and $L$-Lipschitz continuous mapping, $\eta$ is a positive constant and step-size $\lambda \in \left(0, \frac{2\eta}{L^2}\right)$. However, algorithm (1.2) does not work when $A$ is monotone.

The extragradient algorithm of the following type was presented by Korpelevich in [7]: given $z_1 \in C$,

$$\begin{cases}
    y_n = P_C(z_n - \lambda_n A z_n), \\
    z_{n+1} = P_C(z_n - \lambda_n A y_n),
\end{cases}$$  (1.3)

where $\lambda_n \in \left(0, \frac{1}{L}\right)$. $A$ is relaxed to a monotone mapping based on algorithm (1.2). Moreover, it has been shown that the sequence $\{x_n\}$ will eventually arrive at a solution for the (1.1). However, $P_C$ lacks a closed form formula and (1.3) requires calculating $P_C$ twice in each iteration, which will result in an increase in the amount of computing that the procedure requires. There has been a lot of research done in this area by Censor et al ( [8–10]). The issue that it can be tricky to calculate $P_C$ was solved by using the projection onto the half space or intersection of half spaces rather than subset $C$. He first proposed projection and contraction method (PCM) in [11]. Cai et al. in [12] have studied the optimal step size $\eta_n$ for PCM, and the method which takes the following form:

$$\begin{cases}
    y_n = P_C(z_n - \lambda A z_n), \\
    d(z_n, y_n) = z_n - y_n - \lambda (Az_n - Ay_n), \\
    z_{n+1} = z_n - \gamma \eta_n d(z_n, y_n),
\end{cases}$$  (1.4)

where

$$\eta_n = \begin{cases}
    \frac{(z_n - y_n, d(z_n, y_n))}{\|d(z_n, y_n)\|^2}, & \|d(z_n, y_n)\| \neq 0, \\
    0, & \|d(z_n, y_n)\| = 0.
\end{cases}$$  (1.5)

The benefit of this method is that $A$ is as flexible as the algorithm (1.3) and only needs to calculate the projection once. The method’s efficacy will be vastly enhanced by both theoretical and numerical experiments. After adding the optimal step size $\eta_n$, the speed of convergence is enhanced further. The focus of numerous professionals has been drawn to method (1.4) because of its great characteristics and results. Based on the method (1.4), numerous academics have achieved numerous significant advancements (see, e.g., [12–14] and others). Recently, Dong et al. in [13] added inertial to
method (1.4) in order to obtain better convergence effect. In [15], Shehu and Iyiola incorporated alternating inertial and adaptive step-sizes:

\[
\begin{aligned}
    v_n &= \begin{cases} 
        u_n + \alpha_n (u_n - u_{n-1}), & n = \text{odd}, \\
        u_n, & n = \text{even}, 
    \end{cases} \\
    \overline{u}_n &= P_C (v_n - \lambda_n Av_n), \\
    d (v_n, \overline{u}_n) &= v_n - \overline{u}_n - \lambda_n (Av_n - A\overline{u}_n), \\
    u_{n+1} &= v_n - \gamma \eta_n d (v_n, \overline{u}_n),
\end{aligned}
\]

(1.6)

where

\[
\lambda_{n+1} = \begin{cases} 
    \min \left\{ \frac{\mu \|v_n - \overline{u}_n\|}{\|Av_n - A\overline{u}_n\|}, \lambda_n \right\}, & Av_n \neq A\overline{u}_n, \\
    \lambda_n, & \text{otherwise},
\end{cases}
\]

(1.7)

and

\[
\eta_n = \begin{cases} 
    \frac{\langle v_n - \overline{u}_n, d (v_n, \overline{u}_n) \rangle}{\|d (v_n, \overline{u}_n)\|^2}, & \|d (v_n, \overline{u}_n)\| \neq 0, \\
    0, & \|d (v_n, \overline{u}_n)\| = 0.
\end{cases}
\]

(1.8)

When the assumption of mapping \( A \) is relaxed to pseudo-monotone, convergence of the algorithm is proved. Additionally, they gave R-linear convergence analysis when \( A \) is a strongly pseudo-monotone mapping. In numerical experiments, the algorithm with alternating inertial in [15] performs better than the algorithm with general inertial in [13].

A fascinating concept has lately been created by Malitsky in [16] to solve mixed variational inequalities problem: find \( z^* \in C \) satisfied with

\[
\langle Az^*, z - z^* \rangle + g(z) - g(z^*) \geq 0, \quad \forall z \in C,
\]

(1.9)

where \( A \) is monotone mapping, \( g \) is a proper convex lower semicontinuous function. He proposed the following version of the golden ratio algorithm:

\[
\begin{aligned}
    \overline{z}_n &= \frac{(\phi - 1)z_n + z_{n-1}}{\phi}, \\
    z_{n+1} &= \text{prox}_{\lambda g} (\overline{z}_n - \lambda Az_n),
\end{aligned}
\]

(1.10)

where \( \phi \) is golden ratio, i.e. \( \phi = \frac{\sqrt{5} + 1}{2} \). In algorithm (1.10), \( \overline{z}_n \) is actually a convex combination of all the previously generated iterates. It is straightforward to ascertain that when \( g = \iota_C, (1.9) \) is equivalent to (1.1). Then, the algorithm (1.10) may be written equivalently as:

\[
\begin{aligned}
    \overline{z}_n &= \frac{(\phi - 1)z_n + z_{n-1}}{\phi}, \\
    z_{n+1} &= P_C (\overline{z}_n - \lambda Az_n).
\end{aligned}
\]

(1.11)

Numerous inertial algorithms have been published to address the issue of pseudo-monotone variational inequalities. Moreover, the golden ratio algorithms and their convergence have been researched for solving mixed variational inequalities problem when \( A \) is monotone. However, there are
still few results about golden ratio for solving variational inequalities problem (1.1) when $A$ is pseudo-monotone. The algorithm presented by Malitsky is very novel, and it provides us with some inspiration. Under more general circumstances, we hope to solve the variational inequalities problem (1.1) using the convex combination structure in this algorithm.

In this research, we combine the projection contraction method in [11] and golden ratio technique to present a new extrapolation projection contraction algorithm for the pseudo-monotone VI($A, C$) (1.1). To speed up the convergence of the new extrapolation projection contraction algorithm, we also present an alternating extrapolation algorithm. We can greatly expand the selection range of step size in the combination structure, and expanding the range of step size has a significant effect on the results of numerical experiments. Although the golden ratio is not used in our algorithm in the end in [13], considering that this paper is inspired by Malitsky’s golden ratio algorithm, the algorithms proposed in this paper is still recorded as projection contraction algorithms based on the golden ratio. In this paper, we primarily make the following improvements:

- We propose a projection contraction algorithm and an alternating extrapolation projection contraction algorithm based on the golden ratio. Weak convergence of two algorithms are established when $A$ is pseudo-monotone, sequentially weakly continuous and $L$-Lipschitz continuous.
- We get R-linear convergence results of two algorithms when $A$ is strongly pseudo-monotone.
- Our algorithms all use the new self-adaptive step-sizes which is not monotonically decreasing, like (1.7).
- In our algorithms, $A$ is a pseudo-monotone mapping which is weaker than [13, 17, 18]. Additionally, it is not necessary to restrict the extrapolation parameter $\psi$ in $\left(1, \frac{\sqrt{5}+1}{2}\right]$ as in [19,20], it can be to extend the value to $(1, +\infty)$.

