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*Research article*

# Regularity of weak solution of the compressible Navier-Stokes equations with self-consistent Poisson equation by Moser iteration

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**Abstract:** In this paper, we consider the regularity of the weak solution to the compressible Navier-Stokes-Poisson equations in period domain $\Omega \subseteq \mathbb{R}^3$ provided that the density $\rho(t,x)$ with integrability on the space $L^\infty(0,T;L^{q_0}(\Omega))$ where $q_0$ satisfies a certain condition and $T > 0$, by which we could present that $\sup_{t,x}\rho(t,x) < \infty$ and $\inf_{t,x}\rho(t,x) > 0$. Furthermore, we develop the estimate for the velocity $\|u\|_{L^\infty}$ by the Moser iteration method and Gronwall inequality.

**Keywords:** regularity; Moser iteration; Gronwall inequality; Calderon-Zygmund
**Mathematics Subject Classification:** 35L65, 35Q35, 76N10

## 1. Introduction

In this article, we will investigate the regularity of weak solutions of the compressible Navier-Stokes-Poisson (NSP) equations. The compressible (NSP) system is used to simulate the motion of charged particles. It consists of the compressible Navier-Stokes (NS) equations with the electrostatic potential controlled by the self-consistent Poisson equation. A carrier type of NSP system is represented as the following form in $\mathbb{R}^3$:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \mu\Delta u - (\mu+\nu)\nabla \operatorname{div} u - \rho\nabla\Phi = \rho f, \\ \Delta\Phi = \rho - \bar{\rho}, \quad \lim_{|x|\to\infty} \Phi(t,x) = 0. \end{cases} \tag{1.1}$$

Here, $\rho = \rho(t,x) > 0$ is the density, $u(t,x) = (u_1,u_2,u_3)(t,x)$ is the velocity, $p(\rho)$ denotes the pressure and $\Phi$ denotes the electrostatic potential. $\bar{\rho} > 0$ is a constant. $\mu > 0$ and $\nu$ are constant viscous coefficients that satisfy $2\mu + 3\nu \geq 0$. The term $f$ denotes a given external force.

There are many conclusions about NS equations on the condition that $\nabla\Phi = 0$ in (1.1). In the past, people have studied the existence of weak solution to the compressible NS equations. P. L. Lions [12] proved the global existence of finite energy weak solution to NS equations when $p(\rho) = R\rho^\gamma, \gamma > \frac{9}{5}$ in 1998; Feireisl [6] claimed the existence of the weak solution when $p(\rho) = R\rho^\gamma, \gamma > \frac{3}{2}$ in 2001. Afterwards, Choe, Kim and Cho [2, 3] showed the local existence of the strong solution to NS equations. In 2011, Sun, Wang and Zhang [19] proved a blow-up criterion for a strong solution to the compressible viscous heat-conductive flows in $\mathbb{R}^3$ on condition that $\mu > \frac{1}{7}\nu$ in accordance with the upper bound of $(\rho, \rho^{-1}, \theta)$. Valli [24] proved the uniqueness of the solution if the data $\rho_0, u_0$ and $f$ satisfy some condition in bounded smooth domain $\Omega \subseteq \mathbb{R}^3$ . In 2013, Wen and Zhu [25] established a blow-up criterion for strong solutions to the Cauchy problem of compressible isentropic NS equations with vacuum provided $\mu > \frac{3}{29}\nu$ in the light of the integrability of the density. Though Choe [4] proved the regularity to the weak solution, which is defined by D. Hoff [9], it is indeterminate whether the weak solutions is the same to the finite energy weak solution.

However, there are not many conclusions related to the solution to NSP equation. In a series of articles [14, 15, 18], M. Okada et al. [17, 18] studied the vacuum problem of NSP equations which are spherically symmetrical with solid core. They [17] considered a free boundary problem for the equation of the one-dimensional isentropic motion with density-dependent viscosity and proved that there exists a unique weak solution globally in time; Then, Kobayashi and Suzuki [11] proved the existence of weak solution to NSP equations of which the method is similar to the method of [6]. In 2007, Zhang and Tan [27] proved the existence of the global weak solution to the compressible NSP equations where $p(\rho) = a\rho\log^d\rho$ for sufficiently large densities by using suitable Orlicz spaces in $\mathbb{R}^2$ . We may refer to [1, 7, 20–22, 26] for more about NS or NSP equations. As far as we are concerned, to the author's best knowledge, there are not many studies on the regularity for the corresponding weak solution to the compressible NSP equations.

For simplicity, let $f \equiv 0$ of (1.1) in the paper. Thus, the problem we are concerned with in our study is the regularity of weak solutions to the isentropic compressible NSP equations in the periodic domain $\Omega = \mathbb{R}^3 - \mathbb{Z}^3$ as

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \quad \text{for } (t, x) \in (0, T) \times \Omega, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu\Delta u - (\mu + \nu)\nabla \operatorname{div} u + \nabla p(\rho) - \rho\nabla\Phi = 0, \\ \Delta\Phi = \rho - \bar{\rho}, \end{cases} \tag{1.2}$$

with the following initial conditions:

$$\begin{cases} \rho(t, x)|_{t=0} = \rho_0 > 0, \quad u(t, x)|_{t=0} = u_0, \ x \in \Omega, \\ \rho_0 \in W^{1,6}(\Omega), \quad u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ 0 < m \le \rho_0 \le M < +\infty, \end{cases} \tag{1.3}$$

where $\overline{\rho} > 0$ is a constant, $p(\rho) = R\rho^\gamma, R > 0, \gamma > 1$ and $m$ and $M$ both are constants.

**Definition 1.1.** *$(\rho, u)$ is called the finite energy weak solution of the Eq (1.2) for $(t, x) \in \Omega_T = (0, T)\times\Omega$, if $(\rho, u)$ satisfies*

$$\rho \in L^\infty(0, T; L^1(\Omega)) \cap L^\infty(0, T; L^\gamma(\Omega)), \quad u \in L^2(0, T; H^1(\Omega))$$

*and the energy inequality*

$$\frac{d\mathcal{E}(t)}{dt} + \mu \int_\Omega |\nabla u|^2 dx + (\mu + \nu) \int_\Omega |\operatorname{div} u|^2 dx \le 0,$$

*holds in* $\mathcal{D}'((0,\infty))$ *with the finite total energy, where*

$$\mathcal{E}(t) = \int_\Omega \Big(\frac{1}{2}\rho|u|^2 + \frac{p(\rho)}{\gamma - 1} + \frac{1}{2}|\nabla\Phi|^2\Big)dx < \infty, \quad \text{for } t \in (0,\infty).$$

All the priori estimates of this paper depend on the assumption that $\rho$ and $u$ are $C^\infty$ for the time interval as mentioned in [4, 5].

**Theorem 1.1.** *Suppose* $(\rho, u)$ *is the weak solutions denoted by the Definition 1.1 to the Eqs* (1.2) *and* (1.3). *If* $\rho \in L^\infty(0,T; L^{q_0}(\Omega))$ *for time* $T > 0$ *and for* $q_0$ *depending only on* $\gamma$*, then*

$$0 < \inf_{\Omega_T} \rho \le \rho \le \sup_{\Omega_T} \rho < +\infty, \ \sup_{\Omega_T} |u| < +\infty.$$

*Furthermore, we show that* $\rho \in C(0,T; L^q(\Omega)) \cap L^\infty(\Omega_T)$ *for all* $q \in [1,\infty)$*, and* $\sqrt{\rho}u_t \in L^2(\Omega_T), \mathcal{P}u \in L^2(0,T; H^2(\Omega)), G \in L^2(0,T; H^2(\Omega)), \nabla u \in L^\infty(0,T; L^2(\Omega))$*, where* $\mathcal{P}u$ *refers to the divergence-free part of* $u$ *,* $G = (2\mu + \nu)\operatorname{div} u - p(\rho)$*.*

The paper is structured as follows. In Section 2, we give some preliminaries to study the regularity of the weak solution to the NSP equations. In Section 3, one develops the priori estimate of $\rho$ under some condition by many techniques; In Section 4, we shall perfect the proof of Section 3; Finally, in Section 5, we present a prior $L^\infty$ estimate of $u$ by Moser iteration.

## 2. Preliminary

In this section, we collect some auxiliary results which will be used to investigate subsequent studies.

First, we define the divergence-free part of any vector field $u$ as $\mathcal{P}u$, and the gradient part of $u$ as $\mathcal{Q}u$ which means $\mathcal{Q}u = \nabla\Delta^{-1} \operatorname{div} u$. Naturally, the following propositions hold.