The structure of the article is as follows:

Section 2: Related knowledge involved in the paper. Section 3: We give a projection contraction algorithm based on the golden ratio and the proofs of weak and R-linear convergence of the algorithm. Section 4: We also give an alternating extrapolation projection contraction algorithm based on the golden ratio, prove weak and R-linear convergence of the algorithm. Section 5: We give two numerical examples to verify the effectiveness of the algorithms.

2. Preliminaries

Let $\{z_n\}$ be a sequence in $H$. We denote $z_n \rightharpoonup z$ as $\{z_n\}$ weakly converges to $z$, while denote $z_n \rightarrow z$ as $\{z_n\}$ strongly converges to $z$.

**Definition 2.1.** [21] $A : H \rightarrow H$ is known as:
(a) $\eta$-strongly pseudo-monotone if
\[ \langle Av, u - v \rangle \geq 0 \Rightarrow \langle Au, u - v \rangle \geq \eta \|u - v\|^2, \quad \forall u, v \in H, \]
where $\eta > 0$;
(b) pseudo-monotone if
\[ \langle Av, u - v \rangle \geq 0 \Rightarrow \langle Au, u - v \rangle \geq 0, \quad \forall u, v \in H; \]
(c) \( L\)-Lipschitz continuous if there exists a constant \( L > 0 \) such that
\[
\|Au - Av\| \leq L\|u - v\|, \ \forall u, v \in H;
\]
(d) sequentially weakly continuous if for each sequence \( \{u_n\} \) :
\[
u_n \to u \Rightarrow Au_n \to Au.
\]

**Definition 2.2.** [22] \( P_C \) is called the metric projection onto \( C \), if for any point \( u \in H \), there exists a unique point \( P_C u \in C \) such that \( \|u - P_C u\| \leq \|u - v\|, \ \forall u \in C \).

**Definition 2.3.** [23] Suppose a sequence \( \{z_n\} \) in \( H \) converges in norm to \( z^* \in H \). We say that \( \{z_n\} \) converges to \( z^* \) \( R \)-linearly if \( \lim_{n \to \infty} \|z_n - z^*\|^2 < 1 \).

**Lemma 2.1.** [21], [22] \( P_C \) has the following properties:
(i) \( \langle u - v, P_C u - P_C v \rangle \geq \|P_C u - P_C v\|^2, \ \forall u, v \in H \);
(ii) \( P_C u \in C \) and \( \langle v - P_C u, P_C u - u \rangle \geq 0, \ \forall v \in C \).

**Lemma 2.2.** [21] This following equation holds in \( H \) :
\[
\|ou + (1 - \varrho)v\|^2 = \varrho\|u\|^2 + (1 - \varrho)\|v\|^2 - \varrho(1 - \varrho)\|u - v\|^2, \ \forall \varrho \in \mathbb{R}, \ \forall u, v \in H.
\]

**Lemma 2.3.** [24] Suppose \( A \) is pseudo-monotone in \( VI(A, C) \) (1.1) and \( S \) is the solution set of \( VI(A, C) \) (1.1). Then \( S \) is closed, convex and
\[
S = \{z \in C : \langle Aw, w - z \rangle \geq 0, \forall w \in C\}.
\]

**Lemma 2.4.** [25] Let \( \{z_n\} \) be a sequence in \( H \) such that the following two conditions hold:
(i) for any \( z \in C \), \( \lim_{n \to \infty} \|z_n - z\| \) exists;
(ii) \( \omega_w(z_n) \subset S \).
Then \( \{z_n\} \) converges weakly to a point in \( C \).

**Lemma 2.5.** [19] Let \( \{a_n\} \) and \( \{b_n\} \) be non-negative real sequences which meet
\[
a_{n+1} \leq a_n - b_n, \ \forall n > N,
\]
where \( N \) is some non-negative integer. Then \( \lim_{n \to \infty} b_n = 0 \) and \( \lim_{n \to \infty} a_n \) exists.

**Lemma 2.6.** [26] Let \( \{\lambda_n\} \) be non-negative number sequence such that
\[
\lambda_{n+1} \leq \xi_n \lambda_n + \tau_n, \ \forall n \in \mathbb{N},
\]
where \( \{\xi_n\} \) and \( \{\tau_n\} \) meet
\[
\{\xi_n\} \subset [1, +\infty), \sum_{n=1}^{\infty} (\xi_n - 1) < +\infty, \ \tau_n > 0 \text{ and } \sum_{n=1}^{\infty} \tau_n < +\infty.
\]

Then \( \lim_{n \to \infty} \lambda_n \) exists.
3. Projection contraction algorithm based on the golden ratio

We provide a PCM algorithm with a new extrapolation step and the corresponding convergence analyses in this section.

**Assumption 3.1.** In this paper, the following suppositions are true:
(a) A: $H \to H$ is pseudo-monotone, sequentially weakly continuous and L-Lipschitz continuous.
(b) The solution set $S$ is nonempty.
(c) $\{\xi_n\} \subset [1, +\infty), \sum_{n=1}^{\infty} (\xi_n - 1) < +\infty, \tau_n > 0$ and $\sum_{n=1}^{\infty} \tau_n < +\infty$.

**Algorithm 3.1.** Projection contraction algorithm based on the golden ratio.

**Step 0:** Take the iterative parameters $\mu \in (0, 1), \psi \in (1, +\infty), \gamma \in (0, 2), \text{ and } \xi_1, \tau_1, \lambda_1 > 0$. Let $u_1 \in H, v_0 \in H$ be given starting points. Known sequences $\{\xi_n\}, \{\tau_n\}$. Set $n := 1$.

**Step 1:** Compute

$$v_n = \frac{\psi - 1}{\psi} u_n + \frac{1}{\psi} v_{n-1}. \quad (3.1)$$

**Step 2:** Compute

$$\bar{u}_n = P_C (v_n - \lambda_n Av_n), \quad (3.2)$$

where

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\mu \|v_n - \bar{u}_n\|}{\|Av_n - A\bar{u}_n\|}, \xi_n \lambda_n + \tau_n \right\}, & Av_n \neq A\bar{u}_n, \\ \xi_n \lambda_n + \tau_n, & \text{otherwise.} \end{cases} \quad (3.3)$$

If $v_n = \bar{u}_n$, STOP. Otherwise, go to Step 3.

**Step 3:** Compute

$$d (v_n, \bar{u}_n) = (v_n - \bar{u}_n) - \lambda_n (Av_n - A\bar{u}_n), \quad (3.4)$$

$$\varphi_n = \langle v_n - \bar{u}_n, d (v_n, \bar{u}_n) \rangle,$$

$$u_{n+1} = v_n - \gamma \beta_n d (v_n, \bar{u}_n), \quad (3.5)$$

where

$$\beta_n = \begin{cases} \frac{\gamma_n}{\|d (v_n, \bar{u}_n)\|}, & \|d (v_n, \bar{u}_n)\| \neq 0, \\ 0, & \|d (v_n, \bar{u}_n)\| = 0. \end{cases} \quad (3.6)$$

**Step 4:** Set $n \leftarrow n + 1$, and go to Step 1.
**Remark 3.1.** Observe in Algorithm 3.1 that if $Av_n \neq A\bar{u}_n$, then

\[
\frac{\mu \|v_n - \bar{u}_n\|}{\|Av_n - A\bar{u}_n\|} \geq \frac{\mu \|v_n - \bar{u}_n\|}{L \|v_n - \bar{u}_n\|} = \frac{\mu}{L}.
\]

Therefore, $\lambda_n \geq \min\left\{\frac{\mu}{L}, \lambda_1\right\} > 0$. By (3.3), we have $\lambda_{n+1} \leq \xi_n \lambda_n + \tau_n$. From Lemma 2.6 we obtain $\lim_{n \to \infty} \lambda_n = \lambda$ when $\{\xi_n\} \subseteq [1, +\infty)$, $\sum_{n=1}^{\infty} (\xi_n - 1) < +\infty$, and $\sum_{n=1}^{\infty} \tau_n < +\infty$.

**Remark 3.2.** In our algorithms, it is not necessary to restrict the range of $\psi$ to $\left(1, \frac{\sqrt{5} + 1}{2}\right)$ or $(1, 2]$. $\psi$ only needs to be greater than $1$, which greatly relaxes the range of parameter to be chosen.

**Lemma 3.1.** Assume $\{u_n\}$ is the sequence generated by Algorithm 3.1 under the conditions of Assumption 3.1. Then $\{u_n\}$ is bounded and $\lim_{n \to \infty} \|u_n - u^*\|$ exists, where $u^* \in S$.

**Proof.** It is available from the iterative formulate

\[\|u_{n+1} - u^*\|^2 = \|v_n - u^* - \gamma \beta_n d (v_n, \bar{u}_n)\|^2 = \|v_n - u^*\|^2 - 2\gamma \beta_n \langle v_n - u^*, d (v_n, \bar{u}_n) \rangle + \gamma^2 \beta_n^2 \|d (v_n, \bar{u}_n)\|^2. \]  \tag{3.7}

According to (3.2) and Lemma 2.1(i),

\[
\langle \bar{u}_n - u^*, v_n - \bar{u}_n - \lambda_n Av_n \rangle \\
= \langle PC (v_n - \lambda_n Av_n) - PC u^*, v_n - \lambda_n Av_n - u^* + u^* - \bar{u}_n \rangle \\
= \langle PC (v_n - \lambda_n Av_n) - PC u^*, v_n - \lambda_n Av_n - u^* \rangle \\
+ \langle PC (v_n - \lambda_n Av_n) - PC u^*, u^* - \bar{u}_n \rangle \\
\geq \|PC (v_n - \lambda_n Av_n) - PC u^*\|^2 + \langle \bar{u}_n - u^*, u^* - \bar{u}_n \rangle \\
= \|\bar{u}_n - u^*\|^2 - \|\bar{u}_n - u^*\|^2 \\
= 0.
\]

Since $\bar{u}_n \in C$ and $u^* \in S$, and Definition 2.1 (b), we have $\langle A\bar{u}_n, \bar{u}_n - u^* \rangle \geq 0$, thus,

\[\lambda_n \langle A\bar{u}_n, \bar{u}_n - u^* \rangle \geq 0. \]  \tag{3.9}

Making use of (3.8) and (3.9), we gain

\[
\langle \bar{u}_n - u^*, d (v_n, \bar{u}_n) \rangle = \langle \bar{u}_n - u^*, v_n - \bar{u}_n - \lambda_n Av_n + \lambda_n A\bar{u}_n \rangle \geq 0,
\]

so,

\[\langle v_n - u^*, d (v_n, \bar{u}_n) \rangle \geq \varphi_n. \]  \tag{3.10}

Putting (3.10) in (3.7), we get

\[\|u_{n+1} - u^*\|^2 \leq \|v_n - u^*\|^2 - \gamma (2 - \gamma) \beta_n \varphi_n. \]  \tag{3.11}

By (3.5) and (3.6), we gain

\[\beta_n \varphi_n = \|\beta_n d (v_n, \bar{u}_n)\|^2 = \frac{1}{\gamma^2} \|v_n - u_{n+1}\|^2. \]  \tag{3.12}
Putting (3.12) in (3.11),

$$\|u_{n+1} - u^*\|^2 \leq \|v_n - u^*\|^2 - \frac{2 - \gamma}{\gamma} \|v_n - u_{n+1}\|^2.$$  \hfill (3.13)

From (3.1) and Lemma 2.2,

$$\|u_n - u^*\|^2 = \frac{\psi}{\psi - 1} \|v_n - u^*\|^2 - \frac{1}{\psi - 1} \|v_{n-1} - u^*\|^2 + \frac{\psi}{(\psi - 1)^2} \|v_n - v_{n-1}\|^2$$

$$= \frac{\psi}{\psi - 1} \|v_n - u^*\|^2 - \frac{1}{\psi - 1} \|v_{n-1} - u^*\|^2 + \frac{1}{\psi} \|u_n - v_{n-1}\|^2.$$  \hfill (3.14)

Combing (3.13) and (3.14),

$$\|u_{n+1} - u^*\|^2 - \|u_n - u^*\|^2 \leq -\frac{1}{\psi - 1} \|v_n - u^*\|^2 + \frac{1}{\psi - 1} \|v_{n-1} - u^*\|^2$$

$$- \frac{1}{\psi} \|u_n - v_{n-1}\|^2 - \frac{2 - \gamma}{\gamma} \|v_n - u_{n+1}\|^2,$$  \hfill (3.15)

so we can obtain

$$a_{n+1} \leq a_n - b_n,$$

where

$$a_n = \|u_n - u^*\|^2 + \frac{1}{\psi - 1} \|v_{n-1} - u^*\|^2,$$

$$b_n = \frac{2 - \gamma}{\gamma} \|v_n - u_{n+1}\|^2 + \frac{1}{\psi} \|u_n - v_{n-1}\|^2.$$

From the above proof, we have obtained $a_n \geq 0$ and $b_n \geq 0$. According to Lemma 2.5, we can get $\lim_{n \to \infty} a_n = 0$ and $\lim_{n \to \infty} a_n$ exists. Thus, we can get further $\lim_{n \to \infty} \|v_n - u_{n+1}\|^2 = 0$.

Inferring from the definition of $v_n$, we get

$$a_{n+1} = \|u_{n+1} - u^*\|^2 + \frac{1}{\psi - 1} \|v_n - u^*\|^2$$

$$= \frac{\psi}{\psi - 1} \|v_{n+1} - u^*\|^2 + \frac{\psi}{(\psi - 1)^2} \|v_{n+1} - v_n\|^2$$

$$- \frac{1}{\psi - 1} \|v_n - u^*\|^2 + \frac{1}{\psi - 1} \|v_n - u^*\|^2$$

$$= \frac{\psi}{\psi - 1} \|v_{n+1} - u^*\|^2 + \frac{1}{\psi} \|u_{n+1} - v_n\|^2.$$  \hfill (3.16)

We already know that

$$\lim_{n \to \infty} \|v_n - u_{n+1}\|^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} a_n \text{ exists},$$  \hfill (3.17)

we can easily get $\lim_{n \to \infty} \|v_{n+1} - u^*\|^2$ exists. From this it can be concluded that $\lim_{n \to \infty} \|u_{n+1} - u^*\|^2$ exists and $\{u_n\}$, $\{v_n\}$ are bounded. \hfill \Box
Lemma 3.2. Suppose \( \{\overline{u}_n\} \) and \( \{v_n\} \) are generated by Algorithm 3.1. Then under Assumption 3.1, \( \lim_{n \to \infty} \|v_n - \overline{u}_n\| = 0 \).