**Proposition 2.1.**

$$\operatorname{curl}(\mathcal{Q}\mathrm{u}) = 0, \ \operatorname{div}(\mathcal{P}\mathrm{u}) = 0, \ \Delta\mathrm{u} = \nabla(\operatorname{div}\mathrm{u}) - \operatorname{curl}\operatorname{curl}\mathrm{u}, \ \mathcal{P}\mathrm{u} = \mathrm{u} - \mathcal{Q}\mathrm{u}.$$

**Proposition 2.2.** *(Page 67 of [13]). Suppose that* $f \in L^2(\Omega)$*, and*

$$\begin{cases} \Delta v = \nabla p + f, \quad x \in \Omega, \\ \operatorname{div} v = 0, \\ v|_{\partial\Omega} = 0, \end{cases}$$

*then* $v \in W^{2,2}(\Omega)$*, and* $\nabla p \in L^2(\Omega)$*. Furthermore,*

$$\|v\|_{W^{2,2}(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

**Proposition 2.3.** *Suppose that $\rho \in L^\infty(0,T;L^{p_0}(\Omega))$ for $p_0 = \max\{\frac{(2\gamma-1)\alpha+4\gamma}{2},4\}$, where $\alpha$ is a nonnegative real number. Then, we obtain*

$$\int_\Omega \rho|u|^{\alpha+2}dx \lesssim \int_\Omega \rho_0|u_0|^{\alpha+2}dx + \sup_{t\in[0,T]}\|\rho\|^{\frac{(2\gamma-1)\alpha+4\gamma}{2}}_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{2}}(\Omega)} t^{\frac{\alpha+2}{2}} + \|\rho-\overline{\rho}\|^{\alpha+2}_{L^4(\Omega)}\sup_{t\in[0,T]}\|\rho\|_{L^1(\Omega)}t^{\alpha+2}.$$

*Proof.* Let $|u|^\alpha u$ as a test function to (1.2), then one obtain

$$\begin{aligned}\frac{1}{\alpha+2}\frac{d}{dt}\int_\Omega \rho|u|^{\alpha+2}dx + \mu(\alpha+1)\int_\Omega |u|^\alpha|\nabla u|^2dx + \alpha(\mu+\nu)\int_\Omega |u|^\alpha \operatorname{div} u\cdot\nabla u dx + (\mu+\nu)\int_\Omega |\operatorname{div} u|^2|u|^\alpha dx\\ = \int_\Omega R\rho^\gamma|u|^\alpha \operatorname{div} u dx + \alpha\int_\Omega R\rho^\gamma|u|^\alpha\nabla u dx + \int_\Omega \rho\nabla\Phi|u|^\alpha u dx. \qquad (2.1)\end{aligned}$$

Applying Hölder's inequality and Young's inequality to (2.1), implies that

$$\int_\Omega R\rho^\gamma|u|^\alpha \operatorname{div} u dx \lesssim \frac{1}{2}\int_\Omega |u|^\alpha|\nabla u|^2dx + 2R^2\sup_{t\in[0,T]}\|\rho\|^{\frac{(2\gamma-1)\alpha+4\gamma}{\alpha+2}}_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{2}}}\Big(\int_\Omega \rho|u|^{\alpha+2}dx\Big)^{\frac{\alpha}{\alpha+2}}. \qquad (2.2)$$

Similarly, we obtain

$$\alpha\int_\Omega R\rho^\gamma|u|^\alpha\nabla u dx \lesssim \frac{1}{2}\int_\Omega |u|^\alpha|\nabla u|^2dx + 2\alpha^2R^2\sup_{t\in[0,T]}\|\rho\|^{\frac{(2\gamma-1)\alpha+4\gamma}{\alpha+2}}_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{2}}}\Big(\int_\Omega \rho|u|^{\alpha+2}dx\Big)^{\frac{\alpha}{\alpha+2}}. \qquad (2.3)$$

Noting that $\Delta\Phi = \rho-\overline{\rho}$, and using Hölder's inequality and Sobolev inequality, we could have

$$\int_\Omega \rho\nabla\Phi|u|^\alpha u dx \lesssim \|\rho-\overline{\rho}\|_{L^4(\Omega)}\sup_{t\in[0,T]}\|\rho\|^{\frac{1}{\alpha+2}}_{L^1(\Omega)}\Big(\int_\Omega \rho|u|^{\alpha+2}dx\Big)^{\frac{\alpha+1}{\alpha+2}}. \qquad (2.4)$$

Combining (2.2)–(2.4) with (2.1), and letting $U(t) = \int_\Omega \rho|u|^{\alpha+2}dx$, one could have

$$\frac{d}{dt}U(t) \lesssim \sup_{t\in[0,T]}\|\rho\|^{\frac{(2\gamma-1)\alpha+4\gamma}{\alpha+2}}_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{2}}(\Omega)}[U(t)]^{\frac{\alpha}{\alpha+2}} + \|\rho-\overline{\rho}\|_{L^4(\Omega)}\sup_{t\in[0,T]}\|\rho\|^{\frac{1}{\alpha+2}}_{L^1(\Omega)}[U(t)]^{\frac{\alpha+1}{\alpha+2}}.$$

This reduces to

$$U(t) \lesssim U(0) + \sup_{t\in[0,T]}\|\rho\|^{\frac{(2\gamma-1)\alpha+4\gamma}{2}}_{L^{\frac{(2\gamma-1)\alpha+4\gamma}{2}}(\Omega)} t^{\frac{\alpha+2}{2}} + \|\rho-\overline{\rho}\|^{\alpha+2}_{L^4(\Omega)}\sup_{t\in[0,T]}\|\rho\|_{L^1(\Omega)}t^{\alpha+2}.$$

□

## 3. A priori estimates for density $\rho$

### 3.1. The estimates for $\mathcal{P}(H)$, $\mathcal{Q}(H)$ and $G$

Now, applying the operator "divergence-free fields" $\mathcal{P}$ to (1.2) , we consequently obtain that

$$\mathcal{P}(\rho u_t + \rho u\cdot\nabla u - \rho\nabla\Phi) - \mu\Delta\mathcal{P}u = 0. \qquad (3.1)$$

Denote $H = \rho u_t + \rho u\cdot\nabla u - \rho\nabla\Phi$, then (3.1) will be reformulated equivalently as

$$\mathcal{P}(H) - \mu\Delta\mathcal{P}u = 0. \qquad (3.2)$$

Then, by taking divergence operator to the Eq (1.2), one deduces that

$$\operatorname{div}(\rho u_t + \rho u \cdot \nabla u - \rho \nabla \Phi) - \Delta((2\mu + \nu) \operatorname{div} u - p(\rho)) = 0. \tag{3.3}$$

Denote $G = (2\mu + \nu) \operatorname{div} u - p(\rho)$, and we have again by formula (3.3) that

$$G = \Delta^{-1} \operatorname{div}(\rho u_t + \rho u \cdot \nabla u - \rho \nabla \Phi). \tag{3.4}$$

Note that the definition of $\mathcal{Q}u = \nabla \Delta^{-1} \operatorname{div} u$, then (3.4) will be reformulated equivalently as

$$\mathcal{Q}(H) - \nabla G = 0. \tag{3.5}$$

By virtue of (3.2) and (3.5), we get

$$\begin{cases} \mu \Delta \mathcal{P}u + \nabla G = \mathcal{P}(H) + \mathcal{Q}(H), \\ \operatorname{div}(\mathcal{P}u) = 0. \end{cases}$$

Note that $H = \mathcal{P}(H) + \mathcal{Q}(H)$, and from Proposition 2.2 and the Stokes problem (see 2.2 The stokes problem and the operator A of Chapter 2 in Temam [23]), it follows that

$$\|\nabla^2 \mathcal{P}u\|_{L^r(\Omega)} + \|\nabla G\|_{L^r(\Omega)} \leq \|H\|_{L^r(\Omega)}, \quad \text{for} \quad 1 < r < \infty. \tag{3.6}$$

Observe that $H = \rho u_t + \rho u \cdot \nabla u - \rho \nabla \Phi$, then clearly we have

$$\|H\|_{L^r(\Omega)} \leq c(\|\rho u_t\|_{L^r(\Omega)} + \|\rho u \cdot \nabla u\|_{L^r(\Omega)} + \|\rho \nabla \Phi\|_{L^r(\Omega)}), \quad \text{for} \quad 1 < r < \infty, \tag{3.7}$$

where $c = c(N)$, and $N$ denotes the dimension.
By virtue of Proposition 2.1, we can get

$$\|\nabla u\|_{L^r} \leq \|\nabla \mathcal{P}u\|_{L^r} + \|\operatorname{div} u\|_{L^r}, \quad \text{for} \quad 1 < r < \infty. \tag{3.8}$$

From the Gagliardo-Nirenberg-Sobolev (G-N-S) inequality, we have

$$\|u\|_{L^{\frac{3r}{3-r}}} \leq \|\nabla u\|_{L^r}, \quad \text{for} \quad 1 < r < 3. \tag{3.9}$$

From $G = (2\mu + \nu) \operatorname{div} u - p(\rho)$, we obtain

$$\|\operatorname{div} u\|_{L^r} \leq \frac{1}{2\mu + \nu}(\|G\|_{L^r} + R\|\rho\|^{\gamma}_{L^{\gamma r}}). \tag{3.10}$$

Combining (3.8)–(3.10), immediately we have

$$\|\nabla u\|_{L^{\frac{3r}{3-r}}(\Omega)} \leq \|\nabla^2 \mathcal{P}u\|_{L^r(\Omega)} + \frac{1}{2\mu + \nu}(\|\nabla G\|_{L^r(\Omega)} + R\|\rho\|^{\gamma}_{L^{\frac{3r\gamma}{3-r}}}). \tag{3.11}$$

*3.2. The estimates for* $\sup_{\Omega_T}\rho(x,t)$*, and* $\inf_{\Omega_T}\rho(x,t)$

**Lemma 3.1.** *Suppose that* $\rho_0 \in L^\infty(\Omega)$ *and* $u_0 \in H^1(\Omega)$*, and assume that* $\nabla^2\mathcal{P}u$ *and* $\nabla G$ *is bounded, and* $\rho \in L^\infty(0,T;L^{p_1}(\Omega))$*, where* $p_1 = \max\{p_0, 39, 5\gamma, 40\gamma - 19\}$*. Then we could obtain that*

$$\sup_{\Omega_T}\rho(x,t) < \infty, \quad \inf_{\Omega_T}\rho(x,t) > 0.$$

*Proof.* From $\rho_t + \operatorname{div}(\rho u) = 0$, we get the following relations

$$\frac{d\rho(x,t)}{dt} = -\rho(x,t)\operatorname{div} u(x,t).$$

We can rewrite it in differential notation and integrate the equality above from 0 to $t$, then we have

$$\ln\rho(x,t) = \ln\rho_0 - \int_0^t \operatorname{div} u(x,s)ds.$$

Using the equation $G = (2\mu+\nu)\operatorname{div} u - p(\rho)$, one deduces that

$$\ln\rho(x,t) = \ln\rho_0 - \frac{1}{2\mu+\nu}\int_0^t (G + R\rho^\gamma)ds. \tag{3.12}$$

Note that $G = \Delta^{-1}\operatorname{div}\Big((\rho u)_t + \operatorname{div}(\rho u\otimes u) - \rho\nabla\Phi\Big)$. Now, considering the term with respect to $G$ of (3.12), we can infer that

$$\begin{aligned}\int_0^t G\,ds &= \int_0^t \Delta^{-1}\operatorname{div}\left((\rho u)_s + \operatorname{div}(\rho u\otimes u) - \rho\nabla\Phi\right)ds\\ &= \int_0^t (\Delta^{-1}\operatorname{div}((\rho u)_s)ds + \int_0^t \Delta^{-1}\operatorname{div}(\operatorname{div}(\rho u\otimes u))ds - \int_0^t \Delta^{-1}\operatorname{div}(\rho\nabla\Phi)ds.\end{aligned}$$

Note that

$$\int_0^t (\Delta^{-1}\operatorname{div}(\rho u)_s)ds = \Delta^{-1}\operatorname{div}(\rho u) - \Delta^{-1}\operatorname{div}(\rho_0 u_0) - \int_0^t u\cdot\nabla\Delta^{-1}\operatorname{div}(\rho u)ds.$$

Then, one obtains

$$\begin{aligned}\ln\rho(t,x) = {}& \ln\rho_0 - \frac{1}{2\mu+\nu}\Big(\Delta^{-1}\operatorname{div}(\rho u)(x,t) - \Delta^{-1}\operatorname{div}(\rho_0 u_0)\Big) - \frac{1}{2\mu+\nu}\int_0^t R\rho^\gamma ds\\ &+ \frac{1}{2\mu+\nu}\int_0^t \Delta^{-1}\operatorname{div}(\rho\nabla\Phi)ds - \frac{1}{2\mu+\nu}\int_0^t \Big(\Delta^{-1}\operatorname{div}^2\ (\rho u\otimes u) - u\cdot\nabla\Delta^{-1}\operatorname{div}(\rho u)\Big)ds\\ &\triangleq \ln\rho_0 - \frac{1}{2\mu+\nu}\Big(\int_0^t R\rho^\gamma ds + (A_0 - a_0) + \int_0^t (-A_1 + A_2)ds\Big),\end{aligned} \tag{3.13}$$

where $A_0 = \Delta^{-1}\operatorname{div}(\rho u)$, $a_0 = \Delta^{-1}\operatorname{div}(\rho_0 u_0)$, $A_1 = \Delta^{-1}\operatorname{div}(\rho\nabla\Phi)$, $A_2 = \Delta^{-1}\operatorname{div}^2(\rho u\otimes u) - u\cdot\nabla\Delta^{-1}\operatorname{div}(\rho u)$, and $\operatorname{div}^2$ is an operator defined by $\operatorname{div}^2\ M = \partial_{ij}M_{ij}$ for a $3\times 3$ matrix $M = (M_{ij})$.

Therefore, we have

$$\ln\rho(x,t) \lesssim \ln\|\rho_0\|_{L^\infty(\Omega)} + \|A_0 - a_0\|_{L^\infty(\Omega)} + \int_0^t (\|A_1(\cdot,s)\|_{L^\infty(\Omega)} + \|A_2(\cdot,s)\|_{L^\infty(\Omega)})ds, \tag{3.14}$$

$$\ln \rho(x,t) \gtrsim \ln(\inf_{\Omega} \rho_0) - \int_0^t R\|\rho\|^{\gamma}_{L^\infty(\Omega)} ds - \|A_0 - a_0\|_{L^\infty(\Omega)} - \int_0^t (\|A_1(\cdot,s)\|_{L^\infty(\Omega)} + \|A_2(\cdot,s)\|_{L^\infty(\Omega)}) ds. \tag{3.15}$$

Using (3.14), one could obtain that

$$\ln \rho(x,t) - \ln \|\rho_0\|_{L^\infty(\Omega)} \lesssim \|A_0 - a_0\|_{L^\infty(\Omega)} + \int_0^t (\|A_1(\cdot,s)\|_{L^\infty(\Omega)} + \|A_2(\cdot,s)\|_{L^\infty(\Omega)}) ds \triangleq \mathcal{A}(t).$$

Hence, one yields that

$$\rho(t,x) \leq \|\rho_0\|_{L^\infty(\Omega)} \exp\{\mathcal{A}\}.$$

From above inequality, (3.15) implies that

$$\ln \rho(t,x) \geq \ln(\inf_{\Omega} \rho_0) - \mathcal{A} - \int_0^t R\|\rho_0\|^{\gamma}_{L^\infty(\Omega)} \exp\{\gamma \mathcal{A}\} ds.$$

Therefore

$$\rho \geq (\inf_{\Omega} \rho_0) \exp\Big\{ - \mathcal{A} - \int_0^t R\|\rho_0\|^{\gamma}_{L^\infty(\Omega)} \exp(\gamma \mathcal{A}) ds \Big\}.$$

Obviously, once one proves $\sup\limits_{t\in[0,T]} \mathcal{A}(t) < \infty$, we may obtain

$$\sup_{\Omega_T} \rho(x,t) < \infty, \text{and } \inf_{\Omega_T} \rho(x,t) > 0.$$

Therefore, it's necessary to study the estimates of $\mathcal{A}(t)$.

### *3.3. The estimates for $\mathcal{A}(t)$*

First, observe $A_0 = \Delta^{-1} \operatorname{div}(\rho u), A_1 = \Delta^{-1} \operatorname{div}(\rho \nabla \Phi), A_2 = \Delta^{-1} \operatorname{div}^2(\rho u \otimes u) - u \cdot \nabla \Delta^{-1} \operatorname{div}(\rho u)$, then we obtain

$$\Delta A_0 = \operatorname{div}(\rho u), \Delta A_1 = \operatorname{div}(\rho \nabla \Phi).$$

Meanwhile, we have

$$\Delta A_2 = \operatorname{div}(\Delta A_0 \cdot u + \rho(u \cdot \nabla) u + \nabla(u \cdot \nabla A_0)).$$

Using the Calderon-Zygmund theorem (Theorem 9.9 of [8]), we get

$$\begin{aligned}
&\|\nabla A_0\|_{L^p(\Omega)} \leq c\|\rho u\|_{L^p(\Omega)},\\
&\|\nabla A_1\|_{L^p(\Omega)} \leq c\|\rho \nabla \Phi\|_{L^p(\Omega)},\\
&\|\nabla A_2\|_{L^p(\Omega)} \leq c\|(\rho|u| + |\nabla A_0|)|\nabla u|\|_{L^p(\Omega)},
\end{aligned}$$

where $1 < p < \infty$, $c$ depends on $p$. For all $v \in W^{1,4}(\Omega)$, one yields

$$\|v\|_{L^\infty(\Omega)} \leq c\|\nabla v\|_{L^4(\Omega)}.$$

Hence, we have

$$\|A_0\|_{L^\infty(\Omega)} \lesssim \|\nabla A_0\|_{L^4(\Omega)} \lesssim \|\rho u\|_{L^4(\Omega)} \lesssim \|\rho\|_{L^7(\Omega)}^{\frac{7}{8}} \cdot \|\rho^{\frac{1}{8}} u\|_{L^8(\Omega)}. \tag{3.16}$$