Proof. Noting

\[
\varphi_n = \|v_n - \overline{u}_n\|^2 - \lambda_n \langle v_n - \overline{u}_n, Av_n - A\overline{u}_n \rangle \\
\geq \|v_n - \overline{u}_n\|^2 - \lambda_n \|v_n - \overline{u}_n\| \|Av_n - A\overline{u}_n\| \\
\geq \left( 1 - \frac{\lambda_n \mu}{\lambda_{n+1}} \right) \|v_n - \overline{u}_n\|^2.
\] (3.18)

Available from (3.4),

\[
\|d(v_n, \overline{u}_n)\| \leq \|v_n - \overline{u}_n\| + \lambda_n \|Av_n - A\overline{u}_n\| \\
\leq \left( 1 + \frac{\lambda_n}{\lambda_{n+1}} \right) \|v_n - \overline{u}_n\|.
\] (3.19)

Choosing a fixed number \( \rho \) in \((\mu, 1)\). Since \( \lim_{n \to \infty} \lambda_n = \lambda \), we have \( \lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} \mu = \mu < \rho \). Then \( \exists n_0 \) such that \( \frac{\lambda_n}{\lambda_{n+1}} \mu < \rho \), \( \forall n \geq n_0 \). Therefore, \( \forall n \geq n_0 \), we have

\[
\|d(v_n, \overline{u}_n)\| < (1 + \rho) \|v_n - \overline{u}_n\|,
\]

and

\[
\varphi_n > (1 - \rho) \|v_n - \overline{u}_n\|^2.
\]

Thus,

\[
\beta_n = \frac{\varphi_n}{\|d(v_n, \overline{u}_n)\|^2} > \frac{(1 - \rho) \|v_n - \overline{u}_n\|^2}{(1 + \rho)^2 \|v_n - \overline{u}_n\|^2} = \frac{1 - \rho}{(1 + \rho)^2},
\] (3.20)

and so, \( \forall n \geq n_0 \), we can get

\[
\|v_n - \overline{u}_n\|^2 < \frac{1}{1 - \rho} \varphi_n = \frac{1}{(1 - \rho) \beta_n \gamma^2} \|v_n - u_{n+1}\|^2 < \frac{(1 + \rho)^2}{(1 - \rho)^2} \gamma^2 \|v_n - u_{n+1}\|^2.
\] (3.21)

From (3.21), we get \( \lim_{n \to \infty} \|v_n - \overline{u}_n\| = 0 \).

\[
\square
\]

Lemma 3.3. Assume that \( \{u_n\} \) is generated by Algorithm 3.1, then \( \omega_w(u_n) \subset S \).

Proof. Since \( \{u_n\} \) is bounded, \( \omega_w(u_n) \neq \emptyset \). Arbitrarily choose \( q \in \omega_w(u_n) \), then there exists a subsequence \( \{u_{n_k}\} \subset \{u_n\} \) such that \( u_{n_k} \rightharpoonup q \). Then \( \overline{u}_{n_k} \rightharpoonup q \), \( v_{n_k} \rightharpoonup q \). From Lemma 2.1(ii) and (3.2) we have

\[
\langle v_n - \lambda_n Av_n - \overline{u}_n, u - \overline{u}_n \rangle \geq 0, \; \forall u \in C,
\]

thus,

\[
\langle Av_n, u - \overline{u}_n \rangle \geq \frac{1}{\lambda_n} \langle v_n - \overline{u}_n, u - \overline{u}_n \rangle, \; \forall u \in C.
\] (3.22)

From (3.22) we can obtain

\[
\langle Av_n, u - v_n \rangle \geq \langle Av_n, \overline{u}_n - v_n \rangle + \frac{1}{\lambda_n} \langle v_n - \overline{u}_n, u - \overline{u}_n \rangle, \; \forall u \in C.
\]

So
From Definition 2.1(b), we have

\[
\langle Av_{n-1}, u - v_{n-1} \rangle \geq \frac{1}{\lambda_{n-1}} \langle v_{n-1} - \bar{u}_{n-1}, u - \bar{u}_{n-1} \rangle, \quad \forall u \in C. \tag{3.23}
\]

Fixing \( u \in C \) and passing \( k \to \infty \) in (3.23), noting \( \| v_{n_k} - \bar{u}_{n_k} \| \to 0 \), \( \{ \bar{u}_{n_k} \} \) and \( \{ Av_{n_k} \} \) are bounded, we obtain

\[
\lim_{k \to \infty} \langle Av_{n-1}, u - v_{n-1} \rangle \geq 0. \tag{3.24}
\]

Choosing a decreasing sequence \( \{ \epsilon_k \} \) such that \( \epsilon_k > 0 \) and \( \lim_{k \to \infty} \epsilon_k = 0 \). For each \( \epsilon_k \),

\[
Av_{N_k} \neq 0 \quad \text{and} \quad \langle Av_{n-1}, u - v_{n-1} \rangle + \epsilon_k \geq 0, \quad \forall j \geq N_k, \tag{3.25}
\]

where \( N_k \) is smallest non-negative integer that satisfies (3.25). As \( \{ \epsilon_k \} \) is decreasing, \( \{ N_k \} \) is increasing. For simplicity, it is useful to write \( N_k \) as \( n_{N_k} \). Setting

\[
\vartheta_{N_k-1} = \frac{Av_{N_k-1}}{\|Av_{N_k-1}\|^2},
\]

one gets \( \langle Av_{N_k-1}, \vartheta_{N_k-1} \rangle = \left( Av_{N_k-1}, \frac{Av_{N_k-1}}{\|Av_{N_k-1}\|} \right) = 1 \). Then, by (3.25) for each \( k \),

\[
\begin{aligned}
\langle Av_{N_k-1}, u + \epsilon_k \vartheta_{N_k-1} - v_{N_k-1} \rangle \\
= \langle Av_{N_k-1}, u - v_{N_k-1} \rangle + \epsilon_k \langle Av_{N_k-1}, \vartheta_{N_k-1} \rangle \\
\geq 0.
\end{aligned} \tag{3.26}
\]

From Definition 2.1(b), we have

\[
\langle A(u + \epsilon_k \vartheta_{N_k-1}), u + \epsilon_k \vartheta_{N_k-1} - v_{N_k-1} \rangle \geq 0. \tag{3.27}
\]

Since \( v_{n_k} \to q \) as \( k \to \infty \) and Definition 2.1(d), we obtain that \( Av_{n_k} \to Aq \). Suppose \( Aq \neq 0 \) (if \( Aq = 0 \), \( q \in S \)). Following that, employing the norm’s sequentially weakly lower semicontinuity, we gain

\[
0 < \| Aq \| \leq \lim_{k \to \infty} \| Av_{n_k} \|.
\]

Because \( \{ N_k \} \subset \{ n_k \} \), and \( \lim_{k \to \infty} \epsilon_k = 0 \),

\[
0 \leq \lim_{k \to \infty} \| \epsilon_k \vartheta_{N_k-1} \| = \lim_{k \to \infty} \epsilon_k \left( \frac{1}{\| Av_{N_k-1} \|} \right) \leq \lim_{k \to \infty} \epsilon_k \| Av_{N_k-1} \| \leq 0,
\]

and this means \( \lim_{k \to \infty} \| \epsilon_k \vartheta_{N_k-1} \| = 0 \). Inputting \( k \to \infty \) into (3.27), we get

\[
\langle Au, u - q \rangle \geq 0, \quad \forall u \in C.
\]

From Lemma 2.3, \( q \in S \), then \( \omega_u (u_n) \subset S \). \( \square \)
\textbf{Theorem 3.1.} Assume \( \{u_n\} \) is the sequence generated by Algorithm 3.1 under the conditions of Assumption 3.1. There exists \( u^* \in S \) such that \( u_n \to u^* \).