Applying the same method of estimating $\|A_0\|_{L^\infty(\Omega)}$ to estimate $\|A_1\|_{L^\infty(\Omega)}$, we obtain that

$$\|A_1\|_{L^\infty(\Omega)} \lesssim \|\nabla A_1\|_{L^4(\Omega)} \lesssim \|\rho\nabla\Phi\|_{L^4(\Omega)} \lesssim \|\rho\|_{L^8(\Omega)} \cdot \|\nabla\Phi\|_{L^8(\Omega)}.$$

According to $\Delta\Phi = \rho - \overline{\rho}$ and the Calderon-Zygmung inequality [16], we have

$$\|\nabla\Phi\|_{L^{\frac{3\beta}{3-\beta}}(\Omega)} \lesssim \|\rho - \overline{\rho}\|_{L^\beta}, \quad (\beta > 1).$$

Moreover, we estimate that

$$\|A_1\|_{L^\infty(\Omega)} \lesssim \|\rho\|_{L^8(\Omega)} \cdot \|\rho - \overline{\rho}\|_{L^{\frac{24}{11}}(\Omega)} \lesssim \|\rho\|_{L^8(\Omega)}^2 + \|\rho\|_{L^8(\Omega)}. \tag{3.17}$$

Similarly for $A_2$, we could obtain

$$\begin{aligned}
\|A_2\|_{L^\infty(\Omega)} \lesssim \|\nabla A_2\|_{L^4(\Omega)} &\lesssim \|(\rho|u| + |\nabla A_0|)|\nabla u|\|_{L^4(\Omega)} \\
&\lesssim \|(\rho|u| + |\nabla A_0|)\|_{L^{20}(\Omega)} \cdot \|\nabla u\|_{L^5(\Omega)} \\
&\lesssim \|\rho u\|_{L^{20}(\Omega)}^2 + \|\nabla u\|_{L^5(\Omega)}^2 \\
&\triangleq B_1^2 + B_2^2.
\end{aligned} \tag{3.18}$$

Applying Hölder's inequality to $B_1$, we obtain

$$B_1 \lesssim \|\rho u\|_{L^{20}(\Omega)} \lesssim \|\rho\|_{L^{39}(\Omega)}^{\frac{39}{40}} \cdot \|\rho^{\frac{1}{40}} u\|_{L^{40}(\Omega)}.$$

Similarly, by using the Calderon-Zygmung inequality and Hölder's inequality, one has

$$\begin{aligned}
B_2 &\lesssim \|\nabla Pu\|_{L^5(\Omega)} + \|\operatorname{div} u\|_{L^5(\Omega)} \\
&\lesssim \|\nabla Pu\|_{L^5(\Omega)} + (\|G\|_{L^5(\Omega)} + R\|\rho\|_{L^{5\gamma}(\Omega)}^\gamma) \\
&\lesssim \|\nabla^2 Pu\|_{L^{\frac{15}{8}}(\Omega)} + \|\nabla G\|_{L^{\frac{15}{8}}(\Omega)} + R\|\rho\|_{L^{5\gamma}(\Omega)}^\gamma.
\end{aligned}$$

Hence, we get the estimate of $A_2$ by Young inequality,

$$\|A_2\|_{L^\infty(\Omega)} \lesssim \|\rho\|_{L^{39}(\Omega)}^{\frac{39}{10}} + \|\rho^{\frac{1}{40}} u\|_{L^{40}(\Omega)}^4 + \|\nabla^2 Pu\|_{L^{\frac{15}{8}}(\Omega)}^2 + \|\nabla G\|_{L^{\frac{15}{8}}(\Omega)}^2 + R\|\rho\|_{L^{5\gamma}(\Omega)}^{2\gamma}. \tag{3.19}$$

Therefore, combining (3.16), (3.17) and (3.19), one could have

$$\begin{aligned}
\mathcal{A} \lesssim &\int_0^t \|\rho\|_{L^8(\Omega)} \cdot \|\rho - \overline{\rho}\|_{L^{\frac{24}{11}}(\Omega)} ds \\
&+ \int_0^t \Big\{\|\rho^{\frac{1}{40}} u\|_{L^{40}(\Omega)}^4 + \|\rho\|_{L^{39}(\Omega)}^{\frac{39}{10}} + \|\nabla^2 \mathcal{P} u\|_{L^{\frac{15}{8}}(\Omega)}^2 + \|\nabla G\|_{L^{\frac{15}{8}}(\Omega)}^2 + R\|\rho\|_{L^{5\gamma}(\Omega)}^{2\gamma}\Big\} ds \\
&+ \|\rho\|_{L^7(\Omega)}^{\frac{7}{8}} \cdot \|\rho^{\frac{1}{8}} u\|_{L^8(\Omega)}.
\end{aligned} \tag{3.20}$$

Observe Proposition 2.3 and that if $\nabla^2\mathcal{P}u$ and $\nabla G$ is bounded which we will prove in the Lemma 4.3 of the next section, then we could obtain that $\sup_{t\in[0,T]} \mathcal{A}(t) < \infty$. □

## 4. A priori bounds for $\nabla^2\mathcal{P}u$ and $\nabla G$

**Lemma 4.1.** *Let* $1 < r < 2$*,* $\rho \in L^\infty(0,T;L^{p_2}(\Omega))$*, and* $p_2(r) = \max\{p_0, p_1, \frac{5r-2}{2-r}, \frac{r}{2-r}, 4\}$*, then*

$$\int_0^T (\|\nabla^2\mathcal{P}u\|^2_{L^r(\Omega)} + \|\nabla G\|^2_{L^r(\Omega)})dt$$
$$\lesssim \sup_{t\in[0,T]} \|\rho\|_{L^{\frac{r}{2-r}}(\Omega)} \iint_{\Omega_T} \rho|u_t|^2 dxdt + \sup_{t\in[0,T]} \Big\{\|\rho\|^{\frac{5r-2}{2r}}_{L^{\frac{5r-2}{2-r}}(\Omega)} \Big(\int_\Omega \rho|u|^{\frac{4r}{2-r}}\Big)^{\frac{2-r}{2r}} \|\nabla u\|^2_{L^2(\Omega)}\Big\} + \sup_{t\in[0,T]} \|\rho\|_{L^r(\Omega)}\|\rho-\overline{\rho}\|_{L^4(\Omega)}.$$

*Proof.* Using (3.6) and (3.7), we obtain

$$\|\nabla^2\mathcal{P}u\|_{L^r(\Omega)} + \|\nabla G\|_{L^r(\Omega)} \le c(\|\rho u_t\|_{L^r(\Omega)} + \|\rho u\cdot\nabla u\|_{L^r(\Omega)} + \|\rho\nabla\Phi\|_{L^r(\Omega)}),$$

where $1 < r < \infty$, $c = c(N)$ and $N$ is the dimension. For $1 < r < 2$, it follows that

$$\|\rho u_t\|_{L^r(\Omega)} \lesssim \|\rho\|^{\frac12}_{L^{\frac{r}{2-r}}(\Omega)} \|\sqrt{\rho}u_t\|_{L^2(\Omega)},$$
$$\|\rho u\cdot\nabla u\|_{L^r(\Omega)} \lesssim \|\rho\|^{\frac{5r-2}{4r}}_{L^{\frac{5r-2}{2-r}}(\Omega)} \|\rho^{\frac{2-r}{4r}}u\|_{L^{\frac{4r}{2-r}}(\Omega)} \|\nabla u\|_{L^2(\Omega)}.$$

According to $\Delta\Phi = \rho-\overline{\rho}$, we obtain

$$\|\rho\nabla\Phi\|_{L^r(\Omega)} \lesssim \|\rho\|_{L^r(\Omega)}\|\nabla\Phi\|_{L^\infty(\Omega)} \lesssim \|\rho\|_{L^r(\Omega)}\|\Delta\Phi\|_{L^4(\Omega)} \lesssim \|\rho\|_{L^r(\Omega)}\|\rho-\overline{\rho}\|_{L^4(\Omega)}.$$

Hence, if $1 < r < 2$, we obtain

$$\begin{aligned}&\|\nabla^2\mathcal{P}u\|^2_{L^r(\Omega)} + \|\nabla G\|^2_{L^r(\Omega)} \\ \lesssim&\|\rho\|_{L^{\frac{r}{2-r}}(\Omega)}\|\sqrt{\rho}u_t\|^2_{L^2(\Omega)} + \|\rho\|^{\frac{5r-2}{2r}}_{L^{\frac{5r-2}{2-r}}(\Omega)} \|\rho^{\frac{2-r}{4r}}u\|_{L^{\frac{4r}{2-r}}(\Omega)}\|\nabla u\|^2_{L^2(\Omega)} + \|\rho\|^2_{L^r(\Omega)}\|\rho-\overline{\rho}\|^2_{L^4(\Omega)}.\end{aligned} \tag{4.1}$$

Hence, integrating (4.1) over $[0,T]$, we could get

$$\int_0^T (\|\nabla^2\mathcal{P}u\|^2_{L^r(\Omega)} + \|\nabla G\|^2_{L^r(\Omega)})dt$$
$$\lesssim \int_0^T \Big\{\|\rho\|_{L^{\frac{r}{2-r}}(\Omega)}\|\sqrt{\rho}u_t\|^2_{L^2(\Omega)} + \|\rho\|^{\frac{5r-2}{2r}}_{L^{\frac{5r-2}{2-r}}(\Omega)} \|\rho^{\frac{2-r}{4r}}u\|^2_{L^{\frac{4r}{2-r}}(\Omega)}\|\nabla u\|^2_{L^2(\Omega)} + \|\rho\|^2_{L^r(\Omega)}\|\rho-\overline{\rho}\|^2_{L^4(\Omega)}\Big\}dt.$$

□

**Lemma 4.2.** *Let* $\frac65 < r < 2$*, and* $p_3(r) = \max\{4\gamma, \frac{(2\gamma-1)r+1}{r-1}, \frac{6\gamma r}{6-5r}, \frac{3\gamma r}{3-r}, \frac{3}{2r-3}, \frac{2\gamma-1}{r-1} + 2\gamma, \frac{3(2\gamma-1)}{2r-3} + 2\gamma\}$*. Here we assume* $\rho_0 \in L^\infty(0,T;L^{p_3}(\Omega))$*. Then, one chould obtain that*

$$\int_0^T\int_\Omega \rho|u_t|^2(t,x)dxdt + \mu\sup_{t\in[0,T]}\int_\Omega |\nabla u(t,x)|^2dx + \frac{1}{\mu^2}\int_0^T\int_\Omega (R^3(\gamma-1)\rho^{3\gamma})dxdt$$
$$\le 3\varepsilon\int_0^T \Big(\|\nabla^2\mathcal{P}u\|^2_{L^r(\Omega)} + \|\nabla G\|^2_{L^r(\Omega)}\Big)dt + C, \quad \text{for any } \varepsilon > 0,$$

*where* $C = C(\mu, \nu, R, \|u_0\|_{H^1(\Omega)}, \|\rho_0\|_{L^{2\gamma}(\Omega)}, \sup_{t\in[0,T]}\|\rho\|_{L^{p_3}(\Omega)})$.

*Proof.* Multiply $u_t$ to $(1.2)_2$ and integrate over $\Omega_T$, then we could obtain that

$$\begin{aligned}
&\int_0^T\int_\Omega |\rho u_t^2|\,dxdt + \frac{\mu}{2}\sup_{t\in[0,T]}\int_\Omega |\nabla u|^2\,dx + \frac{\mu+\nu}{2}\sup_{t\in[0,T]}\int_\Omega |\operatorname{div} u|^2\,dx + \int_0^T\int_\Omega \nabla p(\rho)\cdot u_t\,dxdt\\
\le&\int_0^T\int_\Omega |\rho\nabla\Phi\cdot u_t|\,dxdt + \frac{\mu}{2}\int_\Omega |\nabla u_0|^2\,dx + \frac{\mu+\nu}{2}\int_\Omega |\operatorname{div} u_0|^2\,dx + \int_0^T\int_\Omega |\rho(u\cdot\nabla u)\cdot u_t|\,dxdt\\
\le&\int_0^T\int_\Omega |\rho\nabla\Phi\cdot u_t|\,dxdt + \frac{\mu}{2}\int_\Omega |\nabla u_0|^2\,dx + \frac{\mu+\nu}{2}\int_\Omega |\operatorname{div} u_0|^2\,dx\\
&+\frac{1}{2}\int_0^T\int_\Omega |\rho u_t^2|\,dxdt + \frac{1}{2}\int_0^T\int_\Omega |\rho u^2(\nabla u)^2|\,dxdt.
\end{aligned}\tag{4.2}$$

Therefore, we have

$$\begin{aligned}
&\int_0^T\int_\Omega |\rho u_t^2|\,dxdt + \mu\sup_{t\in[0,T]}\int_\Omega |\nabla u|^2\,dx + (\mu+\nu)\sup_{t\in[0,T]}\int_\Omega |\operatorname{div} u|^2\,dx + 2\int_0^T\int_\Omega \nabla p(\rho)\cdot u_t\,dxdt\\
\le&2\int_0^T\int_\Omega |\rho\nabla\Phi\cdot u_t\,|dxdt + \mu\int_\Omega |\nabla u_0|^2\,dx + (\mu+\nu)\int_\Omega |\operatorname{div} u_0|^2\,dx + \int_0^T\int_\Omega |\rho u^2|\nabla u|^2|\,dxdt.
\end{aligned}\tag{4.3}$$

Set

$$\mathcal{I} = \int_\Omega \nabla p(\rho)\cdot u_t\,dx, \quad \mathcal{J} = \int_\Omega \rho\nabla\Phi\cdot u_t\,dx.$$

Noting that $\rho_t + \operatorname{div}(\rho u) = 0$, one obtains that

$$(R\rho^\gamma)_t = R\gamma\rho^{\gamma-1}\rho_t = -R\gamma\rho^{\gamma-1}\operatorname{div}(\rho u) = -u\cdot\nabla(R\rho^\gamma) - R\gamma\rho^\gamma\operatorname{div} u.$$

Note that $G = (2\mu+\nu)\operatorname{div} u - p(\rho)$, we deduce that

$$\mathcal{I} = \int_\Omega \nabla(R\rho^\gamma)\cdot u_t dx = -\frac{d}{dt}\int_\Omega R\rho^\gamma\operatorname{div} u dx + \int_\Omega (R\rho^\gamma)_t\operatorname{div} u dx.\tag{4.4}$$

Further, one obtains

$$\begin{aligned}
&\int_\Omega (R\rho^\gamma)_t\operatorname{div} u dx\\
=&-\int_\Omega u\cdot\nabla(R\rho^\gamma)\operatorname{div} u - \int_\Omega R\gamma\rho^\gamma(\operatorname{div} u)^2 dx\\
=&-\int_\Omega (u\cdot\nabla(R\rho^\gamma)\operatorname{div} u + R\rho^\gamma(\operatorname{div} u)^2)dx - R(\gamma-1)\int_\Omega \rho^\gamma(\operatorname{div} u)^2 dx\\
=&-\int_\Omega \operatorname{div}(R\rho^\gamma u)\operatorname{div} u dx - R(\gamma-1)\int_\Omega \rho^\gamma(\operatorname{div} u)^2 dx\\
=&\frac{R}{2\mu+\nu}\int_\Omega \rho^\gamma u\cdot\nabla(G+R\rho^\gamma)dx - \frac{R(\gamma-1)}{(2\mu+\nu)^2}\int_\Omega \rho^\gamma(G^2 - R^2\rho^{2\gamma} + 2(2\mu+\nu)R\rho^\gamma\operatorname{div} u)dx\\
=&\frac{R}{2\mu+\nu}\int_\Omega \rho^\gamma u\cdot\nabla G dx + \frac{R}{2\mu+\nu}\int_\Omega \rho^\gamma u\cdot\nabla(R\rho^\gamma)dx\\
&\quad - \frac{R(\gamma-1)}{(2\mu+\nu)^2}\int_\Omega \rho^\gamma G^2 dx + \frac{R^3(\gamma-1)}{(2\mu+\nu)^2}\int_\Omega \rho^\gamma\rho^{2\gamma}dx - \frac{2R^2(\gamma-1)}{2\mu+\nu}\int_\Omega \rho^{2\gamma}\operatorname{div} u dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{R}{2\mu+\nu}\int_\Omega \rho^\gamma u\cdot\nabla G dx - \frac{R^2}{2(2\mu+\nu)}\int_\Omega \rho^{2\gamma}\operatorname{div} u dx\\
&\quad - \frac{R(\gamma-1)}{(2\mu+\nu)^2}\int_\Omega \rho^\gamma G^2 dx + \frac{R^3(\gamma-1)}{(2\mu+\nu)^2}\int_\Omega \rho^{3\gamma} dx - \frac{2R^2(\gamma-1)}{2\mu+\nu}\int_\Omega \rho^{2\gamma}\operatorname{div} u dx\\
&= \frac{R}{2\mu+\nu}\int_\Omega \rho^\gamma u\cdot\nabla G dx - \frac{R^2(4\gamma-3)}{2(2\mu+\nu)}\int_\Omega \rho^{2\gamma}\operatorname{div} u dx - \frac{R(\gamma-1)}{(2\mu+\nu)^2}\int_\Omega \rho^\gamma G^2 dx + \frac{R^3(\gamma-1)}{(2\mu+\nu)^2}\int_\Omega \rho^{3\gamma} dx.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\mathcal{I} = &- \frac{d}{dt}\int_\Omega R\rho^\gamma \operatorname{div} u dx + \frac{1}{2\mu+\nu}\int_\Omega R\rho^\gamma u\cdot\nabla G dx - \frac{4\gamma-3}{2(2\mu+\nu)}\int_\Omega R^2\rho^{2\gamma}\operatorname{div} u dx\\
&- \frac{1}{(2\mu+\nu)^2}\int_\Omega R(\gamma-1)\rho^\gamma G^2 dx + \frac{1}{(2\mu+\nu)^2}\int_\Omega R^3(\gamma-1)\rho^{3\gamma} dx.
\end{aligned}
$$