\textit{Proof.} From Lemmas 3.1 and 3.3, we get \( \lim_{n \to \infty} \|u_n - u^*\| \) exists and \( \omega_n (u_n) \subset S \). From Lemma 2.4, \( u_n \to u^* \in S \). \( \square \)

\textbf{Theorem 3.2.} Suppose \( \{u_n\} \) is generated by Algorithm 3.1 under the condition of \( A \) is \( \eta \)-strongly pseudo-monotone with \( \eta > 0 \). Then \( \{u_n\} \) converges \( R \)-linearly to the unique solution \( u^* \) of VI\((A, C)\) (1.1).

\textit{Proof.} Since \( \bar{u}_n \in C \), from Definition 2.1(a), we have
\[
\langle A\bar{u}_n, \bar{u}_n - u^* \rangle \geq \eta \|\bar{u}_n - u^*\|^2.
\]
Multiply \( \lambda_n \) on both sides of above inequality, we get
\[
\lambda_n \langle A\bar{u}_n, \bar{u}_n - u^* \rangle \geq \lambda_n \eta \|\bar{u}_n - u^*\|^2. \tag{3.28}
\]
(3.8) plus (3.28), we obtain
\[
\langle \bar{u}_n - u^*, d(v_n, \bar{u}_n) \rangle = \langle \bar{u}_n - u^*, v_n - \bar{u}_n - \lambda_n Av_n + \lambda_n A\bar{u}_n \rangle \geq \lambda_n \eta \|\bar{u}_n - u^*\|^2, \tag{3.29}
\]
so
\[
\langle v_n - u^*, d(v_n, \bar{u}_n) \rangle \geq \varphi_n + \lambda_n \eta \|\bar{u}_n - u^*\|^2. \tag{3.30}
\]
Putting (3.30) into (3.7), we obtain
\[
\|u_{n+1} - u^*\|^2 \leq \|v_n - u^*\|^2 - \gamma (2 - \gamma) \beta_n \varphi_n - 2 \gamma \beta_n \lambda_n \eta \|\bar{u}_n - u^*\|^2. \tag{3.31}
\]
Using (3.18) in (3.31), we have
\[
\|u_{n+1} - u^*\|^2 \leq \|v_n - u^*\|^2 - \gamma (2 - \gamma) \beta_n \left( 1 - \frac{\lambda_n}{\lambda_{n+1}} \right) \|v_n - \bar{u}_n\|^2 - 2 \gamma \beta_n \lambda_n \eta \|\bar{u}_n - u^*\|^2, \tag{3.32}
\]
where
\[
\begin{align*}
&- \gamma (2 - \gamma) \beta_n \left( 1 - \frac{\lambda_n}{\lambda_{n+1}} \right) \|v_n - \bar{u}_n\|^2 - 2 \gamma \beta_n \lambda_n \eta \|\bar{u}_n - u^*\|^2 \\
&\leq - \gamma \beta_n \min \left\{ (2 - \gamma) \left( 1 - \frac{\lambda_n}{\lambda_{n+1}} \right), 2 \lambda_n \eta \right\} \left( \|v_n - \bar{u}_n\|^2 + \|\bar{u}_n - u^*\|^2 \right) \\
&\leq - \gamma \beta_n \min \left\{ \frac{1}{2} (2 - \gamma) \left( 1 - \frac{\lambda_n}{\lambda_{n+1}} \right), \lambda_n \eta \right\} \|v_n - u^*\|^2 \\
&< - \gamma \frac{1 - \rho}{(1 + \rho)^2} \min \left\{ \frac{1}{2} (2 - \gamma) (1 - \rho) \left( 1 - \frac{\lambda_n}{\lambda_{n+1}} \right), \frac{1}{2} \lambda_n \eta \right\} \|v_n - u^*\|^2, \forall n \geq n_0.
\end{align*}
\] (3.33)
The last inequality is true because there exists \( n_0 \) such that \( \beta_n > \frac{1 - \rho}{(1 + \rho)^2} \), \( \lambda_n > \frac{1}{2} \) and \( \left( 1 - \frac{\lambda_n}{\lambda_{n+1}} \right) > 1 - \rho, \forall n \geq n_0 \). Putting (3.33) in (3.32), we get
\[
\|u_{n+1} - u^*\|^2 < \left( 1 - \gamma \frac{1 - \rho}{(1 + \rho)^2} \min \left\{ \frac{1}{2} (2 - \gamma) (1 - \rho) \left( 1 - \frac{\lambda_n}{\lambda_{n+1}} \right), \frac{1}{2} \lambda_n \eta \right\} \right) \|v_n - u^*\|^2. \tag{3.34}
\]
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Since $\beta_n > \frac{1-\rho}{(1+\rho)\gamma}$ and $(1 - \frac{\lambda_n}{\lambda_{n+1}}) > 1 - \rho$, we obtain

$$0 < 1 - \frac{\gamma}{(1 + \rho)^2} \min \left\{ \frac{1}{2} (2 - \gamma) (1 - \rho), \frac{1}{2} \lambda \eta \right\} < 1.$$  

Let $\delta^2 = 1 - \frac{1-\rho}{(1+\rho)^2} \min \left\{ \frac{1}{2} (2 - \gamma) (1 - \rho), \frac{1}{2} \lambda \eta \right\}$, we have $0 < \delta^2 < 1$ and

$$\|u_{n+1} - u^*\|^2 < \delta^2\|v_n - u^*\|^2, \forall n \geq n'.$$  \hspace{1cm} (3.35)

Putting (3.14) into (3.35), after collation, we get

$$\frac{\psi}{\psi - 1} \|v_{n+1} - u^*\|^2 < \left( \delta^2 + \frac{1}{\psi - 1} \right)\|v_n - u^*\|^2, \forall n \geq n'.$$  \hspace{1cm} (3.36)

Since $0 < \delta^2 < 1$, $\delta^2 + \frac{1}{\psi - 1} < 1 + \frac{1}{\psi - 1} = \frac{\psi}{\psi - 1}$. And we can get $0 < \frac{\delta^2 + \frac{1}{\psi - 1}}{\psi} < 1$. Therefore,

$$\|v_{n+1} - u^*\|^2 < r^2\|v_n - u^*\|^2, \forall n \geq n'.$$

where $r = \sqrt{\frac{\delta^2 + \frac{1}{\psi - 1}}{\psi}}$. By induction, we get

$$\|v_{n+1} - u^*\|^2 < r^{2(n-n_0)}\|v_{n_0} - u^*\|^2, \forall n \geq n'.$$

By (3.35),

$$\|u_{n+1} - u^*\|^2 < \delta^2 r^{2(n-n_0)}\|v_{n_0} - u^*\|^2, \forall n \geq n'.$$

And we have

$$\|u_{n+1} - u^*\|^\frac{1}{2} < r^{\frac{n-n_0}{2}}\left( \delta \|v_{n_0} - u^*\| \right)^\frac{1}{2}, \forall n \geq n'.$$

So

$$\lim_{n \to \infty} \|u_{n+1} - u^*\|^\frac{1}{2} \leq r < 1.$$  

Therefore, \{u_n\} converges R-linearly to the unique solution $u^*$.

\[\square\]

4. Alternating extrapolation projection contraction algorithm based on the golden ratio

In this part, we offer an algorithm for settling the problem of variational inequalities based on the golden ratio and provide the proofs of weak and R-linear convergence.

\textbf{Algorithm 4.1.} Alternating extrapolation projection contraction algorithm based on the golden ratio.