Integrate $\mathcal{I}$ over $[0,T]$, then one could obtain

$$
\begin{aligned}
\int_0^T \mathcal{I}(t)dt = &- \int_\Omega R\rho^\gamma \operatorname{div} u(T,x) dx + \int_\Omega R\rho_0^\gamma \operatorname{div} u_0 dx\\
&+ \frac{1}{2\mu+\nu}\int_0^T\int_\Omega (R\rho^\gamma u\cdot\nabla G) dxdt - \frac{4\gamma-3}{2(2\mu+\nu)}\int_0^T\int_\Omega (R^2\rho^{2\gamma}\operatorname{div} u) dxdt\\
&- \frac{1}{(2\mu+\nu)^2}\int_0^T\int_\Omega (R(\gamma-1)\rho^\gamma G^2) dxdt + \frac{1}{(2\mu+\nu)^2}\int_0^T\int_\Omega (R^3(\gamma-1)\rho^{3\gamma}) dxdt. \qquad (4.5)
\end{aligned}
$$

Observing that $\Delta\Phi = \rho - \bar{\rho}$, estimate the term $\mathcal{J}$ , then one obtains

$$
\begin{aligned}
\mathcal{J} &\lesssim \|\nabla\Phi\|_{L^\infty(\Omega)}\|\sqrt{\rho}u_t\|_{L^2(\Omega)}\|\sqrt{\rho}\|_{L^2(\Omega)}\\
&\lesssim \|\rho-\bar{\rho}\|_{L^4(\Omega)}\|\sqrt{\rho}u_t\|_{L^2(\Omega)}\|\sqrt{\rho}\|_{L^2(\Omega)}\\
&\le \frac{1}{4\varepsilon}\|\rho-\bar{\rho}\|_{L^4(\Omega)}^4 + \frac{\sqrt{\varepsilon}}{2}\|\sqrt{\rho}u_t\|_{L^2(\Omega)}^2 + \frac{1}{4\varepsilon}\|\sqrt{\rho}\|_{L^2(\Omega)}^4.
\end{aligned}
$$

Hence, we obtain

$$
\int_0^T \mathcal{J}(t)dt \lesssim \frac{1}{4\varepsilon}\int_0^T\left\{\|\rho-\bar{\rho}\|_{L^4(\Omega)}^4 + \|\rho\|_{L^1(\Omega)}^2\right\}dt + \frac{\sqrt{\varepsilon}}{2}\int_0^T\int_\Omega \rho u_t^2 dxdt. \qquad (4.6)
$$

Combining (4.5) and (4.6), (4.3) can be written as

$$
\begin{aligned}
&\int_0^T\int_\Omega \rho|u_t|^2 dxdt + \sup_{t\in[0,T]}\int_\Omega |\nabla u|^2 dx + \sup_{t\in[0,T]}\int_\Omega |\operatorname{div} u|^2 dx\\
&\qquad + \int_0^T\int_\Omega (R^3(\gamma-1)\rho^{3\gamma}) dxdt\\
&\le \int_\Omega |\nabla u_0|^2 dx + \int_\Omega |\operatorname{div} u_0|^2 dx + \int_\Omega |R\rho_0^\gamma \operatorname{div} u_0| dx\\
&\quad + \sup_{t\in[0,T]}\int_\Omega R|\rho^\gamma \operatorname{div} u| dx + \int_0^T\int_\Omega |R\rho^\gamma u\cdot\nabla G| dxdt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T\int_\Omega |R(\gamma-1)\rho^\gamma G^2|dxdt+\int_0^T\int_\Omega |\rho u^2(\nabla u)^2|dxdt\\
&+\int_0^T\int_\Omega |R^2\rho^{2\gamma}\operatorname{div}u|dxdt+\int_0^T\Big\{\|\rho-\overline{\rho}\|_{L^4(\Omega)}^4+\|\rho\|_{L^1(\Omega)}^2\Big\}dt\\
\triangleq&\sum_{k=1}^{9} i_k.
\end{aligned}
\tag{4.7}
$$

In what follows, estimate $i_1-i_9$. First, based on Cauchy's inequality, we have

$$
i_4\lesssim \sup_{t\in[0,T]}\int_\Omega |R\rho^\gamma \operatorname{div}u|\,dx\le c(\varepsilon)R^2\sup_{t\in[0,T]}\|\rho\|_{L^{2\gamma}}^{2\gamma}+\varepsilon\sup_{t\in[0,T]}\int_\Omega |\operatorname{div}u|^2\,dx.
$$

Similarly, we obtain

$$
\begin{aligned}
i_5&\lesssim \int_0^T\int_\Omega |R\rho^\gamma u\cdot\nabla G|\,dxdt\\
&\le \int_0^T\Big(\|\nabla G\|_{L^r(\Omega)}\cdot\|R\rho^\gamma u\|_{L^{\frac{r}{r-1}}(\Omega)}\Big)dt\\
&\le \varepsilon\int_0^T\|\nabla G\|_{L^r(\Omega)}^2dt+\frac{1}{4\varepsilon}R^2\sup_{t\in[0,T]}\Big(\big(\int_\Omega \rho|u|^{\frac{2r}{r-1}}dx\big)^{\frac{r-1}{r}}\cdot\|\rho\|^{\frac{(2\gamma-1)r+1}{r}}_{L^{\frac{(2\gamma-1)r+1}{r-1}}}\Big)T.
\end{aligned}
$$

Observe that $\|\rho\|_{L^{2\gamma}(\Omega)}^{2\gamma}\lesssim\|\rho\|^{2\gamma}_{L^{\frac{6\gamma r}{6-5r}}(\Omega)}$, and make use of the same method as $i_4$, then one obtains that

$$
\begin{aligned}
i_6&\le \int_0^T\int_\Omega \Big|(R(\gamma-1)\rho^\gamma G^2)\Big|\,dxdt\\
&\le \int_0^T\int_\Omega (\varepsilon G^2+\frac{1}{4\varepsilon}R^2(\gamma-1)^2\rho^{2\gamma}G^2)\,dxdt\\
&\le \varepsilon\int_0^T\|\nabla G\|_{L^2(\Omega)}^2\,dt+\frac{1}{4\varepsilon}R^2(\gamma-1)^2\sup_{t\in[0,T]}\|\rho\|^{2\gamma}_{L^{\frac{6\gamma r}{6-5r}}}\int_0^T\|G\|_{L^2(\Omega)}^2\,dt.
\end{aligned}
$$

Note that $\|\operatorname{div}u\|_{L^q}\le\frac{1}{2\mu+\nu}(\|G\|_{L^q}+R\|\rho\|_{L^{\gamma q}}^\gamma)$ on account of $G=(2\mu+\nu)\operatorname{div}u-p(\rho)$. Make use of (3.11), Cauchy inequality and the interpolation inequality, then one yields

$$
\begin{aligned}
i_7&\le \int_0^T\|\nabla u\|^2_{L^{\frac{2r}{3-r}}(\Omega)}\|\sqrt{\rho}u\|^2_{L^{\frac{2r}{2r-3}}(\Omega)}dt\\
&\le \int_0^T\|\nabla u\|^{\frac{3}{2}}_{L^{\frac{3r}{3-r}}(\Omega)}\|\nabla u\|^{\frac{1}{2}}_{L^{\frac{r}{3-r}}(\Omega)}\|\sqrt{\rho}u\|^2_{L^{\frac{2r}{2r-3}}(\Omega)}dt\\
&\le \frac{3\varepsilon}{4}\int_0^T\|\nabla u\|^2_{L^{\frac{3r}{3-r}}(\Omega)}dt+\frac{1}{4\varepsilon^3}\int_0^T\|\sqrt{\rho}u\|^8_{L^{\frac{2r}{2r-3}}(\Omega)}\|\nabla u\|^2_{L^{\frac{r}{3-r}}(\Omega)}dt\\
&\le \frac{3\varepsilon}{4}\int_0^T\Big(\|\nabla^2\mathcal{P}u\|_{L^r(\Omega)}^2+\frac{1}{2\mu+\nu}(\|\nabla G\|_{L^r(\Omega)}^2+R\|\rho\|^{2\gamma}_{L^{\frac{3r\gamma}{3-r}}(\Omega)})\Big)dt\\
&\quad+\frac{1}{8\varepsilon^3}\int_0^T\Big(\|\sqrt{\rho}u\|^{16}_{L^{\frac{2r}{2r-3}}(\Omega)}+\|\nabla u\|^4_{L^{\frac{r}{3-r}}(\Omega)}\Big)dt
\end{aligned}
$$