\textit{Step 0:} Take the iterative parameters $\mu \in (0, 1), \psi \in (1, +\infty), \gamma \in (0, 2)$ and $\xi_1, \tau_1, \lambda_1 > 0$. Let $u_1 \in H$, $\psi \in (1, +\infty)$, $\gamma \in (0, 2)$ and $\xi_1, \tau_1, \lambda_1 > 0$. Let $u_1 \in H$,
$v_0 \in H$ be given starting points. Known sequences $\{\xi_n\}, \{\tau_n\}$. Set $n := 1$.

**Step 1:** Compute

$$v_n = \begin{cases} \frac{\psi}{2} u_n + \frac{1}{\psi} v_{n-1}, & n \text{ odd}, \\ u_n, & n \text{ even}. \end{cases} \quad (4.1)$$

**Step 2:** Compute

$$\bar{u}_n = P_C (v_n - \lambda_n A v_n), \quad (4.2)$$

where

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\mu \|v_n - \bar{u}_n\|}{\|A v_n - A \bar{u}_n\|}, \xi_n \bar{\mu} + \tau_n \right\}, & A v_n \neq A \bar{u}_n, \\ \xi_n \bar{\mu} + \tau_n, & \text{otherwise}. \end{cases} \quad (4.3)$$

If $v_n = \bar{u}_n$, STOP. Otherwise, go to Step 3.

**Step 3:** Compute

$$d (v_n, \bar{u}_n) = (v_n - \bar{u}_n) - \lambda_n (A v_n - A \bar{u}_n), \quad (4.4)$$

$$u_{n+1} = v_n - \gamma \beta_n d (v_n, \bar{u}_n), \quad (4.5)$$

$$\varphi_n = \langle v_n - \bar{u}_n, d (v_n, \bar{u}_n) \rangle$$

where

$$\beta_n = \begin{cases} \frac{\psi}{\|d (v_n, \bar{u}_n)\|^2}, & \|d (v_n, \bar{u}_n)\| \neq 0, \\ 0, & \|d (v_n, \bar{u}_n)\| = 0. \end{cases} \quad (4.6)$$

**Step 4:** Set $n <- n + 1$, and go to Step 1.

---

**Lemma 4.1.** Assume $\{u_n\}$ is the sequence generated by Algorithm 4.1 under the conditions of Assumption 3.1. Then $\{u_{2n}\}$ is bounded and $\lim_{n \to \infty} \|u_{2n} - u^*\|$ exists, where $u^* \in S$.

**Proof.** Following the proof line (3.7)–(3.13) of Lemma 3.1 and $\|v_{2n} - u^*\|^2 = \|u_{2n} - u^*\|^2$, we obtain

$$\|u_{2n+1} - u^*\|^2 \leq \|u_{2n} - u^*\|^2 - \frac{2 - \gamma}{\gamma} \|v_{2n} - u_{2n+1}\|^2. \quad (4.7)$$

From (3.13) we have

$$\|u_{2n+2} - u^*\|^2 \leq \|v_{2n+1} - u^*\|^2 - \frac{2 - \gamma}{\gamma} \|v_{2n+1} - u_{2n+2}\|^2. \quad (4.8)$$

By the definition of $v_n$,

$$\|v_{2n+1} - u^*\|^2 = \frac{\psi - 1}{\psi} \|u_{2n+1} - u^*\|^2 + \frac{1}{\psi} \|v_{2n} - u^*\|^2 - \frac{\psi - 1}{\psi^2} \|v_{2n} - u_{2n+1}\|^2. \quad (4.9)$$
Combing (4.9) and (4.8), we obtain
\[
\|u_{2n+2} - u^*\|^2 - \|u_{2n} - u^*\|^2 \leq \frac{\psi - 1}{\psi} \left( \|u_{2n+1} - u^*\|^2 - \|u_{2n} - u^*\|^2 \right) - \frac{2 - \gamma}{\gamma} \|v_{2n+1} - u_{2n+2}\|^2 - \frac{\psi - 1}{\psi^2} \|v_{2n} - u_{2n+1}\|^2.
\] (4.10)

From (4.7) we have
\[
\|u_{2n+1} - u^*\|^2 - \|u_{2n} - u^*\|^2 \leq -\frac{2 - \gamma}{\gamma} \|v_{2n} - u_{2n+1}\|^2.
\] (4.11)

Combining (4.10) and (4.11), we get
\[
\|u_{2n+2} - u^*\|^2 - \|u_{2n} - u^*\|^2 \leq -\frac{\psi - 1}{\psi} \left( \|u_{2n+1} - u^*\|^2 - \|u_{2n} - u^*\|^2 \right) - \frac{2 - \gamma}{\gamma} \|v_{2n+1} - u_{2n+2}\|^2 \leq 0.
\] (4.12)

Therefore \(\|u_{2n+2} - u^*\| \leq \|u_{2n} - u^*\|\). This proves that \(\{u_{2n}\}\) is bounded and \(\lim_{n \to \infty} \|u_{2n} - u^*\|\) exists. \(\Box\)

**Lemma 4.2.** Under Assumption 3.1, suppose \(\{u_{2n}\} \) and \(\{\bar{u}_{2n}\} \) are generated by Algorithm 4.1. Then \(\lim_{n \to \infty} \|u_{2n} - \bar{u}_{2n}\| = 0.\)

**Proof.** From (4.12) and \(u_{2n} = v_{2n}\), we get that \(\|u_{2n} - u^*\|\) is bounded and
\[
\lim_{n \to \infty} \|u_{2n} - u_{2n+1}\| = 0.
\]

From (3.18) and (3.19), we have
\[
\varphi_{2n} \geq \left(1 - \frac{\lambda_{2n}}{\beta_{2n+1}}\right) \|v_{2n} - \bar{u}_{2n}\|^2,
\] (4.13)

and
\[
\|d(v_{2n}, \bar{u}_{2n})\| \leq \left(1 + \frac{\lambda_{2n}}{\beta_{2n+1}}\right) \|v_{2n} - \bar{u}_{2n}\|.
\] (4.14)

Combining (4.13) and (4.14), we can obtain
\[
\|u_{2n+1} - v_{2n}\| = \gamma \beta_{2n} \|d(v_{2n}, \bar{u}_{2n})\| = \gamma \varphi_{2n} \|d(v_{2n}, \bar{u}_{2n})\| \geq \gamma \left(1 - \frac{\lambda_{2n}}{\beta_{2n+1}}\right) \|v_{2n} - \bar{u}_{2n}\|^2 \geq \gamma \frac{1 - \rho}{1 + \rho} \|v_{2n} - \bar{u}_{2n}\|, \forall n \geq n_0.
\] (4.15)

Using \(u_{2n} = v_{2n}\) and \(\lim_{n \to \infty} \|u_{2n} - u_{2n+1}\| = 0\), we get
\[
\lim_{n \to \infty} \|u_{2n} - \bar{u}_{2n}\| = 0.
\] \(\Box\)
Lemma 4.3. Assume that \( \{u_{2n}\} \) is generated by Algorithm 4.1, then \( \omega_u (u_{2n}) \subset S \).

Proof. \( \forall p \in \omega_u (u_{2n}) \), then exists a subsequence \( \{u_{2n_k}\} \subset \{u_{2n}\} \), such that \( u_{2n_k} \rightharpoonup p \). By Lemma 2.1(ii) and (4.2) we have

\[
\langle u_{2n_k} - \lambda_{2n_k} Au_{2n_k} - \bar{u}_{2n_k}, u - \bar{u}_{2n_k} \rangle \geq 0, \forall u \in C,
\]

thus,

\[
\langle Au_{2n_k}, u - \bar{u}_{2n_k} \rangle \geq \frac{1}{\lambda_{2n_k}} \langle u_{2n_k} - \bar{u}_{2n_k}, u - \bar{u}_{2n_k} \rangle, \forall u \in C,
\]

and

\[
\frac{1}{\lambda_{2n_k}} \langle u_{2n_k} - \bar{u}_{2n_k}, u - \bar{u}_{2n_k} \rangle + \langle Au_{2n_k}, \bar{u}_{2n_k} - u_{2n_k} \rangle \leq \langle Au_{2n_k}, u - u_{2n_k} \rangle, \forall u \in C. \tag{4.16}
\]

Similar to Lemma 3.3, the following proof steps are omitted as they are redundant. Thus, we come to the conclusion,

\[
\langle Au, u - p \rangle \geq 0, \forall u \in C.
\]

Using Lemma 2.3, we get \( p \in S \). \( \square \)

Theorem 4.1. Assume \( \{u_n\} \) is the sequence generated by Algorithm 4.1 under the conditions of Assumption 3.1. There exists \( q \in S \) such that \( u_n \rightharpoonup q \).

Proof. \( \{u_{2n}\} \) is bounded implies that \( \{u_{2n}\} \) has weakly convergent subsequences. Then, we can choose a subsequence of \( \{u_{2n}\} \), denoted by \( \{u_{2n_k}\} \) such that \( u_{2n_k} \rightharpoonup q \in H \). We obtain \( \lim_{n \to \infty} \|u_{2n} - q\| \) exists and \( q \in S \) from Lemma 4.1 and 4.3. The proof of the whole sequence \( u_{2n} \rightharpoonup q \in S \) which is the same as Lemma 4.4 in [15]. Hence, \( u_n \rightharpoonup q \in S \). \( \square \)

Theorem 4.2. Suppose \( \{u_n\} \) is generated by Algorithm 4.1 under the condition of \( A \) is \( \eta \)-strongly pseudo-monotone with \( \eta > 0 \). Then \( \{u_n\} \) converges \( R \)-linearly to the unique solution \( u^* \) of VI(\( A, C \)) (1.1).

Proof. From (3.35), \( \forall n \geq n'_0 \), we have

\[
\|u_{2n+1} - u^*\|^2 < \delta^2 \|v_{2n} - u^*\|^2 = \delta^2 \|u_{2n} - u^*\|^2, \tag{4.17}
\]

and

\[
\|u_{2n+2} - u^*\|^2 < \delta^2 \|v_{2n+1} - u^*\|^2, \tag{4.18}
\]

where \( \delta^2 = 1 - \gamma \frac{1 - \rho}{(1 + \rho)^2} \min \left\{ \frac{1}{2} \left( 2 - \gamma \right)(1 - \rho), \frac{1}{2} \gamma \rho \right\} \) and \( 0 < \delta^2 < 1 \). Combining (4.9) and (4.18),

\[
\|u_{2n+2} - u^*\|^2 < \delta^2 \left( \frac{\psi - 1}{\psi} \|u_{2n+1} - u^*\|^2 + \frac{1}{\psi} \|u_{2n} - u^*\|^2 - \frac{\psi - 1}{\psi^2} \|v_{2n} - u_{2n+1}\|^2 \right). \tag{4.19}
\]

Putting (4.17) in (4.19), we have

\[
\|u_{2n+2} - u^*\|^2 < \delta^2 \left( \frac{\psi - 1}{\psi} \|u_{2n} - u^*\|^2 + \frac{1}{\psi} \|u_{2n} - u^*\|^2 - \frac{\psi - 1}{\psi^2} \|v_{2n} - u_{2n+1}\|^2 \right) \leq \delta^2 \left( \frac{\psi - 1}{\psi} \|u_{2n} - u^*\|^2 + \frac{1}{\psi} \|u_{2n} - u^*\|^2 \right) \leq \delta^2 \|u_{2n} - u^*\|^2, \forall n \geq n'_0. \tag{4.20}
\]

So
By induction, we have
\[ \|u_{2n+2} - u^*\|^2 < \delta^2 \|u_{2n} - u^*\|^2, \forall n \geq n'_0. \] (4.21)

Thus,
\[ \|u_{2n+3} - u^*\|^2 < \delta^2 \|u_{2n+2} - u^*\|^2 < \|u_{2n+2} - u^*\|^2 < \delta^2 \|u_{2n_0} - u^*\|^2, \forall n \geq n'_0. \] (4.22)

Therefore, \( \{u_n\} \) converges R-linearly to the unique solution \( u^* \).

5. Numerical examples

The following sections provide some computational experiments and comparisons between our algorithms considered in Sections 3 and 4 and other algorithms. All codes were written in MATLAB R2016b and performed on a PC Desktop AMD Ryzen R7-5600U CPU @ 3.00 GHz, RAM 16.00 GB.

We make a comparison of our Algorithm 3.1, Algorithm 4.1, Algorithm 2 in [15] and Algorithm 1 in [27], Time in the table indicates CPU Time. In this section, we set maximum number of iterations \( n_{\max} = 6 \times 10^5 \), \( \xi_n = 1 + \frac{1}{n^2} \) and \( \tau_n = \frac{1}{n^2} \).

Example 5.1. Define \( A : \mathbb{R}^m \rightarrow \mathbb{R}^m \) by
\[ Au = (M + \beta)(Nu + q), \]

where \( M = e^{-u^T Q u}, N \) is a positive semi-definite matrix, \( Q \) is a positive definite matrix, \( q \in \mathbb{R}^m \) and \( \beta > 0 \). In addition to being easy to obtain, \( A \) is pseudo-monotone, differentiable and Lipschitz continuous. Take \( C = \{u \in \mathbb{R}^m | Bu \leq b\} \), where \( B \) is a \( k^* \times m \) matrix and \( b \in \mathbb{R}_{+}^{k^*} \) with \( k^* = 10 \). Select the initial point \( u_1 = (1, 1, \ldots, 1)^T \) for all algorithms. Initial points of Algorithm 3.1 and Algorithm 4.1, \( v_0 \) is generated randomly in \( \mathbb{R}^m \). We take \( \psi = \frac{\sqrt{5}+1}{2}, \mu = 0.6 \) in Algorithm 3.1 and Algorithm 4.1. We take \( \theta_n = \frac{2^{-n}}{1.01n} \) in Algorithm 2 in [15] and \( \theta = 0.45 \) \((1 - \mu)\) in Algorithm 1 in [27]. Thus, we take different values for \( \lambda_1 \) and \( \gamma \) respectively to compare with the algorithms in the other two papers. In this example, we take \( \|u_n - v_n\| < 10^{-3} \) as the stopping criterion.

In Table 1, we give a comparison of our Algorithms 3.1 and 4.1 with Algorithm 1 in [27] and Algorithm 2 in [15] in different dimensions for \( \gamma = 1.5, \lambda_1 = 0.05 \) and a comparison Figure 1 for \( m = 100 \). It is illustrated that our two algorithms have some superiority.
Table 1. Example 5.1 with $\gamma = 1.5, \lambda_1 = 0.05$ and various values of $m$.