$$\leq \frac{3\varepsilon}{4}\int_0^T \Big(\|\nabla^2\mathcal{P}u\|^2_{L^2(\Omega)} + \frac{1}{2\mu+\nu}(\|\nabla G\|^2_{L^2(\Omega)} + R\|\rho\|^{2\gamma}_{L^{\frac{3r\gamma}{3-r}}(\Omega)})\Big)dt$$
$$+ \frac{1}{8\varepsilon^3}\int_0^T \Big(\big(\int_\Omega \rho u^{\frac{4r}{2r-3}}dx\big)^{\frac{4(2r-3)}{r}}\|\rho\|^{\frac{12}{r}}_{L^{\frac{3}{2r-3}}(\Omega)} + \|\nabla u\|^4_{L^2(\Omega)}\Big)dt,$$

where $\frac{3}{2} < r < 2$. Similarly, we have

$$\begin{aligned} i_8 &\leq \frac{4\gamma-3}{\mu}\int_0^T\int_\Omega \Big|(R^2\rho^{2\gamma}\operatorname{div} u)\Big|dxdt \\ &\leq \frac{4\gamma-3}{\mu}R^2\Big(\int_0^T\int_\Omega |\operatorname{div} u|^2dxdt\Big)^{\frac{1}{2}}\Big(\int_0^T\int_\Omega \rho^{4\gamma}dxdt\Big)^{\frac{1}{2}} \\ &\leq \frac{4\gamma-3}{\mu}R^2\Big(\varepsilon\int_0^T\int_\Omega |\nabla u|^2dxdt + \frac{1}{\varepsilon}\int_0^T\int_\Omega \rho^{4\gamma}dxdt\Big). \end{aligned}$$

Combining $i_1 - i_9$ with (4.7), we arrive at the conclusion of the lemma easily. □

Combining Lemma 4.1 with Lemma 4.2, let $\varepsilon$ be small sufficiently, then the following lemma is obtained naturally.

**Lemma 4.3.** *Suppose that $\rho_0$ and $u_0$ satisfy the assumptions of Lemma 4.2. Assume $\rho \in L^\infty(0,T;L^{p_4}(\Omega))$, then we obtain that*

$$\iint_{\Omega_T} \rho u_t^2 dxdt + \sup_{0<\leq t\leq T}\int_\Omega |\nabla u|^2dx \leq C,$$
$$\int_0^T \|\nabla^2\mathcal{P}u\|^2_{L^r(\Omega)} + \|\nabla G\|^2_{L^r(\Omega)}dt \leq C,$$

*where $C = C(\mu, R, T, \|u_0\|_{H^1(\Omega)}, \|\rho_0\|_{L^{2\gamma}(\Omega)}, \sup_{t\in[0,T]}\|\rho\|_{L^{p_4}(\Omega)})$, and $p_4(r) = \max\{p_0(r), p_1(r), p_2(r), p_3(r)\}$.*

Moreover, combining the results of the Lemma 4.3 with (3.20), we could obtain $\sup_{0<t<T}\mathcal{A}(t) < \infty$ easily, which is obvious that we prove that Lemma 3.1 completely.

## 5. The estimate of $\|u\|_{L^\infty}$

In the following, we shall present the $L^\infty$ bound of $u$ by the Moser iteration [4, 10] provided that the result of the Lemma 3.1 holds on. Furthermore, we could get the local $L^\infty$ estimate of $u$ by Gronwall inequality.

**Lemma 5.1.** *Let all conditions satisfy the assumptions of Proposition 2.3 and Lemma 3.1, then we have*

$$\sup_{(t,x)\in\Omega_T} |u(x,t)| \leq C,$$

*where the constant $C = C(T, \sup_{\Omega_T}\rho(x,t), \inf_{\Omega_T}\rho(x,t), \|u_0\|_{L^\infty(\Omega)}) > 0$.*

*Proof.* Integrating the equality (2.1) on $(0,T)$, one could arrive at

$$
\begin{aligned}
&\frac{1}{\alpha+2}\sup_{0\le t\le T}\int_\Omega \rho|u|^{\alpha+2}\,dx+\mu(\alpha+1)\int_0^T\int_\Omega |\nabla u|^2|u|^\alpha\,dxdt\\
&\qquad+\alpha(\mu+\nu)\int_0^T\int_\Omega |u|^\alpha \operatorname{div} u\cdot\nabla u\,dxdt+(\mu+\nu)\int_0^T\int_\Omega |\operatorname{div} u|^2|u|^\alpha\,dxdt\\
&\le \frac{1}{\alpha+2}\int_\Omega \rho_0|u_0|^{\alpha+2}\,dx+\int_0^T\int_\Omega R\rho^\gamma|u|^\alpha \operatorname{div} u\,dxdt\\
&\qquad+\alpha\int_0^T\int_\Omega R\rho^\gamma|u|^\alpha\nabla u\,dxdt+\int_0^T\int_\Omega \rho\nabla\Phi|u|^\alpha u\,dxdt\\
&\triangleq \frac{1}{\alpha+2}\int_\Omega \rho_0|u_0|^{\alpha+2}\,dx+\int_0^T (j_1+j_2+j_3)(t)\,dt.
\end{aligned}\tag{5.1}
$$

Now, we consider the estimates of the right terms of (5.1).

$$
\begin{aligned}
\int_0^T j_1(t)dt &\le c_1\int_0^T\int_\Omega |u|^\alpha|\operatorname{div} u|\,dxdt\\
&\le c_1\int_0^T \left\||u|^{\frac{\alpha}{2}}\operatorname{div} u\right\|_{L^2(\Omega)}\left\||u|^{\frac{\alpha}{2}}\right\|_{L^2(\Omega)}dt\\
&\le c_1\int_0^T \left\||u|^{\frac{\alpha}{2}}\operatorname{div} u\right\|_{L^2(\Omega)}\left\|u\right\|^{\frac{\alpha}{2}}_{L^{\alpha+2}(\Omega)}dt\\
&\le \frac12\iint_{\Omega_T}|u|^\alpha|\nabla u|^2\,dxdt+\frac{c_1^2}{2}\int_0^T\left\|u\right\|^{\alpha}_{L^{\alpha+2}(\Omega)}dt\\
&\le \frac12\iint_{\Omega_T}|u|^\alpha|\nabla u|^2\,dxdt+\frac{c_1^2}{2}\Big(\int_0^T \|u\|^{\alpha+2}_{L^{\alpha+2}(\Omega)}\,dt\Big)^{\frac{\alpha}{\alpha+2}}T^{\frac{2}{\alpha+2}}\\
&= \frac12\iint_{\Omega_T}|u|^\alpha|\nabla u|^2\,dxdt+\frac{c_1^2}{2}\Big(\iint_{\Omega_T}|u|^{\alpha+2}\,dxdt\Big)^{\frac{\alpha}{\alpha+2}}T^{\frac{2}{\alpha+2}},
\end{aligned}
$$

where $c_1 = p(\hat\rho)$ and $\hat\rho=\sup\limits_{\Omega_T}\rho(t,x)$. Similarly, we have

$$
\int_0^T j_2(t)dt \le \frac12\iint_{\Omega_T}|u|^\alpha|\nabla u|^2dxdt+\frac{c_1^2}{2}\Big(\iint_{\Omega_T}|u|^{\alpha+2}dxdt\Big)^{\frac{\alpha}{\alpha+2}}T^{\frac{2}{\alpha+2}}.
$$

Note that $\Delta\Phi=\rho-\overline{\rho}$ and (2.4), then we have

$$
\begin{aligned}
\int_0^T j_3(t)dt &\le \hat\rho\iint_{\Omega_T}|\Phi\operatorname{div}(|u|^\alpha u)|\,dxdt\\
&\lesssim (\hat\rho^2+\hat\rho)\int_0^T\int_\Omega (|u|^\alpha|\operatorname{div} u|+|u|^\alpha|\nabla u|)\,dxdt\\
&\lesssim \frac12\iint_{\Omega_T}|u|^\alpha|\nabla u|^2\,dxdt+\frac{\hat\rho^4+\hat\rho^2}{2}\Big(\iint_{\Omega_T}|u|^{\alpha+2}\,dxdt\Big)^{\frac{\alpha}{\alpha+2}}T^{\frac{2}{\alpha+2}}.
\end{aligned}
$$

Hence, one could get

$$\begin{aligned}&\frac{\tilde{\rho}}{\alpha+2}\sup_{t\in[0,T]}\int_\Omega |u|^{\alpha+2}\,dx + c(\mu,\nu,\alpha)\iint_{\Omega_T}(|\nabla u|^2|u|^\alpha)\,dxdt\\ &\quad\le \frac{M}{\alpha+2}\|u_0\|^{\alpha+2}_{L^\infty(\Omega)} + (p^2(\hat{\rho})+\hat{\rho}^2+\hat{\rho}^4)(\iint_{\Omega_T}|u|^{\alpha+2}\,dxdt)^{\frac{\alpha}{\alpha+2}}T^{\frac{2}{\alpha+2}},\end{aligned}\tag{5.2}$$

where $\tilde{\rho} = \inf\limits_{\Omega_T}\rho(x,t)$. Observe

$$(|a|+|b|)^p \le \begin{cases}|a|^p+|b|^p, & 0\le p\le 1,\\ 2^{p-1}(|a|^p+|b|^p), & p>1.\end{cases}$$

Hence, we could obtain that

$$\begin{aligned}\iint_{\Omega_T}|u(t,x)|^{\frac{5}{3}(\alpha+2)}dxdt &\le \int_0^T \|u^{\frac{2(\alpha+2)}{3}}\|_{L^{\frac{3}{2}}(\Omega)}\|u^{\alpha+2}\|_{L^3(\Omega)}dt\\ &= \int_0^T(\int_\Omega |u|^{\alpha+2}dx)^{\frac{2}{3}}\cdot\|u^{\frac{\alpha+2}{2}}\|^2_{L^6(\Omega)}dt\\ &\lesssim (\sup_{t\in[0,T]}\int_\Omega|u|^{\alpha+2}dx)^{\frac{2}{3}}\cdot\int_0^T\|u^{\frac{\alpha+2}{2}}\|^2_{H^1(\Omega)}dt\\ &\lesssim (\sup_{t\in[0,T]}\int_\Omega|u|^{\alpha+2}dx)^{\frac{2}{3}}\cdot\Big(\iint_{\Omega_T}(\nabla|u|^{\frac{\alpha+2}{2}})^2dxdt+\iint_{\Omega_T}|u|^{\alpha+2}dxdt\Big)\\ &\lesssim (\sup_{t\in[0,T]}\int_\Omega|u|^{\alpha+2}dx)^{\frac{2}{3}}\cdot\Big(\frac{(\alpha+2)^2}{4}\iint_{\Omega_T}|u|^\alpha|\nabla u|^2dxdt+\iint_{\Omega_T}|u|^{\alpha+2}dxdt\Big).\end{aligned}\tag{5.3}$$

Combining (5.2) and (5.3), we could get

$$\iint_{\Omega_T}|u(t,x)|^{\frac{5}{3}(\alpha+2)}dxdt \le C(\alpha+2)^{\frac{8}{3}}\Big(\iint_{\Omega_T}|u|^{\alpha+2}dxdt\Big)^{\frac{5}{3}} + C(\alpha+2)^{\frac{8}{3}},\tag{5.4}$$

where $C = C(T, M, \sup\limits_{\Omega_T}\rho, \inf\limits_{\Omega_T}\rho, \|u_0\|_{L^\infty(\Omega)})$.

Let $r=\frac{5}{3},\ r^k = 2+\alpha_k$, where $k\ge 2$, then the inequality (5.4) is rearranged to be

$$\iint_{\Omega_T}|u|^{r^{k+1}}dxdt \le C_1r^{kC_2}(\iint_{\Omega_T}|u|^{r^k}dxdt)^r + C_1r^{kC_2},\tag{5.5}$$

where $C_1 = C_1(T, M, \sup\limits_{\Omega_T}\rho, \inf\limits_{\Omega_T}\rho, \|u_0\|_{L^\infty(\Omega)}) > 1$, and $C_2 = C_2(T, M, \sup\limits_{\Omega_T}\rho, \inf\limits_{\Omega_T}\rho, \|u_0\|_{L^\infty(\Omega)}) > 1$.

Make use of the inequality (5.5), then we obtain

$$\begin{aligned}&\iint_{\Omega_T}|u(t,x)|^{\frac{5}{3}(\alpha_k+2)}dxdt = \iint_{\Omega_T}|u|^{r^{k+1}}dxdt\\ &\le C_1r^{kC_2}(\iint_{\Omega_T}|u|^{r^k}dxdt)^r + C_1r^{kC_2}\\ &\le 2^{r-1}C_1r^{kC_2}[C_1r^{r(k-1)C_2}(\iint_{\Omega_T}|u|^{r^{k-1}}dxdt)^{r^2} + C_1r^{r(k-1)C_2}] + C_1r^{kC_2}\end{aligned}$$

$$
\begin{aligned}
&= 2^{\frac{2}{3}} C_1^{1+r} r^{C_2(k+r(k-1))} \Big(\iint_{\Omega_T} |u|^{r^{k-1}} dxdt\Big)^{r^2} + 2^{\frac{2}{3}} C_1^{1+r} r^{C_2(k+r(k-1))} + C_1 r^{kC_2} \\
&\le 2^{\frac{2}{3}} C_1^{1+r} r^{C_2(k+r(k-1))} \Big[C_1 r^{(k-2)C_2} \Big(\iint_{\Omega_T} |u|^{r^{k-2}} dxdt\Big)^{r} \\
&\qquad + C_1 r^{(k-2)C_2}\Big]^{r^3} + 2^{\frac{2}{3}} C_1^{1+r} r^{C_2(k+r(k-1))} + C_1 r^{kC_2} \\
&\le \cdots \\
&\le 2^{(r-1)+(r^2-1)+\cdots+(r^{k-1}-1)} C_1^{1+r+\cdots+r^{k-2}} r^{C_2(k+r(k-1)+\cdots+r^{k-2}\cdot 2)} \Big(\iint_{\Omega_T} |u|^{r^2} dxdt\Big)^{r^{k-1}} \\
&\qquad + 2^{(r-1)+(r^2-1)+\cdots+(r^{k-1}-1)} C_1^{1+r+r^2+\cdots+r^{k-2}} r^{C_2(k+r(k-1)+\cdots+r^{k-2}\cdot 2)} \\
&\qquad + 2^{(r-1)+(r^2-1)+\cdots+(r^{k-2}-1)} C_1^{1+r+r^2+\cdots+r^{k-3}} r^{C_2(k+r(k-1)+\cdots+r^{k-3}\cdot 3)} \\
&\qquad + \cdots + C_1 r^{C_2 k}.
\end{aligned}
$$

Next, we will consider the following inequality

$$
\begin{aligned}
\iint_{\Omega_T} |u|^{r^{k+1}} \le\ & 2^{(r-1)+(r^2-1)+\cdots+(r^{k-1}-1)} C_1^{\sum_{l=0}^{k-2} r^l} \cdot r^{C_2 \sum_{l=0}^{k-2}(k-l) r^l} \Big(\iint_{\Omega_T} |u|^{r^2} dxdt\Big)^{r^{k-1}} \\
& + (k-1) 2^{(r-1)+(r^2-1)+\cdots+(r^{k-1}-1)} C_1^{\sum_{l=0}^{k-2} r^l} \cdot r^{C_2 \sum_{l=0}^{k-2}(k-l) r^l}.
\end{aligned}
$$

Denote $a = 2^{(r-1)+(r^2-1)+\cdots+(r^{k-1}-1)}$, then we have the inequality

$$
\begin{aligned}
\|u\|_{L^{r^{k+1}}(\Omega_T)} \le\ & a^{\frac{1}{r^{k+1}}} \cdot C_1^{\sum_{l=0}^{k-2} r^{l-k-1}} \cdot r^{C_2 \sum_{l=0}^{k-2}(k-l) r^{l-k-1}} \Big(\int_{\Omega_T} |u|^{r^2} dxdt\Big)^{r^{-2}} \\
& + (k-1)^{r^{-(k+1)}} a^{\frac{1}{r^{k+1}}} \cdot C_1^{\sum_{l=0}^{k-2} r^{l-k-1}} \cdot r^{C_2 \sum_{l=0}^{k-2}(k-l) r^{l-k-1}}.
\end{aligned} \tag{5.6}
$$

Observe that $b = \sum_{l=1}^{k-1} r^{l-k-2} < +\infty$, $c = \sum_{l=1}^{k-1} (k-l) r^{l-k-2} < +\infty$, $d = (k-1)^{r^{-(k+1)}} < +\infty$. Moreover, we have $2^{\left((r-1)+\cdots+(r^{k-2}-1)\right)\cdot r^{-(k+1)}} < \infty$.

Obviously when $k$ goes to $\infty$, we could obtain that

$$
\|u(x,t)\|_{L^\infty(\Omega_T)} \le 2C_1^b r^{cC_2} \|u\|_{L^{r^2}(\Omega_T)} + 2dC_1^b r^{cC_2}.
$$

Hence, we obtain the inequality

$$
\sup_{\Omega_T} |u| \le 2C_1^b r^{cC_2} \|u\|_{L^{r^2}(\Omega_T)} + 2dC_1^b r^{cC_2}.
$$

Combining the above result and Proposition 2.3, we could obtain that

$$
\sup_{\Omega_T} |u(x,t)| \le C,
$$

where $C = C(T, \sup_{\Omega_T} \rho, \inf_{\Omega_T} \rho, \|u_0\|_{L^\infty(\Omega)})$. □

Finally, let us give the proof of Theorem 1.1.

*Proof.* Under the assumptions of Lemmas 4.1–4.3, we employ the result in Lemma 3.1, Proposition 2.3 and the a prior estimates in Lemma 5.1. Therefore, we could achieve the results of Theorem 1.1 easily. □

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that they have no conflict of interest.

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