<table>
<thead>
<tr>
<th>Problem size</th>
<th>Alg 3.1</th>
<th>Alg 4.1</th>
<th>Alg 2 in [15]</th>
<th>Alg 1 in [27]</th>
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<td>Time</td>
<td>Iter</td>
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<td></td>
<td>1000</td>
<td>3885</td>
<td>9.3913</td>
<td>2332</td>
</tr>
</tbody>
</table>

Figure 1. Relationship between error value and iteration times in Example 5.1 with $k^* = 10, m = 100$.

In Tables 2 and 3, we give a comparison of Algorithm 3.1 and Algorithm 4.1 for the same number of dimensions with different $\gamma$, respectively. We find that the larger $\gamma$ is for both algorithms in the same dimension, the fewer the iterations and the shorter the CPU Time, where $\gamma \in (0, 2)$.

Table 2. Algorithm 3.1 with different $\gamma$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$m = 200$</th>
<th>$m = 400$</th>
<th>$m = 800$</th>
<th>$m = 1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iter</td>
<td>Time</td>
<td>Iter</td>
<td>Time</td>
</tr>
<tr>
<td>0.25</td>
<td>9996</td>
<td>2.4136</td>
<td>14529</td>
<td>1.5986</td>
</tr>
<tr>
<td>0.5</td>
<td>4746</td>
<td>1.1729</td>
<td>7244</td>
<td>3.3542</td>
</tr>
<tr>
<td>1</td>
<td>2438</td>
<td>0.6266</td>
<td>3402</td>
<td>1.7116</td>
</tr>
<tr>
<td>1.25</td>
<td>1939</td>
<td>0.6079</td>
<td>2866</td>
<td>1.4242</td>
</tr>
<tr>
<td>1.5</td>
<td>1546</td>
<td>0.4253</td>
<td>2315</td>
<td>1.1200</td>
</tr>
</tbody>
</table>
Table 3. Algorithm 4.1 with different $\gamma$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$m = 200$</th>
<th>$m = 400$</th>
<th>$m = 800$</th>
<th>$m = 1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iter</td>
<td>Time</td>
<td>Iter</td>
<td>Time</td>
</tr>
<tr>
<td>0.25</td>
<td>7212</td>
<td>1.7554</td>
<td>10193</td>
<td>4.6670</td>
</tr>
<tr>
<td>0.5</td>
<td>3063</td>
<td>6.7968</td>
<td>4836</td>
<td>2.3662</td>
</tr>
<tr>
<td>1</td>
<td>1626</td>
<td>0.4460</td>
<td>2292</td>
<td>1.1557</td>
</tr>
<tr>
<td>1.25</td>
<td>1429</td>
<td>0.7354</td>
<td>1468</td>
<td>0.8067</td>
</tr>
<tr>
<td>1.5</td>
<td>964</td>
<td>0.2697</td>
<td>1300</td>
<td>0.7061</td>
</tr>
</tbody>
</table>

Example 5.2. [28] Define a mapping $A$ by

$$Au = (M^T M + N + P)u.$$ 

The matrices $N$ and $P$ are randomly generated skew-symmetric matrix and positive diagonal matrix, respectively. Assume $C := \{u \in \mathbb{R}^m \mid Mu \leq p\}$, where matrix $M \in \mathbb{R}^{k \times m}$ and vector $p \in \mathbb{R}^k$ are randomly generated. Thus, all entries in $p$ are non-negative. Here $V(A, C)$ (1.1) has a unique solution $u^* = 0$.

Set $\psi = \sqrt{\frac{\sqrt{5}+1}{2}}$, $\mu = \frac{1}{\sqrt{2}}$ in Algorithm 3.1, 4.1. We choose $\theta_n = \frac{2 - \gamma}{1.01\gamma}$ in Algorithm 2 in [15] and $\theta = 0.45 (1 - \mu)$ in Algorithm 1 in [27]. Additionally, we take different values for $\lambda_1$ and $\gamma$, respectively, to compare with the algorithms in the other two papers. We use the stopping criterion $\|u_n - y_n\| \leq 10^{-3}$.

In Table 4, we give a comparison of our Algorithm 3.1 and Algorithm 4.1 with Algorithm 1 in [27] and Algorithm 2 in [15] in different dimensions for $\gamma = 1.5$, $\lambda_1 = 0.05$ and a comparison Figure 2 for $k = 30, m = 60$.

In Figures 3 and 4 we compared the impact of Algorithm 3.1 and Algorithm 4.1 with varying $\psi$.

Table 4. Example 5.2 with $\gamma = 1.5$, $\lambda_1 = 0.05$ and various values of $k, m$.

<table>
<thead>
<tr>
<th>Problem size</th>
<th>Alg 3.1</th>
<th>Alg 4.1</th>
<th>Alg 2 in [15]</th>
<th>Alg 1 in [27]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$m$</td>
<td>Iter</td>
<td>Time</td>
<td>Iter</td>
</tr>
<tr>
<td>60</td>
<td>10</td>
<td>1146</td>
<td>0.2566</td>
<td>437</td>
</tr>
<tr>
<td>30</td>
<td>100</td>
<td>1429</td>
<td>0.3478</td>
<td>510</td>
</tr>
<tr>
<td>50</td>
<td>120</td>
<td>1680</td>
<td>0.4009</td>
<td>586</td>
</tr>
<tr>
<td>70</td>
<td>150</td>
<td>1359</td>
<td>0.4956</td>
<td>501</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>1434</td>
<td>0.5570</td>
<td>476</td>
</tr>
<tr>
<td>150</td>
<td>1439</td>
<td>0.6184</td>
<td>420</td>
<td>0.1832</td>
</tr>
<tr>
<td>200</td>
<td>1399</td>
<td>1.0800</td>
<td>498</td>
<td>0.3947</td>
</tr>
<tr>
<td>300</td>
<td>200</td>
<td>1445</td>
<td>1.3125</td>
<td>405</td>
</tr>
<tr>
<td>500</td>
<td>500</td>
<td>1561</td>
<td>0.7445</td>
<td>448</td>
</tr>
<tr>
<td>1000</td>
<td>1000</td>
<td>887</td>
<td>8.8188</td>
<td>255</td>
</tr>
<tr>
<td>2000</td>
<td>1000</td>
<td>957</td>
<td>18.0159</td>
<td>270</td>
</tr>
</tbody>
</table>
Figure 2. Relationship between error value and iteration times in Example 5.2 with $\gamma = 1.5, \lambda_1 = 0.05, k = 30, m = 60$.

Figure 3. Algorithm 3.1 with varying $\psi$.

Figure 4. Algorithm 4.1 with varying $\psi$. 
Remark 5.1. From Figures 1 and 2, we can see that the projection contraction algorithms based on golden ratio have numerical advantages over inertial extrapolation. Alternating extrapolation projection contraction algorithm is better than projection contraction algorithm based on golden ratio. Thus, it can be seen from Figures 3 and 4 that our algorithms with larger $\psi$ converges faster.

6. Conclusions

We present a projection contraction algorithm and an alternating extrapolation projection contraction algorithm based on the golden ratio for solving pseudo-monotone variational inequalities problem in real Hilbert spaces. We give proofs of weak convergence of the two algorithms when the operator is pseudo-monotone. Thus, we obtain R-linear convergence when $A$ is strongly pseudo-monotone mapping. We have extended the range of the $\psi$ from $\left(1, \frac{\sqrt{5}+1}{2}\right]$ to $(1, +\infty)$, and the proofs of both algorithms are given in the absence of Lipschitz constant. We give numerical examples and show the superiority of our algorithms. Then, we discover that our algorithms suffer less impact under the same unfavorable conditions and has a relatively stable rate of convergence.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References


